Vertex-minors of graphs

Sang-il Oum (엄상일)
IBS Discrete Mathematics Group / KAIST

International Conference on Network Games, Tropical Geometry, and Quantum Communication
Berlin
June 6, 2019

Quantum Information Theory and **Graph Theory**

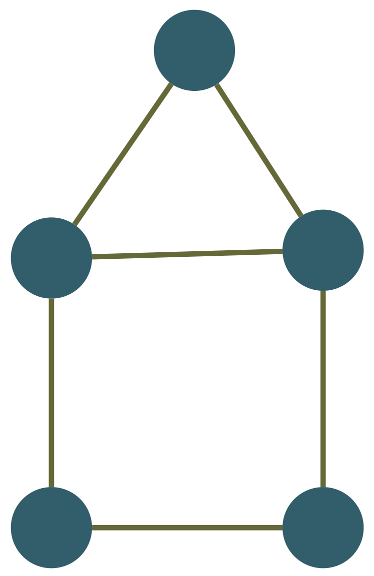
Graph States |G>	**Graphs**
3 operations to reduce some meaningful operations (local Clifford operations, Pauli Z measurements on vertices)	**Vertex-minors** Local complementation deletion
Schmit-rank width	**rank-width**

Plan

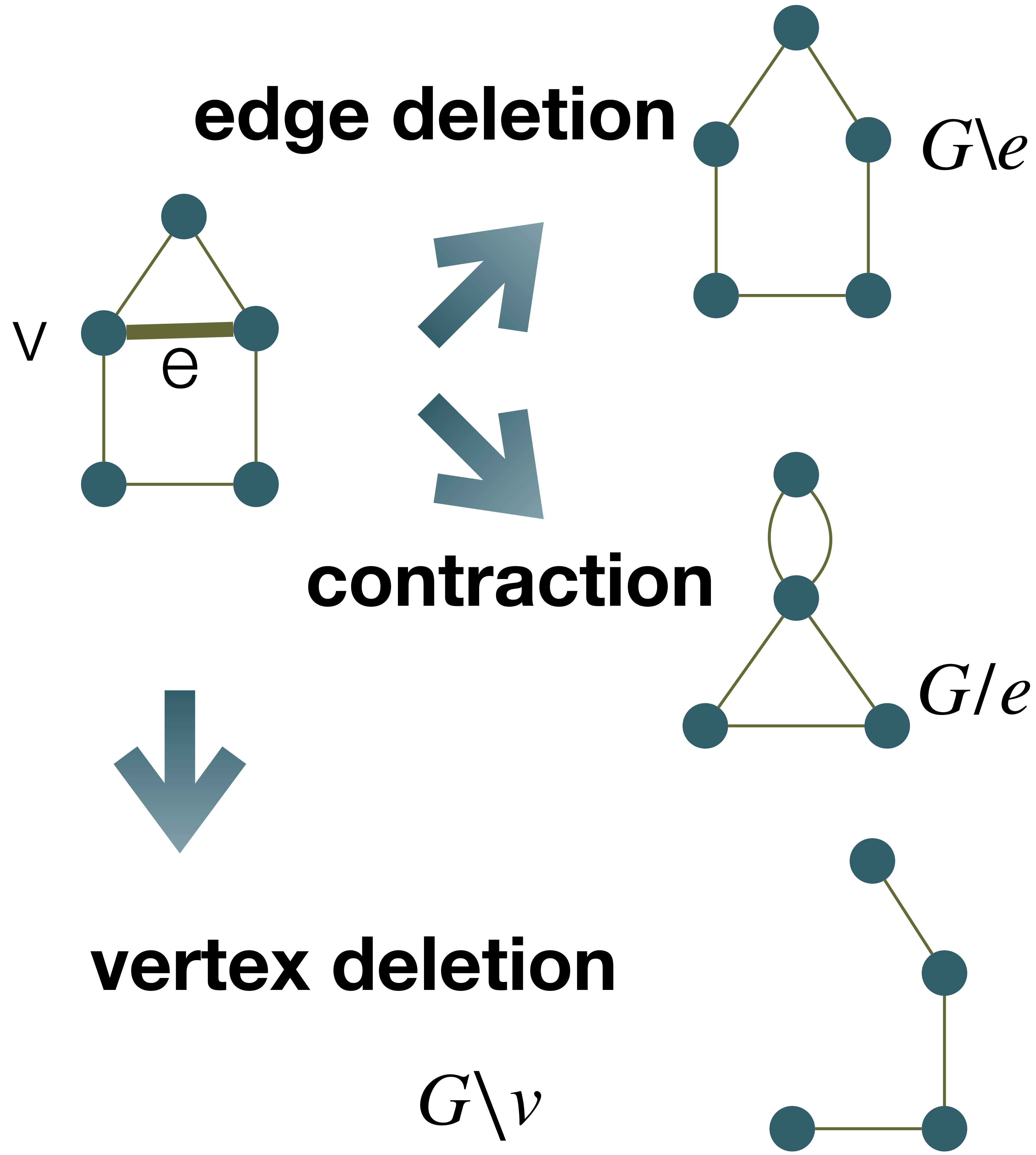
- Planar graphs and minors
- Circle graphs and vertex-minors
- Connectivity Functions
- Width Parameters
- Extending the graph minors project of Robertson and Seymour
- Weakening of conjectures on induced subgraphs

Planar graphs and **minors**

A graph G is **planar** if it can be drawn on the plane with no edge crossing.



Operations that **preserve planarity**

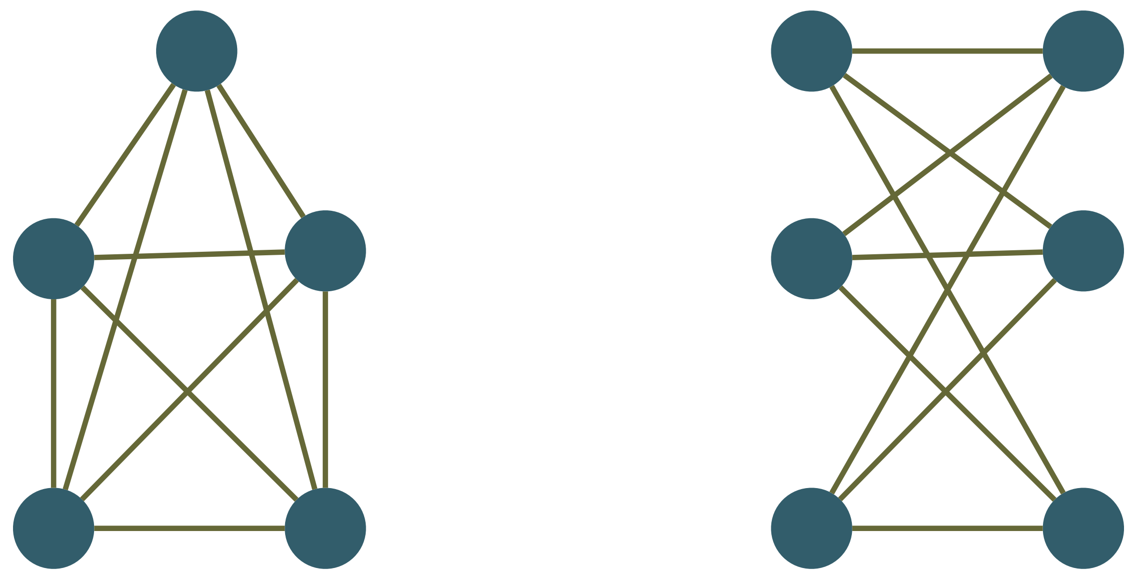


H is a **minor** of G if
H is obtained from G by a sequence of **deleting** edges/vertices and **contracting** edges

Easy Observation:
If H is a **minor** of a planar graph G, then H is planar.

Kuratowski's theorem on planar graphs (1930)

Theorem (Kuratowski 1930 / Wagner 1937)
G is **planar**
if and only if
G has **no minor** isomorphic to $\mathbf{K_5}$ **or** $\mathbf{K_{3,3}}$



