Complexity-Theoretic Barriers for Prices and Mechanisms

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Overview

1. "Why Prices Need Algorithms" (w/Talgam-Cohen, EC '15)
   - from complexity separations to non-existence results for Walrasian (i.e., market-clearing) equilibria
2. "Barriers to Near-Optimal Equilibria" (FOCS '14)
   - from communication lower bounds to lower bounds on the price of anarchy
3. "The Borders of Border's Theorem" (w/Gopalan and Nisan, EC '15)
   - from complexity separations to impossibility results for "nice descriptions" of incentive-compatible mechanisms

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Walrasian Equilibria

Setup: n bidders, m items to allocate. (*indivisible* items)

- bidder i has valuation $v_i(S)$ for each bundle S of items
- allocations ⇔ partitions $S_1,\ldots,S_n$ of items

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Walrasian equilibrium:

- allocation $S_1,\ldots,S_n$ and prices p on items s.t..

(1) every bidder gets favorite bundle

(maximizes $v_i(S) - \sum_{j \in S} p_j$ over bundles S)

(2) market clears (unsold items have price 0)

Non-Existence of Walrasian Equilibria

Easy fact: in general, Walrasian equilibria need not exist.

- 2 bidders (1 an 2), 2 items (A and B)
- "single-minded" bidder: $v_1(AB) = 3$, else $v_1(S)=0$
- "unit-demand" bidder: $v_2(A) = v_2(B) = v_2(AB) = 2$
- in allocation where 1 gets A and B:
  - to deter bidder #2, need prices of A and B at least 2 each
  - then AB too expensive tor #1
- in allocations where 1 doesn't get A and B:
  - similar case analysis

Characterizing Existence

Theorem 1: [Kelso/Crawford 82, Gul/Stacchetti 99] If all v_i's satisfy a "gross substitutes" condition, then a Walrasian equilibrium is guaranteed to exist.

Theorem 2: [Gul/Stacchetti 99, Milgrom 00] partial converse.

Follow-up results: "Tables and chairs" [Sun-Yang'06] and generalizations [Teytelboym'14], GGS [Ben-Zwi/Lavi/Newman '13], complements [Parkes-Ungar'00, Sun-Yang'14], tree valuations [Candogan'15], graphical valuations [Candogan'14], feature-based valuations [Candogan-Pekec'14], ..., demand types w/unimodular vectors [Baldwin-Klemperer'18]

Main Result

Theorem: Suppose that, for a class V of valuations, "welfare maximization" does not reduce to "utility maximization" (polynomial Turing reductions).

Then, there are markets with valuations in V without Walrasian equilibria.

- necessary condition for existence: welfare-maximization no harder than utility-maximization
- connects a purely economic question (existence of equilibria) to a purely algorithmic one

Utility/Welfare Maximization

Utility maximization problem: (with 1 agent)

- input = a valuation v (succinctly described), item prices p
- output = favorite bundle (argmax_S v(S) - $\sum_{j \in S} p_j$)

Welfare maximization problem: (with n agents)

- input = valuations $v_1,\ldots,v_n$ (succinctly described)
- output = optimal allocation (argmax $\sum_i v_i(S_i)$)
- generally only harder than utility-maximization

Examples

Single-minded bidders: agent i only wants the bundle T_i, $v_i(S)$ either v_i (if S includes T_i) or 0.

- utility maximization = trivial (either T_i or the empty set)
- welfare maximization = NP-hard (set packing)

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Budget-additive bidders: for item valuations $v_{i1},\dots,v_{im}$ and a budget b_i, $v_i(S) = \min\{\sum_{j \in S} v_{ij}, b_i\}$

- utility maximization = pseudo-poly-time (Knapsack)
- welfare maximization = strongly NP-hard (bin packing)

Proof Sketch

(Recall: Necessary condition for guaranteed existence – utility maximization as hard as welfare maximization)

1. Assume a Walrasian equilibrium is guaranteed to exist
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Fact 1: [Nisan/Segal 06] *fractional* welfare maximization reduces to utility maximization.

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Fact 1: [Nisan/Segal 06] *fractional* welfare maximization reduces to utility maximization.

Fact 2: [Bikhchandani-Mamer 97] Walrasian equilibrium exists ⇔ optimal fractional allocation = optimal integral allocation

Other Results

- Similar results for oracle models
- With more general anonymous prices Q, efficiently verifiable equilibria exist only when welfare maximization reduces to utility-maximization (with prices in Q)
- Complexity-theoretic explanation for why no useful generalizations of Walrasian equilibria: would require a non-standard polynomial-time algorithm for welfare-maximization

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Equilibria vs. Algorithms

Motivating question: are game-theoretic equilibria more powerful computationally than poly-time algorithms?

Recall: computing a (Nash) equilibrium is hard:

- e.g., computing a mixed Nash equilibrium of a 2-player game is PPAD-complete [Chen/Deng/Teng 06, Daskalakis/Goldberg/Papadimitriou 06]
- even harder with >2 players [Etessami/Yannakakis 07]

Goal: prove fundamental limits on what equilibria can do.

