Combinatorial Convexity

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Contents

0 Some basic and convex facts

0.1 Notation. $\mathbb{R}^n = \{(x_1, \ldots, x_n)^\intercal : x_i \in \mathbb{R}\}$ *denotes the n-dimensional* Euclidean space *equipped with the* Euclidean inner product $\langle x, y\rangle = \sum_{i=1}^n x_i\, y_i$, $x, y \in \mathbb{R}^n$, *and the* Euclidean norm $|x| = \sqrt{\langle x, x\rangle}$.

0.2 Definition [Linear, affine, positive and convex combination]. *Let* $m \in \mathbb{N}$ *and let* $x_i \in \mathbb{R}^n$, $\lambda_i \in \mathbb{R}$, $1 \le i \le m$.

i) $\sum_{i=1}^m \lambda_i\, x_i$ *is called a* linear combination *of* $x_1, \ldots, x_m$.

ii) *If* $\sum_{i=1}^m \lambda_i = 1$ *then* $\sum_{i=1}^m \lambda_i\, x_i$ *is called an* affine combination *of* $x_1, \ldots, x_m$.

iii) *If* $\lambda_i \ge 0$ *then* $\sum_{i=1}^m \lambda_i\, x_i$ *is called a* positive combination *of* $x_1, \ldots, x_m$.

iv) *If* $\lambda_i \ge 0$ *and* $\sum_{i=1}^m \lambda_i = 1$ *then* $\sum_{i=1}^m \lambda_i\, x_i$ *is called a* convex combination *of* $x_1, \ldots, x_m$.

v) *Let* $X \subseteq \mathbb{R}^n$. $x \in \mathbb{R}^n$ *is called* linearly *(*affinely, positively, convexly*)* dependent *of* X, *if* x *is a linear (affine, positive, convex) combination of finitely many points of* X, *i.e., there exist* $x_1, \ldots, x_m \in X$, $m \in \mathbb{N}$, *such that* x *is a linear (affine, positive, convex) combination of the points* $x_1, \ldots, x_m$.

0.3 Definition [Linearly and affinely independent points]. $x_1, \ldots, x_m \in \mathbb{R}^n$ *are called* linearly *(*affinely*)* dependent, *if one of the* x_i *is linearly (*affinely*) dependent of* $\{x_1, \ldots, x_m\}\backslash\{x_i\}$. *Otherwise* $x_1, \ldots, x_m$ *are called* linearly *(*affinely*)* independent.

0.4 Remark. *Let* $x_1, \ldots, x_m \in \mathbb{R}^n$.

i) $x_1, \ldots, x_m$ *are affinely dependent if and only if* $\binom{x_1}{1}, \ldots, \binom{x_m}{1} \in \mathbb{R}^{n+1}$ *are linearly dependent.*

ii) $x_1, \ldots, x_m$ *are affinely dependent if and only if there exist* $\mu_i \in \mathbb{R}$, $1 \le i \le m$, *with* $(\mu_1, \ldots, \mu_m) \ne (0, \ldots, 0)$, $\sum_{i=1}^m \mu_i = 0$ *and* $\sum_{i=1}^m \mu_i\, x_i = 0$.

iii) *If* $m \ge n + 1$ *then* $x_1, \ldots, x_m$ *are linearly dependent.*

iv) *If* $m \ge n + 2$ *then* $x_1, \ldots, x_m$ *are affinely dependent.*

0.5 Definition [Linear subspace, affine subspace, cone and convex set]. $X \subseteq \mathbb{R}^n$ *is called*

i) linear subspace (set) *if it contains all* $x \in \mathbb{R}^n$ *which are linearly dependent of* X,

ii) affine subspace (set) *if it contains all* $x \in \mathbb{R}^n$ *which are affinely dependent of* X,

iii) (convex) cone *if it contains all* $x \in \mathbb{R}^n$ *which are positively dependent of* X,

iv) convex set *if it contains all* $x \in \mathbb{R}^n$ *which are convexly dependent of* X.

0.6 Notation. $\mathcal{C}^n = \{K \subseteq \mathbb{R}^n : K \text{ convex}\}$ *denotes the set of all convex sets in* $\mathbb{R}^n$. *The empty set* $\emptyset$ *is regarded as a convex, linear and affine set.*

0.7 Theorem. $K \subseteq \mathbb{R}^n$ *is convex if and only if*

$$\lambda x + (1-\lambda)\, y \in K, \qquad \text{for all } x, y \in K \text{ and } 0 \leq \lambda \leq 1.$$

0.8 Example. *The closed* n*-dimensional ball* $B_n(a,\rho) = \big\{x \in \mathbb{R}^n : |x-a| \leq \rho\big\}$ *with centre* a *and radius* $\rho > 0$ *is convex. The boundary of* $B_n(a,\rho)$, *i.e.,* $\big\{x \in \mathbb{R}^n : |x-a| = \rho\big\}$ *is non-convex. In the case* $a = 0$ *and* $\rho = 1$ *the ball* $B_n(0,1)$ *is abbreviated by* B_n *and is called* n-dimensional unit ball. *Its boundary is denoted by* S^{n-1}.

