Recent Progress on Equilibria for Dynamic Network Flows

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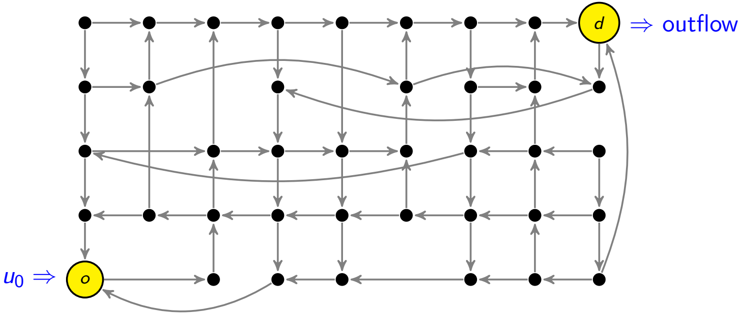
Dynamic routing game

- Nice basic model.
- Optimizing dynamic flows – old results Ford & Fulkerson 58, Gale 59.
- Equilibria for dynamic flows.

  - Old idea Vickrey 69
  - General Existence Zhu, Marcotte 00, Meunier, Wagner 10
  - Equilibrium characterization: Thin flows Koch Skutella 09, 11
  - The extension algorithm Koch Skutella 11
  - Existence of dynamic equilibria Cominetti C. Larre 11, 16
  - Surprising examples Cominetti C. Olver 17
  - Significant recent progress.

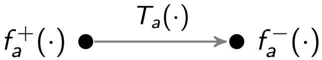
We are given:

- a directed graph $G = (V, A)$
- a source $o \in V$ and a sink $d \in V$
- a constant (time dependent) inflow rate u_0 [pax/sec]
- a flow dynamics for each link $a \in A$



How does the flow propagate from o to d?

Link dynamics



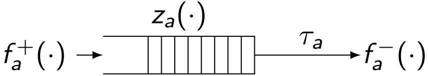
A *link dynamics* is a map that transforms each inflow f_a^+ into an outflow f_a^- and an exit-time function $T_a(t) \geq t$ that depends only on $f_a^+(\cdot)|_{[0,t]}$ and satisfies

$$F_a^+(t) = F_a^-(T_a(t)).$$

$$\left(\int_0^t f_a^+(\tau) = \int_0^t f_a^-(T_a(\tau))\right)$$

FIFO holds: for all $t < s$ we have $T_a(t) < T_a(s)$.

Fluid queue with capacity (ν_a) + Constant delay (τ_a)



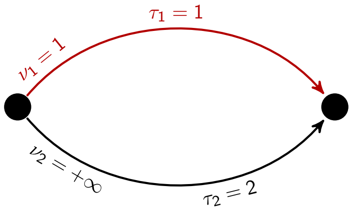
$$z_a(t) = \max_{s \in [0,t]} \int_s^t [f_a^+(\xi) - \nu_a] d\xi$$

$$f_a^-(t + \tau_a) = \begin{cases} \nu_a & \text{if } z_a(t) > 0 \\ \min\{f_a^+(t), \nu_a\} & \text{if } z_a(t) = 0 \end{cases}$$

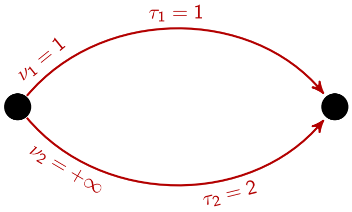
$$T_a(t) = t + \underbrace{\frac{z_a(t)}{\nu_a}}_{\text{queuing}} + \underbrace{\tau_a}_{travel}$$

Simple Example

Consider the follwing situation with $u_0 = 2$.



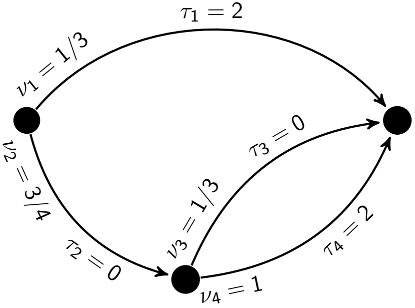
For $t \in [0, 1]$ all flow goes on top. At time 1 the top link has a queue of length 1 so $z_1(1)/\nu_1 = 1$ and the bottom link becomes interesting.



