we have to consider bases and pivots instead of vertices and edges. (See Exercise 3.10 for a brief sketch.) Here we run into problems of degeneracy if the polytope is not simple, or if the linear function is not in general position. One way to treat this is through "perturbation," implicitly or explicitly. For example, if we know an interior point (this is not a natural assumption for practical problems!), then we can rewrite P as $P(A, \mathbf{1})$, and then (implicitly rather than explicitly) optimize over $P(A, \mathbf{1}^\lambda)$ for small enough $\lambda > 0$, which is nondegenerate. This leads to lexicographic pivot rules; see Chvátal [136, pp. 34–36]. Furthermore, to construct a simplex algorithm we have to determine "which edge to take"; this leads to the question of pivot rules. All this is combinatorial geometry. Later in the game, numerical questions dominate the picture.

Anyway, this discussion was only meant as a sketch of the geometric situation — a very simple and special picture of the world according to a discrete geometer.

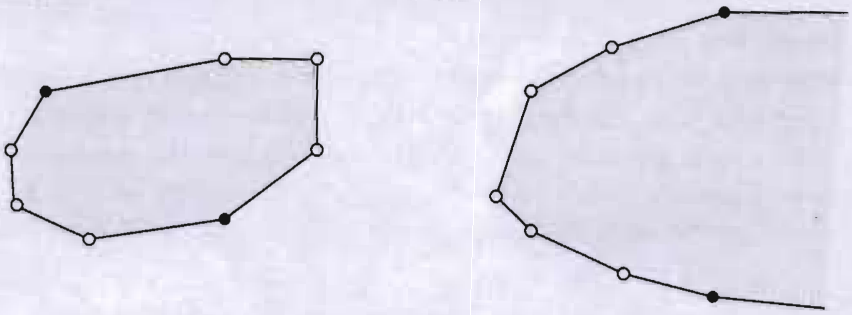
3.3 The Hirsch Conjecture

The *diameter* of a graph G will be denoted by $\delta(G)$: the smallest number δ such that any two vertices in G can be connected by a path with at most δ edges.

For $n > d \geq 2$, let $\Delta(d,n)$ be the maximal diameter of the graph of an d-dimensional polytope P with at most n facets. Similarly, let $\Delta_u(d,n)$ denote this maximal diameter in the unbounded case, for a d-dimensional pointed polyhedron P with at most n facets ($n \geq d \geq 2$). For example,

$$\Delta(2,n) = \lfloor \frac{n}{2} \rfloor, \qquad \Delta_u(2,n) = n-2.$$

Our sketch illustrates the extreme cases for $d = 2$ and $n = 8$.



It is a long-standing problem to determine the behavior of the function $\Delta(d,n)$. The value of $\Delta(d,n)$ is a lower bound for the number of iterations needed for the simplex algorithm with *any* pivot rule. Thus the question of whether $\Delta(d,n)$ grows polynomially in n and d is closely related to the question of whether there is any pivot rule for which the simplex algorithm is a strongly polynomial algorithm for linear programming; see [267, Sect. 3].

A notorious, very specific, question connected with the graphs of polytopes was first posed by Warren M. Hirsch in 1957 (see Dantzig [149, pp. 160, 168]) and has become known as the *Hirsch conjecture.*

Conjecture 3.8 (Hirsch conjecture). [149, p. 168]
For $n > d \geq 2$, let $\Delta(d,n)$ denote the largest possible diameter of the graph of a d-polytope with n facets. Then

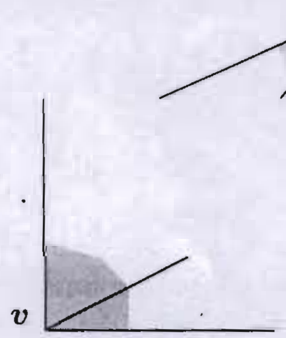
$$\Delta(d,n) \leq n - d.$$

Is this plausible? Here are a few observations, mostly due to Klee & Walkup [271].

