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Upper Bounds on the Maximal Number of Facets of 0/1-Polytopes

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We prove two new upper bounds on the number of facets that a d-dimensional 0/1-polytope can have. The first one is $2(d-1)!+2(d-1)$ (which is the best one currently known for small dimensions), while the second one of $O((d-2)!)$ is the best known bound for large dimensions.

1. INTRODUCTION

Polytopes whose vertices have only coordinates 0 and 1 (*0/1-polytopes*) have been investigated in combinatorial optimization: to any set system over which one wants to optimize, one can naturally associate the 0/1-polytope which is the convex hull of the incidence vectors of all feasible sets. In trying to attack combinatorial optimization problems by linear programming, one needs a description of the facets of the corresponding polytopes. For several 0/1-polytopes coming from combinatorial optimization problems, most notably the travelling salesman polytope, the cut polytope, or the linear ordering polytope, many large classes of facet-defining inequalities have been identified.

Therefore it seems interesting to ask how many facets a d-dimensional 0/1-polytope can have at all [14, Problem 0.15]. A complete census of all 0/1-polytopes with up to five dimensions with regard to various properties was done by Aichholzer [2]. The d-dimensional *cross-polytope* can be realized (combinatorially) as the 0/1-polytope $\text{conv}\{\mathbf{e}_i, \mathbf{1}-\mathbf{e}_i : 1 \le i \le d\}$, where $\mathbf{e}_i$ is the ith canonical unit vector and $\mathbf{1}$ is the all-ones vector, showing that d-dimensional 0/1-polytopes can have as many as 2^d facets. Starting with a special randomly generated 0/1-polytope of dimension 13 with more than 17 million facets (found by Christof [7]), and using some inductive construction due to Kortenkamp *et al.* [11], one can show that the maximal numbers of facets of d-dimensional 0/1-polytopes grow at least as fast as 3.6^d.

On the other hand, Bárány (see [11]) gave a good argument that a d-dimensional 0/1-polytope cannot have more than $d!+2d$ facets, which we will briefly review below (Lemma 2) since we will need it in one of our proofs. Let $f(d)$ be the maximal number of facets that a d-dimensional 0/1-polytope can have. Thus, we know that asymptotically

$$2^{\text{const}\cdot d} \le f(d) \le 2^{\text{const}\cdot d \log d}$$

holds. The most interesting question (in this context) is whether there is an exponential upper bound on $f(d)$ or whether $f(d)$ grows faster than exponentially. In fact, the growth of $f(d)$ in low dimensions indicates that an exponential upper bound is unlikely to exist ([7, 10], see also Table 1).

This paper contains two improved upper bounds. The first one in Section 2 is obtained very easily by a simple observation on projections of 0/1-polytopes and gives an upper bound of $2(d-1)!+2(d-1)$. The second one in Section 3 is obtained by a refinement of the first one

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TABLE 1.
Lower and upper bounds for $f(d)$.

d	Best lower bound	$f(d) \le A_d$	$f(d) \le U_d$	R	U_d/A_d
1	$f(d) = 2 = 2^d$	2	2	1	1.000
2	$= 4 = 2^d$	4	4	2	1.000
3	$= 8 = 2^d$	8	9	2	1.125
4	$= 16 = 2^d$	18	28	2	1.555
5	$= 40 \ge 2.091^d$	56	100	3	1.785
6	$f(d) \ge 121 \ge 2.223^d$	250	469	4	1.876
7	$\ge 432 \ge 2.379^d$	1 452	2 570	5	1.769
8	$\ge 1675 \ge 2.529^d$	10 094	16 328	6	1.617
9	$\ge 6875 \ge 2.669^d$	80 656	118 404	7	1.468
10	$\ge 41\,591 \ge 2.896^d$	725 778	983 516	8	1.355
11	$\ge 250\,279 \ge 3.095^d$	7 257 620	9 044 131	10	1.246
12	$\ge 1\,975{,}935 \ge 3.346^d$	79 833 622	92 580 349	11	1.159
13	$\ge 17\,464\,356 \ge 3.606^d$	958 003 224	1 028 972 176	13	1.074
14		12 454 041 626	12 499 470 015	15	1.003
15		174 356 582 428	164 305 261 217	17	0.942
16		2 615 348 736 030	2 324 510 568 224	19	0.888
17		41 845 579 776 032	35 227 585 773 379	22	0.841
18		711 374 856 192 034	565 675 688 445 291	24	0.795

