Dynamic Flows with Adaptive Route Choice

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The Physical Flow Model

- digraph $G = (V, E)$
The Physical Flow Model

- digraph $G = (V, E)$
- edge $e \in E$ has length $\tau_e \in \mathbb{Z}_+$

Commuters $(s_i, t_i), i \in I$ with $u_i \in [r_i, R_i]$ as constant.
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- digraph $G = (V, E)$
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![Diagram of the Physical Flow Model]
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\[
\nu = 2 \text{ for } \theta \in [0, 1]
\]

\[
\tau_{sv} = 1 \quad \nu_{sv} = 2 \quad \nu_{vt} = 1
\]

\[
\tau_{vt} = 1
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\[ f^+_e(\theta) \quad \text{queue } q_e(\theta) \quad f^-_e(\theta) \quad \text{outflow} \]

\[ v \quad \tau_e \quad w \]

\[ u = 2 \text{ for } \theta \in [0, 1] \]

\[ T_{sv} = 1 \quad \nu_{sv} = 2 \quad T_{vt} = 1 \quad \nu_{vt} = 1 \]
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\[
\begin{align*}
&\nu_e \downarrow u e = 2 \quad \text{for } \theta \in [0, 1] \\
&\tau_{sv} = \frac{1}{1} \\
&\nu_{sv} = 2 \\
&\nu_{vt} = 1 \\
&\tau_{vt} = 1
\end{align*}
\]
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$u = 2$ for $\theta \in [0, 1] 

\[ \begin{align*}
&\quad \text{inflow} \\
&\quad v \\
&\quad \tau_e \\
&\quad w \\
&\quad \text{queue } q_e(\theta) \\
&\quad f_e^+(\theta) \\
&\quad f_e^-(\theta) \\
&\quad \text{outflow} \\
\end{align*} \]
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\begin{align*}
\text{inflow} & \quad \nu \quad \tau_e \quad \downarrow \nu_e \quad w \\
& \quad f_e^+(\theta) \quad \text{queue} \quad q_e(\theta) \quad f_e^-(\theta) \quad \text{outflow}
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![Diagram of flow model]

$u = 2$ for $\theta \in [0, 1]$
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\[ u = 2 \text{ for } \theta \in [0, 1] \]
Current length of a \( v-t \) path \( P \): travel time + waiting times in queues

\[
c(P) = \sum_{e \in P} \tau_e + q_e(\theta)/\nu_e
\]
The Behavioral Model

Current length of a \( v-t \) path \( P \): travel time + waiting times in queues

\[
c(P) = \sum_{e \in P} \tau_e + \frac{q_e(\theta)}{\nu_e}
\]

**Definition (Instantaneous Dynamic Equilibrium (IDE))**

At every point in time: if positive flow enters an edge, the edge must lie on a currently shortest path towards the respective sink.
The Behavioral Model – Single Sink

- let $f$ be a dynamic flow
- total travel time of edge $e$ at time $\theta$: $c_e(\theta) = \tau_e + q_e(\theta)/\nu_e$
- define node labels $\ell_v(\theta)$ measuring the currently shortest travel time to $t$:

$$\ell_t(\theta) = 0, \ell_v(\theta) = \min_{e=vw \in E} c_e(\theta) + \ell_w(\theta) \text{ for all } v \in V \setminus \{t\}.$$ 

**Definition (Active Edges)**

An edge $e = vw \in E$ is active at time $\theta$ if

$$\ell_v(\theta) = \ell_w(\theta) + c_e(\theta).$$

$E_\theta \subseteq E$ set of active edges
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$E_\theta \subseteq E$ set of active edges