K_5 $K_{3,3}$

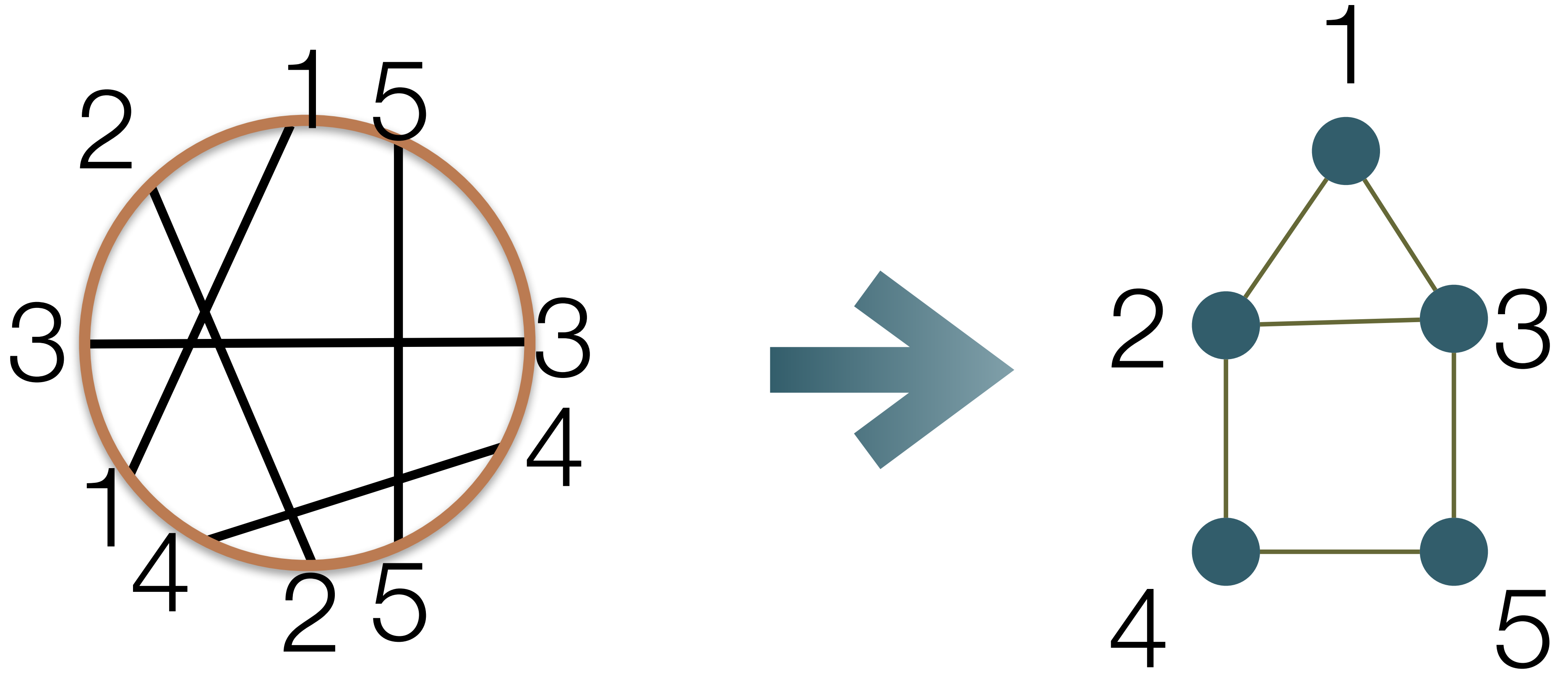
Two minor-minimal non-planar graphs

Circle graphs and **vertex-minors / pivot-minors**

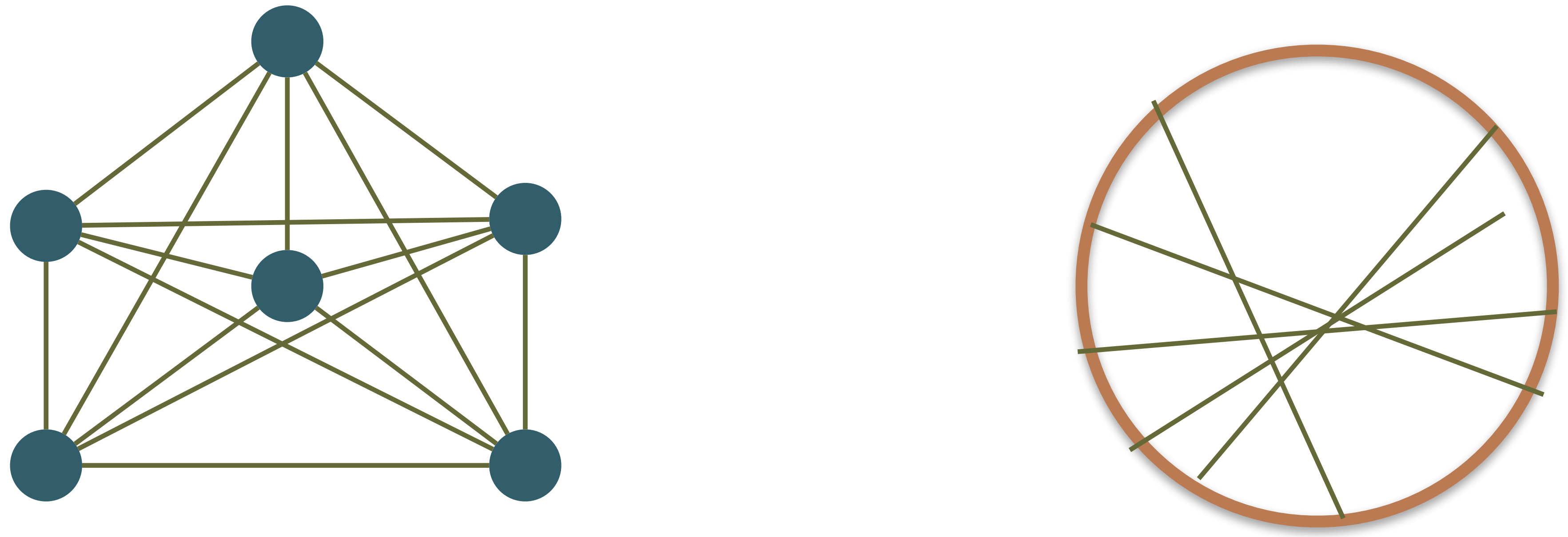
Circle graph = **intersection** graph of **chords** of a circle

chords → vertices

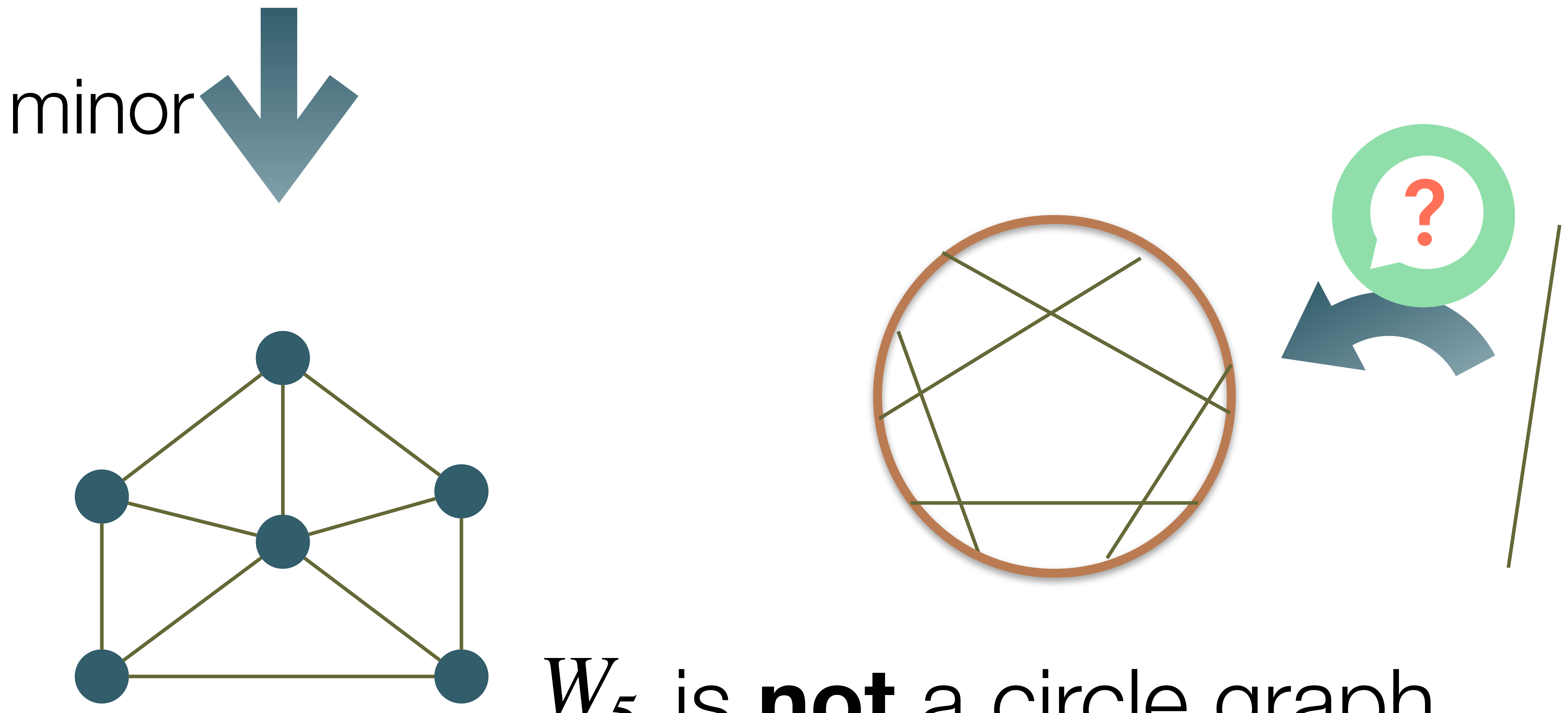
intersect → edges



A minor of a circle graph is **not** necessarily a circle graph.