and yields a bound of $O((d-2)!)$, which is a better bound for higher dimensions. Actually, the arguments that we use there also apply (slightly modified) to integer convex polytopes (i.e., polytopes with integral vertex coordinates) with vertex coordinates in $\{0, \dots, k\}$ for a constant $k \in \mathbb{N}$. Therefore, we prove a more general theorem that bounds the number of facets (and even the numbers of i-dimensional faces for all $0 \le i \le d-1$) of integer convex polytopes with (vertex) coordinates bounded by a constant. In particular, this generalization will enable us also to give some nontrivial upper bounds on the number of i-faces of 0/1-polytopes for intermediate values of i via some kind of 'detour' through more general integer polytopes. In Section 4, we calculate explicit bounds for the number of facets of 0/1-polytopes in low dimensions. Finally, in Section 5, we compare our bounds to some results from the literature, where the number of facets of an integer polytope is bounded in terms of its surface area or volume.

Some definitions and facts By a *polytope* we will always mean a *convex* polytope, i.e., the convex hull of a finite set of points. An *i-face* is the abbreviation for an i-dimensional face of a polytope. The 0-faces are the *vertices* and the $(d-1)$-faces of a d-dimensional polytope are the *facets*. For background information on polytopes, we refer to Ziegler's book [14].

We denote the d-dimensional unit hypercube by C^d. The d-dimensional cross-polytope with diameter $2r$ (or equivalently the l_1-ball of radius r) is

$$B^d(r) := \operatorname{conv}\{r\mathbf{e}_i, -r\mathbf{e}_i : 1 \le i \le d\}.$$

The ith coordinate hyperplane, which is orthogonal to $\mathbf{e}_i$, is denoted by H_i. The orthogonal projection to H_i is

$$\operatorname{pr}_i\colon (x_1, \dots, x_d) \mapsto (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_d).$$

By $C_i^d := \operatorname{pr}_i(C^d)$ we denote the $(d-1)$-dimensional unit hypercube in the coordinate hyperplane H_i.

The euclidean length of a vector $\mathbf{n} = (n_1, n_2, \dots, n_d)$ is $\|\mathbf{n}\|_2 := \sqrt{\sum_{i=1}^d n_i^2}$, while its l_1-norm is $\|\mathbf{n}\|_1 := \sum_{i=1}^d |n_i|$.

The Minkowski sum of sets $A, B \subset \mathbb{R}^d$ is $A + B := \{\mathbf{x}_a + \mathbf{x}_b : \mathbf{x}_a \in A, \mathbf{x}_b \in B\}$; for $k \in \mathbb{R}$ the k-blow up of A is $k \cdot A := \{k \cdot \mathbf{x} : \mathbf{x} \in A\}$ and finally $\mathrm{Vol}^d(A)$ denotes the d-dimensional volume of A. The d-dimensional volume of a parallelotope P spanned by vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d$ is

$$\mathrm{Vol}^d(P) = |\det(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d)|.$$

The volumes of the hypercubes and the cross-polytope are

$$\mathrm{Vol}^d(C^d) = \mathrm{Vol}^{d-1}(C_i^d) = 1$$

(for $1 \le i \le d$) and

$$\mathrm{Vol}^d(B^d(r)) = \frac{2^d r^d}{d!}.$$

Moreover for $\mathbf{x} \in \mathbb{R}^d$ we have $\mathrm{Vol}^d(k \cdot A + \mathbf{x}) = k^d \mathrm{Vol}^d(A)$.

OBSERVATION 1. *The volume* $\mathrm{Vol}^d(P)$ *of any d-dimensional integer polytope P is an integer multiple of* $\frac{1}{d!}$. *In particular,* $\mathrm{Vol}^d(P)$ *is at least* $\frac{1}{d!}$.

