Combinatorial Convexity

Martin Henk

Otto-von-Guericke-Universität Magdeburg
Contents

0 Some basic and convex facts 1
1 Support and separate 5
2 Radon, Helly, Caratheodory and relatives 9
3 Polytopes 11
4 A bit of Gale diagrams and triangulations 19
5 A glimpse of Ehrhart theory 23
6 Borsuk, Hadwiger and planks 29
7 Packings 33

Index 38
0 Some basic and convex facts

0.1 Notation. \( \mathbb{R}^n = \{(x_1, \ldots, x_n)^\top : x_i \in \mathbb{R}\} \) denotes the \( n \)-dimensional Euclidean space equipped with the Euclidean inner product \( \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \), \( x, y \in \mathbb{R}^n \), and the Euclidean norm \(|x| = \sqrt{\langle x, x \rangle}\).

0.2 Definition [Linear, affine, positive and convex combination]. Let \( m \in \mathbb{N} \) and let \( x_i \in \mathbb{R}^n \), \( \lambda_i \in \mathbb{R} \), \( 1 \leq i \leq m \).

i) \( \sum_{i=1}^{m} \lambda_i x_i \) is called a linear combination of \( x_1, \ldots, x_m \).

ii) If \( \sum_{i=1}^{m} \lambda_i = 1 \) then \( \sum_{i=1}^{m} \lambda_i x_i \) is called an affine combination of \( x_1, \ldots, x_m \).

iii) If \( \lambda_i \geq 0 \) then \( \sum_{i=1}^{m} \lambda_i x_i \) is called a positive combination of \( x_1, \ldots, x_m \).

iv) If \( \lambda_i \geq 0 \) and \( \sum_{i=1}^{m} \lambda_i = 1 \) then \( \sum_{i=1}^{m} \lambda_i x_i \) is called a convex combination of \( x_1, \ldots, x_m \).

v) Let \( X \subseteq \mathbb{R}^n \). \( x \in \mathbb{R}^n \) is called linearly (affinely, positively, convexly) dependent of \( X \), if \( x \) is a linear (affine, positive, convex) combination of finitely many points of \( X \), i.e., there exist \( x_1, \ldots, x_m \in X \), \( m \in \mathbb{N} \), such that \( x \) is a linear (affine, positive, convex) combination of the points \( x_1, \ldots, x_m \).

0.3 Definition [Linearly and affinely independent points]. \( x_1, \ldots, x_m \in \mathbb{R}^n \) are called linearly (affinely) dependent, if one of the \( x_i \) is linearly (affinely) dependent of \( \{x_1, \ldots, x_m\} \setminus \{x_i\} \). Otherwise \( x_1, \ldots, x_m \) are called linearly (affinely) independent.

0.4 Remark. Let \( x_1, \ldots, x_m \in \mathbb{R}^n \).

i) \( x_1, \ldots, x_m \) are affinely dependent if and only if \( (x_1^\top, \ldots, x_m^\top) \in \mathbb{R}^{n+1} \) are linearly dependent.

ii) \( x_1, \ldots, x_m \) are affinely dependent if and only if there exist \( \mu_i \in \mathbb{R} \), \( 1 \leq i \leq m \), with \( (\mu_1, \ldots, \mu_m) \neq (0, \ldots, 0) \), \( \sum_{i=1}^{m} \mu_i = 0 \) and \( \sum_{i=1}^{m} \mu_i x_i = 0 \).

iii) If \( m \geq n + 1 \) then \( x_1, \ldots, x_m \) are linearly dependent.

iv) If \( m \geq n + 2 \) then \( x_1, \ldots, x_m \) are affinely dependent.

0.5 Definition [Linear subspace, affine subspace, cone and convex set]. \( X \subseteq \mathbb{R}^n \) is called

i) linear subspace (set) if it contains all \( x \in \mathbb{R}^n \) which are linearly dependent of \( X \),

ii) affine subspace (set) if it contains all \( x \in \mathbb{R}^n \) which are affinely dependent of \( X \),
iii) (convex) cone if it contains all \( x \in \mathbb{R}^n \) which are positively dependent of \( X \),

iv) convex set if it contains all \( x \in \mathbb{R}^n \) which are convexly dependent of \( X \).

