Combinatorial Convexity

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Contents

0 Some basic and convex facts

0.1 Notation. $\mathbb{R}^n = \{(x_1, \dots, x_n)^\intercal : x_i \in \mathbb{R}\}$ *denotes the n-dimensional* Euclidean space *equipped with the* Euclidean inner product $\langle x, y\rangle = \sum_{i=1}^n x_i\, y_i$, $x, y \in \mathbb{R}^n$, *and the* Euclidean norm $|x| = \sqrt{\langle x, x\rangle}$.

0.2 Definition [Linear, affine, positive and convex combination]. *Let* $m \in \mathbb{N}$ *and let* $x_i \in \mathbb{R}^n$, $\lambda_i \in \mathbb{R}$, $1 \le i \le m$.

i) $\sum_{i=1}^m \lambda_i\, x_i$ *is called a* linear combination *of* $x_1, \dots, x_m$.

ii) *If* $\sum_{i=1}^m \lambda_i = 1$ *then* $\sum_{i=1}^m \lambda_i\, x_i$ *is called an* affine combination *of* $x_1, \dots, x_m$.

iii) *If* $\lambda_i \ge 0$ *then* $\sum_{i=1}^m \lambda_i\, x_i$ *is called a* positive combination *of* $x_1, \dots, x_m$.

iv) *If* $\lambda_i \ge 0$ *and* $\sum_{i=1}^m \lambda_i = 1$ *then* $\sum_{i=1}^m \lambda_i\, x_i$ *is called a* convex combination *of* $x_1, \dots, x_m$.

v) *Let* $X \subseteq \mathbb{R}^n$. $x \in \mathbb{R}^n$ *is called* linearly *(*affinely, positively, convexly*)* dependent *of* X, *if* x *is a linear (affine, positive, convex) combination of finitely many points of* X, *i.e., there exist* $x_1, \dots, x_m \in X$, $m \in \mathbb{N}$, *such that* x *is a linear (affine, positive, convex) combination of the points* $x_1, \dots, x_m$.

0.3 Definition [Linearly and affinely independent points]. $x_1, \dots, x_m \in \mathbb{R}^n$ *are called* linearly *(*affinely*)* dependent, *if one of the* x_i *is linearly (*affinely*) dependent of* $\{x_1, \dots, x_m\}\backslash\{x_i\}$. *Otherwise* $x_1, \dots, x_m$ *are called* linearly *(*affinely*)* independent.

0.4 Remark. *Let* $x_1, \dots, x_m \in \mathbb{R}^n$.

i) $x_1, \dots, x_m$ *are affinely dependent if and only if* $\binom{x_1}{1}, \dots, \binom{x_m}{1} \in \mathbb{R}^{n+1}$ *are linearly dependent.*

ii) $x_1, \dots, x_m$ *are affinely dependent if and only if there exist* $\mu_i \in \mathbb{R}$, $1 \le i \le m$, *with* $(\mu_1, \dots, \mu_m) \ne (0, \dots, 0)$, $\sum_{i=1}^m \mu_i = 0$ *and* $\sum_{i=1}^m \mu_i\, x_i = 0$.

iii) *If* $m \ge n + 1$ *then* $x_1, \dots, x_m$ *are linearly dependent.*

iv) *If* $m \ge n + 2$ *then* $x_1, \dots, x_m$ *are affinely dependent.*

0.5 Definition [Linear subspace, affine subspace, cone and convex set]. $X \subseteq \mathbb{R}^n$ *is called*

i) linear subspace (set) *if it contains all* $x \in \mathbb{R}^n$ *which are linearly dependent of* X,

ii) affine subspace (set) *if it contains all* $x \in \mathbb{R}^n$ *which are affinely dependent of* X,

iii) (convex) cone *if it contains all* $x \in \mathbb{R}^n$ *which are positively dependent of* X,

iv) convex set *if it contains all* $x \in \mathbb{R}^n$ *which are convexly dependent of* X.

0.6 Notation. $\mathcal{C}^n = \{K \subseteq \mathbb{R}^n : K \text{ convex}\}$ *denotes the set of all convex sets in* $\mathbb{R}^n$. *The empty set* $\emptyset$ *is regarded as a convex, linear and affine set.*

0.7 Theorem. $K \subseteq \mathbb{R}^n$ *is convex if and only if*

$$\lambda x + (1-\lambda)\, y \in K, \qquad \text{for all } \ x, y \in K \text{ and } 0 \le \lambda \le 1.$$

0.8 Example. *The closed* n*-dimensional ball* $B_n(a,\rho) = \big\{x \in \mathbb{R}^n : |x-a| \le \rho\big\}$ *with centre* a *and radius* $\rho > 0$ *is convex. The boundary of* $B_n(a,\rho)$, *i.e.,* $\big\{x \in \mathbb{R}^n : |x-a| = \rho\big\}$ *is non-convex. In the case* $a = 0$ *and* $\rho = 1$ *the ball* $B_n(0,1)$ *is abbreviated by* B_n *and is called* n-dimensional unit ball. *Its boundary is denoted by* S^{n-1}.

0.9 Corollary. *Let* $K_i \in \mathcal{C}^n$, $i \in I$. *Then* $\bigcap_{i\in I} K_i \in \mathcal{C}^n$.

0.10 Definition [Linear, affine, positive and convex hull, dimension]. *Let* $X \subseteq \mathbb{R}^n$.

i) *The* linear hull $\operatorname{lin} X$ *of* X *is defined by*

$$\operatorname{lin} X = \bigcap_{\substack{L\subseteq\mathbb{R}^n,\ L \text{ linear},\\ X\subseteq L}} L.$$

ii) *The* affine hull $\operatorname{aff} X$ *of* X *is defined by*

$$\operatorname{aff} X = \bigcap_{\substack{A\subseteq\mathbb{R}^n,\ A \text{ affine},\\ X\subseteq A}} A.$$

iii) *The* positive (conic) hull $\operatorname{pos} X$ *of* X *is defined by*

$$\operatorname{pos} X = \bigcap_{\substack{C\subseteq\mathbb{R}^n,\ C \text{ convex cone},\\ X\subseteq C}} C.$$

iv) *The* convex hull $\operatorname{conv} X$ *of* X *is defined by*

$$\operatorname{conv} X = \bigcap_{\substack{K\subseteq\mathbb{R}^n,\ K \text{ convex},\\ X\subseteq K}} K.$$

v) *The* dimension $\dim X$ *of* X *is the dimension of its affine hull, i.e.,* $\dim \operatorname{aff} X$.

0.11 Theorem. *Let $X \subseteq \mathbb{R}^n$. Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{m} \lambda_i \, x_i : m \in \mathbb{N}, x_i \in X, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

0.12 Remark.

i) $\operatorname{conv}\{x, y\} = \big\{\lambda \, x + (1-\lambda) \, y : \lambda \in [0,1]\big\}$.

ii) $\operatorname{lin} X = \big\{\sum_{i=1}^{m} \lambda_i x_i : m \in \mathbb{N}, x_i \in X\big\}$.

iii) $\operatorname{aff} X = \big\{\sum_{i=1}^{m} \lambda_i x_i : m \in \mathbb{N}, x_i \in X, \sum_{i=1}^{m} \lambda_i = 1\big\}$.

iv) $\operatorname{pos} X = \big\{\sum_{i=1}^{m} \lambda_i x_i : m \in \mathbb{N}, x_i \in X, \ \lambda_i \geq 0\big\}$.

0.13 Definition [(Relative) interior point and (relative) boundary point]. *Let $X \subseteq \mathbb{R}^n$.*

i) *$x \in X$ is called an* interior point *of X if there exists a $\rho > 0$ such that $B_n(x, \rho) \subseteq X$. The set of all interior points of X is called the* interior *of X and is denoted by* $\operatorname{int} X$.

ii) *$x \in \mathbb{R}^n$ is called* boundary point *of X if for all $\rho > 0$ hold $B_n(x, \rho) \cap X \neq \emptyset$ and $B_n(x, \rho) \cap (\mathbb{R}^n \backslash X) \neq \emptyset$. The set of all boundary points of X is called the* boundary *of X and is denoted by* $\operatorname{bd} X$.

iii) *Let $A = \operatorname{aff} X$. $x \in X$ is called a* relative interior point *of X if there exists a $\rho > 0$ such that $B_n(x, \rho) \cap A \subseteq X$. The set of all relative interior points is called the* relative interior *of X and is denoted by* $\operatorname{relint} X$.

iv) *Let $A = \operatorname{aff} X$. $x \in A$ is called a* relative boundary point *of X if for all $\rho > 0$ hold $B_n(x, \rho) \cap X \neq \emptyset$ and $B_n(x, \rho) \cap (A \backslash X) \neq \emptyset$. The set of all relative boundary points of X is called* relative boundary *of X and is denoted by* $\operatorname{relbd} X$.