PROOF. A d-dimensional integer polytope can be subdivided into d-dimensional integer simplices, and every d-dimensional integer simplex is the image of the d-dimensional standard simplex (having volume $\frac{1}{d!}$) under an affine transformation with integer coefficients. □

Finally, we need a simple estimate for $\sqrt[d]{d!}$, which we obtain with the help of the inequality between the geometric and harmonic mean.

$$\sqrt[d]{d!} = \left(\sqrt[d]{\sqrt{1}\sqrt{2}\ldots\sqrt{d}}\right)^2 \ge \left(\frac{d}{\sum_{i=1}^{d}\frac{1}{\sqrt{i}}}\right)^2 \ge \left(\frac{d}{\int_0^d \frac{1}{\sqrt{t}}dt}\right)^2 = \left(\frac{d}{2\sqrt{d}}\right)^2 = \frac{d}{4}. \quad (1)$$

(Stirling's formula yields the more precise estimate $\sqrt[d]{d!} = \frac{d}{e} + O(\log d)$.)

2. A SIMPLE UPPER BOUND BY PROJECTION

Let P be a d-dimensional 0/1-polytope. First note that we can assume that P lies in $\mathbb{R}^d$, since every d-dimensional 0/1-polytope $P' \subset \mathbb{R}^{d'}$ (with $d' > d$) is affinely isomorphic to a d-dimensional 0/1-polytope $P \subset \mathbb{R}^d$ by simply 'projecting out' all coordinates that belong to a basis of a nonredundant and complete equation system describing the affine hull of P'. The analogous statement holds for integer polytopes with vertex coordinates in $\{0, \ldots, k\}$.

The following lemma is due to Bárány (see also [14, Problem 0.15], [11]).

LEMMA 2. *A d-dimensional 0/1-polytope* $P \subset \mathbb{R}^d$ *has at most* $d!(1-\mathrm{Vol}^d(P))+2d$ *facets*.

PROOF. If $v \in \{0, 1\}^d \setminus P$ is a vertex of the hypercube that is not a vertex of P, then $\mathrm{conv}(P \cup \{v\})$ is a 0/1-polytope that can be subdivided into P and pyramids with apex v, whose bases are those facets of P which are deleted by the addition of v (i.e., in the terminology of Ziegler [14], the bases are those facets of P *beyond* which v lies). Iterating this process until all vertices of the hypercube are in the convex hull destroys all facets of P except the 'trivial' ones (i.e., those that lie in facets of the hypercube). Thus the total number of facets of P cannot be larger than $d!(1 - \mathrm{Vol}^d(P)) + 2d$. □

Every facet of P is defined by an inequality which is uniquely determined up to multiplication by positive scalars. With respect to some coordinate $i \in \{1, \ldots, d\}$, a facet of P that is defined by an inequality $a^T x \le a_0$ is called a *vertical* facet of P if $a_i = 0$, an *upper* facet if $a_i < 0$, and a *lower* facet if $a_i > 0$. The following facts are well known.

LEMMA 3. *Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope with facets $F^1, \dots, F^t$, and let $i \in \{1, \dots, d\}$. Then the projections of the lower (respectively upper) facets of P with respect to i form a subdivision of $\mathrm{pr}_i(P)$, i.e., their union is $\mathrm{pr}_i(P)$ and they have no common interior points. In particular, we have*

$$\sum_{j=1}^{t} \mathrm{Vol}^{d-1}(\mathrm{pr}_i(F^j)) = 2 \cdot \mathrm{Vol}^{d-1}(\mathrm{pr}_i(P)). \tag{2}$$

From Lemma 3 we derive a simple new upper bound on the number of facets of a 0/1-polytope.

THEOREM 4. *A d-dimensional 0/1-polytope has at most*

$$A_d := 2(d-1)! + 2(d-1)$$

facets, i.e., $f(d) \le 2(d-1)! + 2(d-1)$ holds for every d.

PROOF. Let $P \subset \mathbb{R}^d$ be a d-dimensional 0/1-polytope in $\mathbb{R}^d$. For every lower or upper facet F of P, the projection $\mathrm{pr}_d(F)$ is a $(d-1)$-dimensional 0/1-polytope, which (by Observation 1) has volume at least $\frac{1}{(d-1)!}$. Thus, from Lemma 3, it follows that P cannot have more than $2(d-1)!\mathrm{Vol}^{d-1}(P')$ lower and upper facets, where $P' := \mathrm{pr}_d(P)$.