- The Hirsch conjecture is true for $d \leq 3$ and all n (even in the monotone and unbounded versions discussed below, by Klee [261]), and for $n - d \leq 5$, by Klee & Walkup [271].
- One can show that for Conjecture 3.8 it is sufficient to consider simple polytopes (see Exercise 3.5).
- If $n < 2d$, then any two vertices lie on a common facet. From this we get $\Delta(d,n) \leq \Delta(d-1,n-1)$; iterating this, we get

  $$\Delta(d,n) \leq \Delta(n-d, 2(n-d)) \quad \text{for} \quad n < 2d.$$

  Similarly, we get $\Delta_u(d,n) \leq \Delta_u(n-d, 2(n-d))$. In both cases these inequalities hold with equality [271]: this is quite obvious in the unbounded case. Thus we restrict our attention to the case $n \geq 2d$.
- More surprisingly [271], the Hirsch conjecture would follow if one could prove it in the special case $n = 2d$, which has become known as the *d-step conjecture.*
  Consider two vertices that do not lie on a common facet. Since each of them lies on d facets, we see that the d-step conjecture concerns a very special geometric situation: after a change of coordinates we can assume that the first vertex $\boldsymbol{v}$ is given by $\boldsymbol{v} = \boldsymbol{0}$, where the facets it lies on are given by $x_i \geq 0$, which describes the positive orthant $\boldsymbol{x} \geq \boldsymbol{0}$. Then the other vertex $\boldsymbol{u}$ can be assumed to be $\boldsymbol{u} = \boldsymbol{1}$, and its facets describe an affine image of the positive orthant.



In this situation there are d edges leaving from $\boldsymbol{u}$, whose other endpoints are on the hyperplanes $\{\boldsymbol{x} : x_i = 0\}$. The claim is that we can get from $\boldsymbol{u}$ to $\boldsymbol{v}$ in d steps.

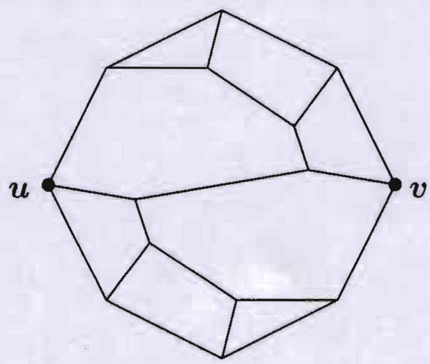
From the special case of the d-cube we get that $\Delta(d, 2d) \geq d$. Thus the bound suggested by the d-step conjecture is certainly the best possible, if it holds. In the general case, the best lower bound known to date seems to be the one due to Adler [4]:

$$\Delta(d, n) \quad \geq \quad \left\lfloor (n-d) - \frac{n-d}{\lfloor 5d/4 \rfloor} \right\rfloor - 1.$$

- If you look for counterexamples, a natural guess would be to consider the polars of cyclic polytopes $C_d(n)^{\Delta}$, or more generally the polars of neighborly polytopes — since they have the largest numbers of vertices for given n and d (according to the upper bound theorem; see Section 8.4). However, Klee [265] has shown that the polars of cyclic polytopes satisfy the Hirsch conjecture. Beyond that, Kalai [247] could prove that if P is the polar neighborly d-polytope with n facets, then one has at least a polynomial diameter bound $\delta(G) \leq d^2(n-d)^d \log(n)$.

- The *nonrevisiting path conjecture*, due to Victor Klee and Philip Wolfe, states the following: for any two vertices $\boldsymbol{u}, \boldsymbol{v}$ of a (simple) polytope, there is a path from $\boldsymbol{u}$ to $\boldsymbol{v}$ that does not revisit any facet it has left before.

  To illustrate this conjecture, the following drawing shows the graph of a simple 3-polytope with nine facets (due to Barnette [28]) in which for two vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ the unique shortest path (of length 3) makes a revisit:



  However, there is a nonrevisiting path: just follow the boundary of the figure.

  It is easy to see that the nonrevisiting path conjecture implies the Hirsch conjecture. In fact, the starting vertex of a nonrevisiting path lies on at least d facets, and with every vertex the path reaches at least one new facet it hasn't visited before. Thus the length of a nonrevisiting path cannot be more than $n - d$.

