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Discrete Geometry II
Tutorial Sheet 3

Please hand in your solutions for this sheet in the next tutorial, which will take place on May 19, 2016.

Exercise T1 (projective transformations) (7 points)
We apply a projective transformation induced by a $(n+1) \times (n+1)$-matrix A with non-negative entries to points in the non-negative orthant of $\mathbb{R}^n$ via

$$[A] \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \left\{ \lambda\, A \cdot \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \;\middle|\; \lambda > 0 \right\} .$$

(a) Let $P \subset \mathbb{R}^2$ be the unbounded polyhedron with outer description:

$$x \geq 0, \qquad y \geq 0\,, \qquad x + y \geq 1$$

Determine an inner and an outer description for P in homogeneous coordinates. Find a projective transformation A that maps P into a dense subset of the unit square $\{(1 : a : b) \mid 0 \leq a, b \leq 1\}$ in homogeneous coordinates. (You may need an additional translation).

(b) Prove that each 4-gon P with vertices $a^{(1)}, a^{(2)}, a^{(3)}, a^{(4)} \in \mathbb{R}^2$ is projectively equivalent to a given 4-gon Q with vertices $b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)} \in \mathbb{R}^2$, i.e. there exist a projective transformation s.t. for each $a \in P$

$$[A] \cdot a = [I] \cdot b = (1 : b_1 : b_2) \text{ with } b \in Q\ .$$

You may assume that both polytopes P and Q are contained in the non-negative ortant.

(c) Find two polytopes that are combinatorially but not projectively equivalent.

Exercise T2 (LP algorithms) (8 points)
Let $\mathrm{P}(D, e) = \{x \in \mathbb{R}^n \mid Dx \leq e\}$ be a rational polytope/polyhedron and $\mathrm{LP}(A, b, c)$ the linear program $\min\langle c, x\rangle$, s. t. $Ax \leq b$. Assume that you have a 'black box' solver for $\mathrm{LP}(A, b, c)$ that gives you the optimal value and an optimal vector for any A, b and c where $Ax \leq b$ is feasible.

(a) Assume that $\mathrm{P}(D, e)$ is full-dimensional. Find A, b and c, such that you can read off a interior point of $\mathrm{P}(D, e)$ from the optimal solution of $\mathrm{LP}(A, b, c)$.

(b) Write an algorithm that computes the dimension of the affine hull $\mathrm{aff}(\mathrm{P}(D, e))$ and a relative interior point in $\mathrm{P}(D, e)$ or a certificate that the polytope/polyhedron is empty. Also show the correctness of the algoritm.

(c) How could your algorithms be used to compute whether $\mathrm{P}(D, e)$ is bounded or unbounded?