Vertical facets of P are projected to facets of P'. Since distinct vertical facets of P are projected to distinct facets of P', the number of vertical facets of P is bounded from above by the number of facets of P'. But by Bárány's argument (Lemma 2), P' has at most

$$(d-1)!(1 - \mathrm{Vol}^{d-1}(P')) + 2(d-1)$$

facets. Therefore, this yields an upper bound of

$$\begin{aligned} f(d) &\le 2(d-1)!\mathrm{Vol}^{d-1}(P') + (d-1)!(1 - \mathrm{Vol}^{d-1}(P')) + 2(d-1) \\ &= (d-1)!\mathrm{Vol}^{d-1}(P') + (d-1)! + 2(d-1) \\ &\le 2(d-1)! + 2(d-1) \end{aligned}$$

on the number of facets of P. □

3. AN IMPROVED UPPER BOUND

In this section, we refine the upper bound A_d of Theorem 4 using two ideas. Instead of projecting only along the dth coordinate, we project along all coordinate directions, and we try to exploit the fact that the projection of a nonvertical facet typically has larger $(d-1)$-volume than $1/(d-1)!$. We need the following fact from linear algebra.

LEMMA 5. *If H is a hyperplane with normal vector $\mathbf{n} = (n_1, n_2, \dots, n_d)$, then*

$$|n_i| = \|\mathbf{n}\|_2 \cdot \mathrm{Vol}^{d-1}(\mathrm{pr}_H(C_i^d)),$$

where pr_H denotes the orthogonal projection to H.

PROOF. Choose $\lambda_j \in \mathbb{R}$ such that $\mathrm{pr}_H(\mathbf{e}_j) = \mathbf{e}_j + \lambda_j \mathbf{n}$. Consider the parallelotope P_i spanned by C_i^d and $\mathbf{n}$. Clearly

$$\begin{aligned} |n_i| &= \mathrm{Vol}^d(P_i) = |\det(\mathbf{n}, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \dots, \mathbf{e}_d)| \\ &= |\det(\mathbf{n}, \mathbf{e}_1 + \lambda_1\mathbf{n}, \dots, \mathbf{e}_{i-1} + \lambda_{i-1}\mathbf{n}, \mathbf{e}_{i+1} + \lambda_{i+1}\mathbf{n}, \dots, \mathbf{e}_d + \lambda_d\mathbf{n})| \\ &= |\det(\mathbf{n}, \mathrm{pr}_H(\mathbf{e}_1), \dots, \mathrm{pr}_H(\mathbf{e}_{i-1}), \mathrm{pr}_H(\mathbf{e}_{i+1}), \dots, \mathrm{pr}_H(\mathbf{e}_d))| \\ &= \|\mathbf{n}\|_2 \cdot \mathrm{Vol}^{d-1}(\mathrm{pr}_H(C_i^d)) \end{aligned}$$

□

COROLLARY 6. *If A lies in a hyperplane H and $\mathrm{Vol}^{d-1}(A)$ is finite and nonzero, then H has a normal vector* $\mathbf{n}$ *of the form*

$$\mathbf{n} = (\pm\mathrm{Vol}^{d-1}(\mathrm{pr}_1(A)), \pm\mathrm{Vol}^{d-1}(\mathrm{pr}_2(A)), \ldots, \pm\mathrm{Vol}^{d-1}(\mathrm{pr}_d(A))).$$

PROOF. Lemma 5 implies that there is a normal vector of H of the form

$$\mathbf{n}' = (\pm\mathrm{Vol}^{d-1}(\mathrm{pr}_H(C_1^d)), \pm\mathrm{Vol}^{d-1}(\mathrm{pr}_H(C_2^d)), \ldots, \pm\mathrm{Vol}^{d-1}(\mathrm{pr}_H(C_d^d))).$$

On the other hand,

$$\frac{\mathrm{Vol}^{d-1}(\mathrm{pr}_H(C_i^d))}{\mathrm{Vol}^{d-1}(C_i^d)} = \frac{\mathrm{Vol}^{d-1}(\mathrm{pr}_i(A))}{\mathrm{Vol}^{d-1}(A)},$$

since there is an isometry exchanging the role of H and H_i. The result follows. □

We prove our main result in a slightly more general setting by extending our subject from 0/1-polytopes to polytopes whose vertices have coordinates in $\{0, \ldots, k\}$ for some constant $k \in \mathbb{N}$. This will enable us to derive some other interesting consequences for 0/1-polytopes later.