0.14 Remark. *Let $X \subseteq \mathbb{R}^n$ be closed. Then $X = \operatorname{relint} X \cup \operatorname{relbd} X$.*

0.15 Theorem. *Let $K \in \mathcal{C}^n$, $x \in \operatorname{relint} K$ and $y \in K$. Then $(1-\lambda)x + \lambda y \in \operatorname{relint} K$ for all $\lambda \in [0, 1)$.*

0.16 Corollary. *Let $K \in \mathcal{C}^n$ be closed. Let $x \in \operatorname{relint} K$ and $y \in \operatorname{aff} K \setminus K$. Then the segment $\operatorname{conv}\{x, y\}$ intersects $\operatorname{relbd} K$ in precisely one point.*

0.17 Definition [Polytope and simplex]. *Let $X \subset \mathbb{R}^n$ of finite cardinality, i.e., $\# X < \infty$.*

i) *$\operatorname{conv} X$ is called a* (convex) polytope.

ii) *A polytope $P \subset \mathbb{R}^n$ of dimension k is called a k-polytope.*

iii) *If X is affinely independent and $\dim X = k$ then $\operatorname{conv} X$ is called a k-simplex.*

0.18 Notation. *$\mathcal{P}^n = \{P \subset \mathbb{R}^n : P$ polytope$\}$ denotes the set of all polytopes in $\mathbb{R}^n$.*

0.19 Lemma. *Let $T = \operatorname{conv}\{x_1, \ldots, x_{k+1}\} \subset \mathbb{R}^n$ be a k-simplex, and let $\lambda_i > 0$, $1 \leq i \leq k+1$, with $\sum \lambda_i = 1$. Then $\sum \lambda_i\, x_i \in \operatorname{relint} T$.*

0.20 Corollary. *Let $K \in \mathcal{C}^n$, $K \neq \emptyset$. Then $\operatorname{relint} K \neq \emptyset$.*

0.21 Theorem. *Let $P = \operatorname{conv}\{x_1, \ldots, x_m\} \in \mathcal{P}^n$. A point $x \in \mathbb{R}^n$ belongs to $\operatorname{relint} P$ if and only if x admits a representation as $x = \sum_{i=1}^m \lambda_i x_i$ with $\lambda_i > 0$, $1 \leq i \leq m$, and $\sum_{i=1}^m \lambda_i = 1$.*

0.22 Notation.

i) *For two sets $X, Y \subseteq \mathbb{R}^n$ the vectorial addition*

$$X + Y = \{x + y : x \in X,\ y \in Y\}$$

is called the Minkowski sum *of X and Y. If X is just a singleton, i.e., $X = \{x\}$, then we write $x + Y$ instead of $\{x\} + Y$.*

ii) *For $\lambda \in \mathbb{R}$ and $X \subseteq \mathbb{R}^n$ we denote by λX the set*

$$\lambda X = \{\lambda x : x \in X\}.$$

1 Support and separate

1.1 Notation. *Let $a \in \mathbb{R}^n$, $a \neq 0$, and $\alpha \in \mathbb{R}$. The* closed halfspaces $H^+(a,\alpha)$, $H^-(a,\alpha) \subset \mathbb{R}^n$ *are given by*

$$H^+(a,\alpha) = \{x \in \mathbb{R}^n : \langle a, x\rangle \geq \alpha\}, \quad H^-(a,\alpha) = \{x \in \mathbb{R}^n : \langle a, x\rangle \leq \alpha\}.$$

The hyperplane $H(a,\alpha)$ *is defined by*

$$H(a,\alpha) = \{x \in \mathbb{R}^n : \langle a, x\rangle = \alpha\}.$$

1.2 Definition [Supporting hyperplane]. *Let $X \subset \mathbb{R}^n$. A hyperplane $H(a,\alpha) \subset \mathbb{R}^n$ is called* supporting hyperplane *of X if:*

$$\text{i) } H(a,\alpha) \cap X \neq \emptyset \quad \text{and} \quad \text{ii) } X \subseteq H^-(a,\alpha).$$

a is called outer normal vector *of X and if, in addition, $|a| = 1$ then it is called* outer unit normal vector *of X.*

1.3 Proposition. *Let $X \subset \mathbb{R}^n$ and let $H(a,\alpha)$ be a supporting hyperplane of X. Then*

$$H(a,\alpha) \cap \operatorname{conv} X = \operatorname{conv}\big(H(a,\alpha) \cap X\big).$$

1.4 Remark. *Let $X \subset \mathbb{R}^n$ be compact and $a \in \mathbb{R}^n \backslash \{0\}$. Then there exists a supporting hyperplane of X with outer normal vector a.*

1.5 Definition [Nearest point map (or metric projection)]. *Let $K \in \mathcal{C}^n$ be closed. The map $\Phi_K : \mathbb{R}^n \to K$, where for $x \in \mathbb{R}^n$ the point $\Phi_K(x) \in K$ is given by $|x - \Phi_K(x)| = \min\{|x-y| : y \in K\}$ is called the* nearest point map *(*metric projection*) with respect to K.*

1.6 Remark. *We prove that the nearest point map is well-defined. Notice that since K is closed, for all $x \in \mathbb{R}^n$ there exist $y_x \in K$ such that $|x - y_x| = \min\{|x-y| : y \in K\}$. We show that y_x is uniquely determined. In fact, if there exists $\overline{y} \in K$, $\overline{y} \neq y_x$, with $|x - \overline{y}| = |x - y_x|$ then we may assume that $x - y_x$ and $x - \overline{y}$ are linearly independent. Hence*

$$\left|x - \frac{y_x + \overline{y}}{2}\right| = \left|\frac{1}{2}(x - y_x) + \frac{1}{2}(x - \overline{y})\right| < \frac{1}{2}\left|x - y_x\right| + \frac{1}{2}\left|x - \overline{y}\right| = \left|x - y_x\right|.$$

Since $(y_x + \overline{y})/2 \in K$ by the convexity of K, it contradicts the minimality of y_x.

1.7 Theorem. *Let $K \in \mathcal{C}^n$ be closed and let $x \in \mathbb{R}^n \setminus K$. Let $a = x - \Phi_K(x)$ and $\alpha = \langle a, \Phi_K(x)\rangle$. Then $H(a,\alpha)$ is a supporting hyperplane of K with outer normal vector a.*

1.8 Corollary. *Let $K \in \mathcal{C}^n$, $K \neq \mathbb{R}^n$, be closed. Then*

$$K = \bigcap_{\substack{H(a,\alpha)\text{ supporting} \\ \text{hyperplane of } K}} H^-(a,\alpha),$$

i.e., K is the intersection of all its "supporting halfspaces".

1.9 Corollary. *Let $X \subset \mathbb{R}^n$ such that* $\operatorname{conv} X$ *is closed and* $\operatorname{conv} X \neq \mathbb{R}^n$. *Then*

$$\operatorname{conv} X = \bigcap_{X \subseteq H^-(a,\alpha)} H^-(a,\alpha),$$

i.e., $\operatorname{conv} X$ *is the intersection of all halfspaces containing X.*

1.10 Lemma [Busemann-Feller Lemma]. *Let $K \in \mathcal{C}^n$ be closed. Then*

$$|\Phi_K(x) - \Phi_K(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}^n$, i.e., the nearest point map does not increase distances. In particular, it is a continuous map.

1.11 Theorem. *Let $K \in \mathcal{C}^n$ be compact and let $\rho > 0$ such that $K \subset$* $\operatorname{int}(\rho B_n)$. *The nearest point map restricted to ρS^{n-1} is surjective, i.e., $\Phi_K : \rho S^{n-1} \to \operatorname{bd} K$ is surjective.*

1.12 Corollary. *Let $K \in \mathcal{C}^n$ be closed and let $x \in \operatorname{relbd} K$. Then there exists a supporting hyperplane $H(a,\alpha)$ of K with $x \in H(a,\alpha)$.*

1.13 Theorem [Separation theorem]. *Let $K_1, K_2 \in \mathcal{C}^n$ with $K_1 \cap K_2 = \emptyset$. Then there exists a* separating hyperplane *$H(a,\alpha)$ of K_1 and K_2, i.e., $K_1 \subseteq H^+(a,\alpha)$ and $K_2 \subseteq H^-(a,\alpha)$.*

If K_1 is closed and K_2 is compact, then there exists even a strictly separating hyperplane *$H(a,\alpha)$ of K_1 and K_2, i.e., $K_1 \subset \operatorname{int} H^+(a,\alpha)$ and $K_2 \subset \operatorname{int} H^-(a,\alpha)$.*