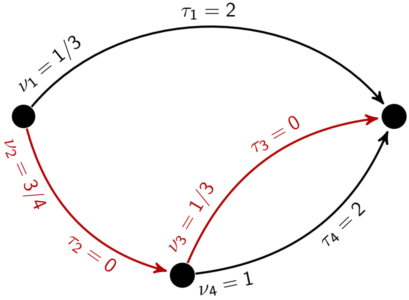
For $t \in [1, \infty]$ the flow splits such that the top queue remains constant.

More involved...

Consider the following instance with unit inflow.



In $[0, 1]$ all flow takes

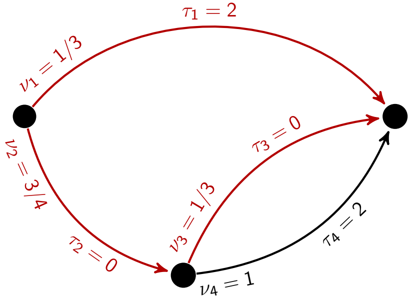


queues end at $\frac{1}{4}, \frac{5}{12}$

last particle experiences $\frac{1}{4}, \frac{5}{9}$

delay $\frac{1/4}{3/4} + 0 + \frac{5/9}{1/3} + 0 = 2$

In $[1, \frac{7}{5}]$ flow splits $\frac{1}{2}, \frac{1}{2}$

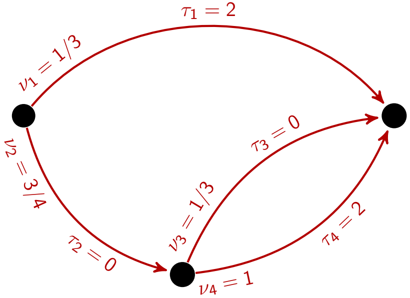


queues end at $\frac{1}{15}, \frac{3}{20}, \frac{7}{12}$

last particle experiences $\frac{1}{15}, \frac{3}{20}, \frac{2}{3}$

delay $\frac{1/15}{1/3} + 2 = 2.2 = \frac{3/20}{3/4} + \frac{2/3}{1/3}$

In $[\frac{7}{5}, 4]$ flow splits $\frac{4}{13}, \frac{4}{13}, \frac{5}{13}$

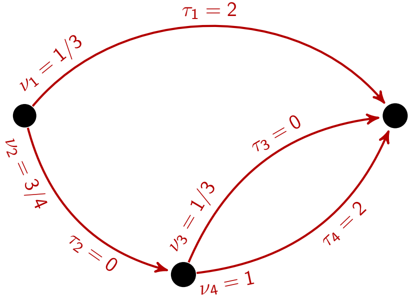


queues end at $0, 0, \frac{2}{3}, 0$

last particle experiences $0, 0, \frac{2}{3}, 0$

Same active; flow pattern changes

In $[4, \infty]$ flow splits $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$



queues stay at $0, 0, \frac{2}{3}, 0$

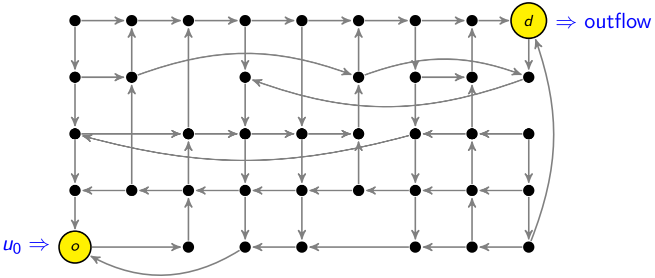
Steady state

Preview of Results

- Equilibrium exists.
  - Very general results based on fixed point theorems.
  - Say very little about the structure.
  - Hopelessly impractical.
  - Uniqueness?
- Constructive approach Koch, Skutella 09, 11 & Cominetti, C. Larre 11, 16
  - Characterize the derivative of an equilibrium (the static flows in the example).
  - These derivatives are the unique solution to some system of equations.
  - Use it to construct a piecewise linear dynamic equilibrium.
  - Uniqueness!
- Recent Progress
  - Long term behavior
  - Computation
  - More general models
  - Price of Anarchy

Dynamic equilibrium: Path view

$\mathcal{H}$ = set of path-flow decompositions $u_0 = \sum_{p \in \mathcal{P}} h^p(t)$ for a.e. $t \geq 0$.