THEOREM 7. *There is a constant $c \in \mathbb{R}$ such that if $P \subset \mathbb{R}^d$ is a convex polytope with vertex coordinates in $\{0, 1, \ldots, k\}$ for some $k \geq 1$, then:*

(a) *P has at most*

$$c \cdot (d-2)! \cdot k^{d(d-1)/(d+1)}$$

facets, for $d \geq 2$, and

(b) *for every i with $0 \leq i < d-1$, P has at most*

$$c \cdot (d-2)! \cdot (2(i+1)k)^{d(d-1)/(d+1)}$$

i-dimensional faces.

PROOF. (a) According to the remark at the beginning of Section 2, we can assume that P is d-dimensional, since the claimed bound is increasing in d. Let $F^1, F^2, \ldots, F^t$ be the facets of P, and define $F_i^j := \mathrm{pr}_i(F^j)$. Corollary 6 implies that each facet F^j has an outer normal vector $\mathbf{n}_j$ of the form

$$\mathbf{n}_j = (d-1)! \cdot (\pm\mathrm{Vol}^{d-1}(F_1^j), \pm\mathrm{Vol}^{d-1}(F_2^j), \ldots, \pm\mathrm{Vol}^{d-1}(F_d^j)),$$

which is integral, by Observation 1. Thus,

$$\sum_{j=1}^{t} \|\mathbf{n}_j\|_1 = (d-1)! \cdot \sum_{j=1}^{t}\sum_{i=1}^{d} \mathrm{Vol}^{d-1}(F_i^j). \tag{3}$$

Applying Lemma 3 we obtain

$$\sum_{j=1}^{t} \mathrm{Vol}^{d-1}(F_i^j) \leq 2 \cdot \mathrm{Vol}^{d-1}(k \cdot C_i^d) = 2 \cdot k^{d-1}.$$

Summation over all coordinate directions i gives an upper bound for (3):

$$\sum_{j=1}^{t} \|\mathbf{n}_j\|_1 \leq 2d! \cdot k^{d-1}. \tag{4}$$

From this relation we will derive our result, using only the fact that $\mathbf{n}_1, \dots, \mathbf{n}_t$ are distinct nonzero integer vectors. For a given small dimension, the largest possible number t of such vectors can be worked out directly. This is done in Section 4 for $k = 1$ (i.e., for 0/1-polytopes). To obtain the general bound that we want to prove, we shall show that

$$t \geq (d-2)! \cdot k^{d(d-1)/(d+1)} \tag{5}$$

implies that the average l_1-norm of $\mathbf{n}_1, \dots, \mathbf{n}_t$ is $\Omega(d^2 \cdot k^{(d-1)/(d+1)})$, see (7) and (9). Let us define

$$\begin{aligned} I^d(r) &:= B^d(r) \cap \mathbb{Z}^d \\ S^d(r) &:= I^d(r) \setminus I^d(r-1) \\ \Sigma^d(r) &:= \sum_{\mathbf{x} \in I^d(r)} \|\mathbf{x}\|_1 = \sum_{i=0}^{r} i \cdot |S^d(i)|. \end{aligned}$$

