**Vertex-minors of graphs**

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Graph States $|G\rangle$

3 operations to reduce some meaningful operations (local Clifford operations, Pauli Z measurements on vertices)

Schmit-rank width

Graphs

Vertex-minors
Local complementation deletion

rank-width
Plan

• Planar graphs and minors
• Circle graphs and vertex-minors
• Connectivity Functions
• Width Parameters
• Extending the graph minors project of Robertson and Seymour
• Weakening of conjectures on induced subgraphs
Planar graphs and minors

A graph $G$ is **planar** if it can be drawn on the plane with no edge crossing.
Operations that **preserve planarity**

- **edge deletion**: $G \setminus e$
- **contraction**: $G/e$
- **vertex deletion**: $G \setminus v$

**H** is a **minor** of **G** if

H is obtained from G by a sequence of **deleting** edges/vertices and **contracting** edges

**Easy Observation:**

If H is a **minor** of a planar graph G, then H is planar.
Kuratowski's theorem on planar graphs (1930)

Theorem (Kuratowski 1930 / Wagner 1937)

G is planar if and only if
G has no minor isomorphic to \(K_5\) or \(K_{3,3}\)

Two minor-minimal non-planar graphs

\[ K_5 \quad K_{3,3} \]
Circle graphs and vertex-minors / pivot-minors

Circle graph = intersection graph of chords of a circle

chords intersect vertices

vertices intersect edges
A minor of a circle graph is **not** necessarily a circle graph.

$K_6$ is a circle graph

$W_5$ is **not** a circle graph

Impossible to characterize circle graphs in terms of forbidden minors.
Local complementation and vertex-minors

If $G$ is a circle graph, then $G^*v$ is a circle graph.

Easy Observation:
If $H$ is a vertex-minor of $G$ and $G$ is a circle graph, then $H$ is a circle graph.
Pivot and pivot-minors

If $G$ is a circle graph, then $G \wedge vw$ is a circle graph.

**Easy Observation:**
If $H$ is a pivot-minor of $G$ and $G$ is a circle graph, then $H$ is a circle graph.

H is a **pivot-minor** of $G$ if $H$ can be obtained from $G$ by a sequence of pivots and vertex deletions.

"pivot an edge 23"
Every pivot-minor is a vertex-minor.

\[ G \xrightarrow{\text{pivot}} G^*3 \xrightarrow{\text{pivot}} G^*3^*2 \xrightarrow{\text{pivot}} G^*3^*2^*3 = G \wedge 23 \]

In general,
\[ G \wedge uv = G^*u^*v^*u \]

Every pivot-minor is a vertex-minor.

(The converse is false.)
Forbidden *vertex-minors* for circle graphs

**Theorem** (Bouchet 1994)

G is a *circle graph*

if and only if

G has no *vertex-minor* isomorphic to

\[ W_5, \quad W_7, \quad BW_3 \]

Three vertex-minor minimal non-circle graphs
Forbidden \textbf{pivot-minors} for circle graphs

\textbf{Theorem} (Geelen, O. 2009)
G is a \textbf{circle graph}
if and only if
G has no \textbf{pivot-minor} isomorphic to

\begin{align*}
\begin{array}{c}
\text{FIGURE 2. Excluded pivot-minors for circle graphs.} \\
\text{FIGURE 3. } H_1, H_2, \text{ and } Q_3.
\end{array}
\end{align*}
Theorem (de Fraysseix 1981)
A bipartite graph is a circle graph if and only if it is a fundamental graph of a planar graph.
Forbidden **pivot-minors** for circle graphs

**Theorem** (Geelen, O. 2009)

G is a **circle graph** if and only if

G has no **pivot-minor** isomorphic to

- 3 bipartite graphs
  - Fundamental graphs of the Fano matroid
    - $M(K_5)$
    - $M(K_{3,3})$

Implies Kuratowski's theorem
Connectivity Functions
**Vertex Connectivity function**

"connectivity function" defined on the edge set of $G$

$$\eta_G(S) = \#\text{vertices meeting both } S \text{ and its complement}$$

If $S$ is a set of edges and $\eta_G(S) \leq k$ then

$$\eta_{G/e}(S - \{e\}) \leq k$$

$$\eta_{G\setminus e}(S - \{e\}) \leq k$$

**Observation:**

The "vertex connectivity function" does not increase while taking minors

"Vertex Connectivity Function" is well studied with respect to minors
Cut-rank function
A connectivity measure, small for simple but possibly dense edge-cuts

\[ \rho_G(X) = \text{rank} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 2 \]

\[ \rho_H(X) = \text{rank} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 3 \]

Cut-rank function of \( G \): rank of a \( X^*(V-X) \) 0-1 matrix over the binary field

The cut-rank function may increase while taking minors!
Cut-rank function and vertex-minors

The cut-rank function is invariant under taking local complementations.