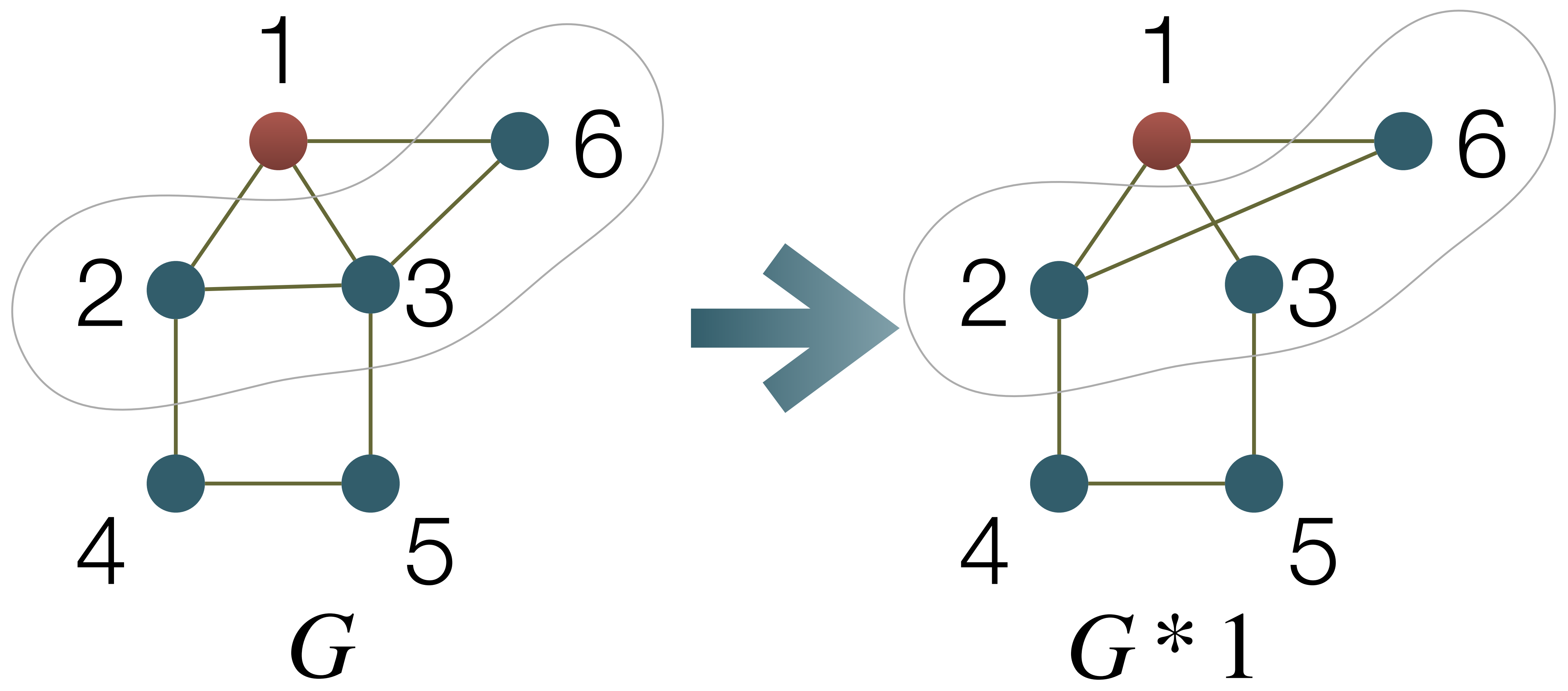
K_6 is a circle graph



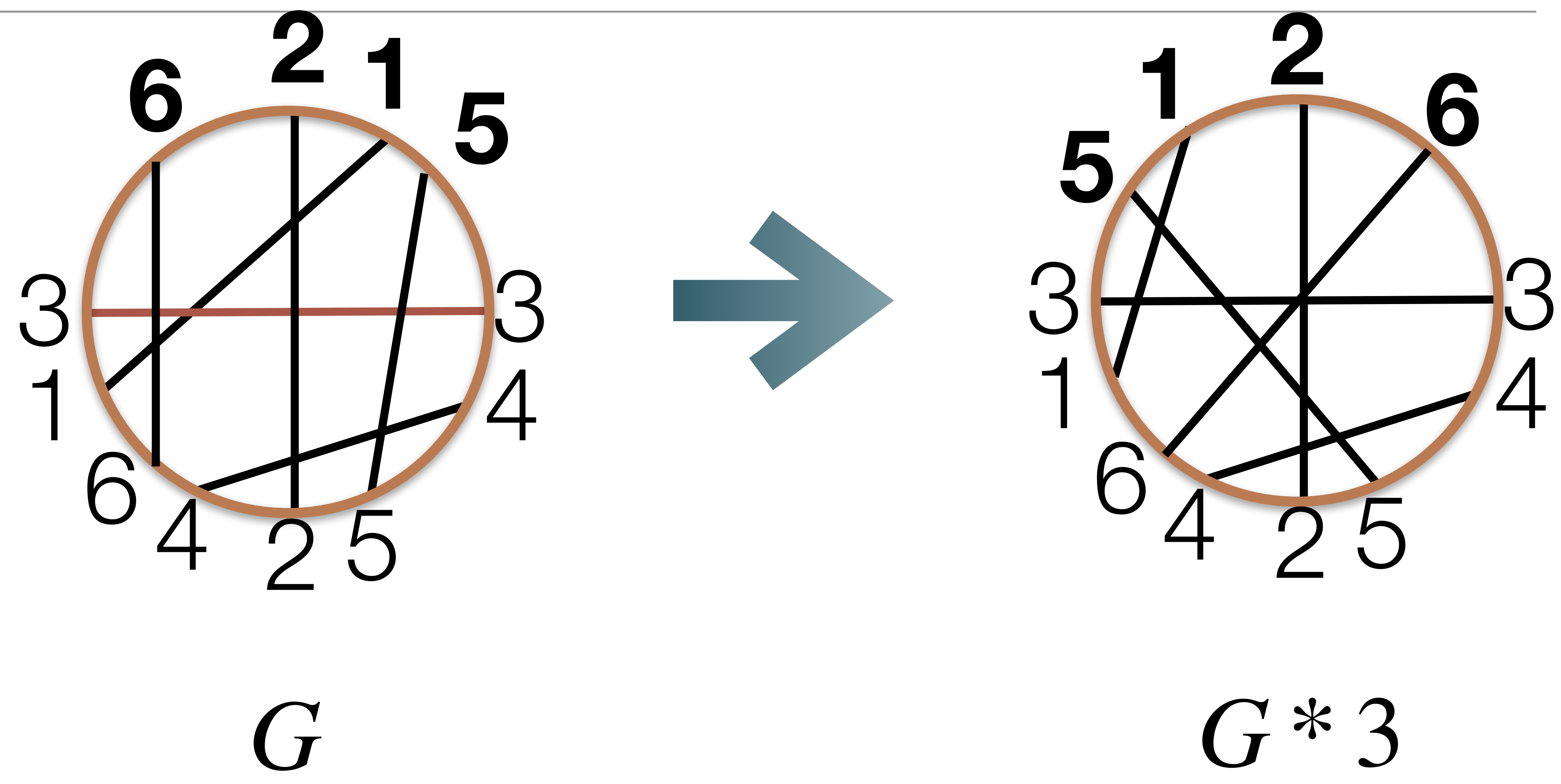
W_5 is **not** a circle graph

Impossible to characterize circle graphs in terms of forbidden minors

Local complementation and **vertex-minors**



"local complementation at 1"

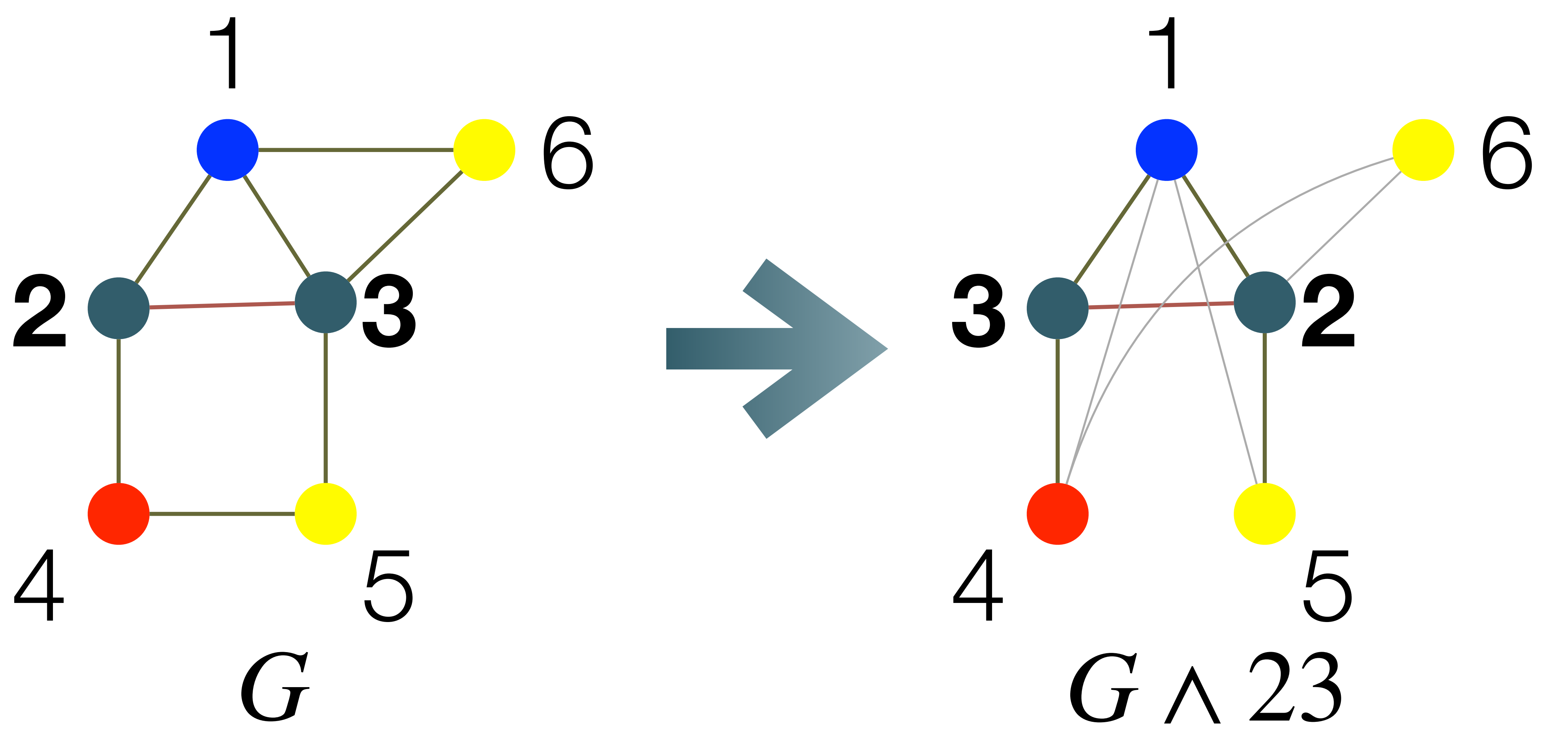


If G is a circle graph,
then G*v is a circle graph.

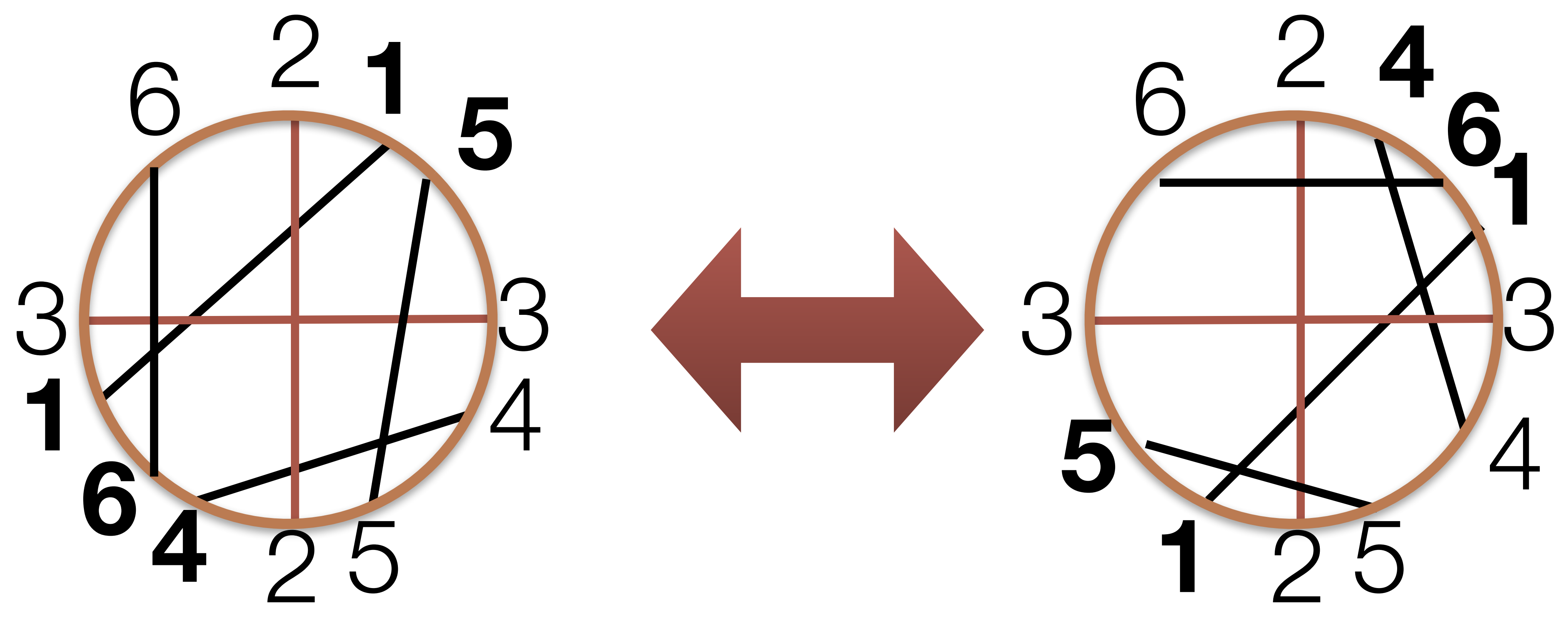
H is a **vertex-minor** of G if H can be obtained from G by a sequence of **local complementations** and **vertex deletions**

Easy Observation:
If H is a vertex-minor of G and G is a circle graph,
then H is a circle graph.

Pivot and **pivot-minors**



"pivot an edge 23"

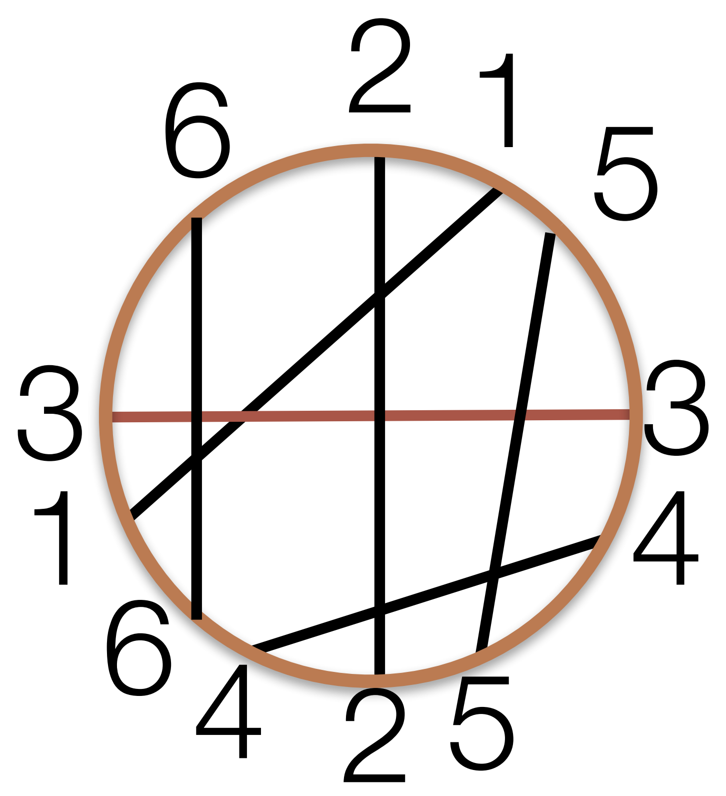


If G is a circle graph,
then G∧vw is a circle graph.

