Combinatorial Convexity

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0 Some basic and convex facts

0.1 Notation. $\mathbb{R}^n = \{(x_1, \ldots, x_n)^\intercal : x_i \in \mathbb{R}\}$ *denotes the* n*-dimensional* Euclidean space *equipped with the* Euclidean inner product $\langle x, y\rangle = \sum_{i=1}^n x_i\, y_i$, $x, y \in \mathbb{R}^n$, *and the* Euclidean norm $|x| = \sqrt{\langle x, x\rangle}$.

0.2 Definition [Linear, affine, positive and convex combination]. *Let* $m \in \mathbb{N}$ *and let* $x_i \in \mathbb{R}^n$, $\lambda_i \in \mathbb{R}$, $1 \le i \le m$.

i) $\sum_{i=1}^m \lambda_i\, x_i$ *is called* a linear combination *of* $x_1, \ldots, x_m$.

ii) *If* $\sum_{i=1}^m \lambda_i = 1$ *then* $\sum_{i=1}^m \lambda_i\, x_i$ *is called an* affine combination *of* x_1, $\ldots, x_m$.

iii) *If* $\lambda_i \ge 0$ *then* $\sum_{i=1}^m \lambda_i\, x_i$ *is called* a positive combination *of* $x_1, \ldots, x_m$.

iv) *If* $\lambda_i \ge 0$ *and* $\sum_{i=1}^m \lambda_i = 1$ *then* $\sum_{i=1}^m \lambda_i\, x_i$ *is called* a convex combination *of* $x_1, \ldots, x_m$.

v) *Let* $X \subseteq \mathbb{R}^n$. $x \in \mathbb{R}^n$ *is called* linearly *(*affinely, positively, convexly*)* dependent *of* X, *if* x *is a linear (affine, positive, convex) combination of finitely many points of* X, *i.e., there exist* $x_1, \ldots, x_m \in X$, $m \in \mathbb{N}$, *such that* x *is a linear (affine, positive, convex) combination of the points* $x_1, \ldots, x_m$.

0.3 Definition [Linearly and affinely independent points]. $x_1, \ldots, x_m \in \mathbb{R}^n$ *are called* linearly *(*affinely*)* dependent, *if one of the* x_i *is linearly (*affinely*) dependent of* $\{x_1, \ldots, x_m\}\backslash\{x_i\}$. *Otherwise* $x_1, \ldots, x_m$ *are called* linearly *(*affinely*)* independent.

0.4 Remark. *Let* $x_1, \ldots, x_m \in \mathbb{R}^n$.

i) $x_1, \ldots, x_m$ *are affinely dependent if and only if* $\binom{x_1}{1}, \ldots, \binom{x_m}{1} \in \mathbb{R}^{n+1}$ *are linearly dependent.*

ii) $x_1, \ldots, x_m$ *are affinely dependent if and only if there exist* $\mu_i \in \mathbb{R}$, $1 \le i \le m$, *with* $(\mu_1, \ldots, \mu_m) \ne (0, \ldots, 0)$, $\sum_{i=1}^m \mu_i = 0$ *and* $\sum_{i=1}^m \mu_i\, x_i = 0$.

iii) *If* $m \ge n+1$ *then* $x_1, \ldots, x_m$ *are linearly dependent.*

iv) *If* $m \ge n+2$ *then* $x_1, \ldots, x_m$ *are affinely dependent.*

0.5 Definition [Linear subspace, affine subspace, cone and convex set]. $X \subseteq \mathbb{R}^n$ *is called*

i) linear subspace (set) *if it contains all* $x \in \mathbb{R}^n$ *which are linearly dependent of* X,

ii) affine subspace (set) *if it contains all* $x \in \mathbb{R}^n$ *which are affinely dependent of* X,

iii) (convex) cone *if it contains all* $x \in \mathbb{R}^n$ *which are positively dependent of* X,

iv) convex set *if it contains all* $x \in \mathbb{R}^n$ *which are convexly dependent of* X.

0.6 Notation. $\mathcal{C}^n = \{K \subseteq \mathbb{R}^n : K \text{ convex}\}$ *denotes the set of all convex sets in* $\mathbb{R}^n$. *The empty set* $\emptyset$ *is regarded as a convex, linear and affine set.*

0.7 Theorem. $K \subseteq \mathbb{R}^n$ *is convex if and only if*

$$\lambda x + (1 - \lambda)\, y \in K, \qquad \text{for all } x, y \in K \text{ and } 0 \leq \lambda \leq 1.$$

0.8 Example. *The closed* n*-dimensional ball* $B_n(a, \rho) = \big\{x \in \mathbb{R}^n : |x - a| \leq \rho\big\}$ *with centre* a *and radius* $\rho > 0$ *is convex. The boundary of* $B_n(a, \rho)$, *i.e.,* $\big\{x \in \mathbb{R}^n : |x - a| = \rho\big\}$ *is non-convex. In the case* $a = 0$ *and* $\rho = 1$ *the ball* $B_n(0, 1)$ *is abbreviated by* B_n *and is called* n-dimensional unit ball. *Its boundary is denoted by* S^{n-1}.

0.9 Corollary. *Let* $K_i \in \mathcal{C}^n$, $i \in I$. *Then* $\bigcap_{i \in I} K_i \in \mathcal{C}^n$.

0.10 Definition [Linear, affine, positive and convex hull, dimension]. *Let* $X \subseteq \mathbb{R}^n$.

i) *The* linear hull $\operatorname{lin} X$ *of* X *is defined by*

$$\operatorname{lin} X = \bigcap_{\substack{L \subseteq \mathbb{R}^n,\, L \text{ linear}, \\ X \subseteq L}} L.$$

ii) *The* affine hull $\operatorname{aff} X$ *of* X *is defined by*

$$\operatorname{aff} X = \bigcap_{\substack{A \subseteq \mathbb{R}^n,\, A \text{ affine}, \\ X \subseteq A}} A.$$

iii) *The* positive (conic) hull $\operatorname{pos} X$ *of* X *is defined by*

$$\operatorname{pos} X = \bigcap_{\substack{C \subseteq \mathbb{R}^n,\, C \text{ convex cone}, \\ X \subseteq C}} C.$$

iv) *The* convex hull $\operatorname{conv} X$ *of* X *is defined by*

$$\operatorname{conv} X = \bigcap_{\substack{K \subseteq \mathbb{R}^n,\, K \text{ convex}, \\ X \subseteq K}} K.$$

v) *The* dimension $\dim X$ *of* X *is the dimension of its affine hull, i.e.,* $\dim \operatorname{aff} X$.

0.11 Theorem. *Let $X \subseteq \mathbb{R}^n$. Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{m} \lambda_i\, x_i : m \in \mathbb{N}, x_i \in X, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

0.12 Remark.

i) $\operatorname{conv}\{x, y\} = \big\{ \lambda\, x + (1-\lambda)\, y : \lambda \in [0,1] \big\}$.

ii) $\operatorname{lin} X = \big\{ \sum_{i=1}^{m} \lambda_i x_i : m \in \mathbb{N}, x_i \in X \big\}$.

iii) $\operatorname{aff} X = \big\{ \sum_{i=1}^{m} \lambda_i x_i : m \in \mathbb{N}, x_i \in X, \sum_{i=1}^{m} \lambda_i = 1 \big\}$.

iv) $\operatorname{pos} X = \big\{ \sum_{i=1}^{m} \lambda_i x_i : m \in \mathbb{N}, x_i \in X,\ \lambda_i \geq 0 \big\}$.

0.13 Definition [(Relative) interior point and (relative) boundary point]. *Let $X \subseteq \mathbb{R}^n$.*

i) *$x \in X$ is called an* interior point *of X if there exists a $\rho > 0$ such that $B_n(x, \rho) \subseteq X$. The set of all interior points of X is called the* interior *of X and is denoted by* $\operatorname{int} X$.

ii) *$x \in \mathbb{R}^n$ is called* boundary point *of X if for all $\rho > 0$ hold $B_n(x, \rho) \cap X \neq \emptyset$ and $B_n(x, \rho) \cap (\mathbb{R}^n \backslash X) \neq \emptyset$. The set of all boundary points of X is called the* boundary *of X and is denoted by* $\operatorname{bd} X$.

iii) *Let $A = \operatorname{aff} X$. $x \in X$ is called a* relative interior point *of X if there exists a $\rho > 0$ such that $B_n(x, \rho) \cap A \subseteq X$. The set of all relative interior points is called the* relative interior *of X and is denoted by* $\operatorname{relint} X$.

iv) *Let $A = \operatorname{aff} X$. $x \in A$ is called a* relative boundary point *of X if for all $\rho > 0$ hold $B_n(x, \rho) \cap X \neq \emptyset$ and $B_n(x, \rho) \cap (A \backslash X) \neq \emptyset$. The set of all relative boundary points of X is called* relative boundary *of X and is denoted by* $\operatorname{relbd} X$.

0.14 Remark. *Let $X \subseteq \mathbb{R}^n$ be closed. Then $X = \operatorname{relint} X \cup \operatorname{relbd} X$.*

0.15 Theorem. *Let $K \in \mathcal{C}^n$, $x \in \operatorname{relint} K$ and $y \in K$. Then $(1-\lambda)x + \lambda y \in \operatorname{relint} K$ for all $\lambda \in [0, 1)$.*

0.16 Corollary. *Let $K \in \mathcal{C}^n$ be closed. Let $x \in \operatorname{relint} K$ and $y \in \operatorname{aff} K \setminus K$. Then the segment $\operatorname{conv}\{x, y\}$ intersects $\operatorname{relbd} K$ in precisely one point.*

0.17 Definition [Polytope and simplex]. *Let $X \subset \mathbb{R}^n$ of finite cardinality, i.e., $\#X < \infty$.*

i) *$\operatorname{conv} X$ is called a* (convex) polytope.

ii) *A polytope $P \subset \mathbb{R}^n$ of dimension k is called a k-polytope.*

iii) *If X is affinely independent and $\dim X = k$ then $\operatorname{conv} X$ is called a k-simplex.*

0.18 Notation. *$\mathcal{P}^n = \{P \subset \mathbb{R}^n : P$ polytope$\}$ denotes the set of all polytopes in $\mathbb{R}^n$.*

0.19 Lemma. *Let $T = \operatorname{conv}\{x_1, \ldots, x_{k+1}\} \subset \mathbb{R}^n$ be a k-simplex, and let $\lambda_i > 0$, $1 \leq i \leq k+1$, with $\sum \lambda_i = 1$. Then $\sum \lambda_i \, x_i \in \operatorname{relint} T$.*

0.20 Corollary. *Let $K \in \mathcal{C}^n$, $K \neq \emptyset$. Then $\operatorname{relint} K \neq \emptyset$.*

0.21 Theorem. *Let $P = \operatorname{conv}\{x_1, \ldots, x_m\} \in \mathcal{P}^n$. A point $x \in \mathbb{R}^n$ belongs to $\operatorname{relint} P$ if and only if x admits a representation as $x = \sum_{i=1}^m \lambda_i x_i$ with $\lambda_i > 0$, $1 \leq i \leq m$, and $\sum_{i=1}^m \lambda_i = 1$.*

0.22 Notation.

i) *For two sets $X, Y \subseteq \mathbb{R}^n$ the vectorial addition*

$$X + Y = \{x + y : x \in X,\ y \in Y\}$$

is called the Minkowski sum *of X and Y. If X is just a singleton, i.e., $X = \{x\}$, then we write $x + Y$ instead of $\{x\} + Y$.*

ii) *For $\lambda \in \mathbb{R}$ and $X \subseteq \mathbb{R}^n$ we denote by λX the set*

$$\lambda X = \{\lambda x : x \in X\}.$$

1 Support and separate

1.1 Notation. *Let* $a \in \mathbb{R}^n$, $a \neq 0$, *and* $\alpha \in \mathbb{R}$. *The* closed halfspaces $H^+(a,\alpha)$, $H^-(a,\alpha) \subset \mathbb{R}^n$ *are given by*

$$H^+(a,\alpha) = \big\{x \in \mathbb{R}^n : \langle a, x\rangle \geq \alpha\big\}, \quad H^-(a,\alpha) = \big\{x \in \mathbb{R}^n : \langle a, x\rangle \leq \alpha\big\}.$$

The hyperplane $H(a,\alpha)$ *is defined by*

$$H(a,\alpha) = \big\{x \in \mathbb{R}^n : \langle a, x\rangle = \alpha\big\}.$$

1.2 Definition [Supporting hyperplane]. *Let* $X \subset \mathbb{R}^n$. *A hyperplane* $H(a,\alpha) \subset \mathbb{R}^n$ *is called* supporting hyperplane *of* X *if:*

$$\text{i) } H(a,\alpha) \cap X \neq \emptyset \quad \text{and} \quad \text{ii) } X \subseteq H^-(a,\alpha).$$

a *is called* outer normal vector *of* X *and if, in addition,* $|a| = 1$ *then it is called* outer unit normal vector *of* X.