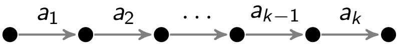


A dynamic equilibrium is a flow-path decomposition $h \in \mathcal{H}$ such that

$$h^p(t) > 0 \Rightarrow T^p(t) \text{ is minimal, a.e. } t \geq 0.$$

How do we compute the exit-time $T^p(t)$ for a path p?

A network loading procedure transforms each $h \in \mathcal{H}$ into unique link inflows $f_a^+(t)$ with corresponding exit-times $T_a(t)$.
The exit-time for a path $p = a_1 a_2 \cdots a_k$ is then given by



$$T^p(t) = T_{a_k} \circ \cdots \circ T_{a_2} \circ T_{a_1}(t).$$

A *dynamic equilibrium* is a path-flow decomposition $h \in \mathcal{H}$ such that

$$h^p(t) > 0 \Rightarrow T^p(t) \text{ minimal for a.e. } t \geq 0.$$

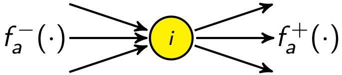
General Existence

- Zhu-Marcotte 00
- Meunier-Wagner 10

Dynamic equilibrium: Link view

A *flow-over-time* is a set of inflows f_a^+ with corresponding outflows f_a^- and link exit-times T_a such that for a.e. $t \geq 0$

$$\sum_{a \in \delta^+(i)} f_a^+(t) - \sum_{a \in \delta^-(i)} f_a^-(t) = \begin{cases} u_0 & \text{for } i = o \\ 0 & \text{for } i \neq o, d \end{cases}$$



Every path-flow decomposition $h \in \mathcal{H}$ induces a flow-over-time.

Dynamic equilibrium: Link view

- The *earliest arrival times* are defined as

$$\ell_j(t) = \min_{p \in \mathcal{P}_o^j} T^p(t) = \text{minimum time to reach } j \text{ starting from } o \text{ at time } t$$

$$= \begin{cases} t & \text{if } j = o \\ \min_{ij \in A} T_{ij}(\ell_i(t)) & \text{if } j \neq o \end{cases}$$

- A'_t = set of active links $ij \in A$ with $\ell_j(t) = T_{ij}(\ell_i(t))$.

A flow-over-time is a dynamic equilibrium if only dynamic shortest paths are used.

$$f_{ij}^+(t) > 0 \Longrightarrow t \in \ell_i(\{\theta : ij \in A'_\theta\})$$

Equilibrium Summary

- Start from inflows $f^+(t)$.
- Compute the queues $z_e(t)$, the out-time functions $T_a(t)$, and the outflows $f^-(t)$.
- Check that $f^+(t)$ and $f^-(t)$ satisfy flow conservation at every vertex at every time.
- Compute the distance labels $\ell_i(t)$ and the active links at time t using the path formulation or bellman.
- Check the equilibrium condition: A link with flow at t was active at the time that particle departed from the origin.

$$f^+ \longrightarrow \text{ link dynamics } \longrightarrow (f^-, T) \longrightarrow \text{ Bellman } \longrightarrow \ell$$

Dynamic equilibrium: Equivalent statement

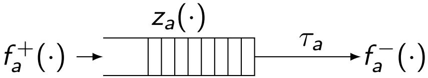
Theorem

A flow-over-time is a dynamic equilibrium *iff for each arc $a = ij \in A$ we have $F_a^+(\ell_i(t)) = F_a^-(\ell_j(t))$.*

$\Rightarrow$ at equilibrium the *cumulative flows* $x_a(t) = F_a^+(\ell_i(t))$ satisfy the static flow conservation equations

$$\sum_{a\in\delta^+(i)} x_a(t) - \sum_{a\in\delta^-(i)} x_a(t) = \begin{cases} U(t) = t \cdot u_0 & \text{for } i = o \\ 0 & \text{for } i \neq o, d \\ -U(t) = -t \cdot u_0 & \text{for } i = d \end{cases}$$