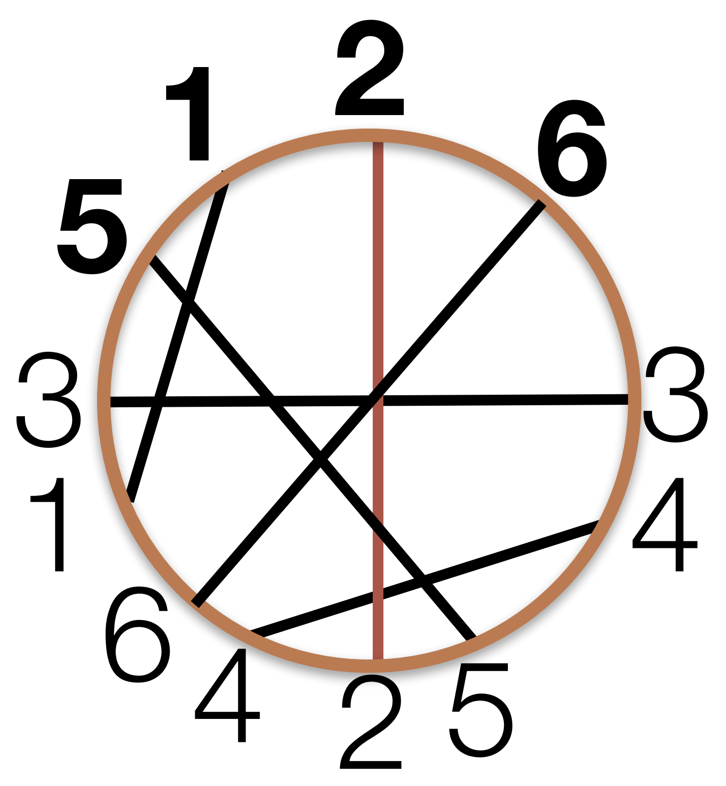
H is a **pivot-minor** of G if H can be obtained from G by a sequence of
pivots and
vertex deletions

Easy Observation:
If H is a pivot-minor of G and G is a circle graph,
then H is a circle graph.

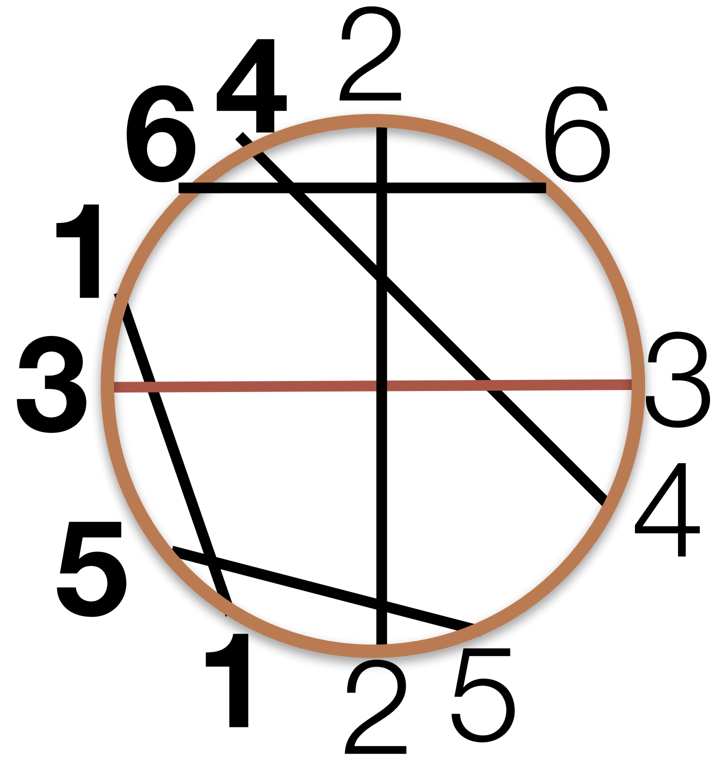
Every pivot-minor is a vertex-minor



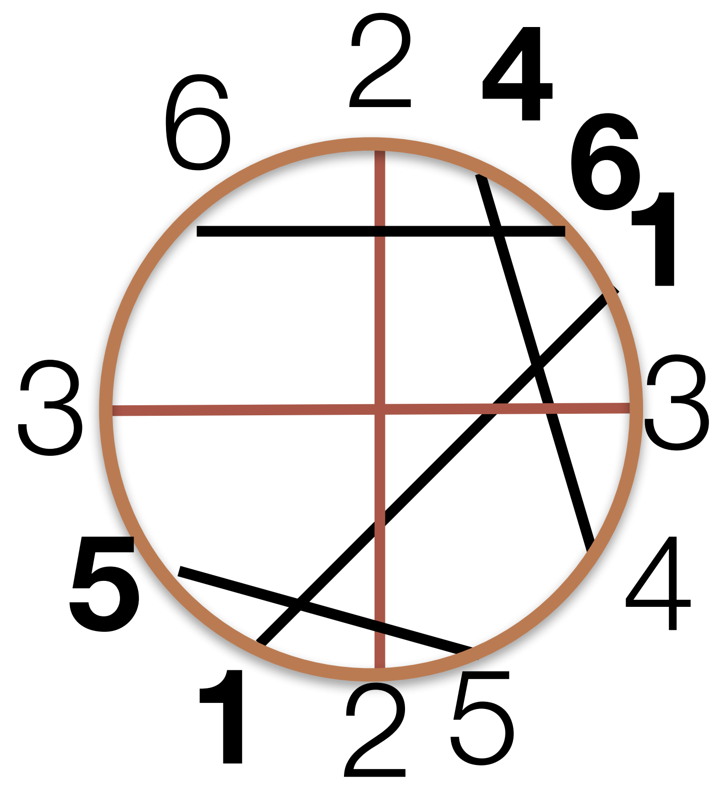
G



$G * 3$



$G * 3 * 2$



$G * 3 * 2 * 3$
$= G \wedge 23$

In general,

$$G \wedge uv = G * u * v * u$$

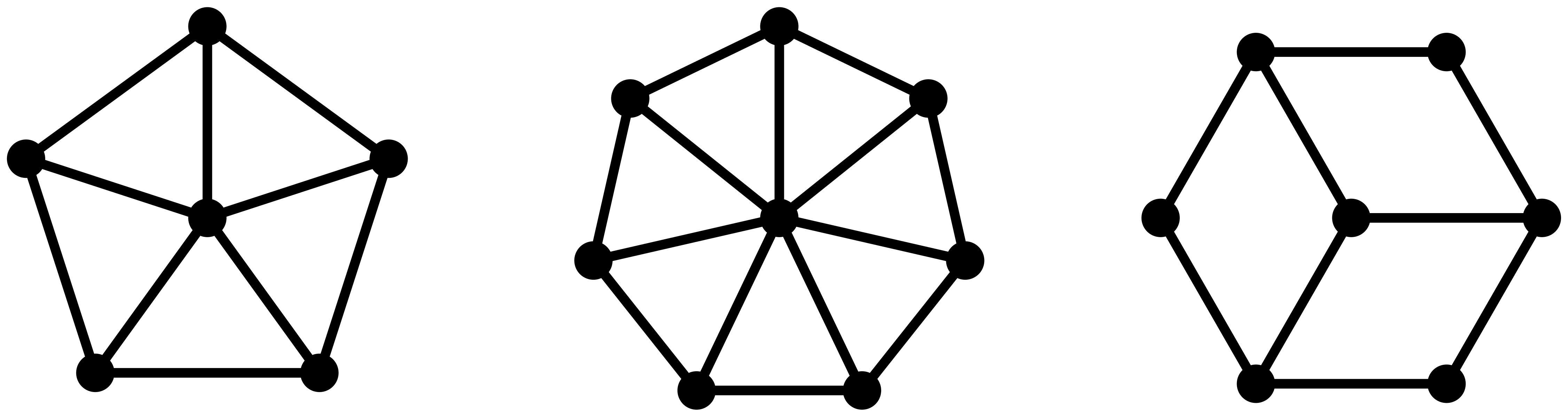
Every pivot-minor is a vertex-minor.

(The converse is false.)

Forbidden **vertex-minors** for circle graphs

Theorem (Bouchet 1994)
G is a **circle graph**
if and only if
G has no **vertex-minor** isomorphic to