1.3 Proposition. *Let* $X \subset \mathbb{R}^n$ *and let* $H(a,\alpha)$ *be a supporting hyperplane of* X. *Then*

$$H(a,\alpha) \cap \operatorname{conv} X = \operatorname{conv}\big(H(a,\alpha) \cap X\big).$$

1.4 Remark. *Let* $X \subset \mathbb{R}^n$ *be compact and* $a \in \mathbb{R}^n \backslash \{0\}$. *Then there exists a supporting hyperplane of* X *with outer normal vector* a.

1.5 Definition [Nearest point map (or metric projection)]. *Let* $K \in \mathcal{C}^n$ *be closed. The map* $\Phi_K : \mathbb{R}^n \to K$, *where for* $x \in \mathbb{R}^n$ *the point* $\Phi_K(x) \in K$ *is given by* $|x - \Phi_K(x)| = \min\big\{|x-y| : y \in K\big\}$ *is called the* nearest point map *(*metric projection*) with respect to* K.

1.6 Remark. *We prove that the nearest point map is well-defined. Notice that since* K *is closed, for all* $x \in \mathbb{R}^n$ *there exist* $y_x \in K$ *such that* $|x - y_x| = \min\big\{|x-y| : y \in K\big\}$. *We show that* y_x *is uniquely determined. In fact, if there exists* $\overline{y} \in K$, $\overline{y} \neq y_x$, *with* $|x - \overline{y}| = |x - y_x|$ *then we may assume that* $x - y_x$ *and* $x - \overline{y}$ *are linearly independent. Hence*

$$\left|x - \frac{y_x + \overline{y}}{2}\right| = \left|\frac{1}{2}(x - y_x) + \frac{1}{2}(x - \overline{y})\right| < \frac{1}{2}\,|x - y_x| + \frac{1}{2}\,|x - \overline{y}| = |x - y_x|\,.$$

Since $(y_x + \overline{y})/2 \in K$ *by the convexity of* K, *it contradicts the minimality of* y_x.

1.7 Theorem. *Let* $K \in \mathcal{C}^n$ *be closed and let* $x \in \mathbb{R}^n \setminus K$. *Let* $a = x - \Phi_K(x)$ *and* $\alpha = \langle a, \Phi_K(x)\rangle$. *Then* $H(a,\alpha)$ *is a supporting hyperplane of* K *with outer normal vector* a.

1.8 Corollary. *Let* $K \in \mathcal{C}^n$, $K \neq \mathbb{R}^n$, *be closed. Then*

$$K = \bigcap_{\substack{H(a,\alpha) \text{ supporting} \\ \text{hyperplane of } K}} H^-(a,\alpha),$$

i.e., K *is the intersection of all its "supporting halfspaces".*

1.9 Corollary. *Let* $X \subset \mathbb{R}^n$ *such that* $\operatorname{conv} X$ *is closed and* $\operatorname{conv} X \neq \mathbb{R}^n$. *Then*

$$\operatorname{conv} X = \bigcap_{X \subseteq H^-(a,\alpha)} H^-(a,\alpha),$$

i.e., $\operatorname{conv} X$ *is the intersection of all halfspaces containing* X.

1.10 Lemma [Busemann-Feller Lemma]. *Let* $K \in \mathcal{C}^n$ *be closed. Then*

$$|\Phi_K(x) - \Phi_K(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}^n$, *i.e., the nearest point map does not increase distances. In particular, it is a continuous map.*

1.11 Theorem. *Let* $K \in \mathcal{C}^n$ *be compact and let* $\rho > 0$ *such that* $K \subset \operatorname{int}(\rho B_n)$. *The nearest point map restricted to* ρS^{n-1} *is surjective, i.e.,* $\Phi_K : \rho\, S^{n-1} \to \operatorname{bd} K$ *is surjective.*

1.12 Corollary. *Let* $K \in \mathcal{C}^n$ *be closed and let* $x \in \operatorname{relbd} K$. *Then there exists a supporting hyperplane* $H(a,\alpha)$ *of* K *with* $x \in H(a,\alpha)$.

1.13 Theorem [Separation theorem]. *Let* $K_1, K_2 \in \mathcal{C}^n$ *with* $K_1 \cap K_2 = \emptyset$. *Then there exists* a separating hyperplane $H(a,\alpha)$ *of* K_1 *and* K_2, *i.e.,* $K_1 \subseteq H^+(a,\alpha)$ *and* $K_2 \subseteq H^-(a,\alpha)$.

If K_1 *is closed and* K_2 *is compact, then there exists even* a strictly separating hyperplane $H(a,\alpha)$ *of* K_1 *and* K_2, *i.e.,* $K_1 \subset \operatorname{int} H^+(a,\alpha)$ *and* $K_2 \subset \operatorname{int} H^-(a,\alpha)$.

1.14 Definition [Support function, breadth]. *Let* $K \in \mathcal{C}^n$. *The function* $h(K,\cdot) : \mathbb{R}^n \to \mathbb{R}$ *given by*

$$h(K,u) = \sup\big\{\langle u, x\rangle : x \in K\big\}$$

is called support function *of* K. *For* $u \in S^{n-1}$ *the* breadth of K in the direction u *is given by* $h(K,u) + h(K,-u)$.

1.15 Remark. *Let* $K \in \mathcal{C}^n$ *be non-empty and compact. Then*

$$K = \bigcap_{u \in S^{n-1}} \big\{x \in \mathbb{R}^n : \langle u, x\rangle \leq h(K,u)\big\}.$$

1.16 Definition [Polar set]. *Let* $X \subseteq \mathbb{R}^n$.

$$X^\star = \big\{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \;\text{ for all } x \in X\big\}$$

is called the polar set *of* X.

1.17 Proposition.

i) $X^\star$ *is a convex and closed set and* $0 \in X^\star$.

ii) *If* $X_1 \subseteq X_2$ *then* $X_2^\star \subseteq X_1^\star$.

iii) *Let* M *be a regular* $n \times n$ *matrix. Then* $(MX)^\star = M^{-\intercal}X^\star$.

iv) *Let* $X_i \subseteq \mathbb{R}^n$, $i \in I$. *Then* $\left(\bigcup_{i\in I} X_i\right)^\star = \bigcap_{i\in I} X_i^\star$.

v) $X \subseteq (X^\star)^\star$.

vi) *Let* $X \subset \mathbb{R}^n$. *Then* $X = X^\star$ *if and only if* $X = B_n$.

1.18 Proposition.

i) *Let* $P = \operatorname{conv}\{x_1, \ldots, x_m\} \subset \mathbb{R}^n$. *Then*

$$P^\star = \big\{y \in \mathbb{R}^n : \langle x_i, y\rangle \leq 1,\, 1 \leq i \leq m\big\}.$$

ii) *Let* $P = \big\{x \in \mathbb{R}^n : \langle a_i, x\rangle \leq 1,\, 1 \leq i \leq m\big\}$ *with* $a_i \in \mathbb{R}^n$. *Then*

$$P^\star = \operatorname{conv}\{0, a_1, \ldots, a_m\}.$$

1.19 Lemma. *Let* $K \in \mathcal{C}^n$ *be closed with* $0 \in K$. *Then* $(K^\star)^\star = K$.

2 Radon, Helly, Caratheodory and relatives

2.1 Theorem [Radon, 1921]. *Let $X \subset \mathbb{R}^n$. If $\#X \geq n+2$ then there exist $X_1, X_2 \subset X$ with $X_1 \cap X_2 = \emptyset$ and $\operatorname{conv} X_1 \cap \operatorname{conv} X_2 \neq \emptyset$.*

2.2 Theorem [Helly, 1913]. *Let $K_1, \ldots, K_m \in \mathcal{C}^n$, $m \geq n+1$, such that for each $(n+1)$-index set $I \subseteq \{1, \ldots, m\}$ we have $\bigcap_{i \in I} K_i \neq \emptyset$. Then all sets K_i have a point in common, i.e., $\bigcap_{j=1}^{m} K_i \neq \emptyset$.*

2.3 Remark.

i) *Without any further restrictions/assumptions Helly's theorem is* not *true for infinitely many convex sets K_i. For instance, let $K_i = (0, \frac{1}{i}]$, $i \in \mathbb{N}$.*

ii) *Helly's theorem, however, can be easily generalised to infinitely many* compact *(bounded and closed) convex sets.*

2.4 Corollary. *Let $C \subset \mathcal{C}^n$ be compact. Then there exists $t \in \mathbb{R}^n$ with*

$$-C \subseteq t + n\,C.$$

2.5 Definition [Centerpoint]. *For a finite point set $X \subset \mathbb{R}^n$ a point $c \in \mathbb{R}^n$ is called* centerpoint *is every closed halfspace containing c containes at least $\lceil \frac{1}{n+1} \#X \rceil$ points of X.*

2.6 Theorem. *Every finite set $X \subset \mathbb{R}^n$ has a centerpoint.*

2.7 Theorem [Carathéodory, 1907]. *Let $X \subset \mathbb{R}^n$. Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{n+1} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1, x_i \in X, i = 1, \ldots, n+1 \right\}.$$

2.8 Remark. *Let $X \subset \mathbb{R}^n$. Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{\dim X+1} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{\dim X+1} \lambda_i = 1, x_i \in X \right\}.$$

As a direct consequence of Carathéodory's Theorem 2.7 we get the following result.

2.9 Corollary. *A polytope is the union of simplices.*

2.10 Corollary. *The convex hull of a compact set is compact.*

2.11 Theorem [(weak)Fractional Helly theorem]. *Let $K_1, \ldots, K_m \in \mathcal{C}^n$, $m \geq n+1$, and let $\alpha \in (0, 1]$ such that for at least $\alpha \binom{m}{n+1}$ of the $(n+1)$-index sets $I \subseteq \{1, \ldots, m\}$ we have $\bigcap_{i \in I} K_i \neq \emptyset$. Then there exists a point in common of at least $\frac{\alpha}{n+1} \cdot m$ sets K_i.*

2.12 Remark. *The (strong and sharp) fractional Helly theorem, which is due to Kalai, gives that* $(1-(1-\alpha)^{1/(n+1)})\cdot m$ *sets have a point in common. Obviously, for* $\alpha = 1$ *we get again the classcial Helly Theorem 2.2.*

2.13 Theorem [Colorful Carathéodory theorem]. *Let* $X_1,\ldots,X_{n+1} \subset \mathbb{R}^n$ *be finite point sets such that* $0 \in \operatorname{conv} X_i$, $1 \le i \le n+1$. *There exist* $x_i \in X_i$, $1 \le i \le n+1$, *such that* $0 \in \operatorname{conv}\{x_1,\ldots,x_{n+1}\}$.

2.14 Theorem [Tverberg's theorem]. *Let* $X \subseteq \mathbb{R}^n$ *and let* $k \in \mathbb{N}_{\ge 1}$. *If* $\#X \ge (k-1)(n+1)+1$, $k \in \mathbb{N}$, *then there exist* k *subsets* $X_1,\ldots,X_k \subset X$ *with* $X_i \cap X_j = \emptyset$, $i \ne j$, *but* $\operatorname{conv} X_1 \cap \operatorname{conv} X_2 \cap \cdots \cap \operatorname{conv} X_k \ne \emptyset$.