Observe that $|I^d(r)| = \mathrm{Vol}^d(I^d(r) + C^d)$ and $I^d(r) + C^d \subset B^d\left(r + \frac{d}{2}\right) + \frac{1}{2} \cdot \mathbf{1}$, yielding

$$|I^d(r)| \leq \frac{(2r+d)^d}{d!}. \tag{6}$$

Observe moreover that for $r_1 < r_2$ we have $|S^d(r_1)| \leq |S^d(r_2)|$, implying

$$\begin{aligned} \Sigma^d(r) &= \frac{1}{2} \sum_{i=0}^{r} \Big[i \cdot |S^d(i)| + (r-i) \cdot |S^d(r-i)| \Big] \\ &\geq \frac{1}{2} \sum_{i=0}^{r} \frac{r}{2} (|S^d(i)| + |S^d(r-i)|) = \frac{r}{2} \sum_{i=0}^{r} |S^d(i)| = \frac{r}{2} |I^d(r)|. \end{aligned}$$

Thus we have

$$\Sigma^d(r) \geq \frac{r}{2} |I^d(r)| \tag{7}$$

for $r \in \mathbb{N}$. (A more careful estimation shows that the constant $\frac{1}{2}$ can be replaced by $\frac{d}{d+1}$.) Choose $R \in \mathbb{N}$ such that

$$|I^d(R)| \leq t < |I^d(R+1)|. \tag{8}$$

Using (5), (6), and (8), we obtain

$$(d-2)! \cdot k^{d(d-1)/(d+1)} < \frac{(2R+d+2)^d}{d!}.$$

By (1), this implies

$$\begin{aligned} R &> \frac{1}{2} \sqrt[d]{d!(d-2)!} \cdot k^{(d-1)/(d+1)} - \frac{d}{2} - 1 \\ &> \frac{d}{8} \left(\frac{d-2}{4} \right)^{\frac{d-2}{d}} \cdot k^{(d-1)/(d+1)} - \frac{d}{2} - 1 > c' d^2 \cdot k^{(d-1)/(d+1)}, \end{aligned} \tag{9}$$

for a certain $1 > c' > 0$ and large enough d. (A more careful analysis reveals that

$$\frac{R}{d^2 \cdot k^{(d-1)/(d+1)}} \geq \frac{1}{2e^2} - O\left(\frac{\log d}{d} \right), \tag{10}$$

as $d \to \infty$.)

To finally estimate t, we bound the left-hand side of inequality (4), using (8), (9), and (7):

$$\begin{aligned} 2d! \cdot k^{d-1} &\geq \sum_{j=1}^{t} \|\mathbf{n}_j\|_1 \geq \Sigma^d(R) + R \cdot (t - |I^d(R)|) \\ &\geq \frac{R}{2}|I^d(R)| + R \cdot (t - |I^d(R)|) \geq \frac{R}{2}(|I^d(R)| + t - |I^d(R)|) \\ &\geq \frac{c'd^2 \cdot k^{(d-1)/(d+1)}}{2} \cdot t. \end{aligned}$$

Thus, there is a $d_0 \in \mathbb{N}$ such that for $d \geq d_0$, (5) implies

$$t \leq \frac{4d!}{c'd^2} \cdot k^{d(d-1)/(d+1)} \leq \frac{4}{c'}(d-2)! \cdot k^{d(d-1)/(d+1)}.$$

Since $c' < 1$, we obtain

$$t \leq c(d-2)! \cdot k^{d(d-1)/(d+1)}$$

for $c := \frac{4}{c'}$ and $d \geq d_0$. By increasing the constant c if necessary, the inequality can be made true for all $d \geq 2$.

(b) First we prove the case $i = 0$, using a construction which is similar to a trick of Andrews [3]. We construct from P another polytope

$$P' := \operatorname{conv}\left\{\tfrac{1}{2}(\mathbf{x} + \mathbf{y}) : \mathbf{x} \text{ and } \mathbf{y} \text{ are different vertices of } P\right\}.$$

No vertex $\mathbf{x}$ of P belongs to P', and any facet of P' that separates $\mathbf{x}$ from P' does not separate any other vertex $\mathbf{z}$ of P from P'. Thus the polytope P' has at least as many facets as P has vertices, and the case $i = 0$ follows from part (a) of the theorem because $2 \cdot P' \subset 2k \cdot C^d$ is an integer polytope. (Andrews [3] used a blow-up factor of 3 instead of 2.)