**0.6 Notation.** \( C^n = \{ K \subseteq \mathbb{R}^n : K \text{ convex} \} \) denotes the set of all convex sets in \( \mathbb{R}^n \). The empty set \( \emptyset \) is regarded as a convex, linear and affine set.

**0.7 Theorem.** \( K \subseteq \mathbb{R}^n \) is convex if and only if

\[
\lambda x + (1 - \lambda) y \in K, \quad \text{for all } x, y \in K \text{ and } 0 \leq \lambda \leq 1.
\]

**0.8 Example.** The closed \( n \)-dimensional ball \( B_n(a, \rho) = \{ x \in \mathbb{R}^n : |x - a| \leq \rho \} \) with centre \( a \) and radius \( \rho > 0 \) is convex. The boundary of \( B_n(a, \rho) \), i.e., \( \{ x \in \mathbb{R}^n : |x - a| = \rho \} \) is non-convex. In the case \( a = 0 \) and \( \rho = 1 \) the ball \( B_n(0, 1) \) is abbreviated by \( B_n \) and is called \( n \)-dimensional unit ball. Its boundary is denoted by \( S^{n-1} \).

**0.9 Corollary.** Let \( K_i \in C^n, i \in I \). Then \( \bigcap_{i \in I} K_i \in C^n \).

**0.10 Definition [Linear, affine, positive and convex hull, dimension].** Let \( X \subseteq \mathbb{R}^n \).

i) The linear hull \( \text{lin} X \) of \( X \) is defined by

\[
\text{lin} X = \bigcap_{L \subseteq \mathbb{R}^n, L \text{ linear}, X \subseteq L} L.
\]

ii) The affine hull \( \text{aff} X \) of \( X \) is defined by

\[
\text{aff} X = \bigcap_{A \subseteq \mathbb{R}^n, A \text{ affine}, X \subseteq A} A.
\]

iii) The positive (conic) hull \( \text{pos} X \) of \( X \) is defined by

\[
\text{pos} X = \bigcap_{C \subseteq \mathbb{R}^n, C \text{ convex cone}, X \subseteq C} C.
\]

iv) The convex hull \( \text{conv} X \) of \( X \) is defined by

\[
\text{conv} X = \bigcap_{K \subseteq \mathbb{R}^n, K \text{ convex}, X \subseteq K} K.
\]

v) The dimension \( \text{dim} X \) of \( X \) is the dimension of its affine hull, i.e., \( \text{dim aff} X \).
0.11 Theorem. Let $X \subseteq \mathbb{R}^n$. Then

$$\text{conv } X = \left\{ \sum_{i=1}^{m} \lambda_i x_i : m \in \mathbb{N}, x_i \in X, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$ 

0.12 Remark.

i) $\text{conv } \{x, y\} = \{\lambda x + (1 - \lambda) y : \lambda \in [0, 1]\}$.

ii) $\text{lin } X = \{\sum_{i=1}^{m} \lambda_i x_i : m \in \mathbb{N}, x_i \in X\}$.

iii) $\text{aff } X = \{\sum_{i=1}^{m} \lambda_i x_i : m \in \mathbb{N}, x_i \in X, \sum_{i=1}^{m} \lambda_i = 1\}$.

iv) $\text{pos } X = \{\sum_{i=1}^{m} \lambda_i x_i : m \in \mathbb{N}, x_i \in X, \lambda_i \geq 0\}$.

0.13 Definition [(Relative) interior point and (relative) boundary point]. Let $X \subseteq \mathbb{R}^n$.

i) $x \in X$ is called an interior point of $X$ if there exists a $\rho > 0$ such that $B_n(x, \rho) \subseteq X$. The set of all interior points of $X$ is called the interior of $X$ and is denoted by $\text{int } X$.

ii) $x \in \mathbb{R}^n$ is called boundary point of $X$ if for all $\rho > 0$ hold $B_n(x, \rho) \cap X \neq \emptyset$ and $B_n(x, \rho) \cap (\mathbb{R}^n \setminus X) \neq \emptyset$. The set of all boundary points of $X$ is called the boundary of $X$ and is denoted by $\text{bd } X$.

iii) Let $A = \text{aff } X$. $x \in X$ is called a relative interior point of $X$ if there exists a $\rho > 0$ such that $B_n(x, \rho) \cap A \subseteq X$. The set of all relative interior points is called the relative interior of $X$ and is denoted by $\text{relint } X$.

iv) Let $A = \text{aff } X$. $x \in A$ is called a relative boundary point of $X$ if for all $\rho > 0$ hold $B_n(x, \rho) \cap X \neq \emptyset$ and $B_n(x, \rho) \cap (A \setminus X) \neq \emptyset$. The set of all relative boundary points of $X$ is called relative boundary of $X$ and is denoted by $\text{relbd } X$.

