MDS Summer School on Discrete Optimization and Network Flows

Problems on Generalized Flow

June 24-26, 2011

1. Prove that the following three statements are equivalent for a proper flow g:

   - g is a maximum proper flow.
   - The residual graph G_g has no generalized augmenting path.
   - There exists a labeling μ such that $\gamma_{ij}^{\mu} \leq 1$ for all arcs (i, j) in the residual graph.

2. In the minimum-cost generalized circulation problem, we are given costs c_{ij} in addition to gains γ_{ij} and capacities u_{ij}. The goal is to find a *generalized circulation* g that minimizes $\sum_{(i,j)\in A} c_{ij} g_{ij}$. A generalized circulation is a generalized pseudoflow that has $e_i^g = 0$ for all $i \in V$.

   For the generalized flow problem we considered in class, we showed that the flow is maximum if and only if there are no generalized augmenting paths. In the case of the minimum-cost generalized circulation problems, the objects of interest are *unit gain cycles* and *bicycles*. A unit gain cycle C has $\gamma(C) = 1$. A bicycle has a flow-generating cycle C_1 connected by a path (possibly trivial) to a flow-absorbing cycle C_2. We say that the unit gain cycle has *negative cost* if there is a generalized circulation f on the arcs of C such that $\sum_{(i,j)\in C} c_{ij} f_{ij}$; a negative-cost bicycle is similar.

   (a) Prove that a minimum-cost generalized circulation g is optimal if and only if there are no negative-cost unit gain cycles and no negative-cost bicycles in the residual graph G_g.

   (b) Suppose that as part of the input we are given b_i, and we must find a minimum-cost generalized pseudoflow such that $e_i^g = b_i$ for all $i \in V$. Determine optimality conditions as above for this problem (that is, prove that the flow is optimal if and only if there are no objects X in the residual graph G_g.)

3. Suppose we have a standard minimum-cost circulation problem, which is the same as the generalized minimum-cost circulation problem given above, except $\gamma_{ij} = 1$ for all $(i, j) \in A$. The goal is to find a circulation of minimum cost. There is a well-known algorithm for the standard minimum-cost circulation problem that involves repeatedly finding a minimum mean-cost cycle in the current residual graph (as long as the mean cost is negative), and canceling it. A minimum mean-cost cycle is a cycle C that minimizes $\sum_{(i,j)\in C} c_{ij}/|C|$. By "canceling" a cycle, we mean that we send as much flow as possible along the arcs of the cycle in the residual graph until the residual capacity of one arc is used up (saturated). The analysis of the algorithm shows that it takes $O(mn)$ time to compute the minimum mean-cost cycle, and that after m cancelations, the (negative) mean cost will have increased by at least a factor

of $(1 - \frac{1}{n})$. If $C = \max_{(i,j)\in A} |c_{ij}|$, then initially the mean cost is at least $-C$. Then after $mn \ln(nC)$ iterations, the mean cost will be at least

$$-C\left(1 - \frac{1}{n}\right)^{n \ln(nC)} > -Ce^{-\ln(nC)} = -\frac{1}{n},$$

using $1 - x < e^{-x}$ for $x > 0$. If the costs c_{ij} are all integers, then any mean cost greater than $-1/n$ must be nonnegative, and so the algorithm can terminate. Hence the algorithm takes $O(m^2 n^2 \ln(nC))$ time.

For this problem, you will need to adapt this problem for the generalized flow problem and cancel all flow generating cycles. We do this by considering costs $c_{ij} = -\log \gamma_{ij}$ and canceling minimum mean-cost cycles as long as they are negative (does this cancel all flow generating cycles? Why?). However, the costs c_{ij} are no longer integral, which was an assumption needed to prove the running time above. Assume that the gains γ_{ij} are ratios of integers that are bounded in absolute value by B. Show that in this case, the running time of the cycle canceling algorithm is $O(m^2 n^3 \log(nB))$.

4. In this problem, we will consider a push/relabel style of algorithm for the generalized flow problem. We say that an arc (i, j) is *admissible* if it has relabeled gain $\gamma^{\mu}_{ij} > 1$ and positive residual capacity. We say that a node i is *active* if it can reach the sink and it has positive excess (that is, $e^{h,\mu}_i > 0$). Now we allow a network G to specify some initial excess $e_i \geq 0$, so that for a flow h its excess at i is its initial excess plus the excess resulting from h: $e^{h,\mu}_i = e^{\mu}_i - \sum_{k:(i,k)\in A} h^{\mu}_{ik}$.

Push/relabel generalized flow

Input: A network G with no flow-generating cycles, excesses $e_i \geq 0$, error parameter ϵ
Output: ϵ-optimal proper flow g

Set $b = (1 + \epsilon)^{1/n}$ and round gains γ to powers of b as in Rounded Truemper
Let H be resulting network; cancel any flow-generating cycles in H
$h \leftarrow 0$
Compute canonical labels μ
While there is an active node i
    If there is an admissible arc (i, j)
        Push: Send $\min(e^{h,\mu}_i, u^{h,\mu}_{ij})$ units of flow on (i, j), update h^{μ}
    Else
        Relabel: $\mu_i \leftarrow \mu_i / b^{1/n}$
Return g^{μ}, the interpretation of flow h^{μ} in G

(a) Prove that the algorithm maintains a flow h and labels μ such that $\gamma^{\mu}_{ij} \leq b^{1/n}$ for all residual arcs $(i, j) \in A_h$.

(b) Prove that during the course of the algorithm, the graph of admissible arcs is acyclic.

(c) Prove that during the course of the algorithm H_h has no flow-generating cycles.

(d) Prove that on termination, the algorithm outputs an ϵ-optimal flow.

(e) Prove that there are at most $O(\frac{1}{\epsilon}n^3 \log B)$ relabels per node.

(f) Prove that there are at most $O(\frac{1}{\epsilon}mn^3 \log B)$ saturating pushes.

(g) Prove that there are at most $O(\frac{1}{\epsilon}mn^4 \log B)$ nonsaturating pushes.

Assuming that we can implement both the push and relabel operations in $O(\log n)$ time, we can thus find an ϵ-optimal flow in $O(\frac{1}{\epsilon}mn^4 \log B \log n)$ time.

(h) Using the above algorithm as a subroutine, and assuming we can cancel flow-generating cycles in $O(mn^3 \log(nB))$ time, obtain an $O(m^2n^4 \log^2 B \log n)$ time algorithm for the maximum generalized flow problem.

5. (Hard) Suppose you are given an instance of the generalized flow problem that in addition has costs on the arcs, but no capacities. Consider a Bellman-Ford style algorithm that tries to determine the minimum-cost method for sending a unit of flow from each vertex i to the sink t. Let y_i^k be the cost of sending a unit of flow along a walk of at most k arcs. Thus we set $y_t^0 = 0$, $y_i^0 = \infty$ for $i \neq t$, and update

$$y_i^k = \min(y_i^{k-1}, \min_{(i,j)\in A}(c_{ij} + \gamma_{ij} y_j^{k-1})).$$

Assume that there are no negative-cost bicycles and no negative-cost unit gain cycles in the graph. Prove that in $O(mn)$ time you can detect whether there is a negative-cost generalized augmenting path.