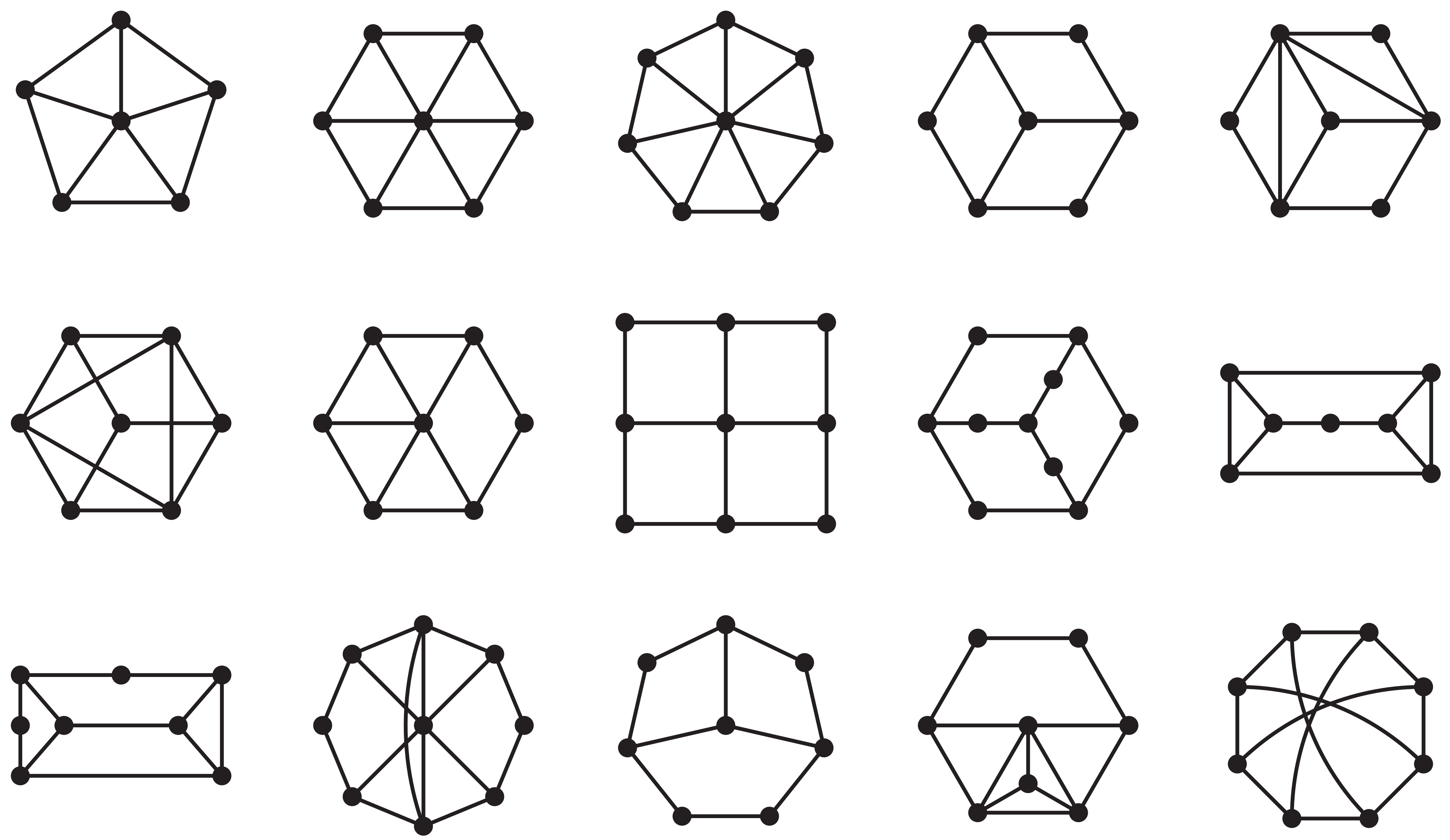
W_5 W_7 BW_3

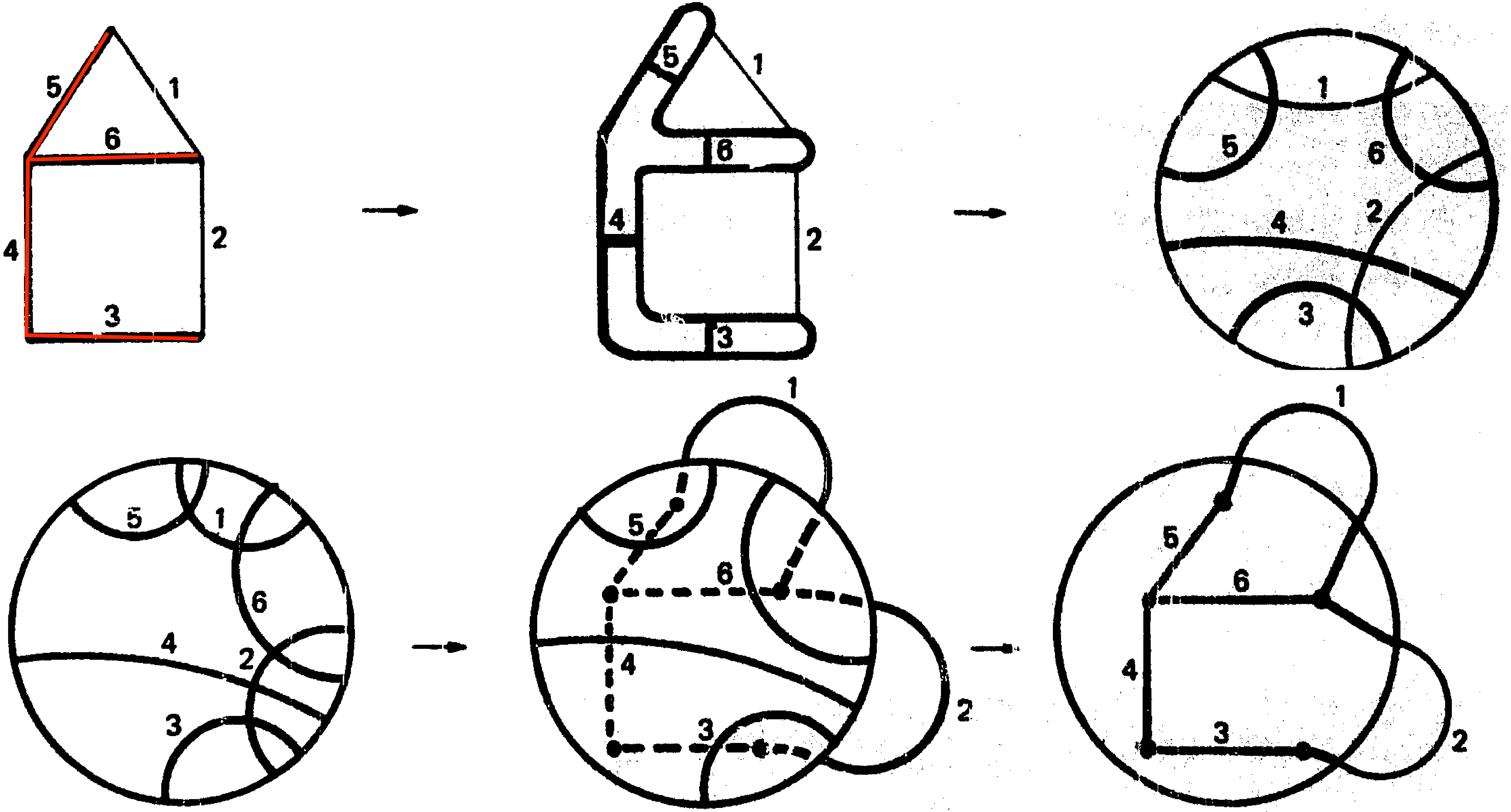


Three vertex-minor minimal non-circle graphs

Forbidden **pivot-minors** for circle graphs

Theorem (Geelen, O. 2009)
G is a **circle graph**
if and only if
G has no **pivot-minor** isomorphic to

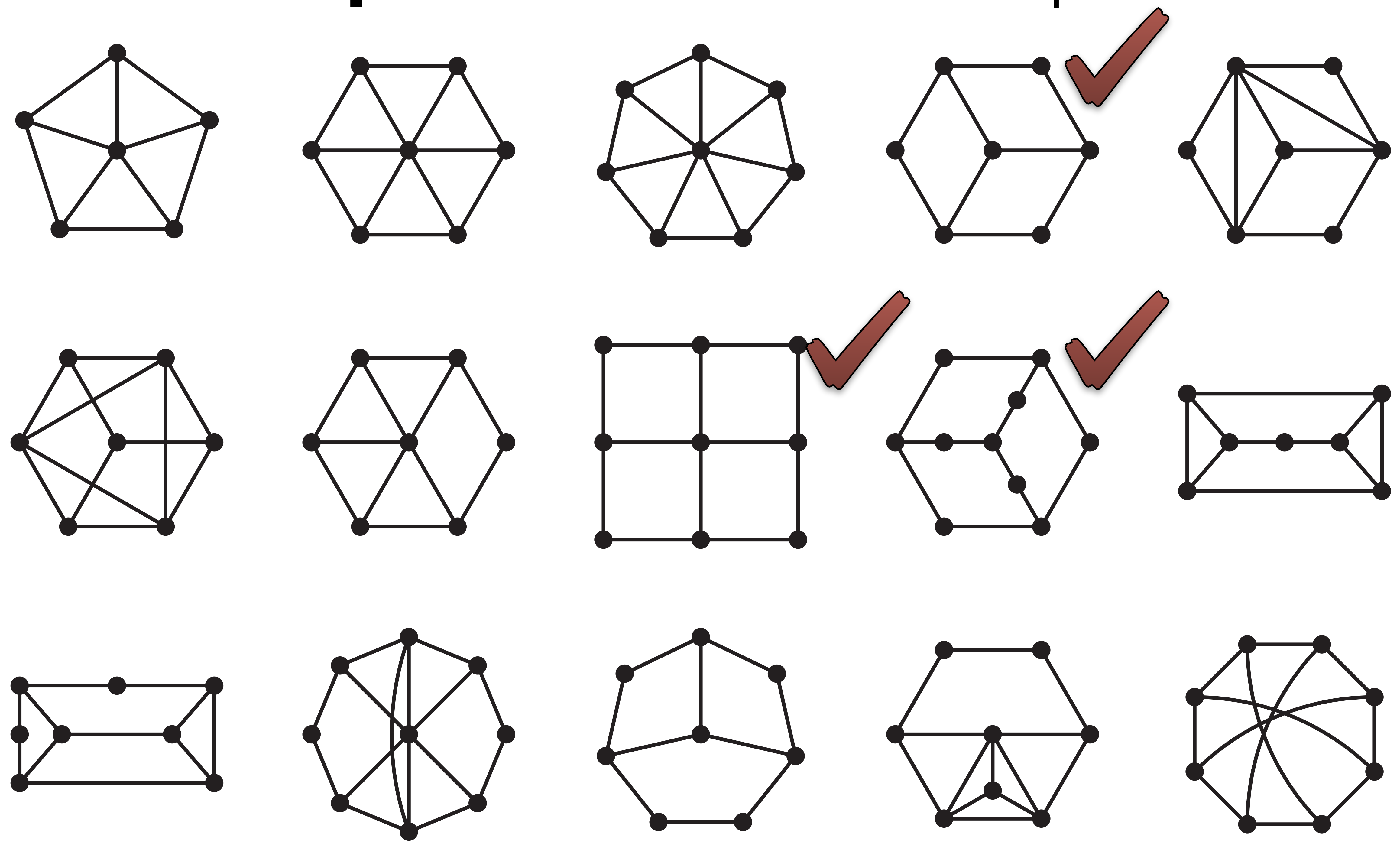




Theorem (de Fraysseix 1981)
A bipartite graph is a circle graph
if and only if it is a **fundamental graph** of a planar graph.

Forbidden **pivot-minors** for circle graphs

Theorem (Geelen, O. 2009)
G is a **circle graph**
if and only if
G has no **pivot-minor** isomorphic to