We reduce the case $1 \leq i < d-1$ to the case $i = 0$ by selecting $i+1$ affinely independent vertices $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{i+1}$ from each i-face F of P. The point $\mathbf{x}_F := \frac{1}{i+1} \cdot \sum_{j=1}^{i+1} \mathbf{x}_j$ lies in the relative interior of F, and therefore all points $\mathbf{x}_F$ are distinct. The points $\mathbf{x}_F$ are the vertices of the polytope

$$P'' := \operatorname{conv}\{\mathbf{x}_F : F \text{ is an } i\text{-face of } P\},$$

since every i-face F of P has a hyperplane H with $H \cap P = F$; it follows that $H \cap P'' = \{\mathbf{x}_F\}$. Thus P'' has a vertex for every i-face of P, and since $(i+1) \cdot P'' \subset (i+1)k \cdot C^d$ is an integer polytope, the result follows from the case $i = 0$. □

For $d = 2$, i. e., for polygons, the precise asymptotic bound of Theorem 7(a) is not difficult to derive, see Thiele [13] or Acketa and Žunić [1]; see also [14, Exercise 4.15, p. 122]. (For the case when the circumference of the polygon is bounded instead of the bounding box, as in Theorem 9 (a),(b) in Section 5 below, the precise asymptotic bound is given in Jarník [8].)

If we set $k = 1$ in Theorem 7, then we get $O((d-2)!)$ bounds for 0/1-polytopes:

COROLLARY 8. *There is a constant $c \in \mathbb{R}$ such that for $d \geq 2$, every d-dimensional 0/1-polytope has at most*

$$c \cdot (d-2)!$$

facets (i.e., $f(d) \in O((d-2)!)$) and at most

$$c \cdot (2(i+1))^{d(d-1)/(d+1)} \cdot (d-2)!$$

i-faces, for every i with $0 \leq i < d-1$.

For small values of i (e.g., $i = 0, 1$) this is not very interesting, since the maximum number of vertices of a 0/1-polytope is of course 2^d, and the number of i-faces is trivially bounded by $\binom{2^d}{i+1}$. But for larger intermediate values of i we get nontrivial bounds.

For the constant c in the bounds of Theorem 7 and Corollary 8, a more thorough analysis shows that, for large dimensions d, one can take

$$c = 4e^2 + O\left(\frac{\log d}{d}\right) \approx 29.55.$$

For the number of facets of 0/1-polytopes, the resulting bound in Corollary 8 should therefore be better than the easy bound A_d of Theorem 4 as soon as d is larger than $c/2 \approx 15$.

4. Explicit Bounds in Low Dimensions

Table 1 gives numerical values of various lower and upper bounds on the number $f(d)$ of facets of a d-dimensional 0/1-polytope. The first column of numbers contains the largest known examples, in terms of the number of facets, for all dimensions up to $d = 13$, from [7]. For $d \leq 5$, these are known to be the true maxima (Aichholzer [2]). The second column gives the easy bound $A_d = 2(d-1)! + 2(d-1)$ of Theorem 4. We see that it is precise for $d \leq 3$, but departs more and more from the lower bounds as d gets higher. The third column U_d is a precise version of the bound in Corollary 8, which is obtained directly from (4). Instead of using the estimates that lead to the proof of Theorem 7, we can enumerate the integer vectors in the successive l_1-spheres $S^d(1), S^d(2), S^d(3), \ldots$ as long as their total l_1-length does not exceed the bound $2d!$ from (4). The number of points in these spheres is given by the formula

$$|S^d(r)| = \sum_{k=1}^{d} 2^k \binom{d}{k} \binom{r-1}{k-1}.$$

The kth term of this sum is the number of vectors $x \in S^d(r)$ with k nonzero coefficients.

The bound can be slightly improved by taking into account that we only have to consider *primitive* vectors as normal vectors of facets, i.e., vectors where the greatest common divisor of its components is one. Each imprimitive vector is a positive multiple of some shorter primitive vector and does therefore not correspond to a new facet direction. The number of nonzero primitive vectors in the l_1-ball $B^d(r)$ can be computed conveniently by the inclusion–exclusion formula

$$(|I^d(r)| - 1) - \sum_{p_i \leq r} \left(\left| I^d\left(\left\lfloor \frac{r}{p_i} \right\rfloor \right) \right| - 1 \right) + \sum_{p_i < p_j \leq r} \left(\left| I^d\left(\left\lfloor \frac{r}{p_i p_j} \right\rfloor \right) \right| - 1 \right) - \ldots b,$$

where $p_1, p_2, \ldots$ is an enumeration of the primes. The number of primitive vectors in $S^d(r)$ is computed easily from these formulas. If the imprimitive vectors were not excluded, the bound on $f(5)$ would be 103 instead of 100. For smaller d, this has no effect, and for larger d it usually means an improvement in U_d somewhere around the middle digit of each figure.

