AN ELEMENTARY PROOF OF GRAM'S THEOREM FOR CONVEX POLYTOPES

G. C. SHEPHARD

In honour of Professor H. S. M. Coxeter on his sixtieth birthday

Let P be a d-polytope (that is, a d-dimensional convex polytope in Euclidean space) and for $0 \leqslant j \leqslant d-1$ let F_i^j $(i = 1, \ldots, f_j(P))$ represent its j-faces. Associated with each face F_i^j is a non-negative number $\phi(P, F_i^j)$, to be defined later, which is called the interior angle of P at the face F_i^j. In this paper we give an elementary proof of the following classical theorem:

GRAM'S THEOREM. *The interior angles $\phi(P, F_i^j)$ of any d-polytope P satisfy the equation*

$$\sum_{j=0}^{d-1} (-1)^j \sum_{i=1}^{f_j(P)} \phi(P, F_i^j) = (-1)^{d-1}. \tag{1}$$

J. P. Gram **(1)** gave the first proof of this theorem in 1874 for the case $d = 3$. In 1927 D. M. Y. Sommerville **(5)** published a proof for general d, and also extended the theorem to give a formula for the volume of a spherical polytope in terms of its interior angles. Recently B. Grünbaum pointed out that part of Sommerville's proof is incorrect, and so, at the present time, there is no published proof for $d > 3$. However, two proofs will appear shortly. The first of these **(2**, §14.1**)** by B. Grünbaum is a correction of Sommerville's proof. Although completely elementary in character, it is long and complicated in detail. The method consists of establishing (1) for d-dimensional convex pyramids, and then extending the result to general d-polytopes by "building up" these polytopes as unions of pyramids. The second proof to appear is by M. A. Perles and the author **(4)**. This is short and simple, but may not be considered "elementary" in that it depends on the methods of integral geometry.

In this paper we present a third proof, which appears to have the merit of both simplicity and also of being completely elementary in character. It begins in the same way as the Sommerville–Grünbaum proof: interior angles are defined as the volumes of sets called "lunes" on the unit $(d-1)$-sphere S^{d-1} centred at the origin o. We shall show, using an idea described in **(4**, §2**)**, that these lunes "fit together" in such a way that they form a simple covering of S^{d-1} and then (1) will follow immediately. The proof is essentially geometrical in character, and no previous knowledge is assumed except for a little elemen-

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tary d-dimensional Euclidean geometry and Euler's theorem on the number of cells in a cell-complex (see, for example, **(3)**).

If the d-polytope P is simplicial (that is, all its faces are simplexes) then it is known that the interior angles of P satisfy a number of linear relations other than (1). For an account of these, and references to the papers in which they appear, see **(2**, §14.2**)**. We remark that the method of proof of Gram's theorem given here can be adapted in an obvious manner to give elementary proofs of all these relations.

The author wishes to record his indebtedness to Professor M. A. Perles and Professor V. Klee for their many stimulating discussions concerning Gram's theorem and related topics.

Proof of Gram's theorem. Let z_i^j be any relative interior point of the face F_i^j such as, for example, its centroid. The polytope P subtends a closed convex polyhedral cone at z_i^j. Apply the translation $-z_i^j$ to this cone so that the apex z_i^j is moved to the origin o, and write $L(P, F_i^j)$ for the intersection of the translated cone with S^{d-1}. The set $L(P, F_i^j)$, which does not depend on the choice of z_i^j, will be called the *lune* associated with the face F_i^j of P, and is a generalized spherical polytope. (We say "generalized" since, unless $j = 0$, it will contain antipodal points of S^{d-1}.) Let μ be a measure on S^{d-1} proportional to the Lebesgue measure and such that $\mu(S^{d-1}) = 1$. Then the *interior angle* $\phi(P, F_i^j)$ of P at the face F_i^j is defined to be $\mu(L(P, F_i^j))$. Hence equation (1) is equivalent to

$$\sum_{j=0}^{d-1} (-1)^j \sum_{i=1}^{f_j(P)} \mu(L(P, F_i^j)) = (-1)^{d-1}. \tag{2}$$

In order to prove this, we shall show that the lunes have the property that if each is counted with the given sign, they form a covering of S^{d-1} with multiplicity $(-1)^{d-1}$, and then (2) will follow from the fact that $\mu(S^{d-1}) = 1$. Thus if $n_j(x)$ is the number of lunes $L(P, F_i^j)$ associated with j-faces of P which contain a given point $x \in S^{d-1}$, then we need to show that

$$\sum_{j=0}^{d-1} (-1)^j n_j(x) = (-1)^{d-1} \tag{3}$$

for all $x \in S^{d-1}$. However, since the boundaries of the lunes are (parts of) a finite number of $(d - 2)$-spheres and so have measure zero, it will be sufficient to establish (3) for those x which do not lie on the boundary of any lune, that is to say, for vectors x which are not parallel to any j-face of P $(1 \leqslant j \leqslant d - 1)$.