2.15 Theorem. *Let* $X \subset \mathbb{R}^n$ *and let* $\#X = m \ge n+1$. *Then there exists a point* $y \in \mathbb{R}^n$ *contained in at least* $\gamma_n\binom{m}{n+1}$ X*-simplices, i.e., simplices of the form* $\operatorname{conv} S$, $S \subseteq X$, $\#S = n+1$. *Here* γ_n *is a positive constant depending only on the dimension, and* X*-simplices* $\operatorname{conv} S_1$, $\operatorname{conv} S_2$ *are considered different if* $S_1 \ne S_2$.

3 Polytopes

3.1 Definition [Polyhedron]. *The intersection of finitely many closed halfspaces is called a* polyhedron.

3.2 Theorem [Minkowski, 1896, Weyl, 1935].

i) *A bounded polyhedron is a polytope.*

ii) *A polytope is a bounded polyhedron.*

3.3 Notation [$\mathcal{V}$-Polytope, $\mathcal{H}$-Polytope]. *A polytope given as the convex hull of finitely many points is called a* $\mathcal{V}$-polytope. *If it is given as the bounded intersection of finitely many closed halfspaces, then it is called an* $\mathcal{H}$-polytope.

3.4 Corollary. *Let $P \in \mathcal{P}^n$.*

i) *Let $A \in \mathbb{R}^{m\times n}$ and $t \in \mathbb{R}^m$. Then $AP + t$ is a polytope.*

ii) *Let $U \subset \mathbb{R}^n$ be an affine subspace. Then $P \cap U$ is a polytope.*

3.5 Definition [Faces]. *Let $K \in \mathcal{C}^n$ be closed and let H be a supporting hyperplane of K. If $j = \dim(K \cap H)$, then $K \cap H$ is called a j-*face *of K. Moreover, K itself is regarded as a $(\dim K)$-face and the empty set $\emptyset$ as (-1)-face of K.*

3.6 Notation [Vertices, edges, facets]. *A 0-face of $K \in \mathcal{C}^n$, K closed, is called* vertex, *an 1-face is called* edge *and a $(\dim K - 1)$-face is called* facet *of K. K itself and the empty set are called* improper faces, *whereas the remaining faces are called* proper faces *of K.*

The set of all vertices of a polytope P is denoted by $\mathrm{vert}\, P$.

3.7 Remark.

i) *Let $K \in \mathcal{C}^n$ be closed. Every (relative) boundary point of K lies in a suitable j-face, $0 \le j \le \dim K - 1$.*

ii) *Let $K \in \mathcal{C}^n$, $\dim K = n$. Let F be a facet of K and H a supporting hyperplane of K with $F = K \cap H$. Then $H = \mathrm{aff}\, F$.*

3.8 Proposition. *Each face of a polytope is a polytope, and a polytope has only finitely many faces.*

3.9 Definition [f-vector]. *For $P \in \mathcal{P}^n$ let $f_i(P)$ be the number of i-faces of P, $-1 \le i \le \dim P$. Furthermore, let $f_i(P) = 0$ for $\dim P + 1 \le i \le n$. The vector $f(P)$ with entries $f_i(P)$, $-1 \le i \le n$, is called the f-*vector *of P.*

3.10 Remark.

i) *Let* $T_n = \mathrm{conv}\,\{0, e_1, \ldots, e_n\}$ *be the so called* standard simplex. *Then* $f_i(T_n) = \binom{n+1}{i+1}$, *i.e., any* $(i+1)$ *subset of the vertices are the vertices of an* i*-face.*

ii) *For any* n*-polytope* $P \in \mathcal{P}^n$ *we have* $\sum_{i=-1}^{n} f_i(P) \geq 2^{n+1}$ *with equality if and only if* P *is an* n*-simplex.*

3.11 Lemma. *Let* $P \in \mathcal{P}^n$.

i) $v \in \mathrm{vert}\,P$ *can not be written as a convex combination of two other points of* P, *i.e.,* $v \notin \mathrm{conv}\,\big(P \backslash \{v\}\big)$.

ii) *If* $P = \mathrm{conv}\,W$, *then* $\mathrm{vert}\,P \subseteq W$.

iii) $P = \mathrm{conv}\,(\mathrm{vert}\,P)$.

3.12 Lemma. *Let* $P \in \mathcal{P}^n$ *be an* n*-polytope with* $0 \in \mathrm{int}\,P$. *For a proper face* F *of* P *let*

$$F^\diamond = \{y \in P^\star : \langle x, y\rangle = 1 \text{ for all } x \in F\}\,.$$

Then

i) $F^\diamond$ *is a face of* $P^\star$.

ii) $F = (F^\diamond)^\diamond$.

iii) *If* G *is a face of* P *and* $F \subseteq G$, *then* $G^\diamond \subseteq F^\diamond$.

iv) $\dim F^\diamond = n - 1 - \dim F$.

3.13 Theorem. *Let* $P \in \mathcal{P}^n$ *be an* n*-polytope with* $0 \in \mathrm{int}\ P$. *Then*

$$f_{n-1-i}(P^\star) = f_i(P), \quad -1 \leq i \leq n.$$

3.14 Theorem. *Let* $P \in \mathcal{P}^n$ *be an* n*-polytope with facets* $F_1, \ldots, F_m$ *and let* $H(a_i, \alpha_i)$, $1 \leq i \leq m$, *be the supporting hyperplanes of* F_i, $1 \leq i \leq m$. *Then*

$$P = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \leq \alpha_i,\ 1 \leq i \leq m\}\,.$$

3.15 Theorem. *Let* $P \in \mathcal{P}^n$ *be an* n*-polytope.*

i) *The boundary of* P *is the union of all its facets.*

ii) *A* k*-face is the intersection of (at least)* $(n-k)$ *facets.*

iii) *An* $(n-2)$*-face is contained in exactly two facets.*

iv) *If* F, G *are faces of* P *with* $F \subseteq G$, *then* F *is a face of* G.

v) *A face of* P *is also a face of a facet of* P.

3.16 Theorem. *Let $P \in \mathcal{P}^n$ be an n-polytope.*

i) *Let G be a face of P and let F be a face of G. Then F is a face of P.*

ii) *Let F_j be a j-face of P and let F_k be a k-face of P with $F_j \subset F_k$. There exist i-faces F_i of P, $j < i < k$, such that*

$$F_j \subset F_{j+1} \subset \cdots \subset F_{k-1} \subset F_k.$$

3.17 Remark. *Let v_0 be a vertex of an n-polytope P and let $\{v_1, \ldots, v_r\}$ be all* adjacent *vertices of v_0, i.e., $\operatorname{conv}\{v_0, v_i\}$ is an edge of P. In other words, $\{v_1, \ldots, v_r\}$ are the neighbours of v_0. Then*

i)
$$P \subset v_0 + \operatorname{pos}\{v_1 - v_0, \ldots, v_r - v_0\}.$$

ii) *Let $c \in \mathbb{R}^n$ with $\langle c, v_0 \rangle \geq \langle c, v_i \rangle$, $1 \leq i \leq r$. Then*

$$\max\{\langle c, x \rangle : x \in P\} = \langle c, v_0 \rangle .$$

3.18 Theorem [Euler-Poincaré formula]. *Let $P \in \mathcal{P}^n$. Then*

$$\sum_{i=-1}^{n} (-1)^i f_i(P) = 0. \tag{3.18.1}$$

In particular, in the 3-dimensional case, i.e., $\dim P = 3$, it holds $f_0 - f_1 + f_2 = 2$.

3.19 Proposition. *The Euler-Poincaré formula is the only linear equation satisfied by the f-vector, i.e., let $\lambda_i \in \mathbb{R}$, such that $\sum_{i=-1}^{n} \lambda_i f_i(P) = 0$ for all $P \in \mathcal{P}^n$. Then there exists a constant $\gamma \in \mathbb{R}$, such that $\lambda_i = \gamma (-1)^i$.*

3.20 Definition [Simple and simplicial polytopes]. *Let $P \in \mathcal{P}^n$.*

i) *P is called* simplicial *if all proper faces are simplices.*

ii) *P is called* simple *if every vertex is contained in exactly $\dim P$ many facets.*

3.21 Lemma. *Let $P \in \mathcal{P}^n$ be an n-polytope with $0 \in \operatorname{int} P$. The following statements are equivalent:*

i) *P is simplicial.*

ii) *All facets of P are simplices.*

iii) *$P^\star$ is simple.*

iv) *Every k-face of $P^\star$ is contained in exactly $n-k$ facets for $k = 0, \ldots, n-1$.*

3.22 Theorem. *Let $P \in \mathcal{P}^n$ be a simple n-polytope. Then*

i) *Every vertex is contained in exactly $\binom{n}{k}$ k-faces of P, $k = 0, \dots, n-1$.*

ii) *The intersection of $k \le n$ facets containing a common vertex is an $(n-k)$-face of P.*

iii) *Let $v_1, \dots, v_n$ be the neighbours of a vertex v_0 of P. For each subset of $k < n$ neighbours $v_{i_1}, \dots, v_{i_k}$ there exists a unique k-face F of P containing $v_0, v_{i_1}, \dots, v_{i_k}$.*

iv) *A face of a simple polytope is simple.*

v) *Every j face of P is contained in exactly $\binom{n-j}{k-j}$ k faces of P.*

3.23 Theorem. *Let $P \in \mathcal{P}^n$ be a simple n-polytope.*

i) $n\, f_0(P) = 2\, f_1(P)$.

ii) $\sum_{k=0}^{n} f_k(P) \le 2^n f_0(P)$.

iii) $f_0(P) \le 2 f_{\lceil n/2 \rceil}(P)$.

Here, for $\rho \in \mathbb{R}$ the number $\lceil \rho \rceil$ is the smallest integer greater or equal than ρ.

3.24 Corollary. *Let P be a simple n-polytope with m facets. Then*

$$f_0(P) \le 2\binom{m}{\lfloor n/2 \rfloor} \quad \text{and} \quad \sum_{k=0}^{n} f_k(P) \le 2^{n+1}\binom{m}{\lfloor n/2 \rfloor}.$$

Or equivalently: Let P be a simplicial n-polytope with m vertices. Then

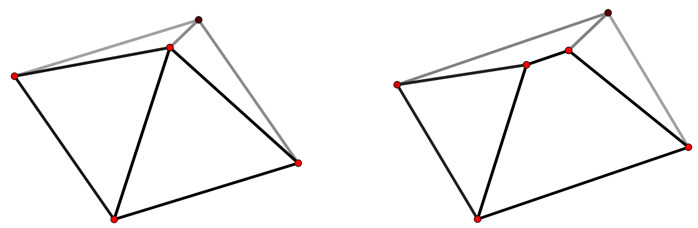
$$f_{n-1}(P) \le 2\binom{m}{\lfloor n/2 \rfloor} \quad \text{and} \quad \sum_{k=0}^{n} f_k(P) \le 2^{n+1}\binom{m}{\lfloor n/2 \rfloor}.$$

Here, for $\rho \in \mathbb{R}$ the number $\lfloor \rho \rfloor$ is the largest integer not greater than ρ.