**Definition (Instantaneous Dynamic Equilibrium)**

For every $\theta \geq 0$: $f_e^+(\theta) > 0 \Rightarrow e \in E_\theta.$
Example:
Example:

\[ \theta = 0.0 \]

\[ u_1(\theta) = 3 \text{ for } \theta \in [0, 1] \]

\[ (1, 2) \]

\[ (3, 1) \]

\[ (1, 1) \]

\[ (\tau_{s_2}, \nu_{s_2}) = (1, 2) \]

\[ u_2(\theta) = 4 \text{ for } \theta \in [1, 2] \]
Example:

\[ \theta = 0.25 \]

\[ u_1(\theta) = 3 \text{ for } \theta \in [0, 1] \]

\[ u_2(\theta) = 4 \text{ for } \theta \in [1, 2] \]
Example:

\[ \theta = 0.5 \]

\[ u_1(\theta) = 3 \text{ for } \theta \in [0, 1] \]

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Example:

\[
\theta = 0.75 \quad u_1(\theta) = 3 \text{ for } \theta \in [0, 1]
\]

\[
(3, 1) \quad (1, 1) \quad (\tau_{v_{s_2}}, \nu_{v_{s_2}}) = (1, 2)
\]

\[
(1, 1) \quad u_2(\theta) = 4 \text{ for } \theta \in [1, 2]
\]
Example:

\[^{1}\theta = 1.0\]

\[^{2}\nu_1(\theta) = 3\text{ for } \theta \in [0, 1]\]

\[^{3}\nu_2(\theta) = 4\text{ for } \theta \in [1, 2]\]

\[^{1}\nu_{S_2}(\theta) = (1, 2)\]
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\[ t \]

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\[ s_2 \]
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\downarrow & \quad \downarrow & \quad \downarrow \\
S_1 & \quad V & \quad S_2
\end{align*} \]

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\[ (1, 2) \]

\[ s_1 \rightarrow v \]

\[ (3, 1) \]

\[ (1, 1) \]

\[ (\tau_{s_2}, \nu_{s_2}) = (1, 2) \]

\[ t \rightarrow s_2 \]

\[ (1, 1) \]

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Example:

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Example:

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\[ (1, 2) \]

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\[ (\tau_{vs_2}, \nu_{vs_2}) = (1, 2) \]

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\[ (3, 1) \rightarrow (1, 1) \]

\[ (1, 2) \rightarrow (1, 1) \]

\[ \tau_{V_2}, \nu_{V_2} = (1, 2) \]

\[ u_2(\theta) = 4 \text{ for } \theta \in [1, 2] \]
Example:

\[
\theta = 7.0 \quad u_1(\theta) = 3 \text{ for } \theta \in [0, 1]
\]

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Related Work:

- fluid queueing model – Nash equilibria
  - Koch and Skutella ('11)
  - Cominetti, Correa and Larre ('15)
  - Cominetti, Correa and Olver ('17)
  - Sering Vargas-Koch ('19)

- IDE for used paths
  - Ran and Boyce ('89, '95)
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<tr>
<td>Single-Sink</td>
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Theorem

For any multi-source single-sink network $G$ there exists an IDE flow.
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**Given:** An IDE flow \( f \) up to some time \( \theta \)
Existence in Single-Sink Networks

**Theorem**

*For any multi-source single-sink network G there exists an IDE flow.*

**Given:** An IDE flow $f$ up to some time $\theta$

**Goal:** Extending $f$ to an IDE flow up to time $\theta + \varepsilon$ for some $\varepsilon > 0$
Existence in Single-Sink Networks

**Theorem**

*For any multi-source single-sink network $G$ there exists an IDE flow.*

**Given:** An IDE flow $f$ up to some time $\theta$

**Goal:** Extending $f$ to an IDE flow up to time $\theta + \varepsilon$ for some $\varepsilon > 0$

Equivalently: For all $v \in V$ distribute arriving flow in $[\theta, \theta + \varepsilon)$ onto active edges at time $\theta$ so that they stay active for some $\varepsilon > 0$. 
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Theorem

For any multi-source single-sink network \( G \) there exists an IDE flow.