Queues and equilibria



Proposition

A'_t = active arcs
A^*_t = arcs with queue when t-th particle reaches it, i.e., $z_a(\ell_i(t)) > 0$.
At equilibrium

$$\begin{aligned} A'_t &= \{a = ij \in A : \ell_j(t) \geq \ell_i(t) + \tau_a\}, \\ A^*_t &= \{a = ij \in A : \ell_j(t) > \ell_i(t) + \tau_a\}. \end{aligned}$$

Derivatives of a dynamic equilibrium

The cumulative flow $x_a(t) = F_a^+(\ell_i(t))$ satisfies flow conservation

$$\sum_{a\in\delta^+(i)} x_a(t) - \sum_{a\in\delta^-(i)} x_a(t) = \begin{cases} u_0 \cdot t & \text{for } i = o \\ 0 & \text{for } i \neq o, d \\ -u_0 \cdot t & \text{for } i = d \end{cases}$$

$$\sum_{a\in\delta^+(i)} x_a'(t) - \sum_{a\in\delta^-(i)} x_a'(t) = \begin{cases} u_0 & \text{for } i = o \\ 0 & \text{for } i \neq o, d \\ -u_0 & \text{for } i = d \end{cases}$$

The node labels satisfy

$$\ell_j(t) = \begin{cases} t & \text{for } j = o \\ \min_{ij\in A} T_{ij}(\ell_i(t)) & \text{for } j \neq o \end{cases}$$

$$\ell_j'(t) = \begin{cases} 1 & \text{for } j = o \\ \min_{ij\in A_t'} T_{ij}'(\ell_i(t))\ell_i'(t) & \text{for } j \neq o \end{cases}$$

$$T_a(t) = t + \frac{z_a(t)}{\nu_a} + \tau_a \quad T_a'(t) = 1 + \frac{z_a'(t)}{\nu_a}$$

Derivatives of a dynamic equilibrium

The cumulative flow $x_a(t) = F_a^+(\ell_i(t))$ satisfies flow conservation

$$\sum_{a\in\delta^+(i)} x_a(t) - \sum_{a\in\delta^-(i)} x_a(t) = \begin{cases} u_0 \cdot t & \text{for } i = o \\ 0 & \text{for } i \neq o, d \\ -u_0 \cdot t & \text{for } i = d \end{cases}$$

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The node labels satisfy

$$\ell_j(t) = \begin{cases} t & \text{for } j = o \\ \min\limits_{ij\in A} T_{ij}(\ell_i(t)) & \text{for } j \neq o \end{cases}$$

$$\ell_j'(t) = \begin{cases} 1 & \text{for } j = o \\ \min\limits_{ij\in A_t'} T_{ij}'(\ell_i(t))\ell_i'(t) & \text{for } j \neq o \end{cases}$$

$$T_a(t) = t + \tfrac{z_a(t)}{\nu_a} + \tau_a \quad T_a'(t) = \begin{cases} \frac{f_a^+(t)}{\nu_a} & \text{if } z_a(t) > 0 \\ \max\{1, \frac{f_a^+(t)}{\nu_a}\} & \text{if } z_a(t) = 0 \end{cases}$$

Thin Flows with Resetting *TFR*

Fix a scalar $u_0 \geq 0$ and link subsets $A^* \subseteq A' \subseteq A$ with A' acyclic.

Definition

A $TFR(u_0, A^*, A')$ is a static o-d flow $x' = (x'_a)_{a \in A}$ of size u_0 with $x'_a = 0$ on every link $a = ij \in A$ such that $\ell'_j < \rho_a(\ell'_i, x'_a)$.

Where

$$\rho_a(\ell'_i, x'_a) = \begin{cases} \frac{x'_a}{\nu_a} & \text{if } a \in A^* \\ \max\{\ell'_i, \frac{x'_a}{\nu_a}\} & \text{if } a \in A' \setminus A^* \\ \infty & \text{if } a \notin A' \end{cases}$$

$$\ell'_j = \begin{cases} 1 & \text{for } j = o \\ \min\limits_{ij \in A'} \rho_{ij}(\ell'_i, x'_a) & \text{for } j \neq o \end{cases}$$