Results in a Nutshell

Meta-theorem: equilibria are generally bound by the same limitations as algorithms with polynomial computation or communication.

Meta-reason: equilibria are still "too easily computable" to overcome typical intractability results.

Caveats: requires that equilibria are

- guaranteed to exist (e.g., mixed Nash equilibria)
- can be efficiently verified

Combinatorial Auctions

Welfare-maximization: n bidders, m non-identical goods

- allocation = partition $S_1, S_2, \ldots, S_n$ of goods
- bidder i has valuation $v_i(S)$ (i.e., max willingness to pay) for each subset S of goods
  - [$\approx 2^m$ parameters]
  - (assume integral + bounded)
- welfare of allocation $S_1, S_2, \ldots, S_n$: $\sum_i v_i(S_i)$
  - goal is to allocate goods to (approximately) maximize this
  - want communication polynomial in n and m

When Do Simple Mechanisms Work Well?

Simultaneous First-Price Auction (S1A): [Bikhchandani 99]

- each bidder submits one bid per item
  - m bids used to summarize 2^m private parameters
- each item sold separately in a first-price auction

Question: what is the worst-case POA of S1A's?

- e.g., for mixed Nash equilibria (pure NE need not exist)
- *"price of anarchy (POA)" = welfare(OPT)/welfare(worst EQ)*

From Protocol Lower Bounds to POA Lower Bounds

Theorem: [Roughgarden 14] Suppose:

- no nondeterministic subexponential-communication protocol approximates the welfare-maximization problem (with valuations V) to within factor of α.
  - i.e., impossible to decide OPT $\geq W^*$ vs. OPT $\leq W^*/\alpha$

Then worst-case POA of ε-approximate mixed Nash equilibria of every "simple" mechanism is at least α.

- "simple" = sub-doubly-exponential number of actions per player
- ε can be as small as inverse sub-exponential in n and m

Point: : reduces lower bounds for equilibria to lower bounds for nondeterministic communication protocols.

Applying the Theorem

Theorem: [Nisan 02] No nondeterministic subexponential protocol approximates welfare with general valuations to any constant factor (as # of items goes to infinity).

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Corollary: no simple mechanism has constant POA for general bidder valuations.

Confirms folklore belief: with sufficiently complex preferences, simple auctions aren't good enough.

Ex: Subadditive Valuations

Theorem: [Dobzinski/Nisan/Schapira 05] No nondeterministic subexponential protocol approximates welfare with subadditive valuations better than a factor of 2.

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Corollary: Worst-case POA of ε-MNE of every simple mechanism (including S1A's) with subadditive bidder valuations is at least 2.

- known for S1A, exact MNE [Christodoulou/Kovacs/Sgouritsa/Tan 14]
- by [Feldman/Fu/Gravin/Lucier 13]: S1A = *optimal* simple mechanism
- contributes to ongoing debates on complex auction formats ("package bidding", etc.)

Why Approximate MNE?

Issue: in an S1A, number of strategies = $(V_{max} + 1)^m$

- valuations, bids assumed integral and poly-bounded

Consequence: can't efficiently guess/verify a MNE.

Theorem: [Lipton/Markakis/Mehta 03] a game with n players and N strategies per player has an ε-approximate mixed Nash equilibrium with support size polynomial in n, log N, and ε^{-1}.

- proof idea based on sampling from an exact MNE

From Protocol Lower Bounds to POA Lower Bounds

Theorem: [Roughgarden 14] Suppose:

- no nondeterministic polynomial-communication protocol approximates the welfare-maximization problem (with valuations V) to within factor of α.
  - i.e., impossible to decide OPT $\geq$ W^* vs. OPT $\leq W^* / \alpha$

Then worst-case POA of ***ε-approximate*** *mixed Nash equilibria of every "simple" mechanism is at least* α.

- ε can be as small as inverse polynomial in n and m

Point: : reduces lower bounds for equilibria to lower bounds for communication protocols.

Proof of Theorem

Suppose worst-case POA of ε-MNE is ϱ<α:

Input: game G
s.t. either (i)
OPT ≥ W* or
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Protocol: "advice" = ε-MNE x with small support (exists by LMM); players verify it privately

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if E[wel(x)] > W*/α then OPT > W*/αso in case (i)

Proof of Theorem

Suppose worst-case POA of ε-MNE is $\varrho<\alpha$:

Input: game G s.t. either (i) $OPT \geq W^*$ or (ii) $OPT \leq W^*/\alpha$

Protocol: "advice" = ε-MNE x with small support (exists by LMM); players verify it privately

if $E[wel(x)] > W^*/\alpha$ then $OPT > W^*/\alpha$ so in case (i)

if $E[wel(x)] \leq W^*/\alpha$ then $OPT \leq (\varrho/\alpha)W^* < W^*$ so in case (ii)

Key point: every ε-MNE is a short, privately verifiable certificate for membership in case (ii).