0.14 Remark. Let $X \subseteq \mathbb{R}^n$ be closed. Then $X = \text{relint } X \cup \text{relbd } X$.

0.15 Theorem. Let $K \in C^n$, $x \in \text{relint } K$ and $y \in K$. Then $(1 - \lambda)x + \lambda y \in \text{relint } K$ for all $\lambda \in [0, 1)$.

0.16 Corollary. Let $K \in C^n$ be closed. Let $x \in \text{relint } K$ and $y \in \text{aff } K \setminus K$. Then the segment $\text{conv } \{x, y\}$ intersects $\text{relbd } K$ in precisely one point.

0.17 Definition [Polytope and simplex]. Let $X \subset \mathbb{R}^n$ of finite cardinality, i.e., $\#X < \infty$.

i) $\text{conv } X$ is called a (convex) polytope.

ii) A polytope $P \subset \mathbb{R}^n$ of dimension $k$ is called a $k$-polytope.

iii) If $X$ is affinely independent and $\dim X = k$ then $\text{conv } X$ is called a $k$-simplex.
0.18 **Notation.** \( \mathcal{P}^n = \{ P \subset \mathbb{R}^n : P \) polytope\} denotes the set of all polytopes in \( \mathbb{R}^n \).

0.19 **Lemma.** Let \( T = \text{conv}\{x_1, \ldots, x_{k+1}\} \subset \mathbb{R}^n \) be a \( k \)-simplex, and let \( \lambda_i > 0, 1 \leq i \leq k+1 \), with \( \sum \lambda_i = 1 \). Then \( \sum \lambda_i x_i \in \text{relint} \, T \).

0.20 **Corollary.** Let \( K \in \mathcal{C}^n, K \neq \emptyset \). Then \( \text{relint} \, K \neq \emptyset \).

0.21 **Theorem.** Let \( P = \text{conv}\{x_1, \ldots, x_m\} \in \mathcal{P}^n \). A point \( x \in \mathbb{R}^n \) belongs to \( \text{relint} \, P \) if and only if \( x \) admits a representation as \( x = \sum_{i=1}^m \lambda_i x_i \) with \( \lambda_i > 0 \), \( 1 \leq i \leq m \), and \( \sum_{i=1}^m \lambda_i = 1 \).

0.22 **Notation.**

i) For two sets \( X, Y \subseteq \mathbb{R}^n \) the vectorial addition

\[ X + Y = \{ x + y : x \in X, y \in Y \} \]

is called the Minkowski sum of \( X \) and \( Y \). If \( X \) is just a singleton, i.e., \( X = \{ x \} \), then we write \( x + Y \) instead of \( \{ x \} + Y \).

ii) For \( \lambda \in \mathbb{R} \) and \( X \subseteq \mathbb{R}^n \) we denote by \( \lambda X \) the set

\[ \lambda X = \{ \lambda x : x \in X \} . \]
1 Support and separate

1.1 Notation. Let \( a \in \mathbb{R}^n, a \neq 0, \) and \( \alpha \in \mathbb{R}. \) The closed halfspaces \( H^+(a, \alpha), \) \( H^-(a, \alpha) \subset \mathbb{R}^n \) are given by
\[
H^+(a, \alpha) = \{ x \in \mathbb{R}^n : \langle a, x \rangle \geq \alpha \}, \quad H^-(a, \alpha) = \{ x \in \mathbb{R}^n : \langle a, x \rangle \leq \alpha \}.
\]
The hyperplane \( H(a, \alpha) \) is defined by
\[
H(a, \alpha) = \{ x \in \mathbb{R}^n : \langle a, x \rangle = \alpha \}.
\]