The column entitled 'R' specifies the l_1-radius R of the U_dth primitive vector. One can check that this value is roughly in accordance with the estimate $R \approx \frac{d^2}{2e^2}$ from (10). The last column is the quotient of the bounds U_d and A_d. The asymptotically stronger bound is never much worse than the easy bound of Theorem 4 and starts to beat it for $d \geq 15$ as predicted at the end of the previous section.

5. Conclusion

Our results are related to a classical theorem of Andrews about vertex numbers of integral polytopes with bounded volume or surface area.

THEOREM 9 (ANDREWS [3,4]). *Let P be a d-dimensional convex polytope with integral vertices.*

(a) *If P has surface area S and t facets, then*

$$t \leq c_d \cdot S^{d/(d+1)}.$$

(b) *If P has surface area S and n vertices, then*

$$n \leq c'_d \cdot S^{d/(d+1)}.$$

(c) *If P has volume V and n vertices, then*

$$n \leq c''_d \cdot V^{(d-1)/(d+1)}.$$

Here, c_d, c'_d, and c''_d are constants that depend the dimension d.

For fixed dimension d, the growth in terms of the 'size' k in our Theorem 7 is of the same order of magnitude as in Theorem 9: the volume and surface area of $k \cdot C_d$, and hence of P, is bounded by k^d and by $2d \cdot k^{d-1}$, respectively.

The proofs of Theorem 9 in the literature pay no attention to the dependence of the bounds on d, although it is not hard to work out expressions for the constants c_d, c'_d, and c''_d from these proofs. (By considering the standard simplex $\operatorname{conv}(\{0\} \cup \{\mathbf{e}_i : 1 \leq i \leq d\})$, one sees immediately that the constants c_d and c'_d have to be at least $(d-1)!/2$; thus, for varying dimensions d, the bounds on the number of facets that one can derive from Theorem 9(a) are weaker than Theorem 7 and Corollary 8.)

The proof of Theorem 9(a) [3] uses a straightforward argument about the area of facets: if a facet F of P has a primitive normal vector $\mathbf{n}$, then its area $\operatorname{Vol}^{d-1}(F)$ is at least $\|\mathbf{n}\|_2/(d-1)!$. Our proofs of Theorems 4 and 7 use area arguments in a similar way. However, in order to get a better dependence on d, it has been advantageous to consider the norm $\|\mathbf{n}\|_1$ of normal vectors instead of their euclidean norm (see Lemma 6).

Andrews [3] derived Theorem 9(b) from Theorem 9 (a) by constructing, from a given polytope P, another polytope P' which has at least as many facets as P has vertices. Schmidt [12, pp. 66–68] and Bárány and Larman [6] considered also bounds on the number of i-faces for i other than 0 and $d-1$, by using extensions of Andrews' construction. Our proof of Theorem 7(b) uses a similar construction, but we tried to keep the dependence on d low.

The proof of Theorem 9(c) is based on Theorem 9(b), but it is much harder. Different proofs are due to Andrews [4], Konyagin and Sevast′yanov [9], Schmidt [12, pp. 64–66], and Bárány and Larman [6]. Bárány and Larman [6] have also proved that the bounds of Theorem 9 are asymptotically tight, by showing that the convex hull of the integer points in a ball of radius k has $\Omega(k^{d(d-1)/(d+1)})$ vertices (and facets), for fixed d and $k \to \infty$. For Theorem 9(c), there was already an easy lower-bound example of Arnol′d [5]: the convex hull of integral points in the paraboloid $x_1^2 + \cdots + x_{d-1}^2 \leq x_d \leq k$.

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