3.25 Lemma. *Let P be an n-polytope.*

i) *There exists a simple n-polytope Q with the same number of facets as P and $f_i(P) \le f_i(Q)$, $0 \le i \le n-2$.*

ii) *There exists a simplical n-polytope $Q^\star$ with the same number of vertices as P and $f_i(P) \le f_i(Q^\star)$, $1 \le i \le n-1$.*



3.26 Corollary. *Let P be an n-polytope with m facets. Then*

$$f_0(P) \leq 2\binom{m}{\lfloor n/2 \rfloor}.$$

Or equivalently: Let P be an n-polytope with m vertices. Then

$$f_{n-1}(P) \leq 2\binom{m}{\lfloor n/2 \rfloor}.$$

3.27 Definition [Cyclic polytopes]. *The curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ given by $\gamma(t) = (t, t^2, t^3, \ldots, t^n)^\intercal$ is called* moment curve. *The convex hull of m points on the moment curve is called a* cyclic polytope *with m vertices and is denoted by $C(n, m)$.*

3.28 Proposition. *Any $n + 1$ points on the moment curve are affinely independent. In particlar, cyclic polytopes are simplicial polytopes.*

3.29 Proposition [Gale's evenness condition]. *Let $t_i \in \mathbb{R}$, $1 \leq i \leq m$, $t_1 < t_2 < \cdots < t_m$, $\gamma(t_i) = (t_i, t_i^2, t_i^3, \ldots, t_i^n)^\intercal$, $1 \leq i \leq m$, and let $S \subset \{1, \ldots, m\}$ be a subset of cardinality n. $F_S = \operatorname{conv}\{\gamma(t_s) : s \in S\}$ is a facet of $C(n, m)$ if and only if $\#\{s \in S : i < s < j\}$ is even for all $i, j \in \{1, \ldots, m\} \setminus S$.*

3.30 Remark. *All points $\gamma(t_i)$ are vertices of $C(n, m)$ and the number of i-faces of $C(n, m)$ is independent of the choice of the m-points on the moment curve.*

3.31 Proposition. *The cyclic polytope $C(n, m)$ is $\lfloor n/2 \rfloor$-neighborly, i.e., the convex hull of any subset of the vertices of cardinality less than or equal $n/2$ is a face.*

3.32 Theorem* [McMullen's Upper Bound Theorem, 1971]. *Let P be an n-polytope with m vertices. Then*

$$f_i(P) \leq f_i(C(n, m)) = \begin{cases} \sum_{j=0}^{(n-1)/2} \frac{i+2}{m-j}\binom{m-j}{j+1}\binom{j+1}{i+1-j}, & n \text{ odd}, \\ \sum_{j=1}^{n/2} \frac{m}{m-j}\binom{m-j}{j}\binom{j}{i+1-j}, & n \text{ even}. \end{cases}$$

In particular,

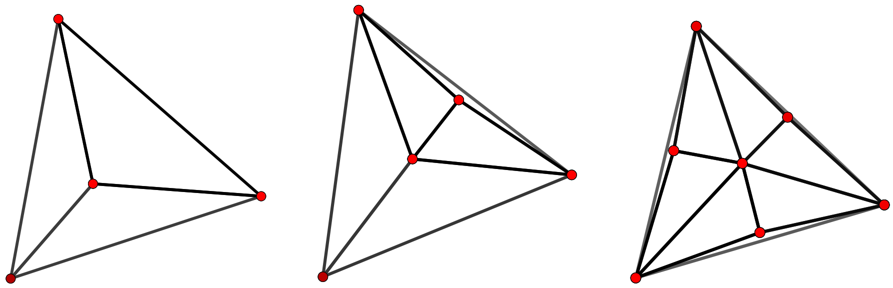
$$f_{n-1}(P) \leq f_{n-1}(C(n, m)) = \begin{cases} 2\binom{m-\lfloor n/2 \rfloor - 1}{\lfloor n/2 \rfloor}, & n \text{ odd}, \\ \binom{m-\lfloor n/2 \rfloor}{\lfloor n/2 \rfloor} + \binom{m-\lfloor n/2 \rfloor - 1}{\lfloor n/2 \rfloor - 1}, & n \text{ even}. \end{cases}$$

For fixed n the right hand sides are of order $m^{\lfloor n/2 \rfloor}$.

3.33 Theorem* [Barnette's Lower Bound Theorem, 1971]. *Let P be a simplicial n-polytope with m vertices. P has at least as many i-faces as the so called* stacked polytopes *$P(n, m)$ with m vertices for which*

$$f_i(P(n, m)) = \begin{cases} m\binom{n}{i} - i\binom{n+1}{i+1}, & 0 \leq i \leq n - 2, \\ n + 1 + (m - (n + 1))(n - 1), & i = n - 1. \end{cases}$$

$P(n, n+1)$ is an n-simplex, and for $m \geq n+2$ an m-vertex stacked n-polytope $P(n, m)$ is the convex hull of an $(m-1)$-vertex stacked polytope with an additional point that is beyond exactly one facet.



3.34 Theorem [Dehn-Sommerville equations, 1905, 1927]. *Let P be a simple n-polytope. Then*

$$f_i(P) = \sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} f_j(P), \quad i = 0, \ldots, n,$$

Or equivalently: Let P be a simplicial n-polytope. Then

$$f_{i-1}(P) = \sum_{j=i}^{n} (-1)^{n-j} \binom{j}{i} f_{j-1}(P), \quad i = 0, \ldots, n.$$

3.35 Definition [h-vector]. *Let $P \in \mathcal{C}^n$ be a simple n-polytope. The vector $h(P) = (h_0(P), \ldots, h_n(P))$ with entries*

$$h_i(P) = \sum_{j=0}^{n} (-1)^{i-j} \binom{j}{i} f_j(P)$$

*is called h-*vector *of P.*

3.36 Remark. *In terms of the h-vector, the Dehn-Sommerville equations become $h_i(P) = h_{n-i}(P)$, $i = 0, \ldots, n$.*

3.37 Theorem* [McMullen's g-Theorem]. *McMullen's g-theorem gives a complete characterization of the f-vectors of simple (or simplicial) polytopes in terms of its g-vector which is given by $g_i(P) = h_i(P) - h_{i-1}(P)$, $i = 1, \ldots, \lfloor n/2 \rfloor$.*

3.38 Remark. *For any n-polytope $P \in \mathcal{P}^n$ we have $n f_0(P) \leq 2 f_1(P)$ with equality iff P simple and $n f_{n-1}(P) \leq 2 f_{n-2}(P)$ with equality iff P simplicial.*

3.39 Theorem [Steinitz, 1906]. *A non-negative integral vector (f_0, f_1, f_2) is the f-vector of a 3-polytope if and only if* i) *$f_0 - f_1 + f_2 = 2$,* ii) *$3 f_0 \leq 2 f_1$, and* iii) *$3 f_2 \leq 2 f_1$.*

3.40 Theorem [Figiel, Lindenstrauss, Milman, 1977]. *Let $P \in \mathcal{P}^n$ be a 0-symmetric n-polytope, i.e., $P = -P$. Then*

$$\ln(f_0(P)) \ln(f_{n-1}(P)) \geq \frac{1}{16} n.$$

3.41 Conjecture [Kalai, 1989]. *Let $P \in \mathcal{P}^n$ be a 0-symmetric n-polytope. Then*

$$\sum_{i=0}^{n} f_i(P) \geq 3^n.$$

Here we have equality, for instance, for the cube C_n and its polar, the cross-poyltope $C_n^\star$, or, more generally, for the class of Hanner-polytopes. Recently, the conjecture has been verified for all $n \leq 4$ (see `http://front.math.ucdavis.edu/0708.3661`).

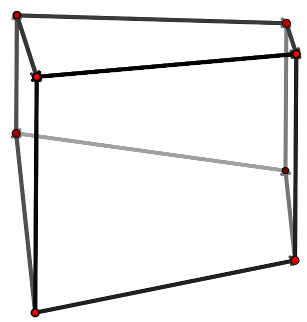
3.42 Theorem [Balinski, 1961]. *Let $P \in \mathcal{P}^n$ be an n-polytope. The graph of P consisting of its vertices and edges is n-connected, i.e., the graph is still connected if $n-1$ vertices and their adjacent edges are removed.*

3.43 Definition [Combinatorial diameter]. *Let $P \in \mathcal{P}^n$. The* combinatorial distance *$\delta_P(v, w)$ between two vertices $v, w \in \operatorname{vert} P$ is the minimum length of an "edge path" connecting v and w, i.e., the minimum length of a sequence $[v, v_{i_1}, \ldots, v_{i_l}, w]$, with $v_{i_j} \in \operatorname{vert} P$ and two consecutive vertices form an edge.*

$\delta(P) = \max\{\delta_P(v, w) : v, w \in \operatorname{vert} P\}$ is called the (combinatorial) diameter *of P. For $n, m \in \mathbb{N}$ let*

$$\Delta(n, m) = \max\left\{\delta(P) : P \in \mathcal{P}^n, \dim P = n \text{ and } f_{n-1}(P) = m\right\}.$$

3.44 Example. *$\delta(T_n) = 1$, $\delta(C_n) = n$ and $\delta(C_n^\star) = 2$.*



3.45 Conjecture [Hirsch, 1957]. *$\Delta(n, m) \leq m - n$.*

3.46 Remark. *It is known that*

i) *the conjecture is true if $n \leq 3$ or $m \leq n + 5$ (Klee&Walkup, 1961/1965),*

ii) *the conjecture would be false for unbounded polyhedra,*

iii) $\Delta(n,m) \leq \frac{1}{3}\,2^{n-2}(m-n+\frac{5}{2})$ *(Barnette, 1974),*

iv) $\Delta(n,m) \leq 2m^{\log(n)+1}$ *(Kalai, 1992),*

v) *it suffices to prove the conjecture for simple polytopes with* $m = 2\,n$ *(Klee&Walkup, 1961/1965)!*

vi) *Disproof of the Hirsch conjecture by Francisco Santos, 2010, see* `http://front.math.ucdavis.edu/1006.2814`

3.47 Definition [0/1**-polytope**]**.** *Let* $[0,1]^n$ *be the* n*-dimensional unit cube with vertices* $\{0,1\}^n = \{(x_1,\ldots,x_n)^\intercal : x_i \in \{0,1\}\}$. $P \in \mathcal{P}^n$ *is called* 0/1-polytope *if* $\operatorname{vert} P \subset \{0,1\}^n$.

3.48 Lemma. *Let* $P \in \mathcal{P}^n$ *be a* 0/1*-polytope and let* $\dim P \leq n-1$. *Then there exists a* 0/1*-polytope* $\widetilde{P} \in \mathcal{P}^{n-1}$ affinely isomorphic *to* P, *i.e., there exists a bijective map between* P *and* $\widetilde{P}$.

3.49 Theorem [Naddef, 1989].

i) *Let* P *be a* 0/1*-polytope. Then* $\delta(P) \leq \dim P$.

ii) *Let* $P \in \mathcal{P}^n$ *be an* n*-dimensional* 0/1*-polytope with* m *facets. Then* $\delta(P) \leq m-n$.

3.50 Remark.

i) $f_{n-1}(P) \leq 2\,n!$ *for a* 0/1*-polytope* $P \in \mathcal{P}^n$.

ii) *There exist* 0/1*-polytopes* $P \in \mathcal{P}^n$ *with*

$$f_{n-1}(P) \geq \left(\frac{c\,n}{\log^2 n}\right)^{\frac{n}{2}},$$

where c *is a universal constant (Gatzouras, Giannopoulos, Markoulakis, 2004).*

4 A bit of Gale diagrams and triangulations

4.1 Definition. *Let $a_1,\dots,a_m \in \mathbb{R}^n$, $m \geq n+1$, such that their affine hull spans $\mathbb{R}^n$, and let $\overline{a}_i = (a_i,1)^\intercal \in \mathbb{R}^{n+1}$, $1 \leq i \leq m$, and let $\overline{A} = (\overline{a}_1,\dots,\overline{a}_m) \in \mathbb{R}^{(n+1)\times m}$. Let $\overline{B} = (\overline{b}_1,\dots,\overline{b}_{m-(n+1)}) \in \mathbb{R}^{m\times(m-(n+1))}$ be a basis of the subspace $\{x \in \mathbb{R}^m : \overline{A}\,x = 0\}$, i.e., of the kernel of $\overline{A}$, and let $\overline{B}^\intercal = (b_1,\dots,b_m) \in \mathbb{R}^{(m-(n+1))\times m}$.*

The vectors $b_1,\dots,b_m \in \mathbb{R}^{m-(n+1)}$ are called the Gale transform *of the points $a_1,\dots,a_m$.*

4.2 Remark.

i) *The Gale transform is unique up to linear isomorphisms of $\mathbb{R}^{(m-(n+1))}$.*

ii) *For the vectors of a Gale transform always hold $\sum_{i=1}^m b_i = 0$.*

4.3 Proposition. *Let $P = \operatorname{conv}\{v_1,\dots,v_m\}$ with vertices v_i, $1 \leq i \leq m$, and for $J \subseteq [m] = \{1,\dots,m\}$ let $V_J = \{v_j : j \in J\}$. Then $\operatorname{conv} V_J$ is a face of P if and only if $\operatorname{conv} V_{[m]\setminus J} \cap \operatorname{aff} V_J = \emptyset$.*

For $\#J = 1$ the statement is also true without the assumption that v_i are vertices.