If $H$ is a vertex-minor of $G$, then for all $X$, $\rho_H(X') \leq \rho_G(X)$ for its corresponding $X'$.
Rank connectivity

G is **k-rank-connected** if $\rho_G(X) < k \Rightarrow \rho_G(X) = \min(|X|, |V - X|)$

Easy: G is **1-rank-connected**
   if and only if
   G is connected

**2-rank-connected**

= "prime with respect to **split decompositions**" (Cunningham 1982)

   Cunningham (1982): Description of a canonical decomposition
   of a connected graph into cuts of cut-rank 1
### Tools for 2-rank-connected graphs

Compare with 3-connected graphs

<table>
<thead>
<tr>
<th><strong>Chain Theorem</strong></th>
<th><strong>Splitter Theorem</strong></th>
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</thead>
</table>
| Every **3-connected** graph $G$ has a minor $H$ such that  
  * $|E(H)| = |E(G)| - 1$ and  
  * $H$ is simple 3-connected  
  unless $G = \text{wheel}$.  
  
  Tutte (1961) | If $H$ is a **3-connected minor** of a 3-connected graph $G$, then $G$ has a minor $G'$ such that  
  * $|E(G')| = |E(G)| - 1$,  
  * $G'$ is 3-connected,  
  * $H$ is isomorphic to a minor of $G'$  
  unless $H = \text{wheel}$ or $|V(G)| = |V(H)|$.  
  
| **2-rank-connected** graph $G$ has a vertex-minor $H$ such that  
  * $|V(H)| = |V(G)| - 1$ and  
  * $H$ is 2-rank-connected  
  unless $|V(G)| \leq 5$.  
  
  Bouchet (1987) | If $H$ is a **2-rank-connected vertex-minor** of a 2-rank-connected graph $G$, then $G$ has a vertex-minor $G'$ such that  
  * $|V(G')| = |V(G)| - 1$,  
  * $G'$ is 2-rank-connected,  
  * $H$ is isomorphic to a vertex-minor of $G'$  
  unless $|V(H)| < 5$ or $|V(G)| = |V(H)|$.  
  
  Bouchet (unpublished), Geelen (1995) |
Applications of Splitter theorems

Graphs with no $K_5$ minor =

- Closure by disjoint union, 1-, 2-, 3-sums
  - planar graphs
    - $K_{3,3}$
    - $V_8$

  Wagner (1937)

Graphs with no $W_5$ vertex-minor =

- Closure by disjoint union and 1-joins
  - circle graphs
    - $W_7$
    - $BW_3$
  - cube

Geelen (1995)
**Unavoidable structure in a large graph**

Every large graph has

\[ K_n \text{ or } \overline{K}_n \]

as an induced subgraph.

Ramsey (1930)

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Every large **3-connected** graph has

\[ W_k \text{ or } K_{3,k} \]

as a minor.

Oporowski, Oxley, Thomas (1993)

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**Theorem** (Kwon, O. 2014)

Every large **2-rank-connected** graph has

\[ C_n \text{ or } K_n \sqsubseteq K_n \]

as a vertex-minor.
Width parameters
**Branch-width** and **Rank-width**

**Branch-width** of $G = \min k$ such that $E(G)$ can be recursively partitioned into subsets of "vertex connectivity" $\leq k$ until each set becomes a singleton

Robertson, Seymour (1991)

"branch-decomposition" of width $k$

**Rank-width** of $G = \min k$ such that $V(G)$ can be recursively partitioned into subsets of "cut-rank" $\leq k$ until each set becomes a singleton

O., Seymour (2006)

"rank-decomposition" of width $k$

Important tool for graph minor theory together with tree-width

Important tool for graph vertex-minor theory
**Algorithmic applications** on graph classes of small rank-width

Meta-theorems

*If the input graph has small and is given with its decomposition of small width, then every problem expressible in can be solved in poly time.*

Courcelle (1990)