✓ 3 bipartite graphs

Fundamental graphs of the Fano matroid

$M(K_5)$

$M(K_{3,3})$

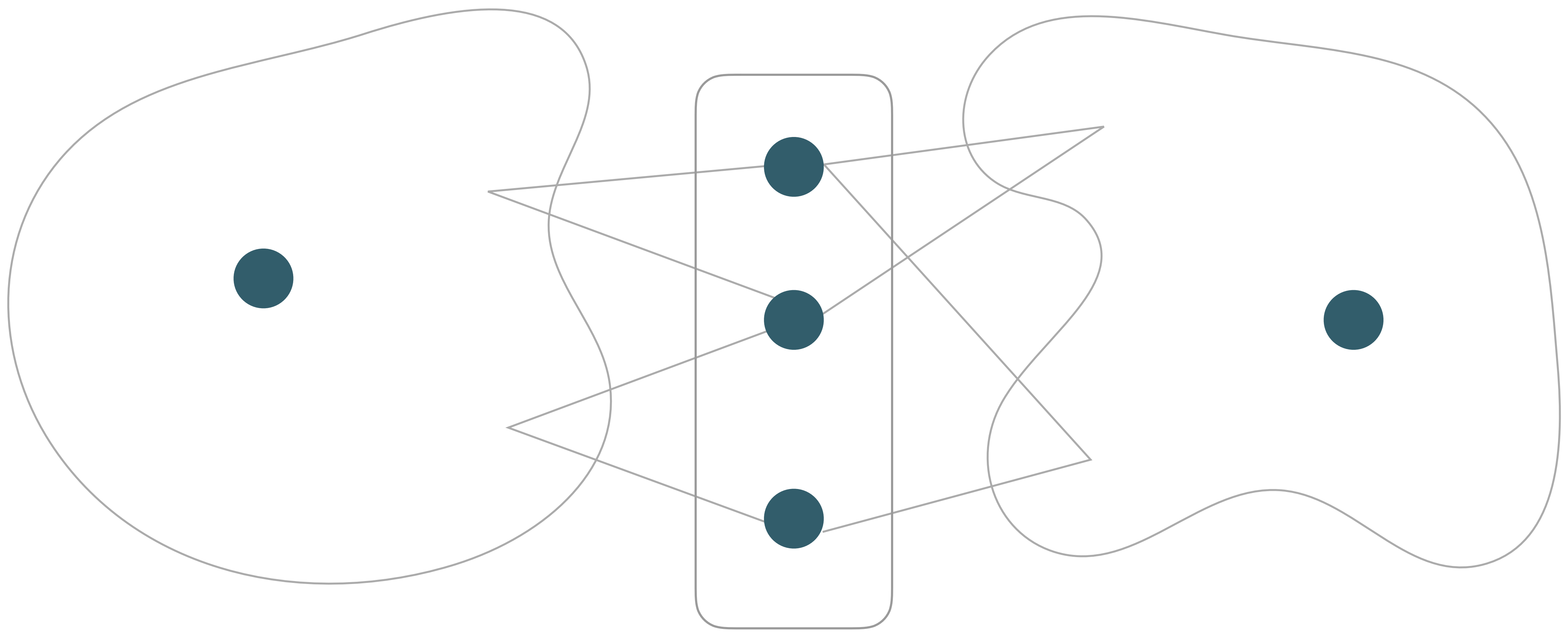
Implies
Kuratowski's theorem

Connectivity Functions

Vertex Connectivity function

"connectivity function" defined on the edge set of G

$\eta_G(S)$=#vertices meeting both S and its complement



If S is a set of edges and $\eta_G(S) \leq k$
then

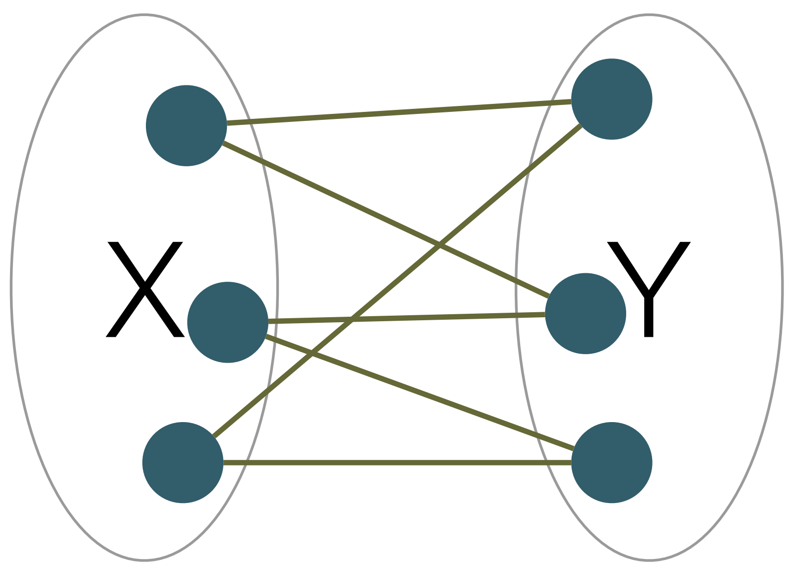
$$\eta_{G/e}(S - \{e\}) \leq k$$
$$\eta_{G\backslash e}(S - \{e\}) \leq k$$

Observation:
The "vertex connectivity function" does not increase while taking minors

"Vertex Connectivity Function" is well studied with respect to minors

Cut-rank function

A connectivity measure, small for simple but possibly dense edge-cuts

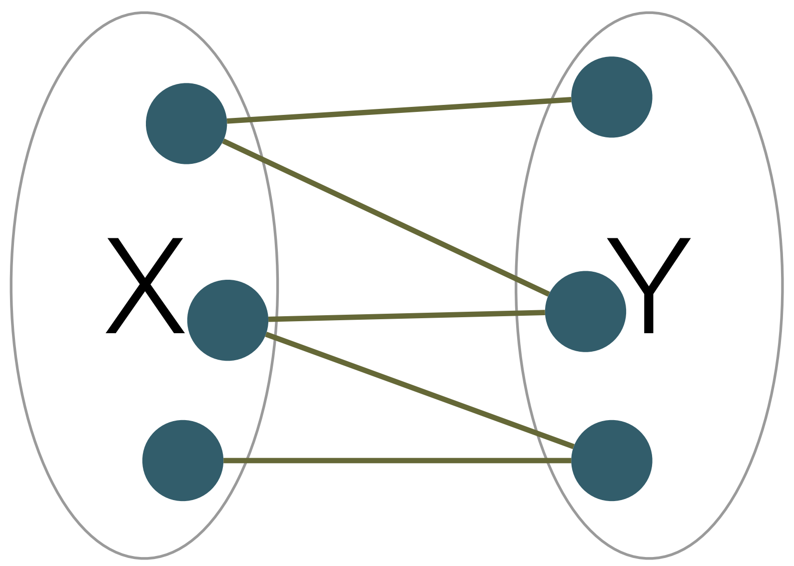


G

$$\rho_G(X) = \text{rank} \begin{array}{cc} & Y \\ X & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \end{array} = 2$$

Cut-rank function of G: rank of a X*(V-X) 0-1 matrix over the binary field

minor

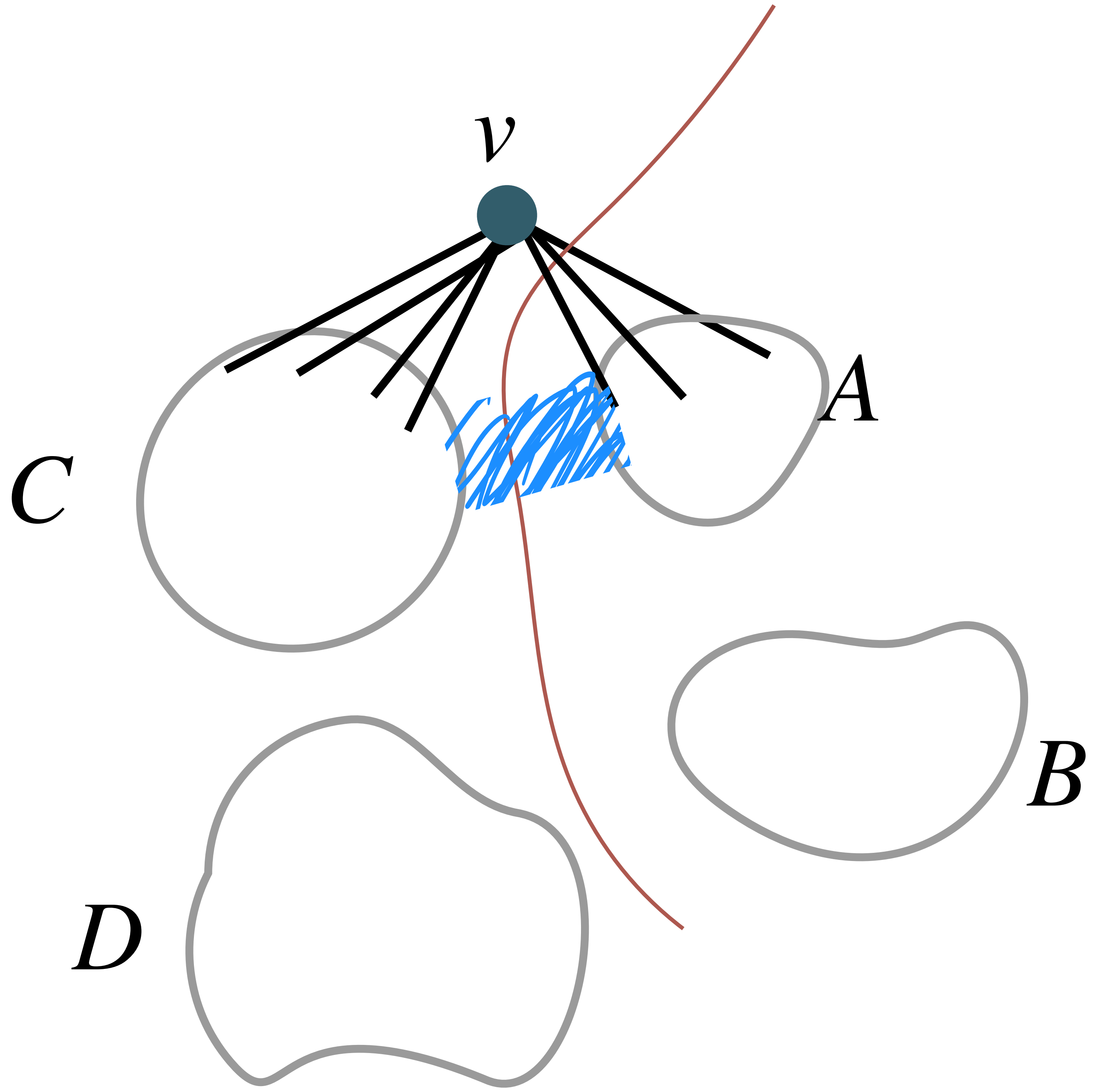


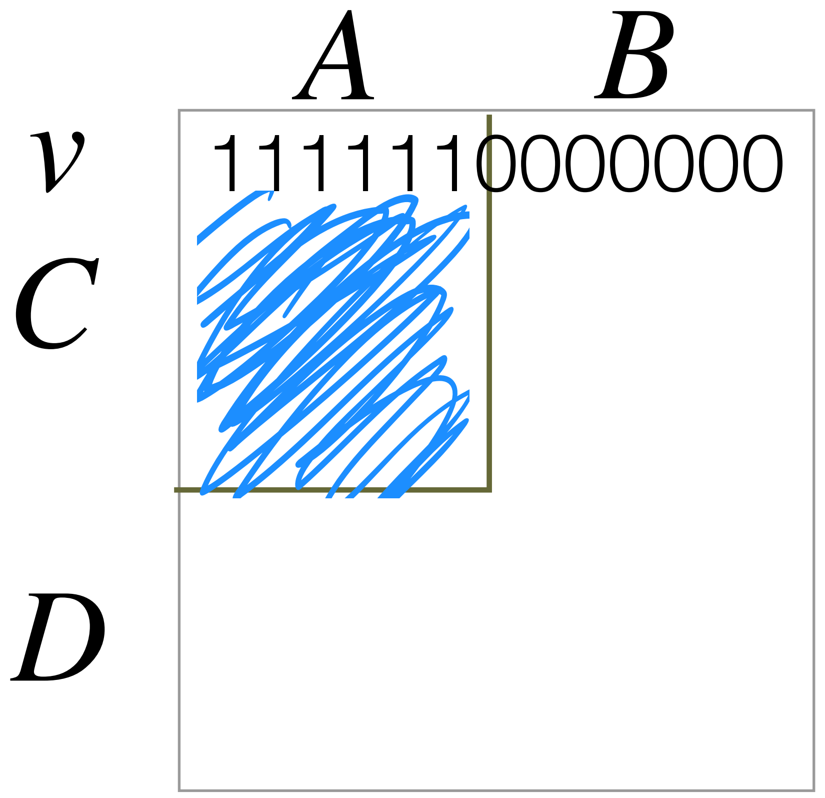
$H = G \backslash e$

$$\rho_H(X) = \text{rank} \begin{array}{cc} & Y \\ X & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array} = 3$$

The cut-rank function may increase while taking minors!