1.2 Definition [Supporting hyperplane]. Let \( X \subset \mathbb{R}^n. \) A hyperplane \( H(a, \alpha) \subset \mathbb{R}^n \) is called supporting hyperplane of \( X \) if:
\[
i) \ H(a, \alpha) \cap X \neq \emptyset \quad \text{and} \quad ii) \ X \subseteq H^-(a, \alpha).
\]
\( a \) is called outer normal vector of \( X \) and if, in addition, \( |a| = 1 \) then it is called outer unit normal vector of \( X. \)

1.3 Proposition. Let \( X \subset \mathbb{R}^n \) and let \( H(a, \alpha) \) be a supporting hyperplane of \( X. \) Then
\[
H(a, \alpha) \cap \text{conv} \, X = \text{conv} \,( H(a, \alpha) \cap X).
\]

1.4 Remark. Let \( X \subset \mathbb{R}^n \) be compact and \( a \in \mathbb{R}^n\setminus\{0\}. \) Then there exists a supporting hyperplane of \( X \) with outer normal vector \( a. \)

1.5 Definition [Nearest point map (or metric projection)]. Let \( K \subset \mathcal{C}^n \) be closed. The map \( \Phi_K : \mathbb{R}^n \to K, \) where for \( x \in \mathbb{R}^n \) the point \( \Phi_K(x) \in K \) is given by
\[
|x - \Phi_K(x)| = \min\{|x - y| : y \in K\}
\]
is called the nearest point map (metric projection) with respect to \( K. \)

1.6 Remark. We prove that the nearest point map is well-defined. Notice that since \( K \) is closed, for all \( x \in \mathbb{R}^n \) there exist \( y_x \in K \) such that
\[
|x - y_x| = \min\{|x - y| : y \in K\}.
\]
We show that \( y_x \) is uniquely determined. In fact, if there exists \( \bar{y} \in K, \bar{y} \neq y_x, \) with \( |x - \bar{y}| = |x - y_x| \) then we may assume that \( x - y_x \) and \( x - \bar{y} \) are linearly independent. Hence
\[
\left| x - y_x + \bar{y} \right| = \frac{1}{2} |x - y_x| + \frac{1}{2} |x - \bar{y}| < \frac{1}{2} |x - y_x| + \frac{1}{2} |x - \bar{y}| = |x - y_x|.
\]
Since \( (y_x + \bar{y})/2 \in K \) by the convexity of \( K, \) it contradicts the minimality of \( y_x. \)

1.7 Theorem. Let \( K \subset \mathcal{C}^n \) be closed and let \( x \in \mathbb{R}^n \setminus K. \) Let \( a = x - \Phi_K(x) \) and \( \alpha = \langle a, \Phi_K(x) \rangle. \) Then \( H(a, \alpha) \) is a supporting hyperplane of \( K \) with outer normal vector \( a. \)

1.8 Corollary. Let \( K \subset \mathcal{C}^n, K \neq \mathbb{R}^n, \) be closed. Then
\[
K = \bigcap_{H(a, \alpha) \text{ supporting hyperplane of } K} H^-(a, \alpha),
\]
i.e., \( K \) is the intersection of all its “supporting halfspaces”. 
1.9 Corollary. Let $X \subseteq \mathbb{R}^n$ such that $\text{conv} \, X$ is closed and $\text{conv} \, X \neq \mathbb{R}^n$. Then

$$\text{conv} \, X = \bigcap_{X \subseteq H^{-}(a, \alpha)} H^{-}(a, \alpha),$$

i.e., $\text{conv} \, X$ is the intersection of all halfspaces containing $X$.

1.10 Lemma [Busemann-Feller Lemma]. Let $K \in \mathcal{C}^n$ be closed. Then

$$|\Phi_K(x) - \Phi_K(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}^n$, i.e., the nearest point map does not increase distances. In particular, it is a continuous map.

1.11 Theorem. Let $K \in \mathcal{C}^n$ be compact and let $\rho > 0$ such that $K \subseteq \text{int} \, (\rho B_n)$. The nearest point map restricted to $\rho S^{n-1}$ is surjective, i.e., $\Phi_K : \rho S^{n-1} \to \text{bd} \, K$ is surjective.

1.12 Corollary. Let $K \in \mathcal{C}^n$ be closed and let $x \in \text{relbd} \, K$. Then there exists a supporting hyperplane $H(a, \alpha)$ of $K$ with $x \in H(a, \alpha)$.