1. branch-width / tree-width
2. monadic second-order logic

Courcelle, Makowsky, Rotics (2000)

1. rank-width / clique-width
2. monadic second-order logic with no edge-set quantification

Example

\[ \exists X_1 \exists X_2 (\forall v \forall w (v \in X_1 \land w \in X_1 \land v \neq w \Rightarrow v \sim w) \land \forall v' \forall w' (v' \in X_1 \land w' \in X_1 \land v' \neq w' \Rightarrow v' \sim w')) \]

For a graph given by a list of edges, how do we find a good decomposition of small width?
Algorithm to find a branch-decomposition / rank-decomposition

**Theorem** (Jeong, Kim, O. 2018+)
An efficient algorithm on "subspace arrangements" to find a branch-decomposition of width $\leq k$ if it exists.

- **Corollary**: $O(f(k)n^3)$-time algorithm to find a branch-decomposition of width $\leq k$, if it exists, for a hypergraph.

- **Corollary**: $O(f(k)n^3)$-time algorithm to find a rank-decomposition of width $\leq k$, if it exists, for a graph.

**Previous algorithm for branch-width:**
Thilikos, Bodlaender (2000): 50-page technical report only for graphs → Not appeared in a journal yet

**Previous algorithm for rank-width:**
Hliněný and O. (2008): Uses a non-trivial result on forbidden vertex-minors for rank-width $\leq k$ and use the meta-theorem of Courcelle et al.
Understanding rank-width in terms of vertex-minors / pivot-minors

**Theorem (O. 2005)**
For all $k$, every pivot-minor-minimal graph of rank-width $> k$ has at most $O(6^k)$ vertices.

**Previous algorithm for rank-width:**
Hliněný and O. (2008): Uses a non-trivial result on forbidden pivot-minors for rank-width $\leq k$ and use the meta-theorem of Courcelle et al.

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**Theorem (Kwon, O. 2014)**
If $G$ has rank-width $\leq k$, then $G$ is a pivot-minor of a graph of tree-width $\leq 2k$. 
Classical simulation versus universality in measurement-based quantum computation

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We investigate for which resource states an efficient classical simulation of measurement-based quantum computation is possible. We show that the Schmidt-rank width, a measure recently introduced to assess universality of resource states, plays a crucial role in also this context. We relate Schmidt-rank width to the optimal description of states in terms of tree tensor networks and show that an efficient classical simulation of measurement-based quantum computation is possible for all states with logarithmically bounded Schmidt-rank width (with respect to the system size). For graph states where the Schmidt-rank width scales in this way, we efficiently construct the optimal tree tensor network descriptions, and provide several examples. We highlight parallels in the efficient description of complex systems in quantum information theory and graph theory.

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Theorem 5. Let $|G\rangle$ be a graph state on $n$ qubits. If the rank width of $G$ grows at most logarithmically with $n$, then any MQC on $|G\rangle$ can efficiently be simulated classically.
Extending the graph minors project of Robertson and Seymour

Why do we have only \textbf{finitely many forbidden} vertex-minors / pivot-minors?

**Theorem (Bouchet 1994)**
G is a \textbf{circle graph}
if and only if
G has no \textbf{vertex-minor} isomorphic to
\begin{align*}
W_5 & \quad & W_7 & \quad & BW_3
\end{align*}

**Theorem (Geelen, O. 2009)**
G is a \textbf{circle graph}
if and only if
G has no \textbf{pivot-minor} isomorphic to
Analogue of the grid theorem

**Theorem** (Robertson and Seymour 1986)
For each **planar graph** $H$, there exists a constant $c = c(H)$ such that every graph $G$ of branch-width $>c$ has $H$ as a **minor**.

**Conjecture**
For each **bipartite circle graph** $H$, there exists a constant $c = c(H)$ such that every graph $G$ of rank-width $>c$ has $H$ as a **pivot-minor**.

**Weaker Conjecture**
For each **circle graph** $H$, there exists a constant $c = c(H)$ such that every graph $G$ of rank-width $>c$ has $H$ as a **vertex-minor**.

Known cases: when $G$ is
- **bipartite graphs** (Courcelle, O. 2007 by a theorem on binary matroids)
- **line graphs** (O. 2009)
- **circle graphs** (O. 2009 by a theorem in Thor Johnson's Ph.D. thesis)

**Solved** by Geelen, Kwon, McCarty, and Wollan (2018+)
Well-quasi-ordering theorems and conjectures

Every class of graphs closed under taking can be characterized in terms of a finite list of forbidden.

