Gram's Equation – A Probabilistic Proof

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If β_v are the angles at the vertices v, $v \in V$, of a convex n-gon in the plane, then $\sum_{v \in V} \beta_v = (n-2)\pi$ – known as Euclid's angle-sum equation. We normalize the full angle to 1, i.e. we set $\alpha_v = \beta_v/(2\pi)$. Then the equation reads as

$$\sum_{v \in V} \alpha_v = \frac{n}{2} - 1 \ . \tag{1}$$

This identity has a generalization to higher dimensions, called Gram's equation. We want to derive this identity by a probabilistic argument. Let us immediately start with the proof, the identity will emerge quite naturally.

Consider a convex polytope $\mathcal{P}$ in 3-space, with vertex set V, edge set E and face set F. For a vertex v, let α_v denote the fraction of an infinitesimally small sphere centered at v that is contained in $\mathcal{P}$. Similarly, for an edge e, let α_e denote the fraction of an infinitesimally small sphere centered at a point in the relative interior of e that is contained in $\mathcal{P}$.

Let us now investigate a random parallel orthogonal projection of $\mathcal{P}$, i.e. we choose a random point uniformly distributed on S^2, and we project $\mathcal{P}$ in the direction specified by this point into a plane orthogonal to the direction. The projection is a convex polygon. What is the expected number of vertices we get? The probability that a vertex v will not project to a vertex in the projected polygon is $2\alpha_v$. Thus the expected number of vertices is

$$\sum_{v \in V} (1 - 2\alpha_v) \ . \tag{2}$$

Similarly, the expected number of edges in the projection is

$$\sum_{e \in E} (1 - 2\alpha_e) \ . \tag{3}$$

Now, since the number of vertices equals the number of edges, (2) equals (3) and we have the equation

$$\sum_{v \in V} \alpha_v - \sum_{e \in E} \alpha_e = \frac{|V| - |E|}{2} = -\frac{|F|}{2} + 1 \ . \tag{4}$$

The formula was known to de Gua (1783) for the case of a tetrahedron. Hopf attributes the result for 3-polytopes to Brianchon (1837), while Grünbaum refers to Gram [4] for a first proof of the result. Grünbaum reports that Gram's paper was forgotten and that meanwhile Dehn [2] and Poincaré [8] contributed to the subject.

Hopf gave another simple proof of the identity. He uses the angle-sum equation for spherical triangles (which we somehow replaced by the probabilistic argument).

The formula generalizes to arbitrary dimensions (see e.g. [5]). For a d-polytope, let α_i, $i = 0, 1, \ldots d$, be the sum of all solid angles at i-faces (defined as above for 0- and 1-faces). In particular, $\alpha_d = 1$, and α_{d-1} is half the number f_{d-1}, where f_i, $0 \leq i \leq d$, denotes the number of i-faces of the polytope ($f_d = 1$). Then

$$\sum_{i=0}^{d}(-1)^i \alpha_i = 0 \tag{5}$$

which looks similar to Euler's formula

$$\sum_{i=0}^{d}(-1)^i f_i = 1 \ , \tag{6}$$

or

$$\sum_{i=0}^{d-2}(-1)^i \alpha_i = (-1)^d\left(\frac{f_{d-1}}{2} - 1\right)$$

to show the similarities to equations (1) and (4) in 2- and 3-space, respectively. The probabilistic argument readily generalizes to a proof of the higher dimensional result. Using arguments as above for the expected number of i-faces in a random projection, by the linearity of expectation and by applying Euler's formula in $d-1$-space, we have

$$\sum_{i=0}^{d-2}(-1)^i (f_i - 2\alpha_i) + (-1)^{d-1} = 1 \ .$$

We rearrange sums

$$\left(\sum_{i=0}^{d}(-1)^i f_i\right) - (-1)^{d-1} f_{d-1} - (-1)^d f_d$$
$$- 2\left(\sum_{i=0}^{d}\alpha_i\right) + 2(-1)^{d-1}\alpha_{d-1} + 2(-1)^d \alpha_d + (-1)^{d-1} = 1 \ .$$

After cancelling terms via $\alpha_{d-1} = f_{d-1}/2$, $\alpha_d = 1$ and $f_d = 1$, we invoke Euler's formula (6) in d dimensions to obtain (5).

A proof of (5) is indicated also by Edelsbrunner in [3] (Remark 3, Section 6). Barnette [1] uses very similar arguments (although in a different terminology) to prove a tight bound of

$$\alpha_i \leq \frac{1}{2}\left(f_i - \binom{d}{i+1}\right) , \qquad 0 \leq i \leq d-2 .$$

The proof exploits the fact that for every projection of a d-polytope the number of i-faces is at least $\binom{d}{i+1}$.

References

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