Equivalently, x' is a u_0-flow supported in A' s.t. $\ell'_o = 1$,

$$\begin{aligned} \ell'_j &= x'_a/\nu_a && \text{for all } a \in A^* \\ \ell'_j &\leq \ell'_i && \text{for all } a \in A' \setminus A^*, x'_a = 0 \\ \ell'_j &= \max\{\ell'_i, x'_a/\nu_a\} && \text{for all } a \in A' \setminus A^*, x'_a > 0 \end{aligned}$$

Derivatives of a dynamic equilibrium

- Then, in a dynamic equilibrium, the derivatives of the cumulative flow $x'_a = \frac{dx_a}{dt}(t)$ and label functions $\ell'_j = \frac{d\ell_j}{dt}(t)$ correspond to a $TFR(u_0, A^*_t, A'_t)$.
- It can be shown that for each $u_0 \geq 0$ and $A^* \subseteq A' \subseteq A$ with A' acyclic, there exists a $TFR(u_0, A^*, A')$. The labels ℓ' are the same in all of them.
- More on Thin Flow with Resetting later.

Construction of dynamic equilibria

For constant inflow u_0 we extend an equilibrium from $[0, \bar{t}]$ to $[0, \bar{t} + \alpha]$:

1. Compute

$$\begin{aligned} A^* &= \{a = ij : \ell_j(\bar{t}) > \ell_i(\bar{t}) + \tau_a\} \\ A' &= \{a = ij : \ell_j(\bar{t}) \geq \ell_i(\bar{t}) + \tau_a\} \end{aligned}$$

2. Find (x', l') a thin flow of value u_0 with resetting on $A^* \subseteq A'$
3. Find the largest $\alpha > 0$ such that the affine extensions

$$\begin{aligned} \ell_j(t) &:= \ell_j(\bar{t}) + (t - \bar{t})\ell'_j && \text{for all } j \in V \\ x_a(t) &:= x_a(\bar{t}) + (t - \bar{t})x'_a && \text{for all } a \in A \end{aligned}$$

   keep the same $A^*_t = A^*$ and $A'_t = A'$ for $t \in [0, \bar{t} + \alpha)$.

   (Queue depletes or arc becomes active.)

Existence and uniqueness of equilibria

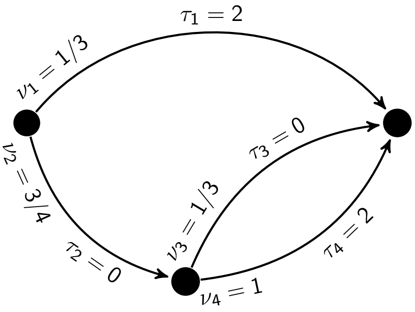
Theorem

- **Existence:** *There exists piecewise constant dynamic equilibrium with corresponding piecewise linear earliest-arrival times* $\ell_i(\cdot)$.
- **Uniqueness:** *In all "right-linear" equilibria the earliest arrival times* $\ell_i(\cdot)$ *are the same.*

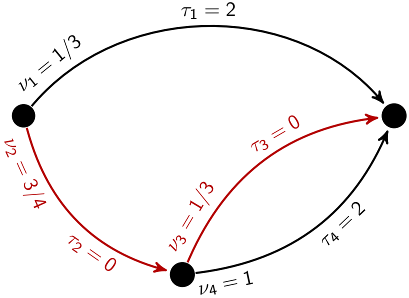
Unexpected Behaviors

- We know that the functions $\ell_v(t)$ are nondecreasing. However the $\ell_v(t) - t$ are not!
- The flow across a cut at a given time can be arbitrarily larger than u_0.
- The queues in the long run can be crazy.

Consider the following instance with unit inflow.