0.9 Corollary. *Let* $K_i \in \mathcal{C}^n$, $i \in I$. *Then* $\bigcap_{i \in I} K_i \in \mathcal{C}^n$.

0.10 Definition [Linear, affine, positive and convex hull, dimension]. *Let* $X \subseteq \mathbb{R}^n$.

i) *The* linear hull $\operatorname{lin} X$ *of* X *is defined by*

$$\operatorname{lin} X = \bigcap_{\substack{L \subseteq \mathbb{R}^n,\, L \text{ linear}, \\ X \subseteq L}} L.$$

ii) *The* affine hull $\operatorname{aff} X$ *of* X *is defined by*

$$\operatorname{aff} X = \bigcap_{\substack{A \subseteq \mathbb{R}^n,\, A \text{ affine}, \\ X \subseteq A}} A.$$

iii) *The* positive (conic) hull $\operatorname{pos} X$ *of* X *is defined by*

$$\operatorname{pos} X = \bigcap_{\substack{C \subseteq \mathbb{R}^n,\, C \text{ convex cone}, \\ X \subseteq C}} C.$$

iv) *The* convex hull $\operatorname{conv} X$ *of* X *is defined by*

$$\operatorname{conv} X = \bigcap_{\substack{K \subseteq \mathbb{R}^n,\, K \text{ convex}, \\ X \subseteq K}} K.$$

v) *The* dimension $\dim X$ *of* X *is the dimension of its affine hull, i.e.,* $\dim \operatorname{aff} X$.

0.11 Theorem. *Let $X \subseteq \mathbb{R}^n$. Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{m} \lambda_i \, x_i : m \in \mathbb{N}, x_i \in X, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

0.12 Remark.

i) $\operatorname{conv}\{x, y\} = \{\lambda\, x + (1-\lambda)\, y : \lambda \in [0,1]\}$.

ii) $\operatorname{lin} X = \{\sum_{i=1}^{m} \lambda_i x_i : m \in \mathbb{N}, x_i \in X\}$.

iii) $\operatorname{aff} X = \{\sum_{i=1}^{m} \lambda_i x_i : m \in \mathbb{N}, x_i \in X, \sum_{i=1}^{m} \lambda_i = 1\}$.

iv) $\operatorname{pos} X = \{\sum_{i=1}^{m} \lambda_i x_i : m \in \mathbb{N}, x_i \in X,\ \lambda_i \geq 0\}$.

0.13 Definition [(Relative) interior point and (relative) boundary point]. *Let $X \subseteq \mathbb{R}^n$.*

i) *$x \in X$ is called an* interior point *of X if there exists a $\rho > 0$ such that $B_n(x, \rho) \subseteq X$. The set of all interior points of X is called the* interior *of X and is denoted by* $\operatorname{int} X$.

ii) *$x \in \mathbb{R}^n$ is called* boundary point *of X if for all $\rho > 0$ hold $B_n(x, \rho) \cap X \neq \emptyset$ and $B_n(x, \rho) \cap (\mathbb{R}^n \backslash X) \neq \emptyset$. The set of all boundary points of X is called the* boundary *of X and is denoted by* $\operatorname{bd} X$.

iii) *Let $A = \operatorname{aff} X$. $x \in X$ is called a* relative interior point *of X if there exists a $\rho > 0$ such that $B_n(x, \rho) \cap A \subseteq X$. The set of all relative interior points is called the* relative interior *of X and is denoted by* $\operatorname{relint} X$.

iv) *Let $A = \operatorname{aff} X$. $x \in A$ is called a* relative boundary point *of X if for all $\rho > 0$ hold $B_n(x, \rho) \cap X \neq \emptyset$ and $B_n(x, \rho) \cap (A \backslash X) \neq \emptyset$. The set of all relative boundary points of X is called* relative boundary *of X and is denoted by* $\operatorname{relbd} X$.

0.14 Remark. *Let $X \subseteq \mathbb{R}^n$ be closed. Then $X = \operatorname{relint} X \cup \operatorname{relbd} X$.*

0.15 Theorem. *Let $K \in \mathcal{C}^n$, $x \in \operatorname{relint} K$ and $y \in K$. Then $(1-\lambda)x + \lambda y \in \operatorname{relint} K$ for all $\lambda \in [0, 1)$.*

0.16 Corollary. *Let $K \in \mathcal{C}^n$ be closed. Let $x \in \operatorname{relint} K$ and $y \in \operatorname{aff} K \setminus K$. Then the segment $\operatorname{conv}\{x, y\}$ intersects $\operatorname{relbd} K$ in precisely one point.*

0.17 Definition [Polytope and simplex]. *Let $X \subset \mathbb{R}^n$ of finite cardinality, i.e., $\#X < \infty$.*

i) $\operatorname{conv} X$ *is called a* (convex) polytope.

ii) *A polytope $P \subset \mathbb{R}^n$ of dimension k is called a k-polytope.*

iii) *If X is affinely independent and $\dim X = k$ then $\operatorname{conv} X$ is called a k-simplex.*

0.18 Notation. $\mathcal{P}^n = \{P \subset \mathbb{R}^n : P$ *polytope*$\}$ *denotes the set of all polytopes in* $\mathbb{R}^n$.

0.19 Lemma. *Let* $T = \operatorname{conv}\{x_1, \ldots, x_{k+1}\} \subset \mathbb{R}^n$ *be a* k*-simplex, and let* $\lambda_i > 0$, $1 \leq i \leq k+1$, *with* $\sum \lambda_i = 1$. *Then* $\sum \lambda_i\, x_i \in \operatorname{relint} T$.

0.20 Corollary. *Let* $K \in \mathcal{C}^n$, $K \neq \emptyset$. *Then* $\operatorname{relint} K \neq \emptyset$.

0.21 Theorem. *Let* $P = \operatorname{conv}\{x_1, \ldots, x_m\} \in \mathcal{P}^n$. *A point* $x \in \mathbb{R}^n$ *belongs to* $\operatorname{relint} P$ *if and only if* x *admits a representation as* $x = \sum_{i=1}^m \lambda_i x_i$ *with* $\lambda_i > 0$, $1 \leq i \leq m$, *and* $\sum_{i=1}^m \lambda_i = 1$.