1.13 Theorem [Separation theorem]. Let $K_1, K_2 \in \mathcal{C}^n$ with $K_1 \cap K_2 = \emptyset$. Then there exists a separating hyperplane $H(a, \alpha)$ of $K_1$ and $K_2$, i.e., $K_1 \subseteq H^+(a, \alpha)$ and $K_2 \subseteq H^-(a, \alpha)$.

If $K_1$ is closed and $K_2$ is compact, then there exists even a strictly separating hyperplane $H(a, \alpha)$ of $K_1$ and $K_2$, i.e., $K_1 \subset \text{int} \, H^+(a, \alpha)$ and $K_2 \subset \text{int} \, H^-(a, \alpha)$.

1.14 Definition [Support function, breadth]. Let $K \in \mathcal{C}^n$. The function $h(K, \cdot) : \mathbb{R}^n \to \mathbb{R}$ given by

$$h(K, u) = \sup \{ \langle u, x \rangle : x \in K \}$$

is called support function of $K$. For $u \in S^{n-1}$ the breadth of $K$ in the direction $u$ is given by $h(K, u) + h(K, -u)$.

1.15 Remark. Let $K \in \mathcal{C}^n$ be non-empty and compact. Then

$$K = \bigcap_{u \in S^{n-1}} \{ x \in \mathbb{R}^n : \langle u, x \rangle \leq h(K, u) \}.$$

1.16 Definition [Polar set]. Let $X \subseteq \mathbb{R}^n$.

$$X^* = \{ y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in X \}$$

is called the polar set of $X$.

1.17 Proposition.

i) $X^*$ is a convex and closed set and $0 \in X^*$. 
ii) If \( X_1 \subseteq X_2 \) then \( X_2^* \subseteq X_1^* \).

iii) Let \( M \) be a regular \( n \times n \) matrix. Then \( (MX)^* = M^{-T}X^* \).

iv) Let \( X_i \subseteq \mathbb{R}^n \), \( i \in I \). Then \( \left( \bigcup_{i \in I} X_i \right)^* = \bigcap_{i \in I} X_i^* \).

v) \( X \subseteq (X^*)^* \).

vi) Let \( X \subset \mathbb{R}^n \). Then \( X = X^* \) if and only if \( X = B_n \).

1.18 Proposition.

i) Let \( P = \text{conv} \{x_1, \ldots, x_m\} \subset \mathbb{R}^n \). Then

\[
P^* = \{y \in \mathbb{R}^n : \langle x_i, y \rangle \leq 1, 1 \leq i \leq m \}.
\]

ii) Let \( P = \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq 1, 1 \leq i \leq m \} \) with \( a_i \in \mathbb{R}^n \). Then

\[
P^* = \text{conv} \{0, a_1, \ldots, a_m\}.
\]

1.19 Lemma. Let \( K \in \mathcal{C}^n \) be closed with \( 0 \in K \). Then \( (K^*)^* = K \).
2 Radon, Helly, Caratheodory and relatives

2.1 Theorem [Radon, 1921]. Let $X \subset \mathbb{R}^n$. If $\# X \geq n + 2$ then there exist $X_1, X_2 \subset X$ with $X_1 \cap X_2 = \emptyset$ and $\text{conv} \ X_1 \cap \text{conv} \ X_2 \neq \emptyset$.

2.2 Theorem [Helly, 1913]. Let $K_1, \ldots, K_m \in \mathcal{C}^n$, $m \geq n + 1$, such that for each $(n + 1)$-index set $I \subseteq \{1, \ldots, m\}$ we have $\bigcap_{i \in I} K_i \neq \emptyset$. Then all sets $K_i$ have a point in common, i.e., $\bigcap_{i=1}^m K_i \neq \emptyset$.

2.3 Remark.

i) Without any further restrictions/assumptions Helly’s theorem is not true for infinitely many convex sets $K_i$. For instance, let $K_i = (0, 1_i]$, $i \in \mathbb{N}$.

ii) Helly’s theorem, however, can be easily generalised to infinitely many compact (bounded and closed) convex sets.