Equivalently; we say C is well-quasi-ordered under if

In every infinite sequence of graphs \( G_1, G_2, \ldots \) (in C) there exist \( i < j \) such that \( G_i \) is a minor of \( G_j \).

**Known cases:** when C is
- bipartite graphs (by binary matroids)
- line graphs (by group-labelled graphs)
- **bounded rank-width** (O. 2008)
- circle graphs (by Graph Minors XXIII)
Complexity of finding minors, vertex-minors, and pivot-minors

Theorem (Robertson and Seymour 1995)
For each fixed graph $H$, we can decide in time $O(n^3)$ whether an input $G$ has a minor isomorphic to $H$.

Corollary
For every class $C$ of graphs closed under taking minors, there exists an $O(n^3)$-time algorithm to decide whether an input graph belongs to $C$.

1. Minor

**Theorem** (Robertson and Seymour 1995)
For each fixed graph $H$, we can decide in time $O(n^3)$ whether an input $G$ has a minor isomorphic to $H$.

1. Vertex-minor: Open

- $P$ if $|V(H)| \leq 6$
- $P$ if $H$ is a circle graph
  - Geelen, Kwon, McCarty, and Wollan (2018+)

1. Pivot-minor: Open

- $P$ if $|V(H)| \leq 4$,
  - $H \notin \{K_4, C_3 + K_1, 4K_1\}$
  - (Dabrowski, Dross, Jeong, Kanté, Kwon, O., Paulusma 2018)
If H is not fixed, ...

If both H and G are given as an input, then it is NP-complete to decide whether H is isomorphic to a

Minor: True
H=Cycle of length n
→ reduction to Hamiltonian cycle problem

Vertex-minor
Axel Dahlberg (personal comm., 2018)

Pivot-minor
Dabrowski, Dross, Jeong, Kanté, Kwon, O., Paulusma 2018
Weakening of conjectures on induced subgraphs
\( \chi \)-boundedness

C: proper class of graphs closed under taking induced subgraphs

Problem (Gyárfás): For which C do we have a function f such that

\[
\chi(G) \leq f(\omega(G))
\]

for all G in C?

Conjecture (Geelen 2009)
True if C is closed under taking vertex-minors.

Equivalent conjecture:
For every graph H, there exists f such that if G has no H vertex-minors, then

\[
\chi(G) \leq f(\omega(G))
\]

Known cases:

- C={circle graphs} (Gyárfás 1985)
- C={rank-width≤k} (Dvořák and Král' 2010)
- H=fan (Ilkyoo Choi, Kwon, O. 2017)
- H=\textbf{wheel} (Hojin Choi, Kwon, O., Wollan 2018+)
- H=circle graph (Geelen, Kwon, McCarty, and Wollan 2018+)
Erdős-Hajnal property

C: proper class of graphs closed under taking induced subgraphs

C has the **Erdős-Hajnal property** if there exists $c > 0$ such that every graph $G$ in $C$ has a clique or a stable set of size $> |V(G)|^c$

**Erdős-Hajnal Conjecture (1989)**
Every such $C$ has the Erdős-Hajnal property.

**Theorem** (Chudnovsky, O. 2018+)
True if $C$ is closed under taking vertex-minors.

**Question:**
Is it true if $C$ is closed under taking pivot-minors?

**Theorem** (Kim, O. 2019+)
True if $C$ is closed under taking pivot-minors and $C$ has no cycle of length $k$. 
Polynomially $\chi$-boundedness

C: proper class of graphs closed under taking induced subgraphs

Problem: For which $C$ do we have a polynomial function $f$ such that
\[ \chi(G) \leq f(\omega(G)) \]
for all $G$ in $C$?

Question (Louis Esperet)
Is every $\chi$-bounded class polynomially $\chi$-bounded?

No counterexample is known yet.

Conjecture:
For every graph $H$, there exists a polynomial $f$ such that
\[ \chi(G) \leq f(\omega(G)) \]
if $G$ has no $H$ vertex-minors, then

Polynomially $\chi$-bounded $\Rightarrow$ Erdős-Hajnal property

Known cases:
- $H=$cycle (Ringi Kim, O-joung Kwon, O., Vaidy Sivaraman 2018+)
Thank you for your attention