0.22 Notation.

i) *For two sets* $X, Y \subseteq \mathbb{R}^n$ *the vectorial addition*

$$X + Y = \{x + y : x \in X,\ y \in Y\}$$

is called the Minkowski sum *of* X *and* Y. *If* X *is just a singleton, i.e.,* $X = \{x\}$, *then we write* $x + Y$ *instead of* $\{x\} + Y$.

ii) *For* $\lambda \in \mathbb{R}$ *and* $X \subseteq \mathbb{R}^n$ *we denote by* λX *the set*

$$\lambda X = \{\lambda x : x \in X\}\,.$$

1 Support and separate

1.1 Notation. *Let* $a \in \mathbb{R}^n$, $a \neq 0$, *and* $\alpha \in \mathbb{R}$. *The* closed halfspaces $H^+(a,\alpha)$, $H^-(a,\alpha) \subset \mathbb{R}^n$ *are given by*

$$H^+(a,\alpha) = \{x \in \mathbb{R}^n : \langle a, x\rangle \geq \alpha\}, \quad H^-(a,\alpha) = \{x \in \mathbb{R}^n : \langle a, x\rangle \leq \alpha\}.$$

The hyperplane $H(a,\alpha)$ *is defined by*

$$H(a,\alpha) = \{x \in \mathbb{R}^n : \langle a, x\rangle = \alpha\}.$$

1.2 Definition [Supporting hyperplane]. *Let* $X \subset \mathbb{R}^n$. *A hyperplane* $H(a,\alpha) \subset \mathbb{R}^n$ *is called* supporting hyperplane *of* X *if:*

$$\text{i) } H(a,\alpha) \cap X \neq \emptyset \quad \text{and} \quad \text{ii) } X \subseteq H^-(a,\alpha).$$

a *is called* outer normal vector *of* X *and if, in addition,* $|a| = 1$ *then it is called* outer unit normal vector *of* X.

1.3 Proposition. *Let* $X \subset \mathbb{R}^n$ *and let* $H(a,\alpha)$ *be a supporting hyperplane of* X. *Then*

$$H(a,\alpha) \cap \operatorname{conv} X = \operatorname{conv}\big(H(a,\alpha) \cap X\big).$$

1.4 Remark. *Let* $X \subset \mathbb{R}^n$ *be compact and* $a \in \mathbb{R}^n \backslash \{0\}$. *Then there exists a supporting hyperplane of* X *with outer normal vector* a.

1.5 Definition [Nearest point map (or metric projection)]. *Let* $K \in \mathcal{C}^n$ *be closed. The map* $\Phi_K : \mathbb{R}^n \to K$, *where for* $x \in \mathbb{R}^n$ *the point* $\Phi_K(x) \in K$ *is given by* $|x - \Phi_K(x)| = \min\{|x-y| : y \in K\}$ *is called the* nearest point map *(*metric projection*) with respect to* K.

1.6 Remark. *We prove that the nearest point map is well-defined. Notice that since* K *is closed, for all* $x \in \mathbb{R}^n$ *there exist* $y_x \in K$ *such that* $|x - y_x| = \min\{|x-y| : y \in K\}$. *We show that* y_x *is uniquely determined. In fact, if there exists* $\overline{y} \in K$, $\overline{y} \neq y_x$, *with* $|x - \overline{y}| = |x - y_x|$ *then we may assume that* $x - y_x$ *and* $x - \overline{y}$ *are linearly independent. Hence*

$$\left|x - \frac{y_x + \overline{y}}{2}\right| = \left|\frac{1}{2}(x - y_x) + \frac{1}{2}(x - \overline{y})\right| < \frac{1}{2}\,|x - y_x| + \frac{1}{2}\,|x - \overline{y}| = |x - y_x|\,.$$

Since $(y_x + \overline{y})/2 \in K$ *by the convexity of* K, *it contradicts the minimality of* y_x.

1.7 Theorem. *Let* $K \in \mathcal{C}^n$ *be closed and let* $x \in \mathbb{R}^n \setminus K$. *Let* $a = x - \Phi_K(x)$ *and* $\alpha = \langle a, \Phi_K(x)\rangle$. *Then* $H(a,\alpha)$ *is a supporting hyperplane of* K *with outer normal vector* a.

1.8 Corollary. *Let* $K \in \mathcal{C}^n$, $K \neq \mathbb{R}^n$, *be closed. Then*

$$K = \bigcap_{\substack{H(a,\alpha) \text{ supporting} \\ \text{hyperplane of } K}} H^-(a,\alpha),$$

i.e., K *is the intersection of all its "supporting halfspaces".*

1.9 Corollary. *Let* $X \subset \mathbb{R}^n$ *such that* $\operatorname{conv} X$ *is closed and* $\operatorname{conv} X \neq \mathbb{R}^n$. *Then*

$$\operatorname{conv} X = \bigcap_{X \subseteq H^-(a,\alpha)} H^-(a,\alpha),$$

i.e., $\operatorname{conv} X$ *is the intersection of all halfspaces containing* X.

1.10 Lemma [Busemann-Feller Lemma]. *Let* $K \in \mathcal{C}^n$ *be closed. Then*

$$|\Phi_K(x) - \Phi_K(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}^n$, *i.e., the nearest point map does not increase distances. In particular, it is a continuous map.*

1.11 Theorem. *Let* $K \in \mathcal{C}^n$ *be compact and let* $\rho > 0$ *such that* $K \subset \operatorname{int}(\rho B_n)$. *The nearest point map restricted to* ρS^{n-1} *is surjective, i.e.,* $\Phi_K : \rho\, S^{n-1} \to \operatorname{bd} K$ *is surjective.*

1.12 Corollary. *Let* $K \in \mathcal{C}^n$ *be closed and let* $x \in \operatorname{relbd} K$. *Then there exists a supporting hyperplane* $H(a,\alpha)$ *of* K *with* $x \in H(a,\alpha)$.