**Given:** An IDE flow \( f \) up to some time \( \theta \)

**Goal:** Extending \( f \) to an IDE flow up to time \( \theta + \varepsilon \) for some \( \varepsilon > 0 \)

Equivalently: For all \( v \in V \) distribute arriving flow in the range \( [\theta, \theta + \varepsilon) \) onto active edges at time \( \theta \) so that they stay active for some \( \varepsilon > 0 \).

**Invariant:** per extension interval, all inflows are constant and so are all outflows as well.
Existence in Single-Sink Networks

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For any multi-source single-sink network $G$ there exists an IDE flow.

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Equivalently: For all $v \in V$ distribute arriving flow in $[\theta, \theta + \varepsilon)$ onto active edges at time $\theta$ so that they stay active for some $\varepsilon > 0$.

**Invariant:** per extension interval, all inflows are constant and so all outflows as well.

**Idea:** Order nodes by increasing labels $\ell_v(\theta)$ (i.e. distance to $t$).
Existence in Single-Sink Networks

**Theorem**

For any multi-source single-sink network $G$ there exists an IDE flow.

**Given:** An IDE flow $f$ up to some time $\theta$

**Goal:** Extending $f$ to an IDE flow up to time $\theta + \varepsilon$ for some $\varepsilon > 0$

Equivalently: For all $v \in V$ distribute arriving flow in $[\theta, \theta + \varepsilon)$ onto active edges at time $\theta$ so that they stay active for some $\varepsilon > 0$.

**Invariant:** per extension interval, all inflows are constant and so all outflows as well.

**Idea:** Order nodes by increasing labels $\ell_v(\theta)$ (i.e. distance to $t$). Distribute inflow node by node starting with smallest labels $[\theta, \theta + \varepsilon)$.
Construction:

\[(v_e, \tau_e) = (2, 3)\]
Construction:

\[ (\nu_e, \tau_e) = (2, 3) \]
Construction:

$(v_e, \tau_e) = (2, 3)$

$V_1$  

$V_2$  

$V_3$  

$V_4$  

$V_5$  

$V_t$
Construction:

\((\nu_e, \tau_e) = (2, 3)\)

\(v_1 \rightarrow v_2\): (2, 1)  
\(v_2 \rightarrow v_3\): (2, 2)  
\(v_3 \rightarrow v_4\): (1, 1)  
\(v_4 \rightarrow t\): (1, 1)  
\(v_3 \rightarrow v_5\): (1, 5)  
\(v_5 \rightarrow t\): (1, 2)  

\(\ell_v\) and \(\ell_t\) graphs are shown for metric calculations.

The diagram illustrates the construction with labeled edges.
Construction:
Construction:
Construction:
How To Fix The Inflow ?

\[ f_{vw}(\theta) > 0 \Rightarrow \ell_v(\theta) = c_{vw}(\theta) + \ell_w(\theta) \]
\[ f_{vw}(\theta) = 0 \Rightarrow \ell_v(\theta) \leq c_{vw}(\theta) + \ell_w(\theta) \]
How To Fix The Inflow?

\[ f_{vw}^+(\theta) > 0 \implies \ell_v(\theta) = c_{vw}(\theta) + \ell_w(\theta) \]
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\[ \iff f_{vw}^+(\theta) > 0 \implies \ell'_v(\theta) = c'_{vw}(\theta) + \ell'_w(\theta) \]
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\[
\min \sum_{e=vw\in\delta_v(\theta)} \int_0^{x_e} \frac{g_e(z)}{\nu_e} + a_w \, dz \quad (\text{OPT-} b_v^-(\theta))
\]

s.t.:
\[
\sum_{e=vw\in\delta_v(\theta)} x_e = b_v^-(\theta)
\]
\[ x_e \geq 0, e \in \delta_v(\theta), \]

\[ g_e(z) := \begin{cases} 
  z - \nu_e, & \text{if } q_e(\theta) > 0 \\
  [z - \nu_e]_+, & \text{if } q_e(\theta) = 0.
\end{cases} \]
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Termination flow terminates = all flow volume $V = \sum_{i} q_i u_i \left( R_i \neq r_i \right)$ reaches sink within finite time.
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### Termination

Flow terminates = all flow volume $V = \sum_{i \in I} u_i \cdot (R_i - r_i)$ reaches sink within finite time
Termination in Single-Sink Networks

Theorem

Within any multi-source single-sink network \( G = (V, E) \) all IDE flows terminate.
Lemma

If, after some time $\theta_0$ the total amount of flow in some subgraph $H$ (including $t$) is always smaller than 1, then all flow arriving at one of the nodes of $H(U)$ will arrive at $t$ after a finite amount of time.