4.4 Theorem. *Let $P = \operatorname{conv}\{v_1,\dots,v_m\}$, $\dim P = n$, with vertices v_i, $1 \leq i \leq m$, and for $J \subseteq [m] = \{1,\dots,m\}$ let $V_J = \{v_j : j \in J\}$. Let $\{b_1,\dots,b_m\}$ be the Gale transform of $\{v_1,\dots,v_m\}$. Then $\operatorname{conv} V_J$ is a face of P if and only if $J = [m]$ or $0 \in \operatorname{relint}(\operatorname{conv}\{b_k : k \notin J\})$.*

For $\#J = 1$ the statement is also true without the assumption that v_i are vertices.

4.5 Corollary. *$\{b_1,\dots,b_m\} \in \mathbb{R}^{m-(n+1)}$ is the Gale transform of the vertex set $\{v_1,\dots,v_m\} \in \mathbb{R}^n$ of a polytope if and only if for each hyperplane $H(a,0) \subset \mathbb{R}^{m-(n+1)}$ the halfspaces $H^+(a,0)$ and $H^-(a,0)$ contain at least two points of $\{b_1,\dots,b_m\}$ in its interior.*

4.6 Definition [Face lattice]. *For a polytope $P \in \mathcal{P}^n$ let $\mathcal{F}(P)$ be the set of all its faces. Together with the inclusion relation "$\subseteq$" on $\mathcal{F}(P)$ the faces form a* partially ordered set (poset) *denoted by $(\mathcal{F}(P),\subseteq)$ and which is called the* face lattice *of P.*

4.7 Definition [Combinatorially isomorphic]. *Two polytopes $P,Q \in \mathcal{P}^n$ are called* combinatorially isomorphic *or* combinatorially equivalent *if the face lattices $(\mathcal{F}(P),\subseteq)$ and $(\mathcal{F}(Q),\subseteq)$ are isomorphic, i.e., there exists an inclusion preserving bijection between the faces of P and Q.*

4.8 Theorem. *There are precisely $\lfloor \frac{1}{4}n^2 \rfloor$ combinatorial different types of n-polytopes with $n+2$ vertices.*

Proof. For such an n-polytope $P = \operatorname{conv}\{v_1, \dots, v_{n+2}\}$ the Gale transform $\{b_1, \dots, b_{n+2}\}$ is one-dimensional. Since $0 \in \operatorname{int}\operatorname{conv}\{b_1, \dots, b_{n+2}\}$ we can classify the Gale transform by the number of points l, e, g which are < 0, $= 0$ and > 0, respectively. Then we have $l + e + g = n + 2$ and according to Corollary 4.5 we have $l, g \geq 2$ and by symmetry we may assume $l \leq g$. So for a fixed $e \in \{0, \dots, n-2\}$ the parameter l is bounded between 2 and $\lfloor (n+2-e)/2 \rfloor$, and so there are $\lfloor (n-e)/2 \rfloor$ possibilities. Altogether we have found

$$\sum_{e=0}^{n-2} \left\lfloor \frac{n-e}{2} \right\rfloor = \left\lfloor \frac{1}{4} n^2 \right\rfloor$$

different possibilities of Gale transforms which correspond by Corollary 4.5 to all n polytopes with $(n+2)$ vertices. It remains to show that these polytopes are combinatorially inequivalent. To this end let (l_i, e_i, g_i) be two different sequences with polytopes P_i, $i = 1, 2$, and assume that the polytopes are combinatorially equivalent. Let ϕ be an isomorphism between the face lattices of P_i which induces an isomorphism on the points of the Gale transforms. First we observe that all points of the groups "e_i" have to be mapped onto each other, since removing all other points yields a face of the polytope in view of Theorem 4.4. So we may assume $e_1 = e_2$ and suppose $1 < l_1 < l_2 \leq g_2 < g_1$. The image of the points of the first group "l_1" cannot consists of all points of l_2 or of all points of g_2. Hence removing the image of the points yields a face of P_2 but not of P_1 according to Theorem 4.4. Hence we get a contradiction. □

4.9 Definition [Point configuration]. *A point configuration in $\mathbb{R}^n$ is a finite set of (perhaps repeated) points with (non-repeated) labels. It is identified with its matrix $A = (a_1, \dots, a_m) \in \mathbb{R}^{n \times m}$.*

4.10 Definition [Triangulation]. *A* triangulation *of a point configuration $A = (a_1, \dots, a_m) \subset \mathbb{R}^{n \times m}$ is a collection $\mathcal{T}$ of simplices, with vertices in A, that satisfies the following properties:*

i) *All faces of simplices of $\mathcal{T}$ are in $\mathcal{T}$.*

ii) *The intersection of any two simplices of $\mathcal{T}$ is a face (possible empty) of both.*

iii) *The union of all simplices of $\mathcal{T}$ equals* conv A.

4.11 Remark.

i) *All vertices of* conv A *are in $\mathcal{T}$.*

ii) *Let F be a face of* conv A. *Then $\mathcal{T}_F := \{\sigma \in \mathcal{T} : \sigma \subset F\}$ is a triangulation of $A \cap F$.*

iii) *The frist two properties of Definition 4.10 are the definition of a (geometric)* simplicial complex.

4.12 Definition [Height function and lower convex hull]. *Let $A = (a_1, \ldots, a_m) \in \mathbb{R}^{n\times m}$ be a point configuration. Let $\eta : A \to \mathbb{R}_{\geq 0}$ be a function and let*

$$A_\eta = \begin{pmatrix} a_1 & a_2 & \ldots & a_m \\ \eta(a_1) & \eta(a_2) & \ldots & \eta(a_m) \end{pmatrix} \in \mathbb{R}^{(n+1)\times m}$$

be the lifted point configuration *(also called a lifting). Then η is called a height function . The union of the faces of* conv A_η *which are visible form below are called the* lower convex hull *of* conv A_η*. More precisely, $y = (y_1, \ldots, y_n, y_{n+1})^\intercal \in$* conv (A_η) *belongs to the lower convex hull if the segment* conv $\{y, (y_1, \ldots, y_n, 0)^\intercal\}$ *intersects* conv A_η *only in y.*

4.13 Definition [Regular triangulation]. *A triangulation of a point configuration $A \in \mathbb{R}^{n\times m}$ is called a* regular triangulation *if it can be obtained by projecting the lower convex hull of a lifting of A.*

4.14 Theorem. *Every point configuration $A \in \mathbb{R}^{n\times m}$ has regular triangulations.*

4.15 Definition [Delaunay triangulation]. *Let $A \in \mathbb{R}^{n\times m}$ be a point configuration containing $n + 1$ affinely independent points. Let $\eta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be the height function given by $\eta(x) = |x|^2$. If the projection of the lower convex hull of* conv A_η *yields a triangulation, then it is called* Delaunay triangulation.

4.16 Lemma. *Let $C \subset \mathbb{R}^{n+1}$ be the paraboloid $C = \{(x, |x|^2)^\intercal : x \in \mathbb{R}^n\}$. Let $H(a, \alpha)$, $a \in \mathbb{R}^{n+1}$, $a_{n+1} \neq 0$, be a hyperplane, and let $\Pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection forgetting the last coordinate. Then $\Pi(C \cap H(a, \alpha))$ is either empty, a point or an n-dimensional sphere.*

4.17 Corollary. *Let $A \in \mathbb{R}^{n\times m}$ be a point configuration containing $n + 1$ affinely independent points. Let $\eta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be the height function given by $\eta(x) = |x|^2$. Let $B \subseteq A$ such that B contains $n + 1$ affinely independent points. Then B corresponds to the vertex set of a facet of the lower convex hull of the lifted points if and only if there is sphere passing through all points of B and leaving all points of $A \setminus B$ outside.*

4.18 Remark. *In particular, simplices in a Delaunay triangulation $\mathcal{T}$ are characterized by the "empty sphere" property: $\sigma \in \mathcal{T}$ if and only if there is an Euclidean sphere with all vertices of σ on the sphere and with the rest of the points outside.*

5 A glimpse of Ehrhart theory

5.1 Notation. *For $S \subset \mathbb{R}^n$, S bounded, we denote by* $\mathrm{G}(S)$ *its* lattice point enumerator, *i.e.,*

$$\mathrm{G}(S) = \#(S \cap \mathbb{Z}^n).$$

Almost all results of this section are valid for arbitrary lattices; for simplification, however, we state most of them only for the standard lattice $\mathbb{Z}^n$.

5.2 Definition [Lattice polytope]. *A polytope $P = \mathrm{conv}\{v_1, \dots, v_m\} \subset \mathbb{R}^n$ is called a* lattice polytope *if $v_i \in \mathbb{Z}^n$, $1 \le i \le m$. The set of all lattice polytopes is denoted by $\mathcal{P}^n_{\mathbb{Z}}$.*

5.3 Lemma. *Let $a_1, \dots, a_n \in \mathbb{Z}^n$ be linearly independent, and let P be the half open parallelepiped $P = \{\rho_1 a_1 + \cdots + \rho_n a_n : 0 \le \rho_i < 1\}$. Then*

$$\mathrm{G}(P) = \mathrm{vol}\,(P) = |\det(a_1, \dots, a_n)|.$$

5.4 Lemma. *Let $T = \mathrm{conv}\{0, v_1, v_2\} \in \mathcal{P}^2_{\mathbb{Z}}$ be a lattice triangle, i.e., $v_1, v_2 \in \mathbb{Z}^2$ are linearly independent. Then*

$$\mathrm{G}(T) = \mathrm{vol}\,(T) + \frac{1}{2}\mathrm{G}(\mathrm{bd}\,T) + 1.$$

5.5 Theorem [Pick]. [1] *Let $P \in \mathcal{P}^2_{\mathbb{Z}}$, $\dim P = 2$. Then*

$$\mathrm{G}(P) = \mathrm{vol}\,(P) + \frac{1}{2}\mathrm{G}(\mathrm{bd}\,P) + 1.$$

5.6 Corollary.

i) *Let $P \in \mathcal{P}^2_{\mathbb{Z}}$, $\dim P = 2$, with edges $F_1, \dots, F_m$. Then*

$$\mathrm{G}(P) = \mathrm{vol}\,(P) + \frac{1}{2}\sum_{i=1}^{m} \frac{\mathrm{vol}_1(F_i)}{\det(\mathrm{aff}\,F_i \cap \mathbb{Z}^2)} + 1.$$

Here $\det(\mathrm{aff}\,F_i \cap \mathbb{Z}^2)$ is the distance of two consecutive lattice points on $\mathrm{aff}\,F_i$.

ii) *Let $K \in \mathcal{K}^2$. Then $\mathrm{G}(K) \le \mathrm{vol}\,(K) + \frac{1}{2}\mathrm{F}(K) + 1$, where $F(K)$ denotes the perimeter of K.*

5.7 Remark.

[1] Georg Alexander Pick; 10.08.1859 (Vienna) – 26.07.1942 (concentration camp Theresienstadt)

i) *Inequality ii) of Corollary 5.6 can not be generalized in a straightforward and best possible way to arbitrary lattices, since the perimeter is not an affine equivariant functional, in contrast to the area.*

ii) *Pick's theorem itself, however, can be generalized in various ways. Its proof is only based on the property that a set S can be subdivided into lattice triangle such that the intersection of any two of them is a face of both. For those sets it was shown by Hadwiger&Wills that*

$$\mathrm{G}(S) = \mathrm{vol}\,(S) + \frac{1}{2}\mathrm{E}(S) + \chi(S).$$

Here $\chi(S)$ is the Euler-Poincaré characteristic of S, and $\mathrm{E}(S)$ is the number of segments between two consecutive lattice points in the boundary of S, where segments are counted twice which are not bordering a 2-dimensional cell.

iii) *By Pick's theorem we immediately get the following polynomial behaviour of the lattice point enumerator of a lattice polygon $P \in \mathcal{P}^2_{\mathbb{Z}}$*

$$\begin{aligned}\mathrm{G}(k\,P) &= \mathrm{vol}\,(P)\,k^2 + \frac{1}{2}\,\mathrm{G}(\mathrm{bd}\,P)\,k + 1,\\ \mathrm{G}\big(\mathrm{int}\,(k\,P)\big) &= \mathrm{vol}\,(P)\,k^2 - \frac{1}{2}\,\mathrm{G}(\mathrm{bd}\,P)\,k + 1.\end{aligned}$$

5.8 Notation. *For integers m, n we denote by*

$$\binom{x+m}{n} = \frac{1}{n!}\prod_{i=0}^{n-1}(x+m-i)$$

the polynomial of degree n with roots $i-m$, $i = 0, \ldots, n-1$, and leading coefficient $1/n!$. In particular, the polynomials $\binom{x+n-i}{n}$, $i = 0, \ldots, n$, form a basis of the space of all polynomials of degree at most n.