In $[0,1]$ all flow takes

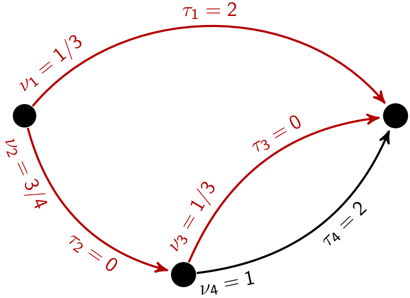


queues end at $\frac{1}{4}, \frac{5}{12}$

$$\ell'_d = 3$$

$$f_1^- + f_3^- + f_4^- = 1/3$$

In $[1, \frac{7}{5}]$ flow splits $\frac{1}{2}, \frac{1}{2}$

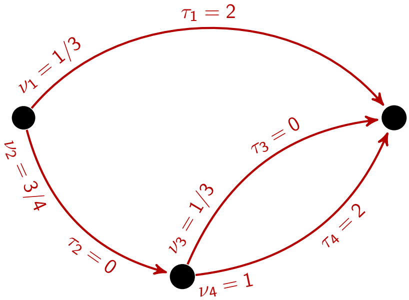


queues end at $\frac{1}{15}, \frac{3}{20}, \frac{7}{12}$

$$\ell'_d = 3/2$$

$$f_1^- + f_3^- + f_4^- = 2/3$$

In $[\frac{7}{5}, 4]$ flow splits $\frac{4}{13}, \frac{4}{13}, \frac{5}{13}$

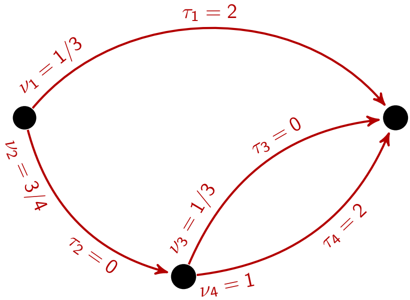


queues end at $0, 0, \frac{2}{3}, 0$

$\ell'_d = 12/13$!!

$f_1^- + f_3^- + f_4^- = 13/12$!!

In $[4, \infty]$ flow splits $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$

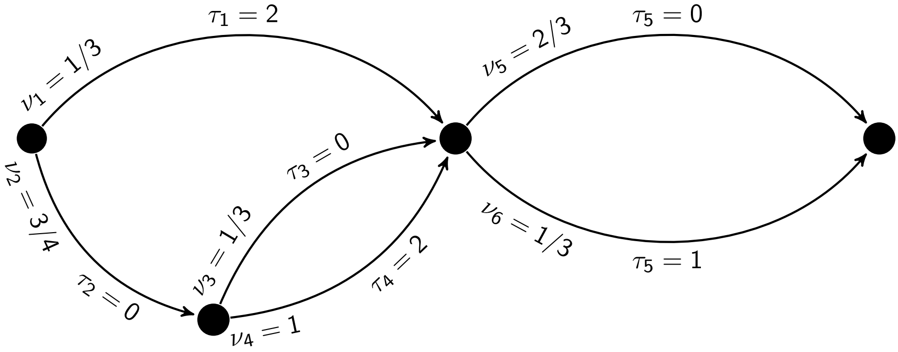


queues stay at $0, 0, \frac{2}{3}, 0$

$\ell'_d = 1$

$f_1^- + f_3^- + f_4^- = 1$

Crazy Queues

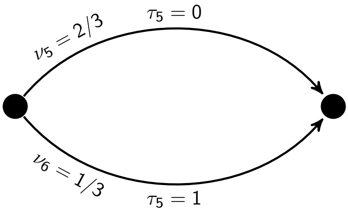


Note that the outflow of our previous example is the inflow of the right part of the instance.

Crazy Queues

So essentially this is like having the following instance with inflow

$$u_0(t) = \begin{cases} 13/12 & \text{for } t \in [0, 2 + 2/5) \\ 1 & \text{for } t \in [2 + 2/5, \infty). \end{cases}$$



Flow takes top link in $[0, 8/5)$, forming a queue of size $2/3$.
Then flow splits in proportions $2/3$, $1/3$, so queues will grow on both links.
At time $2 + 2/5$ steady state is achieved.
Steady state queues will be $32/45$ and $1/45$.

Recent Progress

Long Term Behavior Cominetti, C., Olver IPCO 2017

Steady state $\Leftrightarrow$ $\ell_i'(t) = 1$ *for all* $i \in V$, *for* t *large.*

$\Leftrightarrow$ *all queues become constant* $z_a(t) = z_a^*$.