2.4 Corollary. Let $C \subset \mathcal{C}^n$ be compact. Then there exists $t \in \mathbb{R}^n$ with

$$-C \subseteq t + nC.$$ 

2.5 Definition [Centerpoint]. For a finite point set $X \subset \mathbb{R}^n$ a point $c \in \mathbb{R}^n$ is called centerpoint is every closed halfspace containing $c$ contains at least $\lceil \frac{1}{n+1} \# X \rceil$ points of $X$.

2.6 Theorem. Every finite set $X \subset \mathbb{R}^n$ has a centerpoint.

2.7 Theorem [Carathéodory, 1907]. Let $X \subset \mathbb{R}^n$. Then

$$\text{conv} \ X = \left\{ \sum_{i=1}^{n+1} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1, x_i \in X, i = 1, \ldots, n + 1 \right\}.$$ 

2.8 Remark. Let $X \subset \mathbb{R}^n$. Then

$$\text{conv} \ X = \left\{ \sum_{i=1}^{\dim X+1} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{\dim X+1} \lambda_i = 1, x_i \in X \right\}.$$ 

As a direct consequence of Carathéodory’s Theorem [2.7] we get the following result.

2.9 Corollary. A polytope is the union of simplices.

2.10 Corollary. The convex hull of a compact set is compact.

2.11 Theorem [(weak)Fractional Helly theorem]. Let $K_1, \ldots, K_m \in \mathcal{C}^n$, $m \geq n + 1$, and let $\alpha \in (0, 1]$ such that for at least $\alpha \left( \begin{array}{c} m \\ n + 1 \end{array} \right)$ of the $(n + 1)$-index sets $I \subseteq \{1, \ldots, m\}$ we have $\bigcap_{i \in I} K_i \neq \emptyset$. Then there exists a point in common of at least $\frac{\alpha}{n+1} \cdot m$ sets $K_i$. 

2.12 Remark. The (strong and sharp) fractional Helly theorem, which is due to Kalai, gives that \((1 - (1 - \alpha)^{1/(n+1)}) \cdot m\) sets have a point in common. Obviously, for \(\alpha = 1\) we get again the classical Helly Theorem 2.2.

2.13 Theorem [Colorful Carathéodory theorem]. Let \(X_1, \ldots, X_{n+1} \subset \mathbb{R}^n\) be finite point sets such that \(0 \in \text{conv} X_i, 1 \leq i \leq n + 1\). There exist \(x_i \in X_i, 1 \leq i \leq n + 1\), such that \(0 \in \text{conv} \{x_1, \ldots, x_{n+1}\}\).

2.14 Theorem [Tverberg’s theorem]. Let \(X \subseteq \mathbb{R}^n\) and let \(k \in \mathbb{N}_{\geq 1}\). If \(\#X \geq (k - 1)(n + 1) + 1, k \in \mathbb{N}\), then there exist \(k\) subsets \(X_1, \ldots, X_k \subset X\) with \(X_i \cap X_j = \emptyset, i \neq j\), but \(\text{conv} X_1 \cap \text{conv} X_2 \cap \cdots \cap \text{conv} X_k \neq \emptyset\).

2.15 Theorem. Let \(X \subset \mathbb{R}^n\) and let \(\#X = m \geq n + 1\). Then there exists a point \(y \in \mathbb{R}^n\) contained in at least \(\gamma_n \binom{m}{n+1}\) \(X\)-simplices, i.e., simplices of the form \(\text{conv} S, S \subseteq X, \#S = n + 1\). Here \(\gamma_n\) is a positive constant depending only on the dimension, and \(X\)-simplices \(\text{conv} S_1, \text{conv} S_2\) are considered different if \(S_1 \neq S_2\).
3 Polytopes

3.1 Definition [Polyhedron]. The intersection of finitely many closed halfspaces is called a polyhedron.

3.2 Theorem [Minkowski, 1896, Weyl, 1935].
   i) A bounded polyhedron is a polytope.
   ii) A polytope is a bounded polyhedron.

3.3 Notation [V-Polytope, H-Polytope]. A polytope given as the convex hull of finitely many points is called a V-polytope. If it is given as the bounded intersection of finitely many closed halfspaces, then it is called an H-polytope.

3.4 Corollary. Let $P \in \mathcal{P}^n$.
   i) Let $A \in \mathbb{R}^{m \times n}$ and $t \in \mathbb{R}^m$. Then $AP + t$ is a polytope.
   ii) Let $U \subset \mathbb{R}^n$ be an affine subspace. Then $P \cap U$ is a polytope.