1.13 Theorem [Separation theorem]. *Let* $K_1, K_2 \in \mathcal{C}^n$ *with* $K_1 \cap K_2 = \emptyset$. *Then there exists a* separating hyperplane $H(a,\alpha)$ *of* K_1 *and* K_2, *i.e.,* $K_1 \subseteq H^+(a,\alpha)$ *and* $K_2 \subseteq H^-(a,\alpha)$.

If K_1 *is closed and* K_2 *is compact, then there exists even a* strictly separating hyperplane $H(a,\alpha)$ *of* K_1 *and* K_2, *i.e.,* $K_1 \subset \operatorname{int} H^+(a,\alpha)$ *and* $K_2 \subset \operatorname{int} H^-(a,\alpha)$.

1.14 Definition [Support function, breadth]. *Let* $K \in \mathcal{C}^n$. *The function* $h(K,\cdot) : \mathbb{R}^n \to \mathbb{R}$ *given by*

$$h(K,u) = \sup\big\{\langle u, x\rangle : x \in K\big\}$$

is called support function *of* K. *For* $u \in S^{n-1}$ *the* breadth of K in the direction u *is given by* $h(K,u) + h(K,-u)$.

1.15 Remark. *Let* $K \in \mathcal{C}^n$ *be non-empty and compact. Then*

$$K = \bigcap_{u \in S^{n-1}} \big\{x \in \mathbb{R}^n : \langle u, x\rangle \leq h(K,u)\big\}.$$

1.16 Definition [Polar set]. *Let* $X \subseteq \mathbb{R}^n$.

$$X^\star = \big\{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \ \text{ for all } x \in X\big\}$$

is called the polar set *of* X.

1.17 Proposition.

i) $X^\star$ *is a convex and closed set and* $0 \in X^\star$.

ii) *If* $X_1 \subseteq X_2$ *then* $X_2^\star \subseteq X_1^\star$.

iii) *Let* M *be a regular* $n \times n$ *matrix. Then* $(MX)^\star = M^{-\intercal} X^\star$.

iv) *Let* $X_i \subseteq \mathbb{R}^n$, $i \in I$. *Then* $\left(\bigcup_{i\in I} X_i\right)^\star = \bigcap_{i\in I} X_i^\star$.

v) $X \subseteq (X^\star)^\star$.

vi) *Let* $X \subset \mathbb{R}^n$. *Then* $X = X^\star$ *if and only if* $X = B_n$.

1.18 Proposition.

i) *Let* $P = \operatorname{conv}\{x_1, \dots, x_m\} \subset \mathbb{R}^n$. *Then*

$$P^\star = \left\{y \in \mathbb{R}^n : \langle x_i, y\rangle \le 1,\, 1 \le i \le m\right\}.$$

ii) *Let* $P = \left\{x \in \mathbb{R}^n : \langle a_i, x\rangle \le 1,\, 1 \le i \le m\right\}$ *with* $a_i \in \mathbb{R}^n$. *Then*

$$P^\star = \operatorname{conv}\{0, a_1, \dots, a_m\}.$$

1.19 Lemma. *Let* $K \in \mathcal{C}^n$ *be closed with* $0 \in K$. *Then* $(K^\star)^\star = K$.

2 Radon, Helly, Caratheodory and relatives

2.1 Theorem [Radon, 1921]. *Let $X \subset \mathbb{R}^n$. If $\#X \geq n+2$ then there exist $X_1, X_2 \subset X$ with $X_1 \cap X_2 = \emptyset$ and $\operatorname{conv} X_1 \cap \operatorname{conv} X_2 \neq \emptyset$.*

2.2 Theorem [Helly, 1913]. *Let $K_1, \dots, K_m \in \mathcal{C}^n$, $m \geq n+1$, such that for each $(n+1)$-index set $I \subseteq \{1, \dots, m\}$ we have $\bigcap_{i \in I} K_i \neq \emptyset$. Then all sets K_i have a point in common, i.e., $\bigcap_{j=1}^{m} K_i \neq \emptyset$.*

2.3 Remark.

i) *Without any further restrictions/assumptions Helly's theorem is* not *true for infinitely many convex sets K_i. For instance, let $K_i = (0, \frac{1}{i}]$, $i \in \mathbb{N}$.*

ii) *Helly's theorem, however, can be easily generalised to infinitely many* compact *(bounded and closed) convex sets.*

2.4 Corollary. *Let $C \subset \mathcal{C}^n$ be compact. Then there exists $t \in \mathbb{R}^n$ with*

$$-C \subseteq t + n\,C.$$

2.5 Definition [Centerpoint]. *For a finite point set $X \subset \mathbb{R}^n$ a point $c \in \mathbb{R}^n$ is called* centerpoint *is every closed halfspace containing c containes at least $\lceil \frac{1}{n+1} \#X \rceil$ points of X.*

2.6 Theorem. *Every finite set $X \subset \mathbb{R}^n$ has a centerpoint.*

2.7 Theorem [Carathéodory, 1907]. *Let $X \subset \mathbb{R}^n$. Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{n+1} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1, x_i \in X, i = 1, \dots, n+1 \right\}.$$

2.8 Remark. *Let $X \subset \mathbb{R}^n$. Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{\dim X + 1} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{\dim X + 1} \lambda_i = 1, x_i \in X \right\}.$$

As a direct consequence of Carathéodory's Theorem 2.7 we get the following result.