Proof: The total length of all queues in $H$ is always smaller than 1.

\[ \Delta_t \]

Flow only uses shortest paths within $H$ (i.e. no cycles).
Proof by Contradiction

Let $H$ be maximal and $H' := G \setminus H$. 
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Diagram:

- $H'$ (red dashed box)
- $H$ (blue dashed box)
- $t$ (black node)

The diagram illustrates the relationship between $H'$ and $H$ within the context of the proof.
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Let \( H \) be maximal and \( H' := G \setminus H \).
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For any multi-source multi-sink network there exists an IDE flow.

1. **Given:** An IDE flow $f$ up to some time $\theta_0$
2. **Goal:** Extending $f$ to an IDE flow up to time $\theta_0 + \tau_{\text{min}}$
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For any multi-source multi-sink network there exists an IDE flow.

1. **Given:** An IDE flow $f$ up to some time $\theta_0$
2. **Goal:** Extending $f$ to an IDE flow up to time $\theta_0 + \tau_{\text{min}}$
3. Define the set of possible (not necessary IDE-) extension of $f$:
   \[ K := \{ g \in L^2([\theta_0, \theta_0 + \tau_{\text{min}}] \times E) \mid g \text{ extension of } f \} \]
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4. Consider the mapping $\mathcal{A} : K \rightarrow L^2([\theta_0, \theta_0 + \tau_{\text{min}}])^{I \times E}$, $g = (g_{i,e}) \mapsto h = (h_{i,e})$,
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   where $h_{i,vw}(\theta) := \ell_{i,w}(\theta) + c_{vw}(\theta) - \ell_{i,v}(\theta)$

\[\text{Diagram:}\]

- Node $w$ connected to node $v$
An Existence Theorem

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Then: $g \in K$ is IDE extension $\iff \langle g, \mathcal{A}(g) \rangle = 0$
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   Then: $g \in K$ is IDE extension $\iff \langle g, A(g) \rangle = 0$
   $\iff \langle A(g), g' - g \rangle \geq 0 \text{ f.a. } g' \in K$
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For any multi-source multi-sink network there exists an IDE flow.

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Then: $g \in K$ is IDE extension $\iff \langle g, A(g) \rangle = 0$

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Such a $g$ always exists ($A$ weak-strong-continuous, $K$ non-empty, closed, convex, bounded).
An Existence Theorem

**Theorem**

For any multi-source multi-sink network there exists an IDE flow. (even for more general network inflow functions $u_i$)

1. **Given:** An IDE flow $f$ up to some time $\theta_0$
2. **Goal:** Extending $f$ to an IDE flow up to time $\theta_0 + \tau_{\text{min}}$
3. Define the set of possible (not necessary IDE-) extension of $f$:
   $K := \{ g \in L^2([\theta_0, \theta_0 + \tau_{\text{min}}])^{I \times E} \mid g \text{ extension of } f \}$
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   i.e. $h_{i,\ell,w}(\theta) = 0 \iff \ell_{i,v}(\theta)$

   **Then:** $g \in K$ is IDE extension $\iff \langle g, \mathcal{A}(g) \rangle = 0$
   $\iff \langle \mathcal{A}(g), g' - g \rangle \geq 0$ f.a. $g' \in K$

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There exists a multi-source multi-sink network for which any IDE flow does not terminate.
Non-Termination in Multi-Sink Networks

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