The nonrevisiting path conjecture may seem much stronger than the Hirsch conjecture. However, Klee & Walkup [271] proved that the two conjectures are in fact equivalent.

- The convexity assumption is essential: the Hirsch conjecture is false for some topological cell complexes that are combinatorial spheres, as Mani & Walkup [314] demonstrated. It is also false for simplicial 2-manifolds, see Barnette [36].
- Klee & Walkup [271] showed that the Hirsch conjecture is also false for unbounded polyhedra — although Hirsch's original conjecture was asked for unbounded polyhedra. They proved that for $n \geq 2d$, $\Delta_u(d,n) \geq n - d + \lfloor d/5 \rfloor$. This is the best lower bound known for $\Delta_u(d,n)$.

Even stronger, the *monotone Hirsch conjecture* is false, as Todd [456] demonstrated: it is not true that if $\boldsymbol{cx}$ is a linear function on P and $\boldsymbol{v}$ is a vertex, then there is a monotone path with at most $n - d$ edges from $\boldsymbol{v}$ to a vertex $\boldsymbol{v}_{\max}$ of P that maximizes $\boldsymbol{cx}$.

In fact, consider any d-polyhedron $P \subseteq \mathbb{R}^d$ with at most n facets, and let $\boldsymbol{cx}$ be a linear function. To avoid complications, we will assume for the following that the linear function $\boldsymbol{cx}$ is in general position with respect to P, that $\boldsymbol{cx}$ is bounded on P, and that the polyhedron is pointed (i.e., it has a vertex, and its lineality space is $\mathrm{lineal}(P) = \{\boldsymbol{0}\}$). From these assumptions we get that there is a vertex $\boldsymbol{u}$ of P on which $\boldsymbol{cx}$ achieves its unique maximum.

Now define $H_u(d,n)$ to be the smallest number such that in the situation above, for every vertex $\boldsymbol{v}$ of P, there is a (strictly) monotone path from $\boldsymbol{v}$ to the top, that is, a path from $\boldsymbol{v}$ to $\boldsymbol{u}$ along which $\boldsymbol{cx}$ increases in every single step. Similarly, let $H(d,n)$ be the same number under the additional assumption that P is a polytope.

The monotone (bounded) Hirsch conjecture would require that

$$H_u(d,n) \leq n - d, \qquad \text{respectively} \qquad H(d,n) \leq n - d.$$

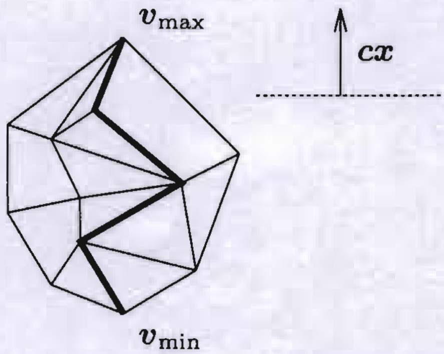
Disproving that, Todd [456] showed that

$$n - d + \min\{\left\lfloor \frac{d}{4} \right\rfloor, \left\lfloor \frac{n-d}{4} \right\rfloor\} \;\leq\; H(d,n) \;\leq\; H_u(d,n).$$

In particular, there is a 4-polytope with $n = 8$ facets for which every monotone path to the top needs at least five steps. However, in Todd's example there is a two-step nonmonotone path, which first goes to the bottom, and then directly to the top! This motivates the following, more restrictive, version of the monotone Hirsch conjecture, which might as well be true and which would imply the Hirsch conjecture (via a simple argument using projective transformations; see Exercise 2.17).

Conjecture 3.9 (Strict monotone Hirsch conjecture).
Let P be a d-dimensional polytope with n facets, and let $\boldsymbol{cx}$ be a linear function that is in general position with respect to P.

Then there is a strictly increasing path with respect to $\boldsymbol{cx}$, from the (unique) vertex $\boldsymbol{v}_{\min}$ that minimizes $\boldsymbol{cx}$, to the (unique) vertex $\boldsymbol{v}_{\max}$ that maximizes $\boldsymbol{cx}$, of length at most $n - d$.