1.14 Definition [Support function, breadth]. *Let $K \in \mathcal{C}^n$. The function $h(K,\cdot) : \mathbb{R}^n \to \mathbb{R}$ given by*

$$h(K,u) = \sup\{\langle u, x\rangle : x \in K\}$$

is called support function *of K. For $u \in S^{n-1}$ the* breadth of K in the direction *u is given by $h(K,u) + h(K,-u)$.*

1.15 Remark. *Let $K \in \mathcal{C}^n$ be non-empty and compact. Then*

$$K = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : \langle u, x\rangle \leq h(K,u)\}.$$

1.16 Definition [Polar set]. *Let $X \subseteq \mathbb{R}^n$.*

$$X^\star = \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } x \in X\}$$

is called the polar set *of X.*

1.17 Proposition.

i) *$X^\star$ is a convex and closed set and $0 \in X^\star$.*

ii) *If* $X_1 \subseteq X_2$ *then* $X_2^\star \subseteq X_1^\star$.

iii) *Let* M *be a regular* $n \times n$ *matrix. Then* $(MX)^\star = M^{-\intercal} X^\star$.

iv) *Let* $X_i \subseteq \mathbb{R}^n$, $i \in I$. *Then* $\left(\bigcup_{i\in I} X_i\right)^\star = \bigcap_{i\in I} X_i^\star$.

v) $X \subseteq (X^\star)^\star$.

vi) *Let* $X \subset \mathbb{R}^n$. *Then* $X = X^\star$ *if and only if* $X = B_n$.

1.18 Proposition.

i) *Let* $P = \operatorname{conv}\{x_1, \ldots, x_m\} \subset \mathbb{R}^n$. *Then*

$$P^\star = \left\{y \in \mathbb{R}^n : \langle x_i, y\rangle \le 1,\ 1 \le i \le m\right\}.$$

ii) *Let* $P = \left\{x \in \mathbb{R}^n : \langle a_i, x\rangle \le 1,\ 1 \le i \le m\right\}$ *with* $a_i \in \mathbb{R}^n$. *Then*

$$P^\star = \operatorname{conv}\{0, a_1, \ldots, a_m\}.$$

1.19 Lemma. *Let* $K \in \mathcal{C}^n$ *be closed with* $0 \in K$. *Then* $(K^\star)^\star = K$.

2 Radon, Helly, Caratheodory and relatives

2.1 Theorem [Radon, 1921]. *Let $X \subset \mathbb{R}^n$. If $\#X \geq n+2$ then there exist $X_1, X_2 \subset X$ with $X_1 \cap X_2 = \emptyset$ and $\operatorname{conv} X_1 \cap \operatorname{conv} X_2 \neq \emptyset$.*

2.2 Theorem [Helly, 1913]. *Let $K_1, \dots, K_m \in \mathcal{C}^n$, $m \geq n+1$, such that for each $(n+1)$-index set $I \subseteq \{1, \dots, m\}$ we have $\bigcap_{i \in I} K_i \neq \emptyset$. Then all sets K_i have a point in common, i.e., $\bigcap_{j=1}^m K_i \neq \emptyset$.*

2.3 Remark.

i) *Without any further restrictions/assumptions Helly's theorem is* not *true for infinitely many convex sets K_i. For instance, let $K_i = (0, \frac{1}{i}]$, $i \in \mathbb{N}$.*

ii) *Helly's theorem, however, can be easily generalised to infinitely many* compact *(bounded and closed) convex sets.*

2.4 Corollary. *Let $C \subset \mathcal{C}^n$ be compact. Then there exists $t \in \mathbb{R}^n$ with*

$$-C \subseteq t + n\,C.$$

2.5 Definition [Centerpoint]. *For a finite point set $X \subset \mathbb{R}^n$ a point $c \in \mathbb{R}^n$ is called* centerpoint *is every closed halfspace containing c containes at least $\lceil \frac{1}{n+1} \#X \rceil$ points of X.*

2.6 Theorem. *Every finite set $X \subset \mathbb{R}^n$ has a centerpoint.*

2.7 Theorem [Carathéodory, 1907]. *Let $X \subset \mathbb{R}^n$. Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{n+1} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1, x_i \in X, i = 1, \dots, n+1 \right\}.$$

2.8 Remark. *Let $X \subset \mathbb{R}^n$. Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{\dim X+1} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{\dim X+1} \lambda_i = 1, x_i \in X \right\}.$$

As a direct consequence of Carathéodory's Theorem 2.7 we get the following result.

2.9 Corollary. *A polytope is the union of simplices.*

2.10 Corollary. *The convex hull of a compact set is compact.*

2.11 Theorem [(weak)Fractional Helly theorem]. *Let $K_1, \dots, K_m \in \mathcal{C}^n$, $m \geq n+1$, and let $\alpha \in (0,1]$ such that for at least $\alpha \binom{m}{n+1}$ of the $(n+1)$-index sets $I \subseteq \{1, \dots, m\}$ we have $\bigcap_{i \in I} K_i \neq \emptyset$. Then there exists a point in common of at least $\frac{\alpha}{n+1} \cdot m$ sets K_i.*

2.12 Remark. *The (strong and sharp) fractional Helly theorem, which is due to Kalai, gives that* $(1-(1-\alpha)^{1/(n+1)})\cdot m$ *sets have a point in common. Obviously, for* $\alpha = 1$ *we get again the classcial Helly Theorem 2.2.*

2.13 Theorem [Colorful Carathéodory theorem]. *Let* $X_1, \ldots, X_{n+1} \subset \mathbb{R}^n$ *be finite point sets such that* $0 \in \operatorname{conv} X_i$, $1 \le i \le n+1$. *There exist* $x_i \in X_i$, $1 \le i \le n+1$, *such that* $0 \in \operatorname{conv}\{x_1, \ldots, x_{n+1}\}$.

2.14 Theorem [Tverberg's theorem]. *Let* $X \subseteq \mathbb{R}^n$ *and let* $k \in \mathbb{N}_{\ge 1}$. *If* $\#X \ge (k-1)(n+1)+1$, $k \in \mathbb{N}$, *then there exist* k *subsets* $X_1, \ldots, X_k \subset X$ *with* $X_i \cap X_j = \emptyset$, $i \ne j$, *but* $\operatorname{conv} X_1 \cap \operatorname{conv} X_2 \cap \cdots \cap \operatorname{conv} X_k \ne \emptyset$.

2.15 Theorem. *Let* $X \subset \mathbb{R}^n$ *and let* $\#X = m \ge n+1$. *Then there exists a point* $y \in \mathbb{R}^n$ *contained in at least* $\gamma_n \binom{m}{n+1}$ X*-simplices, i.e., simplices of the form* $\operatorname{conv} S$, $S \subseteq X$, $\#S = n+1$. *Here* γ_n *is a positive constant depending only on the dimension, and* X*-simplices* $\operatorname{conv} S_1$, $\operatorname{conv} S_2$ *are considered different if* $S_1 \ne S_2$.

3 Polytopes

3.1 Definition [Polyhedron]. *The intersection of finitely many closed halfspaces is called a* polyhedron.

3.2 Theorem [Minkowski, 1896, Weyl, 1935].

i) *A bounded polyhedron is a polytope.*

ii) *A polytope is a bounded polyhedron.*

3.3 Notation [$\mathcal{V}$-Polytope, $\mathcal{H}$-Polytope]. *A polytope given as the convex hull of finitely many points is called a* $\mathcal{V}$-polytope. *If it is given as the bounded intersection of finitely many closed halfspaces, then it is called an* $\mathcal{H}$-polytope.

3.4 Corollary. *Let* $P \in \mathcal{P}^n$.

i) *Let* $A \in \mathbb{R}^{m \times n}$ *and* $t \in \mathbb{R}^m$. *Then* $AP + t$ *is a polytope.*

ii) *Let* $U \subset \mathbb{R}^n$ *be an affine subspace. Then* $P \cap U$ *is a polytope.*

3.5 Definition [Faces]. *Let* $K \in \mathcal{C}^n$ *be closed and let* H *be a supporting hyperplane of* K. *If* $j = \dim(K \cap H)$, *then* $K \cap H$ *is called a* j-face *of* K. *Moreover,* K *itself is regarded as a* $(\dim K)$*-face and the empty set* $\emptyset$ *as* (-1)*-face of* K.

3.6 Notation [Vertices, edges, facets]. *A* 0*-face of* $K \in \mathcal{C}^n$, K *closed, is called* vertex, *an* 1*-face is called* edge *and a* $(\dim K - 1)$*-face is called* facet *of* K. K *itself and the empty set are called* improper faces, *whereas the remaining faces are called* proper faces *of* K.

The set of all vertices of a polytope P *is denoted by* $\operatorname{vert} P$.