3.5 Definition [Faces]. Let $K \in \mathcal{C}^n$ be closed and let $H$ be a supporting hyperplane of $K$. If $j = \dim(K \cap H)$, then $K \cap H$ is called a $j$-face of $K$. Moreover, $K$ itself is regarded as a $(\dim K)$-face and the empty set $\emptyset$ as $(-1)$-face of $K$.

3.6 Notation [Vertices, edges, facets]. A 0-face of $K \in \mathcal{C}^n$, $K$ closed, is called vertex, an 1-face is called edge and a $(\dim K - 1)$-face is called facet of $K$. $K$ itself and the empty set are called improper faces, whereas the remaining faces are called proper faces of $K$.
   The set of all vertices of a polytope $P$ is denoted by $\text{vert } P$.

3.7 Remark.
   i) Let $K \in \mathcal{C}^n$ be closed. Every (relative) boundary point of $K$ lies in a suitable $j$-face, $0 \leq j \leq \dim K - 1$.
   ii) Let $K \in \mathcal{C}^n$, $\dim K = n$. Let $F$ be a facet of $K$ and $H$ a supporting hyperplane of $K$ with $F = K \cap H$. Then $H = \text{aff } F$.

3.8 Proposition. Each face of a polytope is a polytope, and a polytope has only finitely many faces.

3.9 Definition [$f$-vector]. For $P \in \mathcal{P}^n$ let $f_i(P)$ be the number of $i$-faces of $P$, $-1 \leq i \leq \dim P$. Furthermore, let $f_i(P) = 0$ for $\dim P + 1 \leq i \leq n$. The vector $f(P)$ with entries $f_i(P)$, $-1 \leq i \leq n$, is called the $f$-vector of $P$.

3.10 Remark.
i) Let \( T_n = \text{conv} \{0, e_1, \ldots, e_n\} \) be the so called standard simplex. Then \( f_i(T_n) = \binom{n+1}{i+1} \), i.e., any \((i + 1)\) subset of the vertices are the vertices of an \( i \)-face.

ii) For any \( n \)-polytope \( P \in \mathcal{P}^n \) we have \( \sum_{i=-1}^{n} f_i(P) \geq 2^{n+1} \) with equality if and only if \( P \) is an \( n \)-simplex.

3.11 Lemma. Let \( P \in \mathcal{P}^n \).

i) \( v \in \text{vert} P \) can not be written as a convex combination of two other points of \( P \), i.e., \( v \notin \text{conv} (P \setminus \{v\}) \).

ii) If \( P = \text{conv} W \), then \( \text{vert} P \subseteq W \).

iii) \( P = \text{conv} (\text{vert} P) \).

3.12 Lemma. Let \( P \in \mathcal{P}^n \) be an \( n \)-polytope with \( 0 \in \text{int} P \). For a proper face \( F \) of \( P \) let

\[ F^\circ = \{ y \in P^* : \langle x, y \rangle = 1 \text{ for all } x \in F \}. \]

Then

i) \( F^\circ \) is a face of \( P^* \).

ii) \( F = (F^\circ)^\circ \).

iii) If \( G \) is a face of \( P \) and \( F \subseteq G \), then \( G^\circ \subseteq F^\circ \).

iv) \( \dim F^\circ = n - 1 - \dim F \).

3.13 Theorem. Let \( P \in \mathcal{P}^n \) be an \( n \)-polytope with \( 0 \in \text{int} P \). Then

\[ f_{n-1-i}(P^*) = f_i(P), \quad -1 \leq i \leq n. \]

3.14 Theorem. Let \( P \in \mathcal{P}^n \) be an \( n \)-polytope with facets \( F_1, \ldots, F_m \) and let \( H(a_i, \alpha_i), 1 \leq i \leq m \), be the supporting hyperplanes of \( F_i, 1 \leq i \leq m \). Then

\[ P = \{ x \in \mathbb{R}^n : \langle a_i, x \rangle \leq \alpha_i, 1 \leq i \leq m \}. \]

3.15 Theorem. Let \( P \in \mathcal{P}^n \) be an \( n \)-polytope.

i) The boundary of \( P \) is the union of all its facets.

