Chordal structure and polynomial systems

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A polynomial system defined by $m$ equations in $n$ variables:

$$f_i(x_0, \ldots, x_{n-1}) = 0, \quad i = 1, \ldots, m$$

Construct a graph $G$ ("primal graph") with $n$ nodes, as:

- Nodes are variables $\{x_0, \ldots, x_{n-1}\}$.
- For each equation, add a clique connecting the variables appearing in that equation.
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Example:

\[
I = \langle x_0^2 x_1 x_2 + 2x_1 + 1, \quad x_1^2 + x_2, \quad x_1 + x_2, \quad x_2 x_3 \rangle
\]
“Abstracted” the polynomial system to a (hyper)graph.
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- Can the graph structure help solve this system?
- For instance, to optimize, or to compute Groebner bases?
- Or, perhaps we can do something better?
- Preserve graph (sparsity) structure?
- Complexity aspects?
(Hyper)Graphical modelling

Pervasive idea in many areas, in particular: numerical linear algebra, graphical models, constraint satisfaction, database theory, . . .

Key notions: chordality and treewidth.

Many names: Arnborg, Beeri/Fagin/Maier/Yannakakis, Blair/Peyton, Bodlaender, Courcelle, Dechter, Lauritzen/Spiegelhalter, Pearl, Robertson/Seymour, . . .

Reasonably well-known in discrete (0/1) optimization, what happens in the continuous side? (e.g., Waki et al., Lasserre, Bienstock, Jordan/Wainwright, Lavaei, etc)
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Chordality, treewidth, and a meta-theorem

Let $G$ be a graph with vertices $x_0, \ldots, x_{n-1}$.
A vertex ordering $x_0 > x_1 > \cdots > x_{n-1}$ is a \textit{perfect elimination ordering} if for each $x_i$, the set

$$X_i := \{x_i\} \cup \{x_m : x_m \text{ is adjacent to } x_i, \ x_i > x_m\}$$

is such that the restriction $G|_{X_i}$ is a clique.
A graph is \textit{chordal} if it has a perfect elimination ordering.
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is such that the restriction $G|_{X_l}$ is a clique. A graph is chordal if it has a perfect elimination ordering. A chordal completion of $G$ is a chordal graph with the same vertex set as $G$, and which contains all edges of $G$. The treewidth of a graph is the clique number (minus one) of its smallest chordal completion.

Meta-theorem: NP-complete problems are "easy" on graphs of small treewidth.
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**Meta-theorem:**

NP-complete problems are "easy" on graphs of small treewidth.
Bad news? (I)

Subset sum problem, with data $A = \{a_1, \ldots, a_n\} \subset \mathbb{Z}$.
Is there a subset of $A$ that adds up to $S$?

Letting $s_i$ be the partial sums, we can write a polynomial system:

\[
0 = s_0
\]
\[
0 = (s_i - s_{i-1})(s_i - s_{i-1} - a_i)
\]
\[
S = s_n
\]

The graph associated with these equations is a path (treewidth=1)

\[\text{\begin{tikzpicture}
\node (s0) at (0,0) {$s_0$};
\node (s1) at (1,0) {$s_1$};
\node (s2) at (2,0) {$s_2$};
\node (sn) at (4,0) {$s_n$};
\draw (s0) -- (s1); 
\draw (s1) -- (s2); 
\draw (s2) -- (sn); 
\end{tikzpicture} }\]

But, subset sum is NP-complete… :(}
Bad news? (II)

For *linear* equations, “good” elimination preserves graph structure (perfect!)

For polynomials, however, Groebner bases can destroy chordality. Ex: Consider $I = \langle x_0 x_2 - 1, x_1 x_2 - 1 \rangle$, whose associated graph is the path $x_0 - x_2 - x_1$. Every Groebner basis must contain the polynomial $x_0 - x_1$, breaking the sparsity structure.
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Two papers

- Chordal elimination and Groebner bases (arXiv:1411:1745)
  - New *chordal elimination* algorithm, to exploit graphical structure.
  - Conditions under which chordal elimination succeeds.
  - For a certain class, complexity is *linear* in number of variables!
    (exponential in treewidth)
  - Implementation and experimental results

- Computing permanents, hyperdeterminants, and mixed discriminants
  (arXiv:1507:03046)
  - New polynomial time algorithm $O(n2^\omega)$ ($\omega$ is treewidth).
  - Hardness: mixed volume still hard, even with small treewidth.
Chordal elimination (sketch)

Given equations, construct graph $G$, a chordal completion, and a perfect elimination ordering.

Will produce a decreasing sequence of ideals $I = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_{n-1}$.