3.7 Remark.

i) *Let* $K \in \mathcal{C}^n$ *be closed. Every (relative) boundary point of* K *lies in a suitable* j*-face,* $0 \le j \le \dim K - 1$.

ii) *Let* $K \in \mathcal{C}^n$, $\dim K = n$. *Let* F *be a facet of* K *and* H *a supporting hyperplane of* K *with* $F = K \cap H$. *Then* $H = \operatorname{aff} F$.

3.8 Proposition. *Each face of a polytope is a polytope, and a polytope has only finitely many faces.*

3.9 Definition [f-vector]. *For* $P \in \mathcal{P}^n$ *let* $f_i(P)$ *be the number of* i*-faces of* P, $-1 \le i \le \dim P$. *Furthermore, let* $f_i(P) = 0$ *for* $\dim P + 1 \le i \le n$. *The vector* $f(P)$ *with entries* $f_i(P)$, $-1 \le i \le n$, *is called the* f-vector *of* P.

3.10 Remark.

i) *Let* $T_n = \mathrm{conv}\,\{0, e_1, \dots, e_n\}$ *be the so called* standard simplex. *Then* $f_i(T_n) = \binom{n+1}{i+1}$, *i.e., any* $(i+1)$ *subset of the vertices are the vertices of an* i*-face.*

ii) *For any* n*-polytope* $P \in \mathcal{P}^n$ *we have* $\sum_{i=-1}^{n} f_i(P) \geq 2^{n+1}$ *with equality if and only if* P *is an* n*-simplex.*

3.11 Lemma. *Let* $P \in \mathcal{P}^n$.

i) $v \in \mathrm{vert}\, P$ *can not be written as a convex combination of two other points of* P, *i.e.,* $v \notin \mathrm{conv}\,\big(P \backslash \{v\}\big)$.

ii) *If* $P = \mathrm{conv}\, W$, *then* $\mathrm{vert}\, P \subseteq W$.

iii) $P = \mathrm{conv}\,(\mathrm{vert}\, P)$.

3.12 Lemma. *Let* $P \in \mathcal{P}^n$ *be an* n*-polytope with* $0 \in \mathrm{int}\, P$. *For a proper face* F *of* P *let*

$$F^\diamond = \{y \in P^\star : \langle x, y\rangle = 1 \text{ for all } x \in F\}\,.$$

Then

i) $F^\diamond$ *is a face of* $P^\star$.

ii) $F = (F^\diamond)^\diamond$.

iii) *If* G *is a face of* P *and* $F \subseteq G$, *then* $G^\diamond \subseteq F^\diamond$.

iv) $\dim F^\diamond = n - 1 - \dim F$.

3.13 Theorem. *Let* $P \in \mathcal{P}^n$ *be an* n*-polytope with* $0 \in \mathrm{int}\ P$. *Then*

$$f_{n-1-i}(P^\star) = f_i(P), \quad -1 \leq i \leq n.$$

3.14 Theorem. *Let* $P \in \mathcal{P}^n$ *be an* n*-polytope with facets* $F_1, \dots, F_m$ *and let* $H(a_i, \alpha_i)$, $1 \leq i \leq m$, *be the supporting hyperplanes of* F_i, $1 \leq i \leq m$. *Then*

$$P = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \leq \alpha_i,\ 1 \leq i \leq m\}\,.$$

3.15 Theorem. *Let* $P \in \mathcal{P}^n$ *be an* n*-polytope.*

i) *The boundary of* P *is the union of all its facets.*

ii) *A* k*-face is the intersection of (at least)* $(n-k)$ *facets.*

iii) *An* $(n-2)$*-face is contained in exactly two facets.*

iv) *If* F, G *are faces of* P *with* $F \subseteq G$, *then* F *is a face of* G.

v) *A face of* P *is also a face of a facet of* P.

3.16 Theorem. *Let $P \in \mathcal{P}^n$ be an n-polytope.*

i) *Let G be a face of P and let F be a face of G. Then F is a face of P.*

ii) *Let F_j be a j-face of P and let F_k be a k-face of P with $F_j \subset F_k$. There exist i-faces F_i of P, $j < i < k$, such that*

$$F_j \subset F_{j+1} \subset \cdots \subset F_{k-1} \subset F_k.$$

3.17 Remark. *Let v_0 be a vertex of an n-polytope P and let $\{v_1, \ldots, v_r\}$ be all* adjacent *vertices of v_0, i.e., $\operatorname{conv}\{v_0, v_i\}$ is an edge of P. In other words, $\{v_1, \ldots, v_r\}$ are the neighbours of v_0. Then*

i)

$$P \subset v_0 + \operatorname{pos}\{v_1 - v_0, \ldots, v_r - v_0\}.$$

ii) *Let $c \in \mathbb{R}^n$ with $\langle c, v_0\rangle \geq \langle c, v_i\rangle$, $1 \leq i \leq r$. Then*

$$\max\{\langle c, x\rangle : x \in P\} = \langle c, v_0\rangle .$$

3.18 Theorem [Euler-Poincaré formula]. *Let $P \in \mathcal{P}^n$. Then*

$$\sum_{i=-1}^{n} (-1)^i f_i(P) = 0. \tag{3.18.1}$$

In particular, in the 3-dimensional case, i.e., $\dim P = 3$, it holds $f_0 - f_1 + f_2 = 2$.

3.19 Proposition. *The Euler-Poincaré formula is the only linear equation satisfied by the f-vector, i.e., let $\lambda_i \in \mathbb{R}$, such that $\sum_{i=-1}^{n} \lambda_i f_i(P) = 0$ for all $P \in \mathcal{P}^n$. Then there exists a constant $\gamma \in \mathbb{R}$, such that $\lambda_i = \gamma(-1)^i$.*

3.20 Definition [Simple and simplicial polytopes]. *Let $P \in \mathcal{P}^n$.*

i) *P is called* simplicial *if all proper faces are simplices.*

ii) *P is called* simple *if every vertex is contained in exactly $\dim P$ many facets.*

3.21 Lemma. *Let $P \in \mathcal{P}^n$ be an n-polytope with $0 \in \operatorname{int} P$. The following statements are equivalent:*

i) *P is simplicial.*

ii) *All facets of P are simplices.*

iii) *$P^\star$ is simple.*

iv) *Every k-face of $P^\star$ is contained in exactly $n - k$ facets for $k = 0, \ldots, n-1$.*

3.22 Theorem. *Let $P \in \mathcal{P}^n$ be a simple n-polytope. Then*

i) *Every vertex is contained in exactly $\binom{n}{k}$ k-faces of P, $k = 0, \ldots, n-1$.*

ii) *The intersection of $k \leq n$ facets containing a common vertex is an $(n-k)$-face of P.*

iii) *Let $v_1, \ldots, v_n$ be the neighbours of a vertex v_0 of P. For each subset of $k < n$ neighbours $v_{i_1}, \ldots, v_{i_k}$ there exists a unique k-face F of P containing $v_0, v_{i_1}, \ldots, v_{i_k}$.*

iv) *A face of a simple polytope is simple.*

v) *Every j face of P is contained in exactly $\binom{n-j}{k-j}$ k faces of P.*

3.23 Theorem. *Let $P \in \mathcal{P}^n$ be a simple n-polytope.*

i) $n\, f_0(P) = 2\, f_1(P)$.

ii) $\sum_{k=0}^{n} f_k(P) \leq 2^n f_0(P)$.

iii) $f_0(P) \leq 2 f_{\lceil n/2 \rceil}(P)$.

Here, for $\rho \in \mathbb{R}$ the number $\lceil \rho \rceil$ is the smallest integer greater or equal than ρ.

3.24 Corollary. *Let P be a simple n-polytope with m facets. Then*

$$f_0(P) \leq 2\binom{m}{\lfloor n/2 \rfloor} \quad \text{and} \quad \sum_{k=0}^{n} f_k(P) \leq 2^{n+1}\binom{m}{\lfloor n/2 \rfloor}.$$

Or equivalently: Let P be a simplicial n-polytope with m vertices. Then

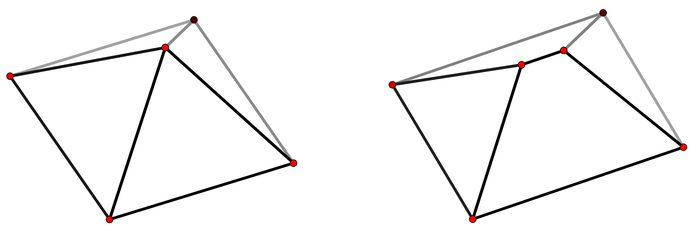
$$f_{n-1}(P) \leq 2\binom{m}{\lfloor n/2 \rfloor} \quad \text{and} \quad \sum_{k=0}^{n} f_k(P) \leq 2^{n+1}\binom{m}{\lfloor n/2 \rfloor}.$$

Here, for $\rho \in \mathbb{R}$ the number $\lfloor \rho \rfloor$ is the largest integer not greater than ρ.