2.9 Corollary. *A polytope is the union of simplices.*

2.10 Corollary. *The convex hull of a compact set is compact.*

2.11 Theorem [(weak)Fractional Helly theorem]. *Let $K_1, \dots, K_m \in \mathcal{C}^n$, $m \geq n+1$, and let $\alpha \in (0,1]$ such that for at least $\alpha \binom{m}{n+1}$ of the $(n+1)$-index sets $I \subseteq \{1, \dots, m\}$ we have $\bigcap_{i \in I} K_i \neq \emptyset$. Then there exists a point in common of at least $\frac{\alpha}{n+1} \cdot m$ sets K_i.*

2.12 Remark. *The (strong and sharp) fractional Helly theorem, which is due to Kalai, gives that* $(1-(1-\alpha)^{1/(n+1)})\cdot m$ *sets have a point in common. Obviously, for* $\alpha = 1$ *we get again the classcial Helly Theorem 2.2.*

2.13 Theorem [Colorful Carathéodory theorem]. *Let* $X_1, \dots, X_{n+1} \subset \mathbb{R}^n$ *be finite point sets such that* $0 \in \operatorname{conv} X_i$, $1 \le i \le n+1$. *There exist* $x_i \in X_i$, $1 \le i \le n+1$, *such that* $0 \in \operatorname{conv}\{x_1, \dots, x_{n+1}\}$.

2.14 Theorem [Tverberg's theorem]. *Let* $X \subseteq \mathbb{R}^n$ *and let* $k \in \mathbb{N}_{\ge 1}$. *If* $\#X \ge (k-1)(n+1)+1$, $k \in \mathbb{N}$, *then there exist* k *subsets* $X_1, \dots, X_k \subset X$ *with* $X_i \cap X_j = \emptyset$, $i \ne j$, *but* $\operatorname{conv} X_1 \cap \operatorname{conv} X_2 \cap \dots \cap \operatorname{conv} X_k \ne \emptyset$.

2.15 Theorem. *Let* $X \subset \mathbb{R}^n$ *and let* $\#X \ge m$, $m \ge n+1$. *Then there exists a point* $y \in \mathbb{R}^n$ *contained in at least* $\gamma_n \binom{m}{n+1}$ X*-simplices, i.e., simplices of the form* $\operatorname{conv} S$, $S \subset X$, $\#S = n+1$. *Here* γ_n *is a positive constant depending only on the dimension, and* X*-simplices* $\operatorname{conv} S_1$, $\operatorname{conv} S_2$ *are considered different if* $S_1 \ne S_2$.

3 Polytopes

3.1 Definition [Polyhedron]. *The intersection of finitely many closed halfspaces is called a* polyhedron.

3.2 Theorem [Minkowski, 1896, Weyl, 1935].

i) *A bounded polyhedron is a polytope.*

ii) *A polytope is a bounded polyhedron.*

3.3 Notation [$\mathcal{V}$-Polytope, $\mathcal{H}$-Polytope]. *A polytope given as the convex hull of finitely many points is called a* $\mathcal{V}$-polytope. *If it is given as the bounded intersection of finitely many closed halfspaces, then it is called an* $\mathcal{H}$-polytope.

3.4 Corollary. *Let $P \in \mathcal{P}^n$.*

i) *Let $A \in \mathbb{R}^{m\times n}$ and $t \in \mathbb{R}^m$. Then $AP + t$ is a polytope.*

ii) *Let $U \subset \mathbb{R}^n$ be an affine subspace. Then $P \cap U$ is a polytope.*

3.5 Definition [Faces]. *Let $K \in \mathcal{C}^n$ be closed and let H be a supporting hyperplane of K. If $j = \dim(K \cap H)$, then $K \cap H$ is called a* j-face *of K. Moreover, K itself is regarded as a $(\dim K)$-face and the empty set $\emptyset$ as (-1)-face of K.*

3.6 Notation [Vertices, edges, facets]. *A 0-face of $K \in \mathcal{C}^n$, K closed, is called* vertex, *an 1-face is called* edge *and a $(\dim K - 1)$-face is called* facet *of K. K itself and the empty set are called* improper faces, *whereas the remaining faces are called* proper faces *of K.*

The set of all vertices of a polytope P is denoted by $\operatorname{vert} P$.

3.7 Remark.

i) *Let $K \in \mathcal{C}^n$ be closed. Every (relative) boundary point of K lies in a suitable j-face, $0 \le j \le \dim K - 1$.*

ii) *Let $K \in \mathcal{C}^n$, $\dim K = n$. Let F be a facet of K and H a supporting hyperplane of K with $F = K \cap H$. Then $H = \operatorname{aff} F$.*

3.8 Proposition. *Each face of a polytope is a polytope, and A polytope has only finitely many faces.*

3.9 Definition [f-vector]. *For $P \in \mathcal{P}^n$ let $f_i(P)$ be the number of i-faces of P, $-1 \le i \le \dim P$. Furthermore, let $f_i(P) = 0$ for $\dim P + 1 \le i \le n$. The vector $f(P)$ with entries $f_i(P)$, $-1 \le i \le n$, is called the* f-vector *of P.*

3.10 Remark.

i) *Let $T_n = \operatorname{conv}\{0, e_1, \ldots, e_n\}$ be the so called* standard simplex. *Then $f_i(T_n) = \binom{n+1}{i+1}$.*

ii) *For any n-polytope $P \in \mathcal{P}^n$ we have $\sum_{i=0}^{n} f_i(P) \geq 2^{n+1}$ with equality if and only if P is an n-simplex.*

3.11 Lemma. *Let $P \in \mathcal{P}^n$.*

i) *$v \in \operatorname{vert} P$ can not be written as a convex combination of two other points of P, i.e., $v \notin \operatorname{conv}\big(P \backslash \{v\}\big)$.*

ii) *If $P = \operatorname{conv} W$, then $\operatorname{vert} P \subseteq W$.*

iii) *$P = \operatorname{conv} \operatorname{vert} P$.*

3.12 Lemma. *Let $P \in \mathcal{P}^n$ be an n-polytope with $0 \in \operatorname{int} P$. For a face F of P let*

$$F^\diamond = \{y \in P^\star : \langle x, y \rangle = 1 \text{ for all } x \in F\}.$$