5.9 Lemma. *Let $T = \mathrm{conv}\,\{0, v_1, \ldots, v_n\} \in \mathcal{P}^n_{\mathbb{Z}}$ be a lattice simplex, i.e., $v_1, \ldots, v_n \in \mathbb{Z}^n$ are linearly independent, and for $0 \le i \le n$ let*

$$\mathrm{a}_i(T) = \#\left\{\sum_{j=1}^n \lambda_j v_j \in \mathbb{Z}^n : 0 \le \lambda_j < 1,\ i-1 < \sum_{j=1}^n \lambda_j \le i\right\}.$$

Then for all $k \in \mathbb{N}$, $k \ge 1$, we have

$$\mathrm{G}(k\,T) = \sum_{i=0}^n \mathrm{a}_i(T)\binom{k+n-i}{n}.$$

5.10 Notation [Characteristic Function]. *For a set $A \subseteq \mathbb{R}^n$ let $\chi_A : \mathbb{R}^n \to \{0,1\}$ with*

$$\chi_A(x) = \begin{cases} 1, & x \in A,\\ 0, & x \notin A.\end{cases}$$

be its characteristic function.

5.11 Lemma [Inclusion-Exclusion Formula]. *Let $A_i \subseteq \mathbb{R}^n$, $1 \le i \le m$, with characteristic functions $\chi(A_i)$. Then*

$$\chi(A_1 \cup A_2 \cup \cdots \cup A_m) = \sum_{\substack{I \subseteq \{1,\ldots,m\} \\ I \neq \emptyset}} (-1)^{\#I-1} \chi\left(\bigcap_{j \in I} A_j\right).$$

5.12 Remark. *By evaluating and summing both sides of Lemma 5.11 over all lattice points $\mathbb{Z}^n$ we find for bounded sets $A_i \subset \mathbb{R}^n$, $1 \le i \le m$*

$$\mathrm{G}(A_1 \cup A_2 \cup \cdots \cup A_m) = \sum_{\substack{I \subseteq \{1,\ldots,m\} \\ I \neq \emptyset}} (-1)^{\#I-1} \mathrm{G}\left(\bigcap_{j \in I} A_j\right).$$

5.13 Theorem [Ehrhart, 1967]. [2] *Let $P \in \mathcal{P}^n_{\mathbb{Z}}$. Then there exist numbers $\mathrm{G}_i(P) \in \mathbb{Q}$, $0 \le i \le n$, depending only on P, such that for all $k \in \mathbb{N}_{\ge 1}$*

$$\mathrm{G}(k\,P) = \sum_{i=0}^{n} \mathrm{G}_i(P)\,k^i.$$

The right hand side is called Ehrhart-polynomial.

5.14 Definition.

i) *Let* $\mathrm{GL}(n,\mathbb{Z}) = \{U \in \mathbb{Z}^{n\times n} : |\det U| = 1\}$ *be the group of* unimodular matrices . *For $U \in \mathrm{GL}(n,\mathbb{Z})$ and $z \in \mathbb{Z}^n$ the map $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(x) = U\,x + z$ is called an* unimodular transformation.

ii) *A functional $g : \mathcal{P}^n_{\mathbb{Z}} \to \mathbb{R}$ with*

$$g(P \cup Q) + g(P \cap Q) = g(P) + g(Q)$$

for all $P, Q \in \mathcal{P}^n_{\mathbb{Z}}$ with $P \cup Q, P \cap Q \in \mathcal{P}^n_{\mathbb{Z}}$ is called additive.

5.15 Proposition. *The unimodular transformations are exactly those affine transformations $f : \mathbb{R}^n \to \mathbb{R}^n$ which leaves $\mathbb{Z}^n$ invariant, i.e., $f(\mathbb{Z}^n) = \mathbb{Z}^n$.*

5.16 Proposition. *Let $P \in \mathcal{P}^n_{\mathbb{Z}}$.*

i) $\mathrm{G}_n(P) = \mathrm{vol}\,(P)$.

ii) *$\mathrm{G}_i : \mathcal{P}^n_{\mathbb{Z}} \to \mathbb{R}$ is homogeneous of degree i, invariant with respect to unimodular transformations and additive, $0 \le i \le n$.*

iii) *$\mathrm{G}_i(P)$ are independent of the dimension of the space in which P is embedded, i.e., let $P \in \mathcal{P}^n_{\mathbb{Z}}$ and let $\tilde{P} = \mathrm{conv}\,\{(v,0)^\intercal : v \in P\} \in \mathcal{P}^{n+1}_{\mathbb{Z}}$. Then $\mathrm{G}_i(P) = \mathrm{G}_i(\tilde{P})$, $i = 0,\ldots,n$.*

[2] Eugéne Ehrhart, 29.04.1906 (Guebwiller (Haut-Rhin, France)) – 17.01.2000 (Strasbourg)

5.17 Theorem* [Betke, Kneser, 1985]. [3] [4] *Every additive and unimodular invariant functional on the space of all lattice polytopes is a linear combination of the $n+1$ functionals $\mathrm{G}_i(\cdot)$.*

5.18 Remark. *Some of the coefficients $\mathrm{G}_i(P)$ might be negative. One family of standard examples in this context are the so called* Reeve-simplices*: let $R_m = \operatorname{conv}\left\{0, e_1, e_2, (1,1,m)^\intercal\right\} \in \mathcal{P}^3_{\mathbb{Z}}$ for $m \in \mathbb{N}$. The only lattice points contained in R_m are the four vertices, the volume, however, can be arbitrarily large. Hence some $\mathrm{G}_i(R_m)$ must be negative for large m. More precisely, it is $\mathrm{G}_3(R_m) = m/6$, $\mathrm{G}_2(R_m) = 1$, $\mathrm{G}_1(R_m) = (12-m)/6$ and $\mathrm{G}_0(R_m) = 1$.*

Of course, we can also rewrite the Ehrhart polynomial in terms of the binomial basis $\binom{k+n-i}{n}$ and get

5.19 Corollary. *Let $P \in \mathcal{P}^n_{\mathbb{Z}}$. Then there exist numbers $\mathrm{a}_i(P) \in \mathbb{Q}$, $0 \le i \le n$, such that for all $k \in \mathbb{N}_{\ge 1}$*

$$\mathrm{G}(k\,P) = \sum_{i=0}^{n} \mathrm{a}_i(P) \binom{k+n-i}{n}.$$

5.20 Example.

i) *Let $T_n = \operatorname{conv}\{0, e_1, e_2, \ldots, e_n\}$. Then*

$$\#(k\,T_n \cap \mathbb{Z}^n) = \binom{n+k}{n},$$

and so we have $\mathrm{a}_i(T_n) = 0$ for $1 \le i \le n$, and $\mathrm{a}_0(T_n) = 1$. The $\mathrm{G}_i(T_n)$ are – up to ± 1 – Stirling numbers of the first kind.

ii) *Let $C_n = [-1,1]^n$. Then $\mathrm{G}(k\,C_n) = (2\,k+1)^n$ and so $\mathrm{G}_i(C_n, \mathbb{Z}^n) = 2^i\binom{n}{i}$, $0 \le i \le n$. Here the $\mathrm{a}_i(C_n)$ are some combinatorial numbers, the so called Eulerian numbers.*

iii) *Let $C_n^\star = \operatorname{conv}\{\pm e_i : 1 \le i \le n\}$. Then*

$$\mathrm{G}(k\,C_n^\star) = \sum_{i=0}^{n} \binom{n}{i}\binom{k+n-i}{n},$$

and so $\mathrm{a}_i(C_n^\star) = \binom{n}{i}$, $i = 0, \ldots, n$.

[3] Martin Kneser, 21.01.1928 (Greifswald) – 16.02.2004 (Göttingen)

[4] Ulrich Betke, ??.??.1948 (???) – 24.05.2008 (Siegen)

iv) *Let* $\widehat{T_2} = \operatorname{conv}\{0, e_1, e_2\} \subset \mathbb{R}^3$ *be the 2-dimensional standard triangle embedded in* $\mathbb{R}^3$*. Then by Pick's Theorem 5.5*

$$\mathrm{G}(k\,\widehat{T_2}) = \frac{1}{2}\,k^2 + \frac{3}{2}\,k + 1 = (-1)\binom{k}{3} + \frac{1}{2}\binom{k+1}{3} - \frac{1}{2}\binom{k+2}{3} + \binom{k+3}{3}.$$

Hence, in this case we have $\mathrm{a}_3(\widehat{T_2}) = -1$, $\mathrm{a}_2(\widehat{T_2}) = 1/2$, $\mathrm{a}_1(\widehat{T_2}) = -1/2$, $\mathrm{a}_0(\widehat{T_2}) = 1$.

Finally, we state without proof some highlights of Ehrhart theory (and which are left to the lecture in the next term).

5.21 Remark.

i) Stanley's Non-Negativity Theorem*: Let* $P \in \mathcal{P}^n_{\mathbb{Z}}$, $\dim P = n$*. Then* $\mathrm{a}_i(P) \in \mathbb{N}_{\geq 0}$, $0 \leq i \leq n$*.*

ii) Stanley's Monotonicity Theorem*: Let* $P, Q \in \mathcal{P}^n_{\mathbb{Z}}$, $\dim P = \dim Q = n$ *with* $P \subseteq Q$*. Then* $\mathrm{a}_i(P, \mathbb{Z}^n) \leq \mathrm{a}_i(Q, \mathbb{Z}^n)$, $0 \leq i \leq n$*.*

iii) Ehrhart-Macdonald Reciprocity*: Let* $P \in \mathcal{P}^n_{\mathbb{Z}}$, $\dim P = n$*. Then*

$$\mathrm{G}(\operatorname{int} k\, P) = (-1)^n \sum_{i=0}^{n} \mathrm{G}_i(P)(-k)^i.$$

iv) *Let* $P \in \mathcal{P}^n_{\mathbb{Z}}$, $\dim P = n$*. Then for* $i \neq n \bmod 2$

$$\mathrm{G}_i(P) = \frac{1}{2}\sum_{j=i}^{n-1}(-1)^{i+j} \sum_{F j-\text{face of } P} \mathrm{G}_i(F).$$

In particular,

$$\mathrm{G}_{n-1}(P) = \frac{1}{2} \sum_{F \text{ facet of } P} \frac{\operatorname{vol}_{n-1}(F)}{\det(\operatorname{aff} F \cap \mathbb{Z}^n)}.$$

6 Borsuk, Hadwiger and planks

6.1 Definition [Diameter]. *Let $X \subset \mathbb{R}^n$ be bounded.*

$$\mathrm{D}(X) = \sup\{|x - y| : x, y \in X\}$$

is called diameter *of X.*

6.2 Remark. *Let $K \in \mathcal{C}^n$ be a convex body with support function $h(K, \cdot)$. Let $x, y \in K$ such that $\mathrm{D}(K) = |x - y|$, and let $u = (x - y)/|x - y|$. The hyperplanes $H(u, \langle u, x\rangle)$ and $H(-u, \langle -u, y\rangle)$ are supporting hyperplanes in x, y, respectively, and it is*

$$\mathrm{D}(K) = \sup_{u \in S^{n-1}} h(K, u) + h(K, -u).$$

6.3 Definition [Convex body, smooth convex body]. *A* convex body *is a convex compact set, $K \in \mathcal{C}^n$. It is called* smooth *if there exists an unique supporting hyperplane for any boundary point of K. The set of all convex bodies is denoted by $\mathcal{K}^n$.*

6.4 Definition [Borsuk number]. *Let $\mathfrak{B}(n)$ be the smallest number $k \in \mathbb{N}$ such that any bounded set $X \subset \mathbb{R}^n$ can be partitioned into k sets having smaller diameter than X.*

6.5 Remark. *A regular simplex shows that $\mathfrak{B}(n) \geq n + 1$ and in 1932 Karol Borsuk[5] asked the question whether $\mathfrak{B}(n) = n + 1$ which led to the so called Borsuk conjecture.*

6.6 Theorem [Borsuk-Ulam]. *Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the boundary of the n-dimensional unit ball B_n. Let $f : S^{n-1} \to \mathbb{R}^n$ be a continuous mapping. Then there exists a $x \in S^n$ with $f(x) = f(-x)$.*

6.7 Proposition. *B_n can be partitioned into $n + 1$ sets of smaller diameter, but not less.*

6.8 Theorem. *Any smooth convex body $K \in \mathcal{C}^n$ can be partitioned into $n+1$ sets of smaller diameter.*

6.9 Proposition. $\mathfrak{B}(n) \leq 2^n$.

6.10 Theorem [Kahn&Kalai]. *For large n it is $\mathfrak{B}(n) \geq (1.1)^{\sqrt{n}}$.*

6.11 Remark. *The smallest known dimension with $\mathfrak{B}(n) > n+1$ is 298. More precisely, Hinrichs&Richter proved $\mathfrak{B}(n) > n + 11$ for all $n \geq 298$.*

[5] Karol Borsuk (8. Mai 1905 in Warschau – 24. Januar 1982 in Warschau)

6.12 Definition [Gohberg-Markus-Hadwiger number]. *Let $\mathfrak{H}(n)$ be smallest number $m \in \mathbb{N}$ such that any convex body $K \subset \mathbb{R}^n$ can be covered by m smaller homothetic copies of K. $\mathfrak{H}(n)$ is called the* Gohberg[6]-Markus-Hadwiger[7] number.