- Let $\bar{\nu} = \sum_{a \in C} \nu_a$ be the min capacity across all cuts C. If $u_0 \leq \bar{\nu}$ then a steady state is attained in finite time.
- So, a steady state always exists.
- Steady state flow solves:

$$\begin{aligned} \min \quad & \sum_{a \in A} \tau_a x_a' \\ s.t. \quad & 0 \leq x_a' \leq \nu_a \ \ \forall a \in A \\ & x' \text{ is a } u_0 - \text{flow} \end{aligned}$$

- Steady state queues obtained from the dual if LP solution is unique.
- Open: $u_0 > \bar{\nu}$ is there a "steady state" in which queues grow linearly?
- Open: Suppose $\bar{\nu} = 2$ and u_0 varies between $1/2$ and 1 is some nice way. Do queues remain bounded?

Recent Progress 2

Computation of Thin Flow with Reseting — Kaiser 2019

Given an acyclic graph $G = (V, A)$ and a subset of edges A^* compute in polynomial time a x' is a u_0-flow and node labels ℓ' s.t.

$$
\begin{aligned}
&\ell'_o = 1 && \\
&0 \leq x'_a \leq \nu_a \ell'_w && \text{for } a = vw \in A \\
&x'_a = \nu_a \ell'_w && \text{for } a = vw \in A^* \\
&x'_a = \nu_a \ell'_w && \text{for } a = vw \in A \setminus A^*, \ell'_w > \ell'_v \\
&x'_a = 0 && \text{for } a = vw \in A \setminus A^*, \ell'_w < \ell'_v.
\end{aligned}
$$

- The problem can be turned into a linear complementarity problem.
- It can be solved in polynomial time for series parallel graphs.
- Open: Polynomial time in general graphs? (implies a polynomial time algorithm for a dynamic equilibrium).

Recent Progress 3

More General Models

Sering, Vargas-Koch SODA 2019, Sering, Skutella ATMOS 2018

- Multi O-D
  - Multiple destinations not so hard. Garrido 17
  - Multiple origins much harder:
    We loose the common clock!
  - But pretty much everything can be done Sering, Skutella 18
  - Open: Multi-commodity Flows.
- Spillover
  - What if queues have limited storage capacity?
  - And flow congestion propagates back!
  - Clever modeling tricks allow for mostly the same results
    Sering, Vargas-Koch 19

Recent Progress 4

Price of Anarchy (PoA) Cristi, C., Oosterwijk EC 2019

Choice of Social Cost:

- Throughput: Maximize amount of flow arriving at d by time T.
- Makespan: Minimize the time it takes for M flow units to arrive to d.
- Earliest Arrival Flow solves both for all T and all M. Gale 59

How well does a Dynamic Equilibrium do?

- PoA measures the worst case ratio.
- For Throughput PoA unbounded Koch, Skutella 09
- For Makespan PoA conjectured to be a small constant Skutella 06
- Capacity reduction leads to PoA $\frac{e}{e-1}$ Bhaskar, Fleischer, Anshelevich 11

Suppose the Dynamic Equilibrium is monotone. I.e., if we reduce the inflow rate then it takes longer to bring u flow units into d.

- Then PoA$=\frac{e}{e-1}$
- Moreover if longest simple path has length k, then PoA$\leq \frac{1}{1-(1-\frac{1}{k+1})^{k+1}}$

Recent Progress 5

Extension algorithm:

1. Compute

$$\begin{aligned} A^* &= \{a = ij : \ell_j(\bar{t}) > \ell_i(\bar{t}) + \tau_a\} \\ A' &= \{a = ij : \ell_j(\bar{t}) \geq \ell_i(\bar{t}) + \tau_a\} \end{aligned}$$

2. Find (x', l') a corresponding TFR
3. Find the largest $\alpha > 0$ such that the affine extensions

$$\begin{aligned} \ell_j(t) &:= \ell_j(\bar{t}) + (t - \bar{t})\ell'_j && \text{for all } j \in V \\ x_a(t) &:= x_a(\bar{t}) + (t - \bar{t})x'_a && \text{for all } a \in A \end{aligned}$$

   keep the same $A^*_t = A^*$ and $A'_t = A'$ for $t \in [0, \bar{t} + \alpha)$.

- This generates a sequence $\alpha_k > 0$
- Is there some ϵ s.t. $\alpha_k > \epsilon$? or can the algorithm get stuck?
- Would imply uniqueness.

Recent Progress 6

Myopic users

Graf, Harks IPCO 2019

- So far users are fully forward looking.
- What if they are myopic and make local decisions?
- Next Talk!