Then

i) *$F^\diamond$ is a face of $P^\star$.*

ii) *$F = (F^\diamond)^\diamond$.*

iii) *If G is a face of P and $F \subseteq G$, then $G^\diamond \subseteq F^\diamond$.*

iv) *$\dim F^\diamond = n - 1 - \dim F$.*

3.13 Theorem. *Let $P \in \mathcal{P}^n$ be an n-polytope with $0 \in \operatorname{int} P$. Then*

$$f_{n-1-i}(P^\star) = f_i(P), \quad -1 \leq i \leq n.$$

3.14 Corollary. *Let $P \in \mathcal{P}^n$ be an n-polytope with facets $F_1, \dots, F_m$ and let $H(a_i, \alpha_i)$, $1 \leq i \leq m$, be the supporting hyperplanes of F_i, $1 \leq i \leq m$. Then*

$$P = \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq \alpha_i,\ 1 \leq i \leq m\}.$$

3.15 Corollary. *Let $P \in \mathcal{P}^n$ be an n-polytope.*

i) *The boundary of P is the union of all its facets.*

ii) *A k-face is the intersection of (at least) $(n-k)$ facets.*

iii) *An $(n-2)$-face is contained in exactly two facets.*

iv) *If F, G are faces of P with $F \subset G$, then F is a face of G.*

v) *A face of P is also a face of a facet of P.*

3.16 Theorem. *Let $P \in \mathcal{P}^n$ be an n-polytope.*

i) *Let G be a face of P and let F be a face of G. Then F is a face of P.*

ii) *Let* F_j *be a* j*-face of* P *and let* F_k *be a* k*-face of* P *with* $F_j \subset F_k$*. There exist* i*-faces* F_i *of* P*,* $j \leq i \leq k$*, such that*

$$F_j \subset F_{j+1} \subset \cdots \subset F_{k-1} \subset F_k.$$

3.17 Remark. *Let* v_0 *be a vertex of an* n*-polytope* P *and let* $\{v_1, \ldots, v_r\}$ *be all* adjacent *vertices of* v_0*, i.e.,* $\operatorname{conv}\{v_0, v_i\}$ *is an edge of* P*. In other words,* $\{v_1, \ldots, v_r\}$ *are the neighbours of* v_0*. Then*

i)

$$P \subset v_0 + \operatorname{pos}\{v_1 - v_0, \ldots, v_r - v_0\}.$$

ii) *Let* $c \in \mathbb{R}^n$ *with* $\langle c, v_0 \rangle \geq \langle c, v_i \rangle$*,* $1 \leq i \leq r$*. Then*

$$\max\{\langle c, x \rangle : x \in P\} = \langle c, v_0 \rangle \,.$$

3.18 Theorem [Euler-Poincaré formula]. *Let* $P \in \mathcal{P}^n$*. Then*

$$\sum_{i=-1}^{n} (-1)^i f_i(P) = 0. \tag{3.18.1}$$

In particular, in the 3-dimensional case, i.e., $\dim P = 3$*, it holds* $f_0 - f_1 + f_2 = 2$.

3.19 Proposition. *The Euler-Poincaré formula is the only linear equation satisfied by the* f*-vector, i.e., let* $\lambda_i \in \mathbb{R}$*, such that* $\sum_{i=-1}^{n} \lambda_i f_i(P) = 0$ *for all* $P \in \mathcal{P}^n$*. Then there exists a constant* $\gamma \in \mathbb{R}$*, such that* $\lambda_i = \gamma (-1)^i$*.*

3.20 Definition [Simple and simplicial polytopes]. *Let* $P \in \mathcal{P}^n$*.*

i) P *is called* simplicial *if all proper faces are simplices.*

ii) P *is called* simple *if every vertex is contained in exactly* $\dim P$ *many facets.*

3.21 Lemma. *Let* $P \in \mathcal{P}^n$ *be an* n*-polytope with* $0 \in \operatorname{int} P$*. The following statements are equivalent:*

i) P *is simplicial.*

ii) *All facets of* P *are simplices.*

iii) $P^\star$ *is simple.*

iv) *Every* k*-face of* $P^\star$ *is contained in exactly* $n - k$ *facets.*

3.22 Theorem. *Let* $P \in \mathcal{P}^n$ *be a simple* n*-polytope. Then*

i) *Every vertex is contained in exactly* $\binom{n}{k}$ k*-faces of* P*.*

ii) *The intersection of $k \leq n$ facets containing a common vertex is an $(n-k)$-face of P.*

iii) *Let $v_1, \ldots, v_n$ be the neighbours of a vertex v_0 of P. For each subset of $k < n$ neighbours $v_{i_1}, \ldots, v_{i_k}$ there exists a unique k-face F of P containing $v_0, v_{i_1}, \ldots, v_{i_k}$.*

iv) *A face of a simple polytope is simple.*

3.23 Theorem. *Let $P \in \mathcal{P}^n$ be a simple n-polytope.*

i) $n\, f_0(P) = 2\, f_1(P)$.

ii) $\sum_{k=0}^{n} f_k(P) \leq 2^n f_0(P)$.

iii) $f_0(P) \leq 2 f_{\lceil n/2 \rceil}(P)$.

Here, for $\rho \in \mathbb{R}$ the number $\lceil \rho \rceil$ is the smallest integer greater or equal than ρ.