To illustrate this for a trivial case, observe that for an n-gon the length of a shortest monotone path "to the top" can be $n - 2 = \Delta_u(2, n)$, but if we start "from the bottom," then we need at most $\lfloor \frac{n}{2} \rfloor = \Delta(2, n)$ steps.

What about upper bounds on $\Delta(d, n)$ and $\Delta_u(d, n)$?

In 1967 Barnette [28, 212] proved that $\Delta_u(d, n) \leq n3^{d-3}$. An improved bound, $\Delta_u(d, n) \leq n2^{d-3}$, was proved in 1970 by Larman [285]. Barnette's and Larman's bounds are linear in n but exponential in the dimension d. After that, nothing happened for a long time. In short, we might summarize the history by saying that the experts thought that the conjecture was plausible until they tried to prove it and couldn't; therefore now they think it is false, and can't prove that. However, in the long run Kalai might prove to be right, when he writes about "the author's guess (which is as good as the reader's)" [248]. The existence of a polynomial (or even linear) bound for $\Delta(d, n)$ is still a major open problem...

However, recently Gil Kalai achieved a substantial breakthrough: in a sequence of papers (each simpler and more striking than the preceding one) he established the first subexponential bounds for the diameter of a polytope. In November 1990 he proved $H_u(d, n) \leq n^{2\sqrt{n}}$ [248, Sect. 3]. In March 1991 he derived a "pseudopolynomial" bound for the diameter problem [248]:

$$\Delta_u(d, n) < n^{2 \log_2(d) + 3}.$$

A substantial simplification, which also strengthened the result slightly to $\Delta_u(d, n) < n^{\log_2(d) + 2}$, was subsequently found by Kalai & Kleitman [251]. The proof we give here is (essentially) the modification of this proof given by Kalai in [249, Sect. 2]. It is equally (surprisingly!) simple but establishes a stronger result: the existence of a "pseudopolynomial" *monotone* path to the top.

Theorem 3.10. (Kalai [249, Sect. 2])
Let $P \subseteq \mathbb{R}^d$ be a d-dimensional polyhedron with at most n facets, and let $\boldsymbol{cx}$ be a generic linear function which achieves its maximum on P in the vertex $\boldsymbol{w}$.

Then from any starting vertex $\boldsymbol{v} \in \mathrm{vert}(P)$, there is a monotone path to the top vertex $\boldsymbol{w}$, whose length is bounded by

$$H_u(d,n) \;\leq\; 2n\binom{d+\lfloor \log_2 n\rfloor - 1}{d-1} \;\leq\; 2\,n^{\log_2(d)+1} \;=\; 2\,(2d)^{\log_2(n)}.$$

Proof. The key to this is the notion of an *active facet*: given any vertex $\boldsymbol{v}$ of a polyhedron P, and a linear function $\boldsymbol{cx}$, a facet of P is active (for $\boldsymbol{v}$) if it contains a point that is higher than $\boldsymbol{v}$ (that is, either the facet is unbounded with respect to $\boldsymbol{cx}$, or it has a top vertex $\boldsymbol{w}$ with $\boldsymbol{cv} < \boldsymbol{cw}$).

For this proof, we also admit problems for which $\boldsymbol{cx}$ is not bounded on P, and where the last step "to the top" takes a ray (unbounded 1-face) on which $\boldsymbol{cx}$ has no upper bound. (You may think of the top as an extra vertex $\boldsymbol{u}_\infty$ in this case, which is adjoined to the directed graph of the problem.)

Let $\bar{H}(d,n)$ be the number of steps that may be required to get to the top vertex if we start from a vertex $\boldsymbol{v}$ for which the polyhedron has at most n *active* facets (and an arbitrary number of nonactive ones!).

Since $H_u(d,n)$ is monotone in n we immediately get

$$\Delta(d,n) \;\leq\; \Delta_u(d,n) \;\leq\; H_u(d,n) \;\leq\; \bar{H}(d,n).$$

Thus it suffices to prove the bounds of the theorem for $\bar{H}(d,n)$. In the following we require $d \geq 2$ and $n \geq 0$. In the "boundary cases" we get

$$\bar{H}(2,n) \;=\; n$$

(all the edges on a monotone path to the top are active facets, and this may be all of them if the problem is not bounded), and

$$\bar{H}(d,0) \;=\; \bar{H}(d,1) \;=\; \dots \;=\; \bar{H}(d,d-2) \;=\; 0,$$

(if $\boldsymbol{v}$ is not the top vertex, then it has an increasing edge, which lies on $d-1$ active facets).

To get a recursion for $\bar{H}(d,n)$, we verify a sequence of four simple facts:

1. *Given any set $\mathcal{F}$ of k active facets of P, we can reach from $\boldsymbol{v}$ either the top vertex, or a vertex in some facet of $\mathcal{F}$, in at most $\bar{H}(d,n-k)$ monotone steps.*

Let "$Ax \le z$" be a minimal system that defines P (having one inequality for each facet of P), and let $P' := P(A', z')$ be the polyhedron obtained by deleting the inequalities that correspond to facets in $\mathcal{F}$. Then v is a vertex of P' (unless v lies on a facet in $\mathcal{F}$, in which case we have nothing to prove), and it has at most $n - k$ active facets in P'.

Now consider a shortest path from v to the top in P'. This path makes at most $\bar{H}(d, n-k)$ steps, by definition. If it touches a facet in $\mathcal{F}$ after at most $\bar{H}(d, n-k)$ steps on P, then we are done. If it doesn't, then the top vertex of P' is also the top vertex of P, and the path to it in P' also yields a path to the top vertex on P, of length at most $\bar{H}(d, n-k)$.

2. *The collection $\mathcal{G}$ of all active facets that we can reach from v by at most $\bar{H}(d, n-k)$ monotone steps contains at least $n - k + 1$ active facets.*

This is true because there are at most n active facets altogether, and $\mathcal{G}$ meets every set $\mathcal{F}$ of k active facets.

3. *Starting at v, we can reach the highest vertex w_0 contained in any facet $F \in \mathcal{G}$ within at most $\bar{H}(d, n-k) + \bar{H}(d-1, n-1)$ monotone steps.*

We need at most $\bar{H}(d, n-k)$ steps to reach any facet of $\mathcal{G}$; this facet (of dimension $d-1$) has at most $n-1$ facets, thus in it we can find a path to its top of length at most $\bar{H}(d-1, n-1)$.

4. *From w_0 we can reach the top in at most $\bar{H}(d, k-1)$ steps.*

This is because none of the facets in $\mathcal{G}$ is active for w_0, and thus w_0 has at most $n - (n-k+1) = k-1$ active facets.

Putting the monotone paths together, we get a bound

$$\bar{H}(d,n) \;\le\; \bar{H}(d, n-k) + \bar{H}(d-1, n-1) + \bar{H}(d, k-1)$$

for the shortest monotone path from v to the top.

Now we choose $k := \lceil \frac{n}{2} \rceil$. Using the fact that by definition $\bar{H}(d,n)$ is a (weakly) increasing function in n, we get

$$\bar{H}(d,n) \;\le\; \bar{H}(d-1, n-1) + 2\bar{H}(d, \left\lfloor \frac{n}{2} \right\rfloor).$$

This recursion reminds us of the recursion for binomial coefficients — and we make a substitution to transform it into that. For this, we define

$$f(d,t) := 2^{-t}\bar{H}(d, 2^t) \qquad \text{for } t \ge 0 \text{ and } d \ge 2,$$

and with this substitution the recursion simplifies to "what we want":

$$f(d,t) \;\le\; f(d-1,t) \;+\; f(d,t-1).$$

From the boundary conditions $f(2,t) = 2^{-t}\bar{H}(2, 2^t) = 2^{-t}2^t = 1 \le \binom{t}{1}$ for $t \ge 1$ and $f(d,0) = \bar{H}(d,1) = 0 = \binom{d-2}{d-1}$ for $d \ge 3$, we obtain