Cut-rank function and **vertex-minors**





$$\rho_G(X) = \rho_{G*v}(X)$$

The cut-rank function is **invariant under** taking **local complementations**

If H is a vertex-minor of G, then for all X,

$$\rho_H(X') \leq \rho_G(X)$$

for its corresponding X'.

Rank connectivity

G is **k-rank-connected** if $\rho_G(X) < k \Rightarrow \rho_G(X) = \min(|X|, |V - X|)$

Easy: G is **1-rank-connected**
if and only if
G is connected

2-rank-connected
= "prime with respect to **split decompositions**" (Cunningham 1982)

Cunningham (1982): Description of a canonical decomposition
of a connected graph into cuts of cut-rank 1

Tools for 2-rank-connected graphs

Compare with 3-connected graphs

Chain Theorem

Every **3-connected** graph G has a minor H such that

* $|E(H)|=|E(G)|-1$ and
* H is simple 3-connected

unless G=wheel.

Tutte (1961)

Every **2-rank-connected** graph G has a vertex-minor H such that

* $|V(H)|=|V(G)|-1$ and
* H is 2-rank-connected

unless $|V(G)|\leq 5$.

Bouchet (1987)
Allys (1994)

Splitter Theorem

If H is a **3-connected minor** of a 3-connected graph G, then G has a minor G' such that

* $|E(G')|=|E(G)|-1$,
* G' is 3-connected,
* H is isomorphic to a minor of G'

unless H=wheel or $|V(G)|=|V(H)|$

Seymour (1980), Negami (1982)

If H is a **2-rank-connected** vertex-minor of a 2-rank-connected graph G, then G has a vertex-minor G' such that

* $|V(G')|=|V(G)|-1$,
* G' is 2-rank-connected,
* H is isomorphic to a vertex-minor of G'

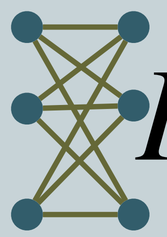
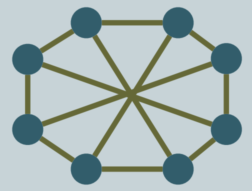
unless $|V(H)|<5$ or $|V(G)|=|V(H)|$.

Bouchet (unpublished), Geelen (1995)

Applications of Splitter theorems

Graphs with no K_5 minor =

Closure by disjoint union, 1-, 2-, 3-sums

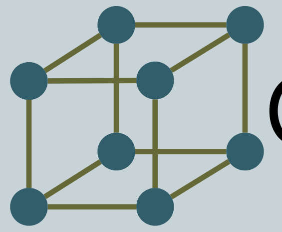
planar graphs  $K_{3,3}$  V_8

Wagner (1937)

Graphs with no W_5 vertex-minor =

Closure by disjoint union and 1-joins

circle graphs W_7 BW_3

 cube

Geelen (1995)

Unavoidable structure in a large graph

Every large graph has

K_n or $\overline{K_n}$

as an induced subgraph.

Ramsey (1930)

Every large **3-connected** graph has

W_k or $K_{3,k}$

as a minor.

Oporowski, Oxley, Thomas (1993)

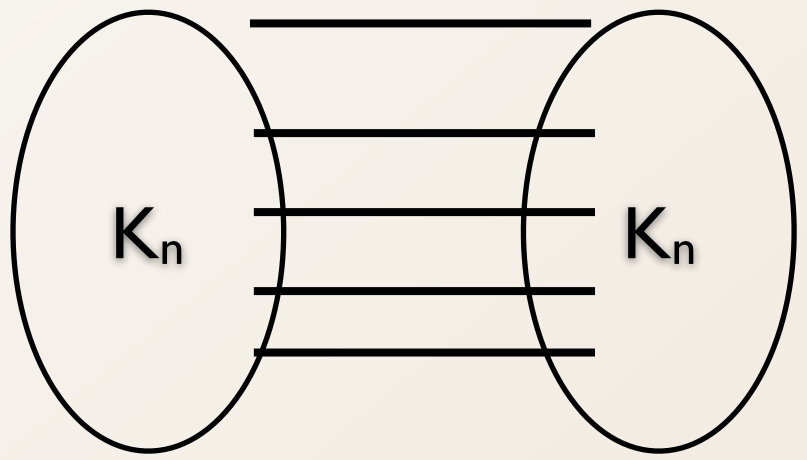
Theorem (Kwon, O. 2014)

Every large **2-rank-connected** graph has

C_n or $K_n ⊟ K_n$

as a vertex-minor.

$K_n ⊟ K_n$



Width parameters

Branch-width and **Rank-width**

Branch-width of G= min k such that E(G) can be recursively partitioned into subsets of "**vertex connectivity**" $\leq k$ until each set becomes a singleton

Robertson, Seymour (1991)

"**branch-decomposition**" of width k

Important tool for graph minor theory together with tree-width

Rank-width of G = min k such that V(G) can be recursively partitioned into subsets of "**cut-rank**" $\leq k$ until each set becomes a singleton

O., Seymour (2006)

"**rank-decomposition**" of width k

Important tool for graph vertex-minor theory

Algorithmic applications on graph classes of small rank-width

Meta-theorems

*If the input graph has **small** 1*
*and **is given with its decomposition of small width**, then*
every problem expressible in 2 can be solved in poly time.

Courcelle (1990)

1 branch-width / tree-width

2 monadic second-order logic

Courcelle, Makowsky, Rotics (2000)

1 rank-width / clique-width

2 monadic second-order logic with no edge-set quantification

Example

$$\exists X_1 \exists X_2 (\forall v \forall w (v \in X_1 \wedge w \in X_1 \wedge v \neq w \Rightarrow v \nsim w) \wedge \forall v' \forall w' (v' \in X_1 \wedge w' \in X_1 \wedge v' \neq w' \Rightarrow v \nsim w'))$$

For a graph given by a list of edges,
how do we find a good decomposition of small width?

Algorithm to find a branch-decomposition / rank-decomposition

Theorem (Jeong, Kim, O. 2018+)

An efficient algorithm on "*subspace arrangements*"
to find a branch-decomposition of width≤k
if it exists.

- **Corollary**: $O(f(k)n^3)$-time algorithm to find a **branch-decomposition** of width≤k, if it exists, for a hypergraph.

- **Corollary**: $O(f(k)n^3)$-time algorithm to find a **rank-decomposition** of width≤k, if it exists, for a graph.

Previous algorithm for branch-width:
Thilikos, Bodlaender (2000): 50-page technical report only for graphs
→ Not appeared in a journal yet

Previous algorithm for rank-width:
Hliněný and O. (2008): Uses a non-trivial result on forbidden vertex-minors for rank-width≤k
and use the meta-theorem of Courcelle et al.

Understanding rank-width in terms of vertex-minors / pivot-minors

Theorem (O. 2005)
For all k, every pivot-minor-minimal graph of rank-width>k
has at most $O(6^k)$ vertices.

Previous algorithm for rank-width:
Hliněný and O. (2008): Uses a non-trivial result on forbidden pivot-minors for rank-width≤k
and use the meta-theorem of Courcelle et al.

Theorem (Kwon, O. 2014)
If G has rank-width≤k, then
G is a pivot-minor of a graph of tree-width≤2k.

PHYSICAL REVIEW A **75**, 012337 (2007)

Classical simulation versus universality in measurement-based quantum computation

M. Van den Nest,[1] W. Dür,[1,2] G. Vidal,[3] and H. J. Briegel[1,2]

[1] *Institut für Quantenoptik und Quanteninformation der Österreichischen Akademie der Wissenschaften, Innsbruck, Austria*

[2] *Institut für Theoretische Physik, Universität Innsbruck, Technikerstraße 25, A-6020 Innsbruck, Austria*

[3] *School of Physical Sciences, University of Queensland, QLD 4072, Australia*

(Received 19 September 2006; published 31 January 2007)

We investigate for which resource states an efficient classical simulation of measurement-based quantum computation is possible. We show that the *Schmidt-rank width*, a measure recently introduced to assess universality of resource states, plays a crucial role in also this context. We relate Schmidt-rank width to the optimal description of states in terms of tree tensor networks and show that an efficient classical simulation of measurement-based quantum computation is possible for all states with logarithmically bounded Schmidt-rank width (with respect to the system size). For graph states where the Schmidt-rank width scales in this way, we efficiently construct the optimal tree tensor network descriptions, and provide several examples. We highlight parallels in the efficient description of complex systems in quantum information theory and graph theory.

DOI: 10.1103/PhysRevA.75.012337 PACS number(s): 03.67.Lx, 75.10.Hk, 75.10.Pq

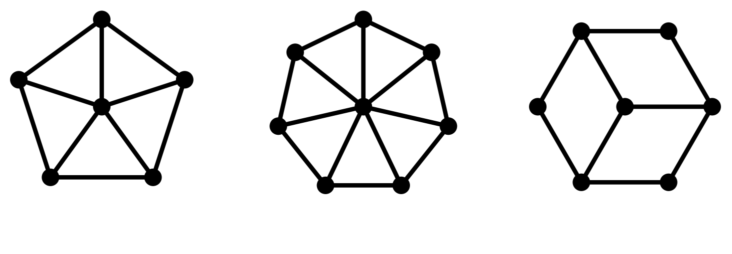
Theorem 5. Let $|G\rangle$ be a graph state on n qubits. If the rank width of G grows at most logarithmically with n, then any MQC on $|G\rangle$ can efficiently be simulated classically.

Extending the graph minors project of Robertson and Seymour

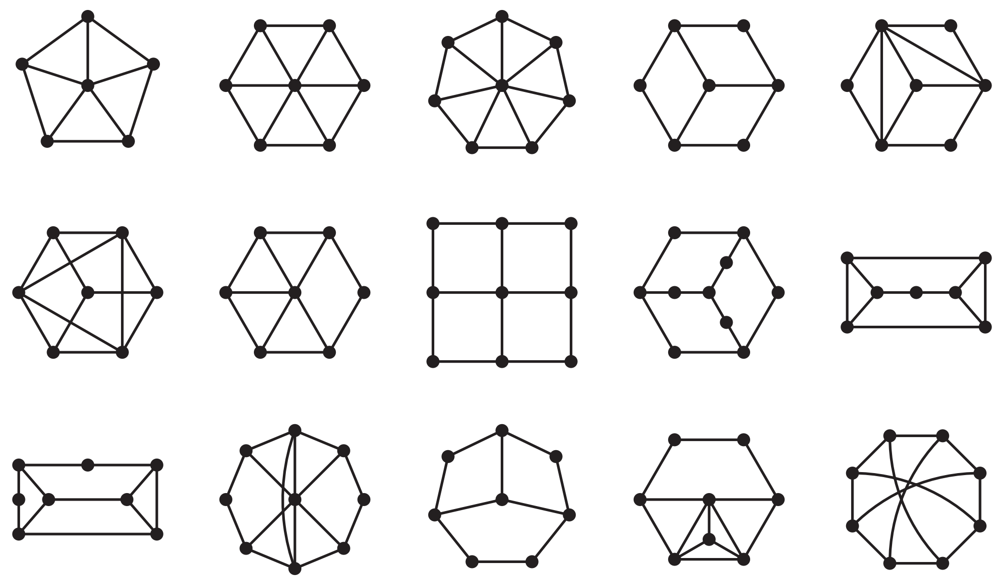
Why do we have only **finitely many forbidden** vertex-minors / pivot-minors?

