On the Structure of Graphs of Minimum Degree at least 4

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We outline a proof that for every vertex x of a 4-connected graph G there exists a subgraph H in G isomorphic to a subdivision of the complete graph K_4 on four vertices such that $G - V(H)$ is connected and contains x. This implies an affirmative answer to a question of J.-I. Itoh and W. Kühnel whether every 4-connected graph G contains a subdivision of K_4 such that $G - V(H)$ is connected. A generalized, "induced" version of the statement is the key ingredience for their proof that every 4-connected graph has an embedding in $\mathbb{R}^3$ such that the intersection with every open halfspace is connected.

The motor for our induction is a result of Fontet and Martinov stating that every 4-connected graph can be reduced to a smaller one by contracting a single edge, unless the graph is the square of a cycle or the line graph of a cubic graph. It turns out that this is the only ingredience of the proof where 4-connectedness is used. We then generalize our result to connected graphs of minimum degree at least 4, by developing the respective motor, a structure theorem for the class of simple connected graphs of minimum degree at least 4.

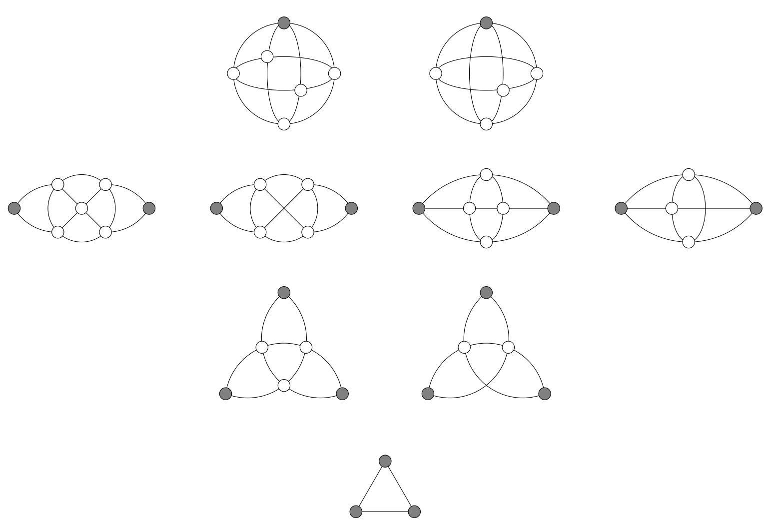

Figure 1: The nine bricks. Vertices of attachment are displayed solid.

A simple connected graph G of minimum degree 4 can not be reduced to a smaller such graph by deleting a single edge or contracting a single edge and simplifying if and only if it is the square of a cycle or the edge disjoint union of copies of certain *bricks* as follows: The bricks are K_3, K_5, $K_{2,2,2}$, K_5^-, $K_{2,2,2}^-$, and the four graphs $K_5^{\triangledown}$, $K_{2,2,2}^{\triangledown}$, $K_5^{\triangleright\triangleleft}$, $K_{2,2,2}^{\triangleright\triangleleft}$ obtained from K_5 and $K_{2,2,2}$ by deleting the edges of a triangle, or replacing a vertex x by two new vertices and adding four edges to the endpoints of two disjoint edges of its former neighborhood, respectively (see Figure 1). Bricks isomorphic to K_5 or $K_{2,2,2}$ share exactly one vertex with the other bricks of the decomposition; vertices of degree 4 in any other brick are not contained in any further brick of the decomposition; the vertices of a brick isomorphic to K_3 must have degree 4 in G and its neighbors are pairwise distinct; and at least one vertex of degree 2 in each brick isomorphic to $K_5^{\triangledown}$ must have degree 4 in G.