3.24 Corollary. *Let P be a simple n-polytope with m facets. Then*

$$f_0(P) \leq 2\binom{m}{\lfloor n/2 \rfloor} \quad \text{and} \quad \sum_{k=0}^{n} f_k(P) \leq 2^{n+1}\binom{m}{\lfloor n/2 \rfloor}.$$

Or equivalently: Let P be a simplicial n-polytope with m vertices. Then

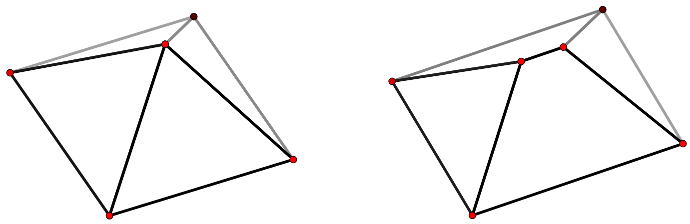
$$f_{n-1}(P) \leq 2\binom{m}{\lfloor n/2 \rfloor} \quad \text{and} \quad \sum_{k=0}^{n} f_k(P) \leq 2^{n+1}\binom{m}{\lfloor n/2 \rfloor}.$$

Here, for $\rho \in \mathbb{R}$ the number $\lfloor \rho \rfloor$ is the largest integer not greater than ρ.

3.25 Lemma. *Let P be an n-polytope.*

i) *There exists a simple n-polytope Q with the same number of facets as P and $f_i(P) \leq f_i(Q)$, $0 \leq i \leq n-2$.*

ii) *There exists a simplical n-polytope $Q^\star$ with the same number of vertices as P and $f_i(P) \leq f_i(Q^\star)$, $1 \leq i \leq n-1$.*



3.26 Corollary. *Let P be an n-polytope with m facets. Then*

$$f_0(P) \leq 2\binom{m}{\lfloor n/2 \rfloor}.$$

Or equivalently: Let P be an n-polytope with m vertices. Then

$$f_{n-1}(P) \leq 2\binom{m}{\lfloor n/2 \rfloor}.$$

3.27 Definition [Cyclic polytopes]. *The curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ given by $\gamma(t) = (t, t^2, t^3, \dots, t^n)^\intercal$ is called* moment curve. *The convex hull of m points on the moment curve is called a* cyclic polytope *with m vertices and is denoted by $C(n, m)$.*

3.28 Proposition. *Any $n+1$ points on the moment curve are affinely independent. In particlar, cyclic polytopes are simplicial polytopes.*

3.29 Proposition [Gale's evenness condition]. *Let $t_i \in \mathbb{R}$, $1 \le i \le m$, $t_1 < t_2 < \cdots < t_m$, $v(t_i) = (t_i, t_i^2, t_i^3, \dots, t_i^n)^\intercal$, $1 \le i \le m$, and let $S \subset \{1, \dots, m\}$ be a subset of cardinality n.*

$F_S = \operatorname{conv}\{v(t_s). s \in S\}$ is a facet of $C(n, m)$ if and only if $\#\{s \in S : i < s < j\}$ is even for all $i, j \in \{1, \dots, m\} \setminus S$ with $i < j$.

3.30 Remark. *All points $v(t_i)$ are vertices of $C(n, m)$ and the number of i-faces of $C(n, m)$ is independent of the choice of the m-points on the moment curve.*

3.31 Theorem [McMullen's Upper Bound Theorem, 1971]. *Let P be an n-polytope with m vertices. Then*

$$f_i(P) \le f_i(C(n, m)) = \begin{cases} \sum_{j=0}^{(n-1)/2} \frac{i+2}{m-j} \binom{m-j}{j+1} \binom{j+1}{i+1-j}, & n \text{ odd}, \\ \sum_{j=1}^{n/2} \frac{m}{m-j} \binom{m-j}{j} \binom{j}{i+1-j}, & n \text{ even}. \end{cases}$$

In particular,

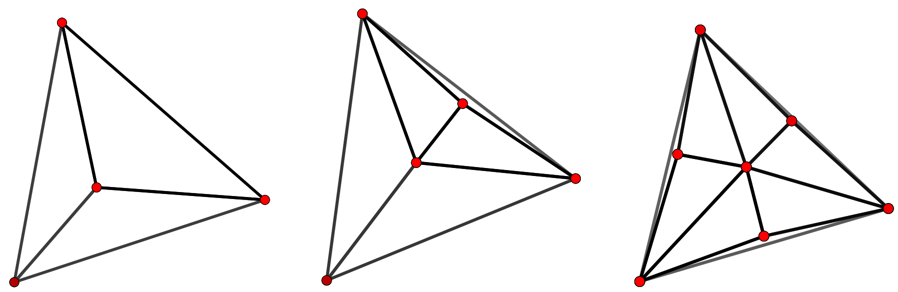
$$f_{n-1}(P) \le f_{n-1}(C(n, m)) = \begin{cases} 2\binom{m-\lfloor n/2 \rfloor - 1}{\lfloor n/2 \rfloor}, & n \text{ odd}, \\ \binom{m-\lfloor n/2 \rfloor}{\lfloor n/2 \rfloor} + \binom{m-\lfloor n/2 \rfloor - 1}{\lfloor n/2 \rfloor - 1}, & n \text{ even}. \end{cases}$$

For fixed n the right hand sides are of order $m^{\lfloor n/2 \rfloor}$.

3.32 Theorem [Barnette's Lower Bound Theorem, 1971]. *Let P be a simplicial n-polytope with m vertices. P has at least as many i-faces as the so called* stacked polytopes *$P(n, m)$ with m vertices for which*

$$f_i(P(n, m)) = \begin{cases} m\binom{n}{i} - i\binom{n+1}{i+1}, & 0 \le i \le n-2, \\ n + 1 + (m - (n+1))(n-1), & i = n-1. \end{cases}$$

$P(n, n+1)$ is an n-simplex, and for $m \ge n+2$ an m-vertex stacked n-polytope $P(n, m)$ is the convex hull of an $(m-1)$-vertex stacked polytope with an additional point that is beyond exactly one facet.