More Applications

- optimality results for "simple" auctions with other valuation classes (general, XOS)
- analogous results for combinatorial auctions with succinct valuations (assuming coNP not in MA)
- analogous results for routing and scheduling games (assuming PLS not in P)
  - e.g., tolls don't reduce the POA in atomic routing games
- unlikely to reduce planted clique to ε-Nash hardness

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Single-Item Auctions

Bayesian assumption: bidders' valuations $v_1,\ldots,v_n$ drawn independently from distributions $F_1,\ldots,F_n$.

- F_i's known to seller, v_i's unknown

Goal: find auction that maximizes expected revenue.

(Sealed-Bid) Auction:	Example:
• collect one bid per bidder • decide on a winner (if any) • decide on a selling price	2nd price auction with reserve r. • winner = highest bidder above r (if any) • price = r or 2nd-highest bid, whichever is larger

Optimal Single-Item Auctions

[Myerson 81]: characterized the optimal auction, as a function of the prior distributions $F_1,\ldots,F_n$.

- e.g., for i.i.d. valuations (all F_i's the same), optimal auction = second price with suitable reserve

[Maskin/Riley 84]: to generalize to harder problems, can optimization help?

- want to express "feasible region" via linear constraints
- assume finite-support distributions

A Naive Linear Program

- *decision variable* $x_i(\mathbf{b})$ = probability that bidder i wins when the bids are **b**
- *decision variable* $p_i(\mathbf{b})$ = bidder i's payment to seller when the bids are **b**

A Naive Linear Program

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- *incentive constraints:* truthful bidding an equilibrium
- *individual rationality constraints*: truthful bidding guarantees non-negative expected utility
- *feasibility*: can only sell one item ($\sum_i x_i(\mathbf{b}) \leq 1$)

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Problem: way too big! (exponentially many **b**'s)

A Projected Linear Program

- variable $y_i(b_i)$ (intent: $y_i(b_i) = E_{\mathbf{b}_{-i} \sim F_{-i}}[x_i(b_i, \mathbf{b}_{-i})]$)
- variable $q_i(b_i)$ (intent: $q_i(b_i) = E_{\mathbf{b}_{-i} \sim F_{-i}}[p_i(b_i, \mathbf{b}_{-i})]$)

- can express constraints "truthful bidding an equilibrium" and "truthful bidding guarantees non-negative expected utility" in these variables
- number of variables ≈ sum of support sizes

A Projected Linear Program

- variable $y_i(b_i)$ (intent: $y_i(b_i) = \underset{\mathbf{b}_{-i} \sim F_{-i}}{E}[x_i(b_i, \mathbf{b}_{-i})]$)
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- number of variables ≈ sum of support sizes

Problem: feasibility constraints $\sum_i x_i(\mathbf{b}) \leq 1$ (for all **b**)

- can these be expressed purely in terms of the y_i's?

Interim Feasibility

Key question: given $y_i(b_i)$'s, are they *interim feasible* --- are they induced by some set of $x_i(\mathbf{b})$'s?

- are given marginals consistent with some joint distribution?

Interim Feasibility

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"No" certificate: pick subsets $A_1, \ldots, A_n$ of bidders' supports, call i *special* if v_i in A_i.

- if Pr[winning bidder is special]

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- if Pr[winning bidder is special] > Pr[exists special bidder]

  (Pr[winning bidder is special]: sum of some $y_i(b_i)$'s; Pr[exists special bidder]: constant (depending on prior))

then $y_i(b_i)$'s cannot be interim feasible.

Border’s Theorem

Theorem: [Border 91] $y_i(b_i)$’s are interim feasible if and only if, for all subsets $A_1,\ldots,A_n$ of bidders’ supports,

Pr[winning bidder is special] ≤ Pr[exists special bidder].

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Theorems: [Alaei/Fu/Haghpanah/Hartline/Malekian 11], [Cai/Daskalakis/Weinberg 11], [Che/Kim/Mierendorff 13]

- extend Border's theorem to slightly more general settings (multi-unit auctions or additive valuations)
- quite general $(1+\varepsilon)$-approximate versions

Question: can we extend Border's theorem (exactly) significantly beyond single-item auctions?

More Formally...

Border-like theorem: a characterization of feasible interim allocation rules by a set of easy-to-verify linear inequalities.

- weaker goal than polynomial-time separation

Theorem: Unless $P^{NP}=\#P$, there is no Border-like theorem for

- Public Projects (e.g., build a bridge or not?)
- Multi-item auctions with unit-demand bidders
- <your favorite setting here>

Proof Structure

1) If a Border-like characterization exists for a certain mechanism design problem then the computational problem of recognizing feasible interim allocations is in P^{NP}. (via ellipsoid)

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2) But, for public projects (and other mechanism design tasks) the computational problem of recognizing feasible interim allocations is #P-hard. (enough to show computing the optimal revenue is #P-hard, prove this via reduction, case-by-case)

Take-Aways

- computational and communication complexity explain several "barriers" in proving desirable economic results
  - existence of Walrasian and more general price equilibria
  - simple auctions with near-optimal equilibria
  - tractable descriptions of the (interim) auction design space
- research direction #1: characterize the tractable vs. intractable frontier (e.g.. optimal simple auctions) research direction #2: make impossibility results unconditional (e.g., extension complexity of auctions)
- research direction #3: identify more such barriers!

FIN