3.25 Lemma. *Let P be an n-polytope.*

i) *There exists a simple n-polytope Q with the same number of facets as P and $f_i(P) \leq f_i(Q)$, $0 \leq i \leq n-2$.*

ii) *There exists a simplical n-polytope $Q^\star$ with the same number of vertices as P and $f_i(P) \leq f_i(Q^\star)$, $1 \leq i \leq n-1$.*



3.26 Corollary. *Let P be an n-polytope with m facets. Then*

$$f_0(P) \le 2\binom{m}{\lfloor n/2 \rfloor}.$$

Or equivalently: Let P be an n-polytope with m vertices. Then

$$f_{n-1}(P) \le 2\binom{m}{\lfloor n/2 \rfloor}.$$

3.27 Definition [Cyclic polytopes]. *The curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ given by $\gamma(t) = (t, t^2, t^3, \ldots, t^n)^\intercal$ is called* moment curve. *The convex hull of m points on the moment curve is called a* cyclic polytope *with m vertices and is denoted by $C(n, m)$.*

3.28 Proposition. *Any $n + 1$ points on the moment curve are affinely independent. In particlar, cyclic polytopes are simplicial polytopes.*

3.29 Proposition [Gale's evenness condition]. *Let $t_i \in \mathbb{R}$, $1 \le i \le m$, $t_1 < t_2 < \cdots < t_m$, $\gamma(t_i) = (t_i, t_i^2, t_i^3, \ldots, t_i^n)^\intercal$, $1 \le i \le m$, and let $S \subset \{1, \ldots, m\}$ be a subset of cardinality n. $F_S = \operatorname{conv}\{\gamma(t_s) : s \in S\}$ is a facet of $C(n, m)$ if and only if $\#\{s \in S : i < s < j\}$ is even for all $i, j \in \{1, \ldots, m\} \setminus S$.*

3.30 Remark. *All points $\gamma(t_i)$ are vertices of $C(n, m)$ and the number of i-faces of $C(n, m)$ is independent of the choice of the m-points on the moment curve.*

3.31 Proposition. *The cyclic polytope $C(n, m)$ is $\lfloor n/2 \rfloor$-neighborly, i.e., the convex hull of any subset of the vertices of cardinality less than or equal $n/2$ is a face.*

3.32 Theorem* [McMullen's Upper Bound Theorem, 1971]. *Let P be an n-polytope with m vertices. Then*

$$f_i(P) \le f_i(C(n, m)) = \begin{cases} \sum_{j=0}^{(n-1)/2} \frac{i+2}{m-j}\binom{m-j}{j+1}\binom{j+1}{i+1-j}, & n \text{ odd}, \\ \sum_{j=1}^{n/2} \frac{m}{m-j}\binom{m-j}{j}\binom{j}{i+1-j}, & n \text{ even}. \end{cases}$$

In particular,

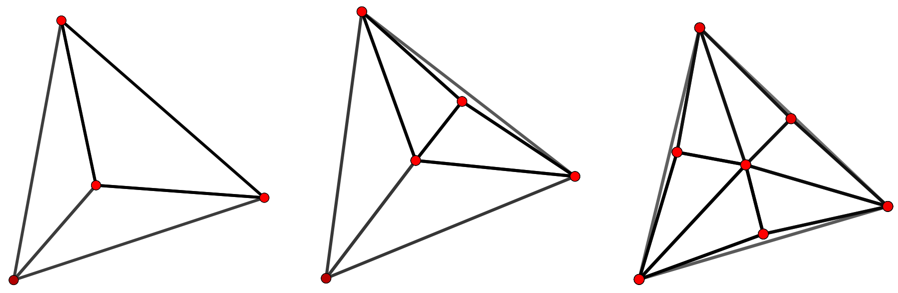
$$f_{n-1}(P) \le f_{n-1}(C(n, m)) = \begin{cases} 2\binom{m-\lfloor n/2 \rfloor-1}{\lfloor n/2 \rfloor}, & n \text{ odd}, \\ \binom{m-\lfloor n/2 \rfloor}{\lfloor n/2 \rfloor} + \binom{m-\lfloor n/2 \rfloor-1}{\lfloor n/2 \rfloor-1}, & n \text{ even}. \end{cases}$$

For fixed n the right hand sides are of order $m^{\lfloor n/2 \rfloor}$.

3.33 Theorem* [Barnette's Lower Bound Theorem, 1971]. *Let P be a simplicial n-polytope with m vertices. P has at least as many i-faces as the so called* stacked polytopes *$P(n, m)$ with m vertices for which*

$$f_i(P(n, m)) = \begin{cases} m\binom{n}{i} - i\binom{n+1}{i+1}, & 0 \le i \le n-2, \\ n + 1 + (m - (n+1))(n-1), & i = n-1. \end{cases}$$

$P(n,n+1)$ is an n-simplex, and for $m \geq n+2$ an m-vertex stacked n-polytope $P(n,m)$ is the convex hull of an $(m-1)$-vertex stacked polytope with an additional point that is beyond exactly one facet.



3.34 Theorem [Dehn-Sommerville equations, 1905, 1927]. *Let P be a simple n-polytope. Then*

$$f_i(P) = \sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} f_j(P), \quad i = 0, \ldots, n,$$

Or equivalently: Let P be a simplicial n-polytope. Then

$$f_{i-1}(P) = \sum_{j=i}^{n} (-1)^{n-j} \binom{j}{i} f_{j-1}(P), \quad i = 0, \ldots, n.$$

3.35 Definition [h-vector]. *Let $P \in \mathcal{C}^n$ be a simple n-polytope. The vector $h(P) = (h_0(P), \ldots, h_n(P))$ with entries*

$$h_i(P) = \sum_{j=0}^{n} (-1)^{i-j} \binom{j}{i} f_j(P)$$

*is called h-*vector *of P.*

3.36 Remark. *In terms of the h-vector, the Dehn-Sommerville equations become $h_i(P) = h_{n-i}(P)$, $i = 0, \ldots, n$.*

3.37 Theorem* [McMullen's g-Theorem]. *McMullen's g-theorem gives a complete characterization of the f-vectors of simple (or simplicial) polytopes in terms of its g-vector which is given by $g_i(P) = h_i(P) - h_{i-1}(P)$, $i = 1, \ldots, \lfloor n/2 \rfloor$.*

3.38 Remark. *For any n-polytope $P \in \mathcal{P}^n$ we have $n f_0(P) \leq 2 f_1(P)$ with equality iff P simple and $n f_{n-1}(P) \leq 2 f_{n-2}(P)$ with equality iff P simplicial.*

3.39 Theorem [Steinitz, 1906]. *A non-negative integral vector (f_0, f_1, f_2) is the f-vector of a 3-polytope if and only if* i) *$f_0 - f_1 + f_2 = 2$,* ii) *$3 f_0 \leq 2 f_1$, and* iii) *$3 f_2 \leq 2 f_1$.*

3.40 Theorem [Figiel, Lindenstrauss, Milman, 1977]. *Let $P \in \mathcal{P}^n$ be a 0-symmetric n-polytope, i.e., $P = -P$. Then*

$$\ln(f_0(P))\ln(f_{n-1}(P)) \geq \frac{1}{16}n.$$

3.41 Conjecture [Kalai, 1989]. *Let $P \in \mathcal{P}^n$ be a 0-symmetric n-polytope. Then*

$$\sum_{i=0}^{n} f_i(P) \geq 3^n.$$

Here we have equality, for instance, for the cube C_n and its polar, the cross-poyltope $C_n^\star$, or, more generally, for the class of Hanner-polytopes. Recently, the conjecture has been verified for all $n \leq 4$ (see `http://front.math.ucdavis.edu/0708.3661`*).*

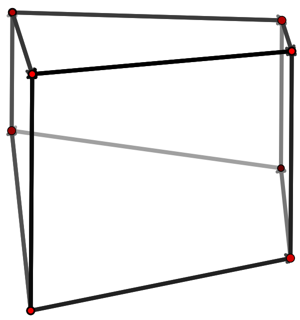
3.42 Theorem [Balinski, 1961]. *Let $P \in \mathcal{P}^n$ be an n-polytope. The graph of P consisting of its vertices and edges is n-connected, i.e., the graph is still connected if $n-1$ vertices and their adjacent edges are removed.*

3.43 Definition [Combinatorial diameter]. *Let $P \in \mathcal{P}^n$. The* combinatorial distance $\delta_P(v, w)$ *between two vertices $v, w \in \operatorname{vert} P$ is the minimum length of an "edge path" connecting v and w, i.e., the minimum length of a sequence $[v, v_{i_1}, \ldots, v_{i_l}, w]$, with $v_{i_j} \in \operatorname{vert} P$ and two consecutive vertices form an edge.*