6.13 Remark.

i) *The cube C_n shows that $\mathfrak{H}(n) \geq 2^n$, and it is conjectured that this lower bound is extremal, i.e., $\mathfrak{H}(n) = 2^n$. The problem or conjecture is known as the Gohberg-Markus-Hadwiger problem/conjecture.*

ii) *So far the conjecture has only be confirmed for $n \leq 2$ (and possible for $n \leq 3$) and for some special classes of convex bodies. In particular for smooth bodis, where always $n+1$ smaller homothetic copies suffice.*

iii) *A boundary point x of a convex body $K \in \mathcal{K}^n$ is illuminated by a direction $v \in \mathbb{R}^n$, if the line $x + \mathbb{R}\,v$ intersects the interior of K. The minimal number m needed to illuminate any convex body in $\mathbb{R}^n$ by m directions is called the* illumination number *and it is denoted by $\mathfrak{I}(n)$. It was shown by Boltjanski that $\mathfrak{H}(n) = \mathfrak{I}(n)$.*

6.14 Definition [Minimal Width, plank].

i) *Let $K \in \mathcal{K}^n$ be convex body with support function $h(K, \cdot)$. The minimal breadth of K, i.e.,*

$$\min_{u \in S^{n-1}} h(K, u) + h(K, -u)$$

is called the minimal width *of K and will be denoted by $\Delta(K)$.*

ii) *For a vector $u \in \mathbb{R}^n \setminus \{0\}$ and $\alpha, \beta \in \mathbb{R}$ the set*

$$S(u, \alpha, \beta) = \{x \in \mathbb{R}^n : \beta \leq \langle u, x \rangle \leq \alpha\}$$

is called a (parallel) strip *or* plank *of width $(\alpha - \beta)/|u|$.*

6.15 Remark [Plank problem]. *Tarski[8] asked the following: Let $K \in \mathcal{K}^n$ be a convex body of minimal width $\Delta(K)$ which is entirely covered by m planks of width μ_i, say. Is it true that*

$$\sum_{i=1}^{m} \mu_i \geq \Delta(K)?$$

6.16 Lemma. *Let $K \in \mathcal{K}^n$ be a convex body and let $v \in \mathbb{R}^n \setminus \{0\}$. Then there exists a chord in direction v with parallel supporting hyperplanes at its ends, i.e., there exists a $t \in \mathbb{R}^n$ such that the two boundary points $\{v, w\} = \{t + \mathbb{R}\,v\} \cap \operatorname{bd} K$ have parallel supporting hyperplanes.*

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[6] Israel Gohberg, 23. August 1928 – 12. Oktober 2009

[7] Hugo Hadwiger, 23. Dezember 1908 – 29. Oktober 1981

[8] Alfred Tarski bzw. ursprünglich Alfred Tajtelbaum oder Teitelbaum, 14. Januar 1901 – 26. Oktober 1983

6.17 Lemma. *Let $K \in \mathcal{K}^n$ be a convex body of minimal width $\Delta(K)$ and let $y \in \mathbb{R}^n$ with $|y| < \Delta(K)/2$. Then $L_y = (K - y) \cap (K + y) \in \mathcal{K}^n$ is non-empty with $\Delta(L_y) \geq \Delta(K) - 2\,|y|$.*

6.18 Notation.

i) *In the following we will assume that for a plank $S(u, \alpha, \beta)$ the length of u is $1/2$ of its width, i.e., $2\,|u|^2 = \alpha - \beta$ and so we may write $\alpha = |u|^2 - \gamma$ and $\beta = -\,|u|^2 - \gamma$ for a certain constant γ. In this way, a plank is uniquely determined by u and γ and therefore we may write $S(u, \gamma)$ for a plank.*

ii) *For a set of planks $S(u_i, \gamma_i)$ of widths $2\,|u_i|$, $1 \leq i \leq m$, and for a sign vecor $\epsilon \in \{-1, 1\}^m$ let*

$$P_\epsilon = \left\{ x \in \mathbb{R}^n : \langle u_i, x\rangle \begin{cases} \geq |u_i|^2 - \gamma_i, & \epsilon_i = 1 \\ \leq -\,|u_i|^2 - \gamma_i, & \epsilon_i = -1 \end{cases}, \; 1 \leq i \leq m \right\}$$
$$= \left\{ x \in \mathbb{R}^n : \langle \epsilon_i\, u_i, x\rangle \geq |u_i|^2 - \epsilon_i \gamma_i,\ 1 \leq i \leq m \right\}$$

be a polyhedron "outside the planks".

6.19 Lemma. *Let $S(u_i, \gamma_i) \subset \mathbb{R}^n$ be planks of widths $2\,|u_i|$, $1 \leq i \leq m$. Then*

$$\bigcup_{\epsilon \in \{-1,1\}^m} \left(P_\epsilon - \sum_{i=1}^m \epsilon_i\, u_i \right) = \mathbb{R}^n.$$

6.20 Theorem [Bang]. *The answer to the Plank problem is "Yes", i.e., let $K \in \mathcal{K}^n$ be a convex body of minimal width $\Delta(K)$ which is entirely covered by m planks $S(u_i, \alpha_i, \beta_i)$ of width μ_i, say. Then $\sum_{i=1}^m \mu_i \geq \Delta(K)$.*

6.21 Remark. *Bang also raised the following stronger question: Let $K \in \mathcal{K}^n$ be a convex body with support function $h(K, \cdot)$, and suppose K is covered by m planks $S(u_i, \alpha_i, \beta_i)$ with $u_i \in S^{n-1}$ and width $\alpha_i - \beta_i$, $1 \leq i \leq m$. Is it true that*

$$\sum_{i=1}^m \frac{\alpha_i - \beta_i}{h(K, u_i) + h(K, -u_i)} \geq 1?$$

For centrally symmetric bodies it was proven by Keith Ball in 1991.

7 Packings

7.1 Definition [Packing sets]. *A subset $D \subset \mathbb{R}^n$ is called* a packing set *of $K \in \mathcal{K}^n$ if for all $x, y \in D$, $x \neq y$,*

$$(x + \operatorname{int} K) \cap (y + \operatorname{int} K) = \emptyset.$$

The family of all packing sets of K is denoted by $\mathcal{P}(K)$.

7.2 Definition [Density of a Packing]. *Let $K \in \mathcal{K}^n$ and $D \in \mathcal{P}(K)$.*

$$\delta(K, D) = \limsup_{\lambda \to \infty} \frac{\operatorname{vol}(K) \# \{x \in D : x + K \subset \lambda\, C_n\}}{\operatorname{vol}\big(\lambda\, C_n\big)}$$

is called the density of the packing $D + K$ *(with respect to the gauge body $C_n = [-1, 1]^n$).*

7.3 Remark. *$\delta(K, D)$ depends on the gauge body C_n. For instance, let $D = \{z \in \mathbb{Z}^n : z \geq 0\} \in \mathcal{P}([0, 1]^n)$. Then $\delta\big([0, 1]^n, D\big) = \frac{1}{2^n}$. Now let $H_n = C_n \cap \{x \in \mathbb{R}^n : |x_1 - x_2 + x_3 + \cdots + x_n| \leq n - 1\}$. Then $[0, 1]^n \subset H_n$, $\operatorname{vol}(H_n) = 2^n - 2/n!$ and we get*

$$\limsup_{\lambda \to \infty} \frac{\operatorname{vol}([0, 1]^n) \# \{x \in D : x + [0, 1]^n \subset \lambda H_n\}}{\operatorname{vol}(\lambda\, H_n)} = \frac{1}{2^n - \frac{2}{n!}}.$$

7.4 Theorem. *Let $K \in \mathcal{K}^n$. The supremum $\sup\big\{\delta(K, D) : D \in \mathcal{P}(K)\big\}$ is independent of the chosen gauge body, and there exists a packing set $D_K \in \mathcal{P}(K)$ such that*

$$\sup\big\{\delta(K, D) : D \in \mathcal{P}(K)\big\} = \delta(K, D_K).$$

7.5 Definition [Density of a Densest Packing]. *Let $K \in \mathcal{K}^n$.*

$$\delta(K) = \sup\big\{\delta(K, D) : D \in \mathcal{P}(K)\big\}$$

is called the density of a densest packing of K *and a set $D_K \in \mathcal{P}(K)$ with $\delta(K) = \delta(K, D_K)$ is called* a densest packing set of K.

7.6 Proposition. *Let $K \in \mathcal{K}^n$. The following properties hold:*

i) $0 < \delta(K) \leq 1$.

ii) $\delta(t + A\, K) = \delta(K)$ *for all $A \in \mathrm{GL}(n, \mathbb{R})$ and $t \in \mathbb{R}^n$.*

iii) *Let $K \in \mathcal{K}_0^n$. Then $D \in \mathcal{P}(K)$ if and only if $|x - y|_K \geq 2$ for all $x, y \in D$, $x \neq y$.*

iv) *Let* $K \in \mathcal{K}^n$ *and* $D \in \mathcal{P}(K)$. *Then*

$$\delta(K, D) = \limsup_{\lambda \to \infty} \frac{\mathrm{vol}\,(K)\,\#\big\{D \cap \lambda C_n\big\}}{\mathrm{vol}\,\big(\lambda C_n\big)}.$$

v) $\mathcal{P}(K) = \mathcal{P}\left(\frac{1}{2}(K - K)\right)$ *and for* $D \in \mathcal{P}(K)$ *we have*

$$\delta(K, D) = \delta\left(\tfrac{1}{2}(K - K), D\right) \frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,\left(\frac{1}{2}(K - K)\right)},$$

and consequently

$$\delta(K) = \delta\left(\tfrac{1}{2}(K - K)\right) \frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,\left(\frac{1}{2}(K - K)\right)}.$$

7.7 Lemma. *Let* $S \subset \mathbb{R}^n$ *be a bounded and measurable set with* $\mathrm{vol}\,(S) > 0$ *and let* $D \in \mathcal{P}(K)$. *Then there exist* $v, w \in \mathbb{R}^n$ *such that*

$$\frac{\mathrm{vol}\,(K)\,\#\big\{(w + S) \cap D\big\}}{\mathrm{vol}\,(S)} \leq \delta(K, D) \leq \frac{\mathrm{vol}\,(K)\,\#\big\{(v + S) \cap D\big\}}{\mathrm{vol}\,(S)}.$$

7.8 Remark. *Let* $K \in \mathcal{K}^n$.

$$\mathrm{R}(K) = \min\{R > 0 : \exists\, x \in \mathbb{R}^n \text{ with } K \subseteq x + R\, B_n\}$$

is called the circumradius *of* K. *The point* $t_c \in \mathbb{R}^n$ *with* $K \subseteq t_c + \mathrm{R}(K)\, B_n$ *is called* circumcenter *and it is uniquely determined. Moreover, for some* $k \in \{1, \ldots, n\}$, *there exist* $k + 1$ *points* $x_0, \ldots, x_k \in \mathrm{bd}\, K \cap \mathrm{bd}\,\big(t_c + \mathrm{R}(K)\, B_n\big)$ *and* $\lambda_i > 0$, $0 \leq i \leq k$, *with* $\sum_{i=0}^{k} \lambda_i = 1$ *such that* $t_c = \sum_{i=0}^{k} \lambda_i\, x_i$.