For such an x let H_x be any hyperplane perpendicular to x, π_x be orthogonal projection onto H_x, and $P_x = \pi_x(P)$. Let $\mathscr{S}_x$ be the shadow boundary of the projection, that is to say, the set of faces F_i^j of P for which $\pi_x(F_i^j) \subset \partial P_x$, the boundary of P_x (see Fig. 1). Write $\mathscr{F}_x$ for the set of faces F_i^j of P which have the property that the open half-line $\{z_i^j + \lambda x | \lambda > 0\}$ meets the interior of P. Then it is clear from the definition that $F_i^j \in \mathscr{F}_x$ if and only if $x \in L(P, F_i^j)$, and therefore $n_j(x)$ is the number of j-faces of P belonging to $\mathscr{F}_x$.

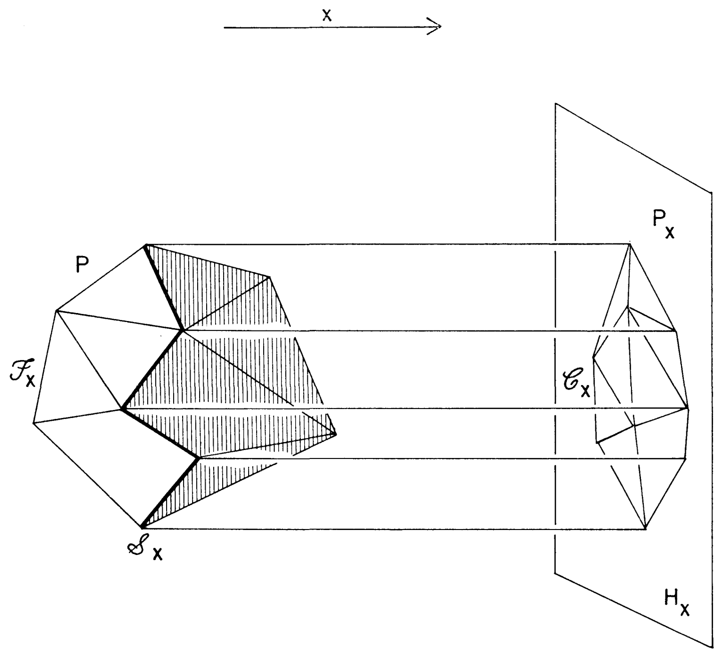

FIGURE 1

Now each point on the boundary ∂P of P projects into a uniquely determined point of P_x, and each point y in the interior of P_x is the image under π_x of two points y_1, y_2 of ∂P. These two points may be distinguished by the fact that for one of them, say y_1, the open half-line $\{y_1 + \lambda x | \lambda > 0\}$ meets the interior of P (and then y_1 is a relative interior point of some face of $\mathscr{F}_x$), and for the other point y_2, the open half-line $\{y_2 + \lambda x | \lambda > 0\}$ does not meet the interior of P. Consequently, the projection π_x induces a one-to-one mapping between the set of relative interior points of the faces $F_i^j \in \mathscr{F}_x$, and the interior P_x. Thus if $F_i^j \in \mathscr{F}_x$, $\pi_x(F_i^j)$ is a j-polytope (or j-cell) in P_x. Now the set of all faces in $\mathscr{F}_x \cup \mathscr{S}_x$ has the property that the intersection of any two faces is either empty or is a face of P belonging to $\mathscr{F}_x \cup \mathscr{S}_x$. Consequently, the cells $\pi_x(F_i^j)$ $(F_i^j \in \mathscr{F}_x \cup \mathscr{S}_x)$ have the same property and so form a cell-complex $\mathscr{C}_x$ whose point-set (that is, the union of all its cells) is P_x. We deduce that $n_j(x)$ is the number of j-cells of $\mathscr{C}_x$ whose interiors lie in the interior of P_x, and so the total number of j-cells in $\mathscr{C}_x$ is $n_j(x) + f_j(P_x)$. Applying Euler's theorem (**3**, Theorem 2.3) to the cell-complex $\mathscr{C}_x$ we obtain

$$\sum_{j=0}^{d-1} (-1)^j (n_j(x) + f_j(P_x)) = 1. \tag{4}$$

But by Euler's theorem for the $(d-1)$-polytope P_x (**2**, §8.1),

$$\sum_{j=0}^{d-1} (-1)^j f_j(P_x) = 1 + (-1)^d. \tag{5}$$

Equalities (4) and (5) yield (3) and so the proof of the theorem is completed.

References

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University of East Anglia,
England