Given current ideal $I_l$, split the generators

$$I_l = \underbrace{J_l}_{\in \mathbb{K}[X_l]} + \underbrace{K_{l+1}}_{\notin \mathbb{K}[X_l]}$$

and eliminate variable $x_l$

$$I_{l+1} = \text{elim}_{l+1}(J_l) + K_{l+1}$$

“Ideally” (!), $I_l$ should be the $l$-th elimination ideal $\text{elim}_l(I)$...

Notice that by chordality, graph structure is always preserved!
When does chordal elimination succeed?

We need conditions for this to work, i.e., for $\mathbf{V}(I_l) = \mathbf{V}(\text{elim}_l(I))$.

**Thm 1:** Let $I$ be an ideal and assume that for each $l$ such that $X_l$ is a maximal clique of $G$, the ideal $J_l \subseteq \mathbb{K}[X_l]$ is zero dimensional. Then, chordal elimination succeeds.

In particular, finite fields $\mathbb{F}_q$, and 0/1 problems.
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**Def:** A polynomial $f$ is *simplicial* if for each variable $x_l$, the monomial $m_l$ of largest degree in $x_l$ is unique and has the form $m_l = x_{d_l}^l$.

**Thm 2:** Let $I = \langle f_1, \ldots, f_s \rangle$ be an ideal such that for each $1 \leq i \leq s$, $f_i$ is generic simplicial. Then, chordal elimination succeeds.
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We need conditions for this to work, i.e., for $V(I_l) = V(\text{elim}_l(I))$.

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[Intuition: interaction of (iterated) “closure/extension thm” + chordality]

[Intuition: variety has “small” coordinate projections, can compute those, and glue them]
Complexity

For “nice” cases, complexity is linear in number of variables $n$, number of equations $s$, and exponential in treewidth $\kappa$.

**Thm:** Let $I$ be such that each (maximal) $\tilde{H}^i$ is $q$-dominated. The complexity of computing $I_i$ is $\tilde{O}(s + lq^\alpha \kappa)$. We can find all elimination ideals in $\tilde{O}(nq^\alpha \kappa)$.

E.g., we recover known results on linear-time colorability for bounded treewidth:

**Cor:** Let $G$ be a graph and $\bar{G}$ a chordal completion with largest clique of size $\kappa$. We can describe all $q$-colorings of $G$ in $\tilde{O}(nq^\alpha \kappa)$. 
Implementation and examples

Implemented in Sage, using Singular and PolyBoRi (for $\mathbb{F}_2$).

- Graph colorings (counting $q$-colorings)
- Cryptography (“baby” AES, Cid et al.)
- Sensor Network localization
- Discretization of polynomial equations
Results: Crypto - AES variant (Cid et al.) - $\mathbb{F}_2[x]$

Performance on $SR(n, 1, 2, 4)$ for chordal elimination, and computing (lex/degrevlex) Gröbner bases (PolyBoRi).

<table>
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<th>$n$</th>
<th>Variables</th>
<th>Equations</th>
<th>Seed</th>
<th>ChordElim</th>
<th>LexGB</th>
<th>DegrevlexGB</th>
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- For small problems standard Gröbner bases outperform chordal elimination, particularly using degrevlex order.
- Nevertheless, chordal elimination scales better, being faster than both methods for $n = 10$.
- In addition, standard Gröbner bases have higher memory requirements, which is reflected in the many experiments that aborted for this reason.
Results: Sensor network localization - $\mathbb{Q}[x]$

Find positions, given a few known fixed anchors and pairwise distances.

Comparison with Singular: DegrevlexGB, LexFGLM

- Natural graph structure
  \[
  \|x_i - x_j\|^2 = d_{ij}^2 \quad ij \in A \\
  \|x_i - a_k\|^2 = e_{ij}^2 \quad ik \in B
  \]

- Simplicial, therefore exact elimination

- Underconstrained regime: chordal is much better

- Overconstrained regime: competitive (plot)
The *permanent* of a matrix is

\[
\text{perm}(M) := \sum_{\pi} \prod_{i=1}^{n} M_{i,\pi(i)}
\]

where the sum is over all permutations \( \pi \in S_n \).

Very difficult (\#P-hard).

What happens under small treewidth?

What about generalizations (e.g., mixed discriminants, mixed volumes, etc)?
New tree-decomposition (DP) algorithms for permanents, mixed discriminants and hyperdeterminants

Hardness results for mixed volumes and above.
(Hyper)graphical structure *may* simplify optimization/solving

- Under assumptions (treewidth + algebraic structure), tractable!
- Yields practical, competitive, implementable algorithms
- Ongoing and future work: other polynomial solving approaches (e.g., triangular sets, homotopies, full numerical algebraic geometry...)

If you want to know more:

Thanks for your attention!
Summary

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Thanks for your attention!