$\delta(P) = \max\{\delta_P(v, w) : v, w \in \operatorname{vert} P\}$ *is called the* (combinatorial) diameter *of P. For $n, m \in \mathbb{N}$ let*

$$\Delta(n, m) = \max\left\{\delta(P) : P \in \mathcal{P}^n, \dim P = n \text{ and } f_{n-1}(P) = m\right\}.$$

3.44 Example. $\delta(T_n) = 1$, $\delta(C_n) = n$ *and* $\delta(C_n^\star) = 2$.



3.45 Conjecture [Hirsch, 1957]. $\Delta(n, m) \leq m - n$.

3.46 Remark. *It is known that*

i) *the conjecture is true if $n \leq 3$ or $m \leq n + 5$ (Klee&Walkup, 1961/1965),*

ii) *the conjecture would be false for unbounded polyhedra,*

iii) $\Delta(n, m) \leq \frac{1}{3}\, 2^{n-2}(m - n + \frac{5}{2})$ *(Barnette, 1974),*

iv) $\Delta(n, m) \leq 2m^{\log(n)+1}$ *(Kalai, 1992),*

v) *it suffices to prove the conjecture for simple polytopes with* $m = 2\,n$ *(Klee&Walkup, 1961/1965)!*

vi) *Disproof of the Hirsch conjecture by Francisco Santos, 2010, see* `http://front.math.ucdavis.edu/1006.2814`

3.47 Definition [0/1**-polytope**]**.** *Let* $[0, 1]^n$ *be the* n*-dimensional unit cube with vertices* $\{0, 1\}^n = \{(x_1, \ldots, x_n)^\intercal : x_i \in \{0, 1\}\}$. $P \in \mathcal{P}^n$ *is called* 0/1-polytope *if* $\operatorname{vert} P \subset \{0, 1\}^n$.

3.48 Lemma. *Let* $P \in \mathcal{P}^n$ *be a* 0/1*-polytope and let* $\dim P \leq n - 1$*. Then there exists a* 0/1*-polytope* $\widetilde{P} \in \mathcal{P}^{n-1}$ affinely isomorphic *to* P*, i.e., there exists a bijective map between* P *and* $\widetilde{P}$*.*

3.49 Theorem [Naddef, 1989].

i) *Let* P *be a* 0/1*-polytope. Then* $\delta(P) \leq \dim P$*.*

ii) *Let* $P \in \mathcal{P}^n$ *be an* n*-dimensional* 0/1*-polytope with* m *facets. Then* $\delta(P) \leq m - n$*.*

3.50 Remark.

i) $f_{n-1}(P) \leq 2\,n!$ *for a* 0/1*-polytope* $P \in \mathcal{P}^n$*.*

ii) *There exist* 0/1*-polytopes* $P \in \mathcal{P}^n$ *with*

$$f_{n-1}(P) \geq \left(\frac{c\,n}{\log^2 n}\right)^{\frac{n}{2}},$$

where c *is a universal constant (Gatzouras, Giannopoulos, Markoulakis, 2004).*

4 A bit of Gale diagrams and triangulations

4.1 Definition. *Let $a_1, \dots, a_m \in \mathbb{R}^n$, $m \geq n+1$, such that their affine hull spans $\mathbb{R}^n$, and let $\overline{a}_i = (a_i, 1)^\intercal \in \mathbb{R}^{n+1}$, $1 \leq i \leq m$, and let $\overline{A} = (\overline{a}_1, \dots, \overline{a}_m) \in \mathbb{R}^{(n+1)\times m}$. Let $\overline{B} = (\overline{b}_1, \dots, \overline{b}_{m-(n+1)}) \in \mathbb{R}^{m\times(m-(n+1))}$ be a basis of the subspace $\{x \in \mathbb{R}^m : \overline{A}\, x = 0\}$, i.e., of the kernel of $\overline{A}$, and let $\overline{B}^\intercal = (b_1, \dots, b_m) \in \mathbb{R}^{(m-(n+1))\times m}$.*

The vectors $b_1, \dots, b_m \in \mathbb{R}^{m-(n+1)}$ are called the Gale transform *of the points $a_1, \dots, a_m$.*

4.2 Remark.

i) *The Gale transform is unique up to linear isomorphisms of $\mathbb{R}^{(m-(n+1))}$.*

ii) *For the vectors of a Gale transform always hold $\sum_{i=1}^m b_i = 0$.*

4.3 Proposition. *Let $P = \operatorname{conv}\{v_1, \dots, v_m\}$ with vertices v_i, $1 \leq i \leq m$, and for $J \subseteq [m] = \{1, \dots, m\}$ let $V_J = \{v_j : j \in J\}$. Then $\operatorname{conv} V_J$ is a face of P if and only if $\operatorname{conv} V_{[m]\setminus J} \cap \operatorname{aff} V_J = \emptyset$.*

For $\#J = 1$ the statement is also true without the assumption that v_i are vertices.

4.4 Theorem. *Let $P = \operatorname{conv}\{v_1, \dots, v_m\}$, $\dim P = n$, with vertices v_i, $1 \leq i \leq m$, and for $J \subseteq [m] = \{1, \dots, m\}$ let $V_J = \{v_j : j \in J\}$. Let $\{b_1, \dots, b_m\}$ be the Gale transform of $\{v_1, \dots, v_m\}$. Then $\operatorname{conv} V_J$ is a face of P if and only if $J = [m]$ or $0 \in \operatorname{relint}(\operatorname{conv}\{b_k : k \notin J\})$.*

For $\#J = 1$ the statement is also true without the assumption that v_i are vertices.

4.5 Corollary. *$\{b_1, \dots, b_m\} \in \mathbb{R}^{m-(n+1)}$ is the Gale transform of the vertex set $\{v_1, \dots, v_m\} \in \mathbb{R}^n$ of a polytope if and only if for each hyperplane $H(a, 0) \subset \mathbb{R}^{m-(n+1)}$ the halfspaces $H^+(a, 0)$ and $H^-(a, 0)$ contain at least two points of $\{b_1, \dots, b_m\}$ in its interior.*

4.6 Definition [Face lattice]. *For a polytope $P \in \mathcal{P}^n$ let $\mathcal{F}(P)$ be the set of all its faces. Together with the inclusion relation "$\subseteq$" on $\mathcal{F}(P)$ the faces form a* partially ordered set (poset) *denoted by $(\mathcal{F}(P), \subseteq)$ and which is called the* face lattice *of P.*

4.7 Definition [Combinatorially isomorphic]. *Two polytopes $P, Q \in \mathcal{P}^n$ are called* combinatorially isomorphic *or* combinatorially equivalent *if the face lattices $(\mathcal{F}(P), \subseteq)$ and $(\mathcal{F}(Q), \subseteq)$ are isomorphic, i.e., there exists an inclusion preserving bijection between the faces of P and Q.*

4.8 Theorem. *There are precisely $\lfloor \frac{1}{4} n^2 \rfloor$ combinatorial different types of n-polytopes with $n+2$ vertices.*

Proof. For such an n-polytope $P = \operatorname{conv}\{v_1, \ldots, v_{n+2}\}$ the Gale transform $\{b_1, \ldots, b_{n+2}\}$ is one-dimensional. Since $0 \in \operatorname{int}\operatorname{conv}\{b_1, \ldots, b_{n+2}\}$ we can classify the Gale transform by the number of points l, e, g which are < 0, $= 0$ and > 0, respectively. Then we have $l + e + g = n + 2$ and according to Corollary 4.5 we have $l, g \geq 2$ and by symmetry we may assume $l \leq g$. So for a fixed $e \in \{0, \ldots, n-2\}$ the parameter l is bounded between 2 and $\lfloor (n+2-e)/2 \rfloor$, and so there are $\lfloor (n-e)/2 \rfloor$ possibilities. Altogether we have found

$$\sum_{e=0}^{n-2} \left\lfloor \frac{n-e}{2} \right\rfloor = \left\lfloor \frac{1}{4} n^2 \right\rfloor$$

different possibilities of Gale transforms which correspond by Corollary 4.5 to all n polytopes with $(n+2)$ vertices. It remains to show that these polytopes are combinatorially inequivalent. To this end let (l_i, e_i, g_i) be two different sequences with polytopes P_i, $i = 1, 2$, and assume that the polytopes are combinatorially equivalent. Let ϕ be an isomorphism between the face lattices of P_i which induces an isomorphism on the points of the Gale transforms. First we observe that all points of the groups "e_i" have to be mapped onto each other, since removing all other points yields a face of the polytope in view of Theorem 4.4. So we may assume $e_1 = e_2$ and suppose $1 < l_1 < l_2 \leq g_2 < g_1$. The image of the points of the first group "l_1" cannot consists of all points of l_2 or of all points of g_2. Hence removing the image of the points yields a face of P_2 but not of P_1 according to Theorem 4.4. Hence we get a contradiction. □

4.9 Definition [Point configuration]. *A point configuration in $\mathbb{R}^n$ is a finite set of (perhaps repeated) points with (non-repeated) labels. It is identified with its matrix $A = (a_1, \ldots, a_m) \in \mathbb{R}^{n \times m}$.*

4.10 Definition [Triangulation]. *A* triangulation *of a point configuration $A = (a_1, \ldots, a_m) \subset \mathbb{R}^{n \times m}$ is a collection $\mathcal{T}$ of simplices, with vertices in A, that satisfies the following properties:*

i) *All faces of simplices of $\mathcal{T}$ are in $\mathcal{T}$.*

ii) *The intersection of any two simplices of $\mathcal{T}$ is a face (possible empty) of both.*

iii) *The union of all simplices of $\mathcal{T}$ equals* $\operatorname{conv} A$.