3.33 Theorem [Dehn-Sommerville equations, 1905, 1927]. *Let P be a simple n-polytope. Then*

$$f_i(P) = \sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} f_j(P), \quad i = 1, \dots, n,$$

Or equivalently: Let P be a simplicial n-polytope. Then

$$f_{i-1}(P) = \sum_{j=i}^{n} (-1)^{n-j} \binom{j}{i} f_{j-1}(P), \quad i = 0, \dots, n-1.$$

3.34 Theorem [McMullen's g-Theorem]. *McMullen's g-theorem gives a complete characterization of the f-vectors of simple (or simplicial) polytopes.*

3.35 Theorem [Steinitz, 1906]. *A non-negative integral vector (f_0, f_1, f_2) is the f-vector of a 3-polytope if and only if* i) $f_0 - f_1 + f_2 = 2$, ii) $3\,f_0 \le 2\,f_1$, *and* iii) $3\,f_2 \le 2 f_1$.

3.36 Theorem [Figiel, Lindenstrauss, Milman, 1977]. *Let $P \in \mathcal{P}^n$ be a 0-symmetric n-polytope, i.e., $P = -P$. Then*

$$\ln(f_0(P)) \ln(f_{n-1}(P)) \ge \frac{1}{16} n.$$

3.37 Conjecture [Kalai, 1989]. *Let $P \in \mathcal{P}^n$ be a 0-symmetric n-polytope. Then*

$$\sum_{i=0}^{n} f_i(P) \ge 3^n.$$

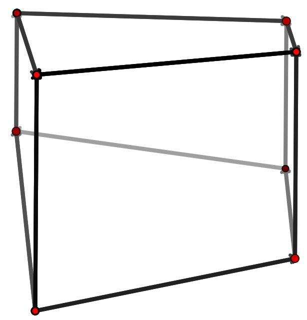
Here we have equality, for instance, for the cube C_n and its polar, the cross-poyltope $C_n^\star$, or, more generally, for the class of Hanner-polytopes. Recently, the conjecture has been verified for all $n \le 4$ (see `arXiv:0708.3661`*).*

3.38 Definition [Combinatorial diameter]. *Let $P \in \mathcal{P}^n$. The* combinatorial distance $\delta_P(v, w)$ *between two vertices $v, w \in \operatorname{vert} P$ is the minimum length of a path along the edges connecting v and w, i.e., the minimum length of a sequence $[v, v_{i_1}, \dots, v_{i_l}, w]$, with $v_{i_j} \in \operatorname{vert} P$ and two consecutive vertices form an edge.*

$\delta(P) = \max\{\delta_P(v, w) : v, w \in \operatorname{vert} P\}$ *is called the* (combinatorial) diameter *of P. For $n, m \in \mathbb{N}$ let*

$$\Delta(n, m) = \max\left\{\delta(P) : P \in \mathcal{P}^n, \dim P = n \text{ and } f_{n-1}(P) = m\right\}.$$

3.39 Example. $\delta(T_n) = 1$, $\delta(C_n) = n$ *and* $\delta(C_n^\star) = 2$.



3.40 Conjecture [Hirsch, 1957]. $\Delta(n, m) \leq m - n$.

3.41 Remark. *It is known that*

i) *the conjecture is true if* $n \leq 3$ *or* $m \leq n + 5$ *(Klee&Walkup, 1961/1965),*

ii) *the conjecture would be false for unbounded polyhedra,*

iii) $\Delta(n, m) \leq \frac{1}{3}\, 2^{n-2}(m - n + \frac{5}{2})$ *(Barnette, 1974),*

iv) $\Delta(n, m) \leq 2m^{\log(n)+1}$ *(Kalai, 1992),*

v) *it suffices to prove the conjecture for simple polytopes with* $m = 2\,n$ *(Klee&Walkup, 1961/1965)!*

3.42 Definition [0/1-polytope]. *Let* $[0, 1]^n$ *be the* n*-dimensional unit cube with vertices* $\{0, 1\}^n = \{(x_1, \ldots, x_n)^\intercal : x_i \in \{0, 1\}\}$. $P \in \mathcal{P}^n$ *is called* 0/1-polytope *if* $\operatorname{vert} P \subset \{0, 1\}^n$.

3.43 Lemma. *Let* $P \in \mathcal{P}^n$ *be a 0/1-polytope and let* $\dim P \leq n - 1$. *Then there exists a 0/1-polytope* $\widetilde{P} \in \mathcal{P}^{n-1}$ affinely isomorphic *to* P, *i.e., there exists a bijective map between* P *and* $\widetilde{P}$.

3.44 Theorem [Naddef, 1989].

i) *Let* P *be a 0/1-polytope. Then* $\delta(P) \leq \dim P$.

ii) *Let* $P \in \mathcal{P}^n$ *be an* n*-dimensional 0/1-polytope with* m *facets. Then* $\delta(P) \leq m - n$.

3.45 Remark.

i) $f_{n-1}(P) \leq 2\,n!$ *for a 0/1-polytope* $P \in \mathcal{P}^n$.

ii) *There exist 0/1-polytopes* $P \in \mathcal{P}^n$ *with*

$$f_{n-1}(P) \geq \left(\frac{c\,n}{\log^2 n}\right)^{\frac{n}{2}},$$

where c *is a universal constant (Gatzouras, Giannopoulos, Markoulakis, 2004).*

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