7.9 Corollary. *Let* $K \in \mathcal{K}_0^n$. *Then*

i) $\delta(K) \geq 2^{-n}$,

ii) $\delta(B_n) \leq (n + 1)\, \sqrt{2}^{\,-n}$.

7.10 Remark. *Chronologically, the first upper bound is due to Blichfeldt (1928), who proved*

$$\delta(B_n) \leq \frac{n + 2}{2} \sqrt{2}^{\,-n}.$$

This was slightly improved by Rogers (1958) by a factor of 2/e, *roughly speaking. In 1973/74 Sidelnikov showed that*

$$\delta(B_n) \leq 2^{-(0.509 + o(1))n}, \quad n \text{ large}.$$

Subsequently, this bound was improved by Levenštein (1975), Kabatjanskiĭ and Levenštein (1978) to

$$\delta(B_n) \leq 2^{-(0.599 + o(1))n}, \quad n \text{ large},$$

which is still the best known bound.

7.11 Definition [Lattice]. *Let $b_1, \dots, b_n \in \mathbb{R}^n$ be linearly independent. The set*

$$\Lambda = \{z_1 b_1 + z_2 b_2 + \cdots + z_n b_n : z_i \in \mathbb{Z},\ 1 \le i \le n\}$$

is called lattice. *The set of generating vectors $\{b_1, \dots, b_n\}$ or the matrix $B = (b_1, \dots, b_n)$ with columns b_i is called* basis *of Λ. An element $b \in \Lambda$ is called* lattice point *of Λ. The set of all lattices in $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$.*

7.12 Definition [Unimodular matrix]. *An integral matrix $U \in \mathbb{Z}^{n\times n}$ is called* unimodular *iff $|\det U| = 1$. The group of all unimodular matrices is denoted by $\mathrm{GL}(n, \mathbb{Z})$.*

Oberserve that a matrix is unimodular if and only the matrix and its inverse are integral matrices.

7.13 Proposition. $\mathrm{GL}(n, \mathbb{Z}) = \{U \in \mathbb{R}^{n\times n} : U\mathbb{Z}^n = \mathbb{Z}^n\}$.

7.14 Lemma. *Let $\Lambda = B\mathbb{Z}^n \in \mathcal{L}^n$. $A = (a_1, \dots, a_n)$ is a basis of Λ iff there exists a $U \in \mathrm{GL}(n, \mathbb{Z})$ such that $A = B\,U$.*

7.15 Definition [Determinant, fundamental cell]. *Let $\Lambda \in \mathcal{L}^n$ with basis $B = (b_1, \dots, b_n)$.*

i) $\det \Lambda = |\det B|$ *is called* determinant *of Λ.*

ii) $P_B = \{\rho_1 b_1 + \cdots + \rho_n b_n : 0 \le \rho_i < 1,\ 1 \le i \le n\} = B\,[0, 1)^n$ *is called* fundamental cell *or* fundamental parallelepiped *of Λ (w.r.t. the basis B).*

7.16 Remark.

i) $\det \Lambda$ *is independent of the choice of the basis (cf. Lemma 7.14).*

ii) $\det \Lambda = \mathrm{vol}\,(P_B)$ *and* $\det(\mu\Lambda) = |\mu|^n \det \Lambda$, $\mu \in \mathbb{R}$.

iii) $\det \Lambda \le |b_1|\,|b_2| \cdot \ldots \cdot |b_n|$, *with equality if and only if the vectors b_i are pairwise orthogonal (Hadamard inequality).*

iv) $P_B \cap \Lambda = \{0\}$. *Since $(P_B - P_B) = B\,(-1, 1)^n$ we even have $(P_B - P_B) \cap \Lambda = \{0\}$.*

7.17 Proposition. *Let $\Lambda \in \mathcal{L}^2$ and let $a_1, a_2 \in \Lambda$ be linearly independent. Then*

$$a_1, a_2 \text{ basis of } \Lambda \Leftrightarrow \mathrm{conv}\,\{0, a_1, a_2\} \cap \Lambda = \{0, a_1, a_2\}.$$

7.18 Remark. *An analogous statement to Lemma 7.17 does not exist in dimension ≥ 3. For $n \ge 3$ and $m \in \mathbb{N}$ let $b(m) = (1, \dots, 1, m)^{\intercal} \in \mathbb{R}^n$ and $T^n(m) = \mathrm{conv}\,\{0, e_1, \dots, e_{n-1}, b(m)\}$. Then*

$$T^n(m) \cap \mathbb{Z}^n = \{0, e_1, \dots, e_{n-1}, b(m)\},$$

but the determinant of the lattice with basis $\{e_1, \dots, e_{n-1}, b(m)\}$ is m. $T^n(m)$ are called Reeve simplices.

7.19 Proposition. *Let $\Lambda = B\mathbb{Z}^n \in \mathcal{L}^n$. Then*

$$\mathbb{R}^n = \dot{\bigcup}_{b\in\Lambda} (b + P_B),$$

i.e., $\mathbb{R}^n$ is the pairwise disjunct union of the lattice translates $b + P_B$.

7.20 Corollary. *Let $K \in \mathcal{K}^n$ and $\Lambda \in \mathcal{L}^n \cap \mathcal{P}(K)$. Then*

$$\delta(K, \Lambda) = \frac{\mathrm{vol}\,(K)}{\det \Lambda}.$$

7.21 Definition [Density of a densest Lattice Packing]. *For $K \in \mathcal{K}^n$ the set $\mathcal{P}_{\mathcal{L}}(K) = \mathcal{L}^n \cap \mathcal{P}(K)$ is called the family of all* packing lattices *of K. For $\Lambda \in \mathcal{P}_{\mathcal{L}}(K)$ the arrangement $\Lambda + K$ is called a* lattice packing *of K and*

$$\delta_{\mathcal{L}}(K) = \sup\big\{\delta(K, \Lambda) : \Lambda \in \mathcal{P}_{\mathcal{L}}(K)\big\}$$

is called the density of a densest lattice packing of K.

7.22 Definition [Critical determinant and admissible lattices]. *Let $K \in \mathcal{K}_0^n$. A lattice Λ is called* admissible for K *(or* K-admissible*) if* $\mathrm{int}\, K \cap \Lambda = \{0\}$.

$$\Delta(K) = \inf\{\det \Lambda : \Lambda \text{ admissible for } K\}$$

is called the critical determinant of K.

7.23 Remark. *Let $K \in \mathcal{K}_0^n$ and $\Lambda \in \mathcal{L}^n$. Then $\big(1/\lambda_1(K, \Lambda)\big)\,\Lambda$ is admissible for K.*

7.24 Proposition [Critical lattice]. *For $K \in \mathcal{K}_0^n$ there exists a K-admissible lattice Λ_K with $\det \Lambda_K = \Delta(K)$. Such a lattice will be called a* critical lattice of K.

7.25 Proposition. *Let $K \in \mathcal{K}^n$. Then*

$$\delta_{\mathcal{L}}(K) = \frac{\mathrm{vol}\,(K)}{\Delta(K - K)}.$$

7.26 Remark.

i) $0 < \delta_{\mathcal{L}}(K) \le \delta(K) \le 1$.

ii) $\delta_{\mathcal{L}}(A\,K + t) = \delta_{\mathcal{L}}(K)$ *for all* $A \in \mathrm{GL}(n, \mathbb{R})$ *and* $t \in \mathbb{R}^n$.

iii) *For $K \in \mathcal{K}_0^n$ we have* $\delta_{\mathcal{L}}(K) = 2^{-n}\mathrm{vol}\,(K)/\Delta(K)$.

7.27 Theorem [Minkowski-Hlawka, 1943]. *Let $S \subset \mathbb{R}^n$ be a bounded Jordan-measurable set with $\mathrm{vol}\,(S) < 1$. Then there exists a lattice $\Lambda \in \mathcal{L}^n$ with*

$$\det \Lambda = 1 \quad \textit{and} \quad S \cap \Lambda \setminus \{0\} = \emptyset.$$

7.28 Remark. *Theorem 7.27 remains true for Jordan-measurable, unbounded, closed sets.*

7.29 Corollary. *Let $S \subset \mathbb{R}^n$ be a bounded Jordan-measurable set with $S = -S$ and with $\mathrm{vol}\,(S) < 2$. Then there exists a lattice $\Lambda \in \mathcal{L}^n$ with*

$$\det \Lambda = 1 \quad \textit{and} \quad S \cap \Lambda \setminus \{0\} = \emptyset.$$

7.30 Corollary. *Let $K \in \mathcal{K}_0^n$. Then*

$$\delta_{\mathcal{L}}(K) \geq 2^{-(n-1)} \qquad \left[\Leftrightarrow \Delta(K) \leq \frac{\mathrm{vol}\,(K)}{2}\right].$$

7.31 Theorem. *Let $S \subset \mathbb{R}^n$, $n \geq 2$, be a bounded ray set (i.e., if $x \in S$ then $\lambda\, x \in S$ for all $\lambda \in [0,1]$) with $\mathrm{vol}\,(S) < \zeta(n)$. Then there exists a lattice $\Lambda \in \mathcal{L}^n$ with $\det \Lambda = 1$ and $S \cap \Lambda \setminus \{0\} = \emptyset$.*
Here $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ denotes the (Riemann) zeta function (ζ-function).

7.32 Theorem [K. Ball, 1992]. $\delta(B_n) \geq (n-1)2^{-(n-1)}\zeta(n)$.

7.33 Remark. *The densest lattice packings of balls are only known in dimensions $n \leq 8$ (Lagrange 1773, Gauß 1840, Korkine-Zolotareff 1872/73, Blichfeldt, 1934) and $n = 24$ (Cohn& Kumar, 2004).*

7.34 Theorem [Swinnerton-Deyer, 1953]. *Let $K \in \mathcal{K}_0^n$ and let $\Lambda_K \in \mathcal{L}^n$ be a critical lattice of K. Then*

$$\#\big(K \cap \Lambda_K \setminus \{0\}\big) \geq n(n+1).$$

7.35 Theorem. *Let $K \in \mathcal{K}_0^2$.*

i) *Let $b_1, b_2 \in \mathrm{bd}\, K$ such that $b_2 - b_1 \in \mathrm{bd}\, K$. Then the lattice $\Lambda = (b_1, b_2)\mathbb{Z}^2$ is admissible for K.*

ii) *Let $\Lambda \in \mathcal{L}^2$ be a critical lattice of K. Then there exists a basis b_1, b_2 of Λ such that $b_1, b_2, b_2 - b_1 \in \mathrm{bd}\, K$.*

7.36 Corollary. *Let* $K \in \mathcal{K}_0^2$ *and* H_K *be an affinely regular hexagon of minimal volume with vertices on* $\operatorname{bd} K$*. Then*

$$\delta_{\mathcal{L}}(K) = \frac{3}{4}\frac{\operatorname{vol}(K)}{\operatorname{vol}(H_K)} \qquad \left[\Leftrightarrow \Delta(K) = \frac{1}{3}\operatorname{vol}(H_K)\right].$$

7.37 Theorem [Fejes Tóth, 1950; Rogers, 1951]. *Let* $K \in \mathcal{K}^2$*. Then*

$$\delta(K) = \delta_{\mathcal{L}}(K).$$

7.38 Theorem [Hales, 1998/2005].

$$\delta(B_3) = \delta_{\mathcal{L}}(B_3) = \frac{\pi}{3\sqrt{2}}.$$

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