4.11 Remark.

i) *All vertices of* $\operatorname{conv} A$ *are in $\mathcal{T}$.*

ii) *Let F be a face of* $\operatorname{conv} A$. *Then $\mathcal{T}_F := \{\sigma \in \mathcal{T} : \sigma \subset F\}$ is a triangulation of $A \cap F$.*

iii) *The frist two properties of Definition 4.10 are the definition of a (geometric)* simplicial complex.

4.12 Definition [Height function and lower convex hull]. *Let $A = (a_1, \dots, a_m) \in \mathbb{R}^{n\times m}$ be a point configuration. Let $\eta : A \to \mathbb{R}_{\geq 0}$ be a function and let*

$$A_\eta = \begin{pmatrix} a_1 & a_2 & \dots & a_m \\ \eta(a_1) & \eta(a_2) & \dots & \eta(a_m) \end{pmatrix} \in \mathbb{R}^{(n+1)\times m}$$

be the lifted point configuration *(also called a lifting). Then η is called a height function . The union of the faces of* conv A_η *which are visible form below are called the* lower convex hull *of* conv A_η*. More precisely, $y = (y_1, \dots, y_n, y_{n+1})^\intercal \in$* conv (A_η) *belongs to the lower convex hull if the segment* conv $\{y, (y_1, \dots, y_n, 0)^\intercal\}$ *intersects* conv A_η *only in y.*

4.13 Definition [Regular triangulation]. *A triangulation of a point configuration $A \in \mathbb{R}^{n\times m}$ is called* a regular triangulation *if it can be obtained by projecting the lower convex hull of a lifting of A.*

4.14 Theorem. *Every point configuration $A \in \mathbb{R}^{n\times m}$ has regular triangulations.*

4.15 Definition [Delaunay triangulation]. *Let $A \in \mathbb{R}^{n\times m}$ be a point configuration containing $n+1$ affinely independent points. Let $\eta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be the height function given by $\eta(x) = |x|^2$. If the projection of the lower convex hull of* conv A_η *yields a triangulation, then it is called* Delaunay triangulation.

4.16 Lemma. *Let $C \subset \mathbb{R}^{n+1}$ be the paraboloid $C = \{(x, |x|^2)^\intercal : x \in \mathbb{R}^n\}$. Let $H(a, \alpha)$, $a \in \mathbb{R}^{n+1}$, $a_{n+1} \neq 0$, be a hyperplane, and let $\Pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection forgetting the last coordinate. Then $\Pi(C \cap H(a, \alpha))$ is either empty, a point or an n-dimensional sphere.*

4.17 Corollary. *Let $A \in \mathbb{R}^{n\times m}$ be a point configuration containing $n+1$ affinely independent points. Let $\eta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be the height function given by $\eta(x) = |x|^2$. Let $B \subseteq A$ such that B contains $n+1$ affinely independent points. Then B corresponds to the vertex set of a facet of the lower convex hull of the lifted points if and only if there is sphere passing through all points of B and leaving all points of $A \setminus B$ outside.*

4.18 Remark. *In particular, simplices in a Delaunay triangulation $\mathcal{T}$ are characterized by the "empty sphere" property: $\sigma \in \mathcal{T}$ if and only if there is an Euclidean sphere with all vertices of σ on the sphere and with the rest of the points outside.*

5 A glimpse of Ehrhart theory

Almost all results of this section are valid for arbitrary lattices; for simplification, however, we state most of them only for the standard lattice $\mathbb{Z}^n$.

5.1 Definition [Lattice polytope]. *A polytope* $P = \operatorname{conv}\{v_1, \ldots, v_m\} \subset \mathbb{R}^n$ *is called a* lattice polytope *if* $v_i \in \mathbb{Z}^n$, $1 \leq i \leq m$. *The set of all lattice polytopes is denoted by* $\mathcal{P}^n_{\mathbb{Z}}$.

5.2 Notation. *For* $S \subset \mathbb{R}^n$ *we denote by* $\mathrm{G}(S)$ *its* lattice point enumerator, *i.e.,*

$$\mathrm{G}(S) = \#(S \cap \mathbb{Z}^n).$$

5.3 Lemma. *Let* $a_1, \ldots, a_n \in \mathbb{Z}^n$ *be linearly independent, and let* P *be the half open parallelepiped* $P = \{\rho_1 a_1 + \cdots + \rho_n a_n : 0 \leq \rho_i < 1\}$. *Then*

$$\mathrm{G}(P) = \operatorname{vol}(P).$$

5.4 Lemma. *Let* $T = \operatorname{conv}\{0, v_1, v_2\} \in \mathcal{P}^2_{\mathbb{Z}}$ *be a lattice triangle, i.e.,* $v_1, v_2 \in \mathbb{Z}^2$ *are linearly independent. Then*

$$\mathrm{G}(T) = \operatorname{vol}(T) + \frac{1}{2}\mathrm{G}(\operatorname{bd} T) + 1.$$

5.5 Theorem [Pick]. [1] *Let* $P \in \mathcal{P}^2_{\mathbb{Z}}$, $\dim P = 2$. *Then*

$$\mathrm{G}(P) = \operatorname{vol}(P) + \frac{1}{2}\mathrm{G}(\operatorname{bd} P) + 1.$$

5.6 Corollary.

i) *Let* $P \in \mathcal{P}^2_{\mathbb{Z}}$, $\dim P = 2$, *with edges* $F_1, \ldots, F_m$. *Then*

$$\mathrm{G}(P) = \operatorname{vol}(P) + \frac{1}{2}\sum_{i=1}^{m} \frac{\operatorname{vol}_1(F_i)}{\det(\operatorname{aff} F_i \cap \mathbb{Z}^2)} + 1.$$

Here $\det(\operatorname{aff} F_i \cap \mathbb{Z}^2)$ *is the distance of two consecutive lattice points on* $\operatorname{aff} F_i$.

ii) *Let* $K \in \mathcal{K}^2$. *Then* $\mathrm{G}(K) \leq \operatorname{vol}(K) + \frac{1}{2}\mathrm{F}(K) + 1$, *where* $F(K)$ *denotes the perimeter of* K.

5.7 Remark.

[1] Georg Alexander Pick; 10.08.1859 (Vienna) – 26.07.1942 (concentration camp Theresienstadt)

i) *Inequality ii) of Corollary 5.6 can not be generalized in a straightforward and best possible way to arbitrary lattices, since the perimeter is not an affine equivariant functional, in contrast to the area.*

ii) *Pick's theorem itself, however, can be generalized in various ways. Its proof is only based on the property that a set S can be subdivided into lattice triangle such that the intersection of any two of them is a face of both. For those sets it was shown by Hadwiger&Wills that*

$$\mathrm{G}(S) = \mathrm{vol}\,(S) + \frac{1}{2}\mathrm{E}(S) + \chi(S).$$

Here $\chi(S)$ is the Euler-Poincaré characteristic of S, and $\mathrm{E}(S)$ is the number of segments between two consecutive lattice points in the boundary of S, where segments are counted twice which are not bordering a 2-dimensional cell.

iii) *By Pick's theorem we immediately get the following polynomial behaviour of the lattice point enumerator of a lattice polygon $P \in \mathcal{P}^2_{\mathbb{Z}}$*

$$\mathrm{G}(k\,P) = \mathrm{vol}\,(P)\,k^2 + \frac{1}{2}\,\mathrm{G}(\mathrm{bd}\,P)\,k + 1,$$
$$\mathrm{G}\big(\mathrm{int}\,(k\,P)\big) = \mathrm{vol}\,(P)\,k^2 - \frac{1}{2}\,\mathrm{G}(\mathrm{bd}\,P)\,k + 1.$$

5.8 Notation. *For integers m, n we denote by*

$$\binom{x+m}{n} = \frac{1}{n!}\prod_{i=0}^{n-1}(x+m-i)$$

the polynomial of degree n with roots $i-m$, $i=0,\ldots,n-1$, and leading coefficient $1/n!$. In particular, the polynomials $\binom{x+n-i}{n}$, $i=0,\ldots,n$, form a basis of the space of all polynomials of degree at most n.