Theorem (Bouchet 1994)
G is a **circle graph**
if and only if
G has no **vertex-minor** isomorphic to

W_5 W_7 BW_3



Theorem (Geelen, O. 2009)
G is a **circle graph**
if and only if
G has no **pivot-minor** isomorphic to



Analogue of the **grid theorem**

Theorem (Robertson and Seymour 1986)
For each **planar graph** H,
there exists a constant c=c(H) such that
every graph of branch-width>c has H as a **minor**.

Conjecture
For each **bipartite circle graph** H,
there exists a constant c=c(H) such that
every graph G of rank-width>c has H as a **pivot-minor**.

Weaker Conjecture
For each **circle graph** H,
there exists a constant c=c(H) such that
every graph of rank-width>c has H as a **vertex-minor**.

Solved by Geelen, Kwon, McCarty, and Wollan (2018+)

Known cases: when **G** is

- **bipartite graphs** (Courcelle, O. 2007 by a theorem on binary matroids)
- **line graphs** (O. 2009)
- **circle graphs** (O. 2009 by a theorem in Thor Johnson's Ph.D. thesis)

Well-quasi-ordering theorems and conjectures

Every class of graphs closed under taking 1
*can be characterized in terms of a **finite** list of forbidden* 1

Equivalently; we say C is **well-quasi-ordered** under 1 if

In every infinite sequence of graphs G_1, G_2, ... (in C)
there exist $i<j$ such that G_i is a 1 *of G_j.*

1 minor — **Graph minor theorem**
(Robertson and Seymour. Graph minors XX. 2004)

1 pivot-minor — Conjecture

1 vertex-minor — Weaker Conjecture

Jim Geelen Bert Gerards Geoff Whittle

Known cases: when C is

- bipartite graphs (by binary matroids)
- line graphs (by group-labelled graphs)
- **bounded rank-width** (O. 2008)
- circle graphs (by Graph Minors XXIII)

Complexity of finding minors, vertex-minors, and pivot-minors

For each fixed graph H, we can decide in polynomial time whether an input G has a 1 *isomorphic to H.* ?

1 Minor

Theorem (Robertson and Seymour 1995)
For each fixed graph H,
we can decide in time O(n^3)
whether an input G has a minor isomorphic to H.

Corollary
For every class C of graphs closed under taking minors, there exists an O(n^3)-time algorithm to decide whether an input graph belongs to C.

1 Vertex-minor: Open

P if $|V(H)| \leq 6$
Kang, Kwon, O. (2018+)

P if H is a circle graph
Geelen, Kwon, McCarty, and Wollan (2018+)

1 Pivot-minor: Open

P if $|V(H)| \leq 4$,
$H \notin \{K_4, C_3 + K_1, 4K_1\}$
(Dabrowski, Dross, Jeong, Kanté, Kwon, O., Paulusma 2018)

If H is not fixed, ...

If both H and G are given as an input, then it is NP-complete to decide whether H is isomorphic to a **1** *of G.*

1 Minor: True

H=Cycle of length n

→ reduction to Hamiltonian cycle problem

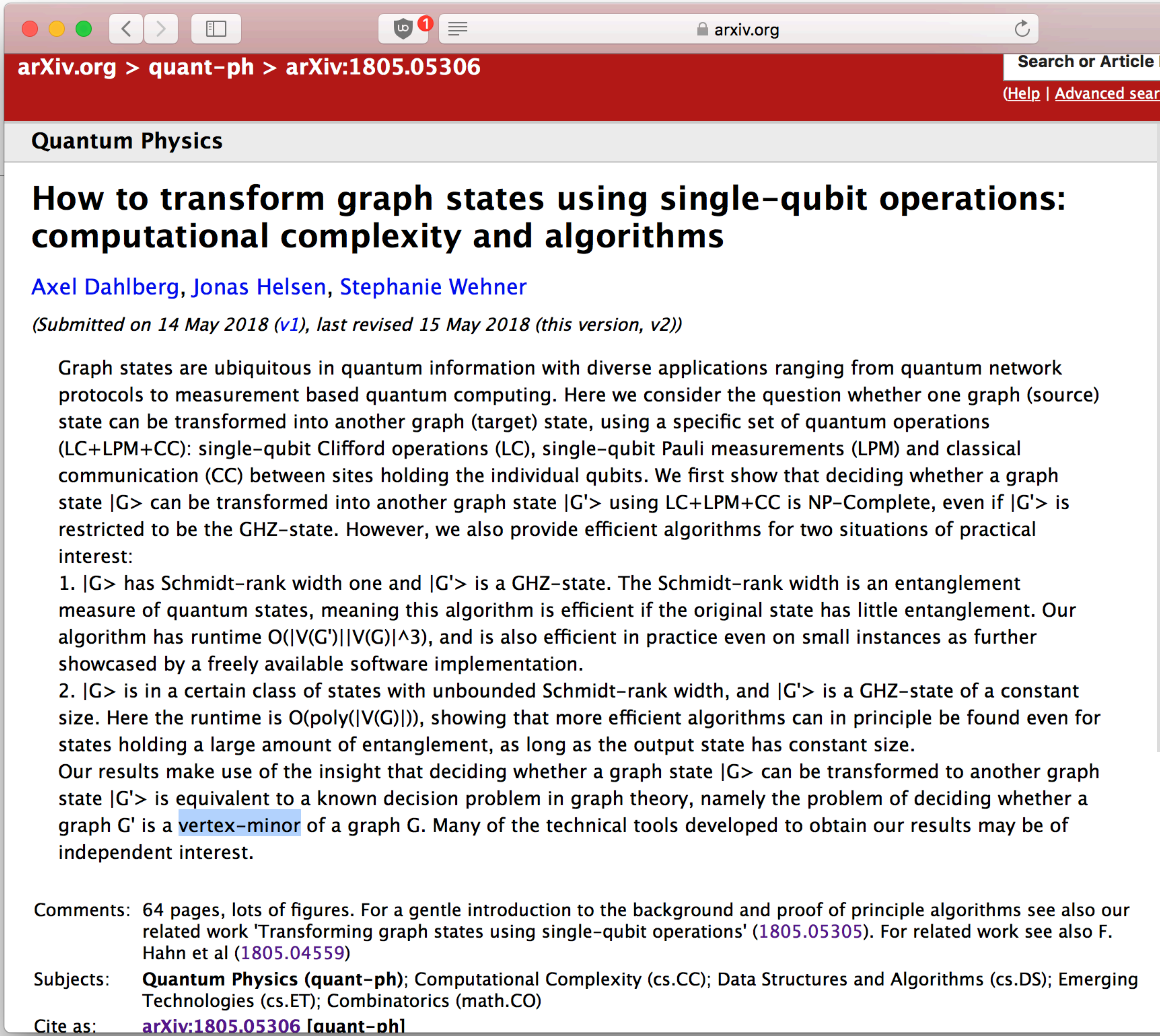
1 Vertex-minor

Axel Dahlberg (personal comm., 2018)

related: arXiv:1805.05306 (2018)

1 Pivot-minor

Dabrowski, Dross, Jeong, Kanté, Kwon, O., Paulusma 2018



arxiv.org

arXiv.org > quant-ph > arXiv:1805.05306

Search or Article

(Help | Advanced sear

Quantum Physics

How to transform graph states using single-qubit operations: computational complexity and algorithms

Axel Dahlberg, Jonas Helsen, Stephanie Wehner

(Submitted on 14 May 2018 (v1), last revised 15 May 2018 (this version, v2))

Graph states are ubiquitous in quantum information with diverse applications ranging from quantum network protocols to measurement based quantum computing. Here we consider the question whether one graph (source) state can be transformed into another graph (target) state, using a specific set of quantum operations (LC+LPM+CC): single-qubit Clifford operations (LC), single-qubit Pauli measurements (LPM) and classical communication (CC) between sites holding the individual qubits. We first show that deciding whether a graph state |G> can be transformed into another graph state |G'> using LC+LPM+CC is NP-Complete, even if |G'> is restricted to be the GHZ-state. However, we also provide efficient algorithms for two situations of practical interest:

1. |G> has Schmidt-rank width one and |G'> is a GHZ-state. The Schmidt-rank width is an entanglement measure of quantum states, meaning this algorithm is efficient if the original state has little entanglement. Our algorithm has runtime O(|V(G')||V(G)|^3), and is also efficient in practice even on small instances as further showcased by a freely available software implementation.

2. |G> is in a certain class of states with unbounded Schmidt-rank width, and |G'> is a GHZ-state of a constant size. Here the runtime is O(poly(|V(G)|)), showing that more efficient algorithms can in principle be found even for states holding a large amount of entanglement, as long as the output state has constant size.

Our results make use of the insight that deciding whether a graph state |G> can be transformed to another graph state |G'> is equivalent to a known decision problem in graph theory, namely the problem of deciding whether a graph G' is a vertex-minor of a graph G. Many of the technical tools developed to obtain our results may be of independent interest.

Comments: 64 pages, lots of figures. For a gentle introduction to the background and proof of principle algorithms see also our related work 'Transforming graph states using single-qubit operations' (1805.05305). For related work see also F. Hahn et al (1805.04559)

Subjects: **Quantum Physics (quant-ph)**; Computational Complexity (cs.CC); Data Structures and Algorithms (cs.DS); Emerging Technologies (cs.ET); Combinatorics (math.CO)

Cite as: arXiv:1805.05306 [quant-ph]

Weakening of conjectures on induced subgraphs

χ-boundedness

C: proper class of graphs closed under taking induced subgraphs

Problem (Gyárfás): For which C do we have a function f such that

$$\chi(G) \leq f(\omega(G))$$

for all G in C?

"χ-bounded"

Conjecture (Geelen 2009)
True if C is closed under taking vertex-minors.

Equivalent conjecture:
For every graph H, there exists f such that if G has no H vertex-minors, then

$$\chi(G) \leq f(\omega(G))$$

Known cases:

- C={circle graphs} (Gyárfás 1985)
- C={rank-width≤k} (Dvořák and Král' 2010)
- H=fan (Ilkyoo Choi, Kwon, O. 2017)
- H=**wheel** (Hojin Choi, Kwon, O., Wollan 2018+)
- H=circle graph (Geelen, Kwon, McCarty, and Wollan 2018+)

Erdős-Hajnal property

C: proper class of graphs closed under taking induced subgraphs

C has the **Erdős-Hajnal property** if there exists c>0 such that every graph G in C has a clique or a stable set of size $> |V(G)|^c$

Erdős-Hajnal Conjecture (1989)
Every such C has the Erdős-Hajnal property.

Theorem (Chudnovsky, O. 2018+)
True if C is closed under taking vertex-minors.

Question:
Is it true if C is closed under taking pivot-minors?

Theorem (Kim, O. 2019+)
True if C is closed under taking pivot-minors and C has no cycle of length k.

Polynomially χ-boundedness

C: proper class of graphs closed under taking induced subgraphs

Problem : For which C do we have a **polynomial** function f such that

$$\chi(G) \leq f(\omega(G))$$

for all G in C?

"**polynomially** χ-bounded"

Question (Louis Esperet)

Is every χ-bounded class polynomially χ-bounded?

Polynomially χ-bounded

$\Rightarrow$ Erdős-Hajnal property

No counterexample is known yet.

Conjecture:

For every graph H, there exists a **polynomial f** such that if G has no H vertex-minors, then

$$\chi(G) \leq f(\omega(G))$$

Known cases:

- H=**cycle** (Ringi Kim, O-joung Kwon, O., Vaidy Sivaraman 2018+)

Thank you for your attention