ii) A \( k \)-face is the intersection of (at least) \( (n - k) \) facets.

iii) An \((n - 2)\)-face is contained in exactly two facets.

iv) If \( F, G \) are faces of \( P \) with \( F \subseteq G \), then \( F \) is a face of \( G \).

v) A face of \( P \) is also a face of a facet of \( P \).
3.16 Theorem. Let $P \in \mathcal{P}^n$ be an $n$-polytope.

i) Let $G$ be a face of $P$ and let $F$ be a face of $G$. Then $F$ is a face of $P$.

ii) Let $F_j$ be a $j$-face of $P$ and let $F_k$ be a $k$-face of $P$ with $F_j \subset F_k$. There exist $i$-faces $F_i$ of $P$, $j < i < k$, such that $F_j \subset F_{j+1} \subset \cdots \subset F_{k-1} \subset F_k$.

3.17 Remark. Let $v_0$ be a vertex of an $n$-polytope $P$ and let $\{v_1, \ldots, v_r\}$ be all adjacent vertices of $v_0$, i.e., $\text{conv} \{v_0, v_i\}$ is an edge of $P$. In other words, $\{v_1, \ldots, v_r\}$ are the neighbours of $v_0$. Then

i) $P \subset v_0 + \text{pos} \{v_1 - v_0, \ldots, v_r - v_0\}$.

ii) Let $c \in \mathbb{R}^n$ with $\langle c, v_0 \rangle \geq \langle c, v_i \rangle$, $1 \leq i \leq r$. Then $\max \{\langle c, x \rangle : x \in P\} = \langle c, v_0 \rangle$.

3.18 Theorem [Euler-Poincaré formula]. Let $P \in \mathcal{P}^n$. Then

$$\sum_{i=-1}^{n} (-1)^i f_i(P) = 0. \quad (3.18.1)$$

In particular, in the 3-dimensional case, i.e., $\dim P = 3$, it holds $f_0 - f_1 + f_2 = 2$.

3.19 Proposition. The Euler-Poincaré formula is the only linear equation satisfied by the $f$-vector, i.e., let $\lambda_i \in \mathbb{R}$, such that $\sum_{i=-1}^{n} \lambda_i f_i(P) = 0$ for all $P \in \mathcal{P}^n$. Then there exists a constant $\gamma \in \mathbb{R}$, such that $\lambda_i = \gamma (-1)^i$.

3.20 Definition [Simple and simplicial polytopes]. Let $P \in \mathcal{P}^n$.

i) $P$ is called simplicial if all proper faces are simplices.

ii) $P$ is called simple if every vertex is contained in exactly $\dim P$ many facets.

3.21 Lemma. Let $P \in \mathcal{P}^n$ be an $n$-polytope with $0 \in \text{int} P$. The following statements are equivalent:

i) $P$ is simplicial.

ii) All facets of $P$ are simplices.

iii) $P^*$ is simple.

iv) Every $k$-face of $P^*$ is contained in exactly $n-k$ facets for $k = 0, \ldots, n-1$. 
3.22 Theorem. Let $P \in \mathcal{P}^n$ be a simple $n$-polytope. Then

i) Every vertex is contained in exactly $\binom{n}{k}$ $k$-faces of $P$, $k = 0, \ldots, n - 1$.

ii) The intersection of $k \leq n$ facets containing a common vertex is an $(n-k)$-face of $P$.

iii) Let $v_1, \ldots, v_n$ be the neighbours of a vertex $v_0$ of $P$. For each subset of $k < n$ neighbours $v_{i_1}, \ldots, v_{i_k}$ there exists a unique $k$-face $F$ of $P$ containing $v_0, v_{i_1}, \ldots, v_{i_k}$.

iv) A face of a simple polytope is simple.

v) Every $j$ face of $P$ is contained in exactly $\binom{n-j}{k-j}$ $k$ faces of $P$.

3.23 Theorem. Let $P \in \mathcal{P}^n$ be a simple $n$-polytope.

i) $n f_0(P) = 2 f_1(P)$.

ii) $\sum_{k=0}^n f_k(P) \leq 2^n f_0(P)$.

iii) $f_0(P) \leq 2 f_{\lfloor n/2 \rfloor}(P)$.

Here, for $\rho \in \mathbb{R}$ the number $\lceil \