5.9 Lemma. *Let $T = \mathrm{conv}\,\{0, v_1, \ldots, v_n\} \in \mathcal{P}^n_{\mathbb{Z}}$ be a lattice simplex, i.e., $v_1, \ldots, v_n \in \mathbb{Z}^n$ are linearly independent, and for $0 \leq i \leq n$ let*

$$\mathrm{a}_i(T) = \#\left\{\sum_{j=1}^{n}\lambda_j v_j \in \mathbb{Z}^n : 0 \leq \lambda_j < 1,\ i-1 < \sum_{j=1}^{n}\lambda_j \leq i\right\}.$$

Then for all $k \in \mathbb{N}$, $k \geq 1$, we have

$$\mathrm{G}(k\,T) = \sum_{i=0}^{n}\mathrm{a}_i(T)\binom{k+n-i}{n}.$$

5.10 Lemma [Inclusion-Exclusion Formula]. *Let $A_i \subseteq \mathbb{R}^n$, $1 \leq i \leq m$, with characteristic functions $\chi(A_i)$. Then*

$$\chi(A_1 \cup A_2 \cup \cdots \cup A_m) = \sum_{\substack{I \subset \{1,\ldots,m\} \\ I \neq \emptyset}} (-1)^{\#I-1}\chi\left(\bigcap_{j\in I} A_j\right).$$

5.11 Theorem [Ehrhart, 1967]. [2] *Let $P \in \mathcal{P}^n_{\mathbb{Z}}$. Then there exist numbers $\mathrm{G}_i(P) \in \mathbb{Q}$, $0 \le i \le n$, depending only on P, such that for all $k \in \mathbb{N}_{\ge 1}$*

$$\mathrm{G}(k\,P) = \sum_{i=0}^{n} \mathrm{G}_i(P)\,k^i.$$

The right hand side is called Ehrhart-polynomial.

5.12 Proposition. *Let $P \in \mathcal{P}^n_{\mathbb{Z}}$.*

i) $\mathrm{G}_n(P) = \mathrm{vol}\,(P)$.

ii) $\mathrm{G}_i : \mathcal{P}^n_{\mathbb{Z}} \to \mathbb{R}$ *is homogeneous of degree i, invariant with respect to unimodular transformations and additive, $0 \le i \le n$.*

iii) $\mathrm{G}_i(P)$ *are independent of the dimension of the space in which P is embedded, i.e., let $P \in \mathcal{P}^n_{\mathbb{Z}}$ and let $\tilde{P} = \mathrm{conv}\,\{(v,0)^\intercal : v \in P\} \in \mathcal{P}^{n+1}_{\mathbb{Z}}$. Then $\mathrm{G}_i(P) = \mathrm{G}_i(\tilde{P})$, $i = 0, \ldots, n$.*

5.13 Theorem* [Betke, Kneser, 1985]. [3] [4] *Every additive and unimodular invariant functional on the space of all lattice polytopes is a linear combination of the $n+1$ functionals $\mathrm{G}_i(\cdot)$.*

5.14 Remark. *Some of the coefficients $\mathrm{G}_i(P)$ might be negative. One family of standard examples in this context are the so called* Reeve-simplices*: let $R_m = \mathrm{conv}\left\{0, e_1, e_2, (1,1,m)^\intercal\right\} \in \mathcal{P}^3_{\mathbb{Z}}$ for $m \in \mathbb{N}$. The only lattice points contained in R_m are the four vertices, the volume, however, can be arbitrarily large. Hence some $\mathrm{G}_i(R_m)$ must be negative for large m. More precisely, it is $\mathrm{G}_3(R_m) = m/6$, $\mathrm{G}_2(R_m) = 1$, $\mathrm{G}_1(R_m) = (12-m)/6$ and $\mathrm{G}_0(R_m) = 1$.*

Of course, we can also rewrite the Ehrhart polynomial in terms of the binomial basis $\binom{k+n-i}{n}$ and get

5.15 Corollary. *Let $P \in \mathcal{P}^n_{\mathbb{Z}}$. Then there exist numbers $\mathrm{a}_i(P) \in \mathbb{Q}$, $0 \le i \le n$, such that for all $k \in \mathbb{N}_{\ge 1}$*

$$\mathrm{G}(k\,P) = \sum_{i=0}^{n} \mathrm{a}_i(P) \binom{k+n-i}{n}.$$

5.16 Example.

[2] Eugéne Ehrhart, 29.04.1906 (Guebwiller (Haut-Rhin, France)) – 17.01.2000 (Strasbourg)
[3] Martin Kneser, 21.01.1928 (Greifswald) – 16.02.2004 (Göttingen)
[4] Ulrich Betke, ??.??.1948 (???) – 24.05.2008 (Siegen)

i) *Let* $T_n = \operatorname{conv}\{0, e_1, e_2, \dots, e_n\}$. *Then*

$$\#(k\,T_n \cap \mathbb{Z}^n) = \binom{n+k}{n},$$

and so we have $\mathrm{a}_i(T_n) = 0$ *for* $1 \le i \le n$, *and* $\mathrm{a}_0(T_n) = 1$. *The* $\mathrm{G}_i(T_n)$ *are – up to* ± 1 *– Stirling numbers of the first kind.*

ii) *Let* $C_n = [-1, 1]^n$. *Then* $\mathrm{G}(k\,C_n) = (2\,k+1)^n$ *and so* $\mathrm{G}_i(C_n, \mathbb{Z}^n) = 2^i\binom{n}{i}$, $0 \le i \le n$. *Here the* $\mathrm{a}_i(C_n)$ *are some combinatorial numbers, the so called Eulerian numbers.*

iii) *Let* $C_n^\star = \operatorname{conv}\{\pm e_i : 1 \le i \le n\}$. *Then*

$$\mathrm{G}(k\,C_n^\star) = \sum_{i=0}^{n} \binom{n}{i}\binom{k+n-i}{n},$$

and so $\mathrm{a}_i(C_n^\star) = \binom{n}{i}$, $i = 0, \dots, n$.

iv) *Let* $\widehat{T_2} = \operatorname{conv}\{0, e_1, e_2\} \subset \mathbb{R}^3$ *be the 2-dimensional standard triangle embedded in* $\mathbb{R}^3$. *Then by Pick's Theorem 5.5*

$$\mathrm{G}(k\,\widehat{T_2}) = \frac{1}{2}\,k^2 + \frac{3}{2}\,k + 1 = (-1)\binom{k}{3} + \frac{1}{2}\binom{k+1}{3} - \frac{1}{2}\binom{k+2}{3} + \binom{k+3}{3}.$$

Hence, in this case we have $\mathrm{a}_3(\widehat{T_2}) = -1$, $\mathrm{a}_2(\widehat{T_2}) = 1/2$, $\mathrm{a}_1(\widehat{T_2}) = -1/2$, $\mathrm{a}_0(\widehat{T_2}) = 1$.

Finally, we state without proof some highlights of Ehrhart theory (and which are left to the lectures in the next term)

5.17 Remark.

i) Stanley's Non-Negativity Theorem*: Let* $P \in \mathcal{P}^n_{\mathbb{Z}}$, $\dim P = n$. *Then* $\mathrm{a}_i(P) \in \mathbb{N}_{\ge 0}$, $0 \le i \le n$.

ii) Stanley's Monotonicity Theorem*: Let* $P, Q \in \mathcal{P}^n_{\mathbb{Z}}$, $\dim P = \dim Q = n$ *with* $P \subseteq Q$. *Then* $\mathrm{a}_i(P, \mathbb{Z}^n) \le \mathrm{a}_i(Q, \mathbb{Z}^n)$, $0 \le i \le n$.

iii) Ehrhart-Macdonald Reciprocity*: Let* $P \in \mathcal{P}^n_{\mathbb{Z}}$, $\dim P = n$. *Then*

$$\mathrm{G}(\operatorname{int} k\,P) = (-1)^n \sum_{i=0}^{n} \mathrm{G}_i(P)(-k)^i.$$

iv) *Let* $P \in \mathcal{P}^n_{\mathbb{Z}}$, $\dim P = n$. *Then for* $i \ne n \bmod 2$

$$\mathrm{G}_i(P) = \frac{1}{2}\sum_{j=i}^{n-1}(-1)^{i+j} \sum_{F\, j-\text{face of } P} \mathrm{G}_i(F).$$

In particular,

$$\mathrm{G}_{n-1}(P) = \frac{1}{2} \sum_{F \text{ facet of } P} \frac{\operatorname{vol}_{n-1}(F)}{\det(\operatorname{aff} F \cap \mathbb{Z}^n)}.$$

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