$$f(d,t) \;\le\; \binom{d+t-2}{d-1}$$

for $(d,t) \neq (2,0)$, by induction for $t \geq 0$ and $d \geq 2$. From this we derive

$$\begin{aligned} H_u(d,n) \;\leq\; \bar{H}(d,n) &\leq \bar{H}(d, 2^{1+\lfloor \log_2 n \rfloor}) \\ &= 2^{1+\lfloor \log_2 n \rfloor} f(d, 1 + \lfloor \log_2 n \rfloor) \\ &\leq 2n \binom{d + \lfloor \log_2 n \rfloor - 1}{d-1} \\ &\leq 2n\, d^{\log_2(n)} \;=\; 2\, n^{1+\log_2(d)}, \end{aligned}$$

for $n, d \geq 2$, using the inequality $\binom{a+b}{b} \leq (a+1)^b$, which follows by induction over $a \geq 0$ and $b \geq 0$.

(In fact, there are various ways to derive bounds on $\bar{H}(d,n)$ from the recursion. This is a standard type of gymnastics for which you should get training at your "analysis of algorithms" class. Here is another way to proceed, which obtains the original Kleitman-Kalai bound. We use the starting values $\bar{H}(2,n) = n$ and $\bar{H}(d,0) = 0$. Since $\bar{H}(d,n)$ grows monotonically in n, we get a simple recursion

$$\bar{H}(d,n) \;\leq\; \bar{H}(d-1,n) + 2\bar{H}(d, \lfloor \tfrac{n}{2} \rfloor)$$

for $n > 0$ and $d \geq 3$. This we can iterate, to get

$$\begin{aligned} \bar{H}(d,n) &\leq \bar{H}(2,n) + 2\sum_{i=3}^{d} \delta(i, \lfloor \tfrac{n}{2} \rfloor) \\ &\leq n \;+\; 2(d-2)\cdot(2d)^{\log(n/2)} \\ &\leq 2d \cdot (2d)^{\log(n)-1} \;=\; (2d)^{\log(n)}, \end{aligned}$$

using $n < 4^{\log_2 n} \leq 4\,(2d)^{\log_2 n - 1}$.) □

It would be tricky and probably unnatural to formulate this proof in such a way that it stays within the family of polytopes: even if P is a polytope, the polyhedron P' will not, in general, be bounded. This is why this theorem and proof were done in the generality of polyhedra.

Also the proof does not stay within the class of polyhedra P with only $2\dim(P)$ facets, as considered by the d-step conjecture (see Exercise 3.7). However, we can specialize the result to fit this situation, and get $\Delta(d,2d) \leq (2d)^{\log_2 d+1}$. In fact, in the special case of $n = 2d$ one can modify/sharpen the computation of upper bounds to get

$$\Delta(d,2d) \;\leq\; d^{\log_9 d+2},$$

according to Kalai [249]. Still; this is far away from the conjectured bound of $\Delta(d,2d) = d$.

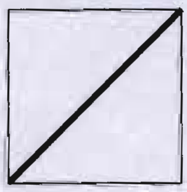
What's the problem? Why can't we do much better? There is some evidence in Matoušek's work [315] that the above analysis is essentially the best possible, that is, any proof for a substantially better upper bound has to use more of the specific geometry of the problem. Not much of the geometry was used in the preceding proof. (In fact, Kalai [248, Sect. 4] indicates a very general abstract framework, of a "simplicial complex with a fixed shelling order" (see Lecture 8), in which such upper bounds can be proved.)

Finally, let us mention that the diameter bounds can indeed (not quite directly) be used to construct algorithms for linear programming. In his research, Kalai [249, Sect. 3] found randomized pivot rules for linear programming that roughly require an expected number of $n^{4\sqrt{d}}$ arithmetic operations for every linear programming problem of dimension d with n facets. See Exercise 3.9(ii) for a simple sketch.

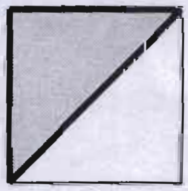
Very similar results were reached independently and nearly simultaneously (on a completely different path) by Matoušek, Sharir & Welzl [316], in the setting of a "dual simplex method."

For 0/1-polytopes (Example 1.11), the Hirsch conjecture is quite trivial — however, it took quite a time until Naddef [349] realized this. A more general result that also bounds the diameter of integral polytopes was given by Kleinschmidt & Onn [276]. We will give a slightly sharpened form of Naddef's theorem.

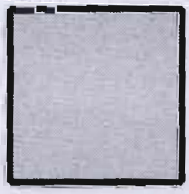
Before that, here are some examples of 0/1-polytopes P in $\mathbb{R}^d$: for each of them we list the space dimension d, the dimension $k = \dim(P)$ of the polytope itself, and the diameter $\delta(G(P))$.



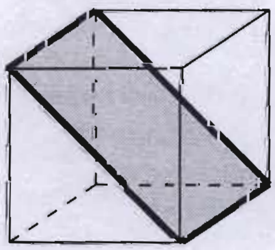
$d = 2$
$\dim(P) = 1$
$\delta(G(P)) = 1$



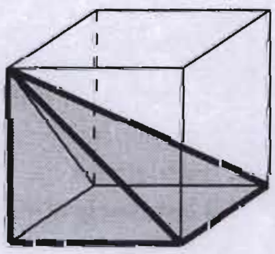
$d = 2$
$\dim(P) = 2$
$\delta(G(P)) = 1$



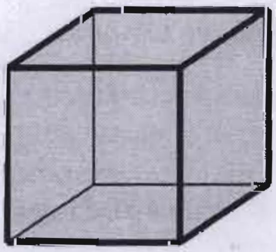
$d = 2$
$\dim(P) = 2$
$\delta(G(P)) = 2$



$d = 3$
$\dim(P) = 2$
$\delta(G(P)) = 2$



$d = 3$
$\dim(P) = 3$
$\delta(G(P)) = 2$



$d = 3$
$\dim(P) = 3$
$\delta(G(P)) = 3$

Theorem 3.11. *Let $P = \text{conv}(V)$ be a 0/1-polytope, $V \subseteq \{0,1\}^d$. Then P satisfies the Hirsch conjecture. In fact, the diameter of $G(P)$ is bounded by*

$$\delta(G(P)) \;\leq\; \dim(P),$$

with equality if and only if P is affinely isomorphic to a regular cube.

Proof. Let P have two vertices $\boldsymbol{v}, \boldsymbol{u}$ of distance $\delta(\boldsymbol{u}, \boldsymbol{v}) \geq d$. We use the symmetry of the cube

$$I_d := [0,1]^d = \text{conv}(\{0,1\}^d)$$

to reduce to the case where $\boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{u} \in \{0,1\}^d$.

Using induction on the dimension d we can assume that P is full-dimensional: otherwise let $\boldsymbol{a}\boldsymbol{x} = z$ be an equation that is valid for P. We get $z = 0$ from $\mathbf{0} \in P$, and thus $\boldsymbol{a} \neq 0$. By permuting coordinates we may assume $a_d \neq 0$. Then the projection map

$$\pi : \mathbb{R}^d \longrightarrow \mathbb{R}^{d-1}, \qquad \begin{pmatrix} \boldsymbol{x} \\ x_{d+1} \end{pmatrix} \longmapsto \boldsymbol{x}$$

(deleting the last coordinate) maps the 0/1-polytope $P \subseteq \mathbb{R}^d$ to an affinely isomorphic 0/1-polytope $\pi(P) \subseteq \mathbb{R}^{d-1}$. Thus we may assume $P \subseteq \mathbb{R}^d$ with $\dim(P) = d$.

Now assume that $u_i = 0$ for some i. Then $\mathbf{0}$ and $\boldsymbol{u}$ are both vertices of the face $F_{(i)} := P \cap \{\boldsymbol{x} \in \mathbb{R}^d : x_i = 0\}$ of P, which corresponds to the valid inequality $x_i \geq 0$. Thus we get

$$\delta(\mathbf{0}, \boldsymbol{u}) \;\leq\; \delta(G(F_{(i)})) \;\leq\; d-1$$

by induction on d. Therefore we may assume that $\boldsymbol{u} = \mathbf{1}$.

Now if any neighbor $\boldsymbol{w} \in N(\mathbf{1})$ of $\mathbf{1}$ has $k > 1$ components that are 0, then we get

$$\delta(\mathbf{0}, \mathbf{1}) \;\leq\; \delta(\mathbf{0}, \boldsymbol{w}) + \delta(\boldsymbol{w}, \mathbf{1}) \;\leq\; (d-k) + 1 < d,$$

where we use that the face

$$F_{\boldsymbol{w}} \;:= P \cap \{\boldsymbol{x} \in \mathbb{R}^d : x_i = 0 \text{ whenever } v_i = 0\}$$

has diameter at most $d - k$, by induction.

Thus if $\delta(\mathbf{0}, \mathbf{1}) \geq d$, then all the neighbors of $\mathbf{1}$ have exactly one 0-component. Since $\mathbf{1}$ has at least d neighbors (see Lemma 3.6), we find that $N(\mathbf{1}) = \{\mathbf{1} - \boldsymbol{e}_i : 1 \leq i \leq d\}$.

Also, again considering the faces

$$F_{(i)} = P \cap \{\boldsymbol{x} \in \mathbb{R}^d : x_i = 0\}$$

of P, we get that $\mathbf{0}$ and $\mathbf{1}-\boldsymbol{e}_i$ have distance $d-1$ in $G(F_{(i)})$, so by induction on d we get

$$F_{(i)} = \mathrm{conv}\{\boldsymbol{x} \in \{0,1\}^d : x_i = 0\}.$$

Collecting all the vertices that we now know have to be in P, we get $P = \mathrm{conv}(\{0,1\}^d) = I_d$ and $\delta(G(P)) = d$. □

The bound $\delta(G(P)) \leq \dim(P)$ can also be proved (with the same kind of argument) in the monotone version, where we ask for the shortest path "to the top" with respect to a given linear function. If we restrict to the strictly monotone version of Conjecture 3.9, then the characterization of the equality case also remains valid (with the same proof).

3.4 Kalai's Simple Way to Tell a Simple Polytope from Its Graph

In this section we consider simple polytopes and their graphs. Our treatment is based on a striking (and strikingly simple) paper by Gil Kalai. To be honest — the situation is even worse: the following is copied quite directly from his paper "**A simple way to tell a simple polytope from its graph**" [242].

Let P be a simple d-dimensional polytope and let $G(P)$ be the graph of P. Thus, $G(P)$ is an abstract graph defined on the set of vertices vert(P) of P. Two vertices $\boldsymbol{v}$ and $\boldsymbol{u}$ in vert(P) are adjacent in $G(P)$ if $[\boldsymbol{v}, \boldsymbol{u}]$ is a one-dimensional face of P.

Perles [362] conjectured the following result.

Theorem 3.12. (Blind & Mani [92])
If P is a simple polytope, then the graph $G(P)$ determines the entire combinatorial structure of P.

In other words, if two simple polytopes have isomorphic graphs, then their face lattices are isomorphic as well.

Proof. Here is Kalai's [242] simple proof of this result.

We consider the set of *all* acyclic orientations (i.e., edge orientations with no oriented cycles) of $G(P)$. We will not distinguish between an orientation O of $G(P)$ and the partial order induced by O on vert(P), which is defined by $\boldsymbol{v} \leq_O \boldsymbol{u}$ whenever there is an O-directed path from $\boldsymbol{v}$ to $\boldsymbol{u}$.

Note that if O is an acyclic orientation of $G(P)$, then the restriction of $G(P)$ to any nonempty subset A of vert(P) has a sink (an element with out-degree zero) with respect to O.

An acyclic orientation O of $G(P)$ is called *good* if for every nonempty face F of P, the graph $G(F)$ has exactly one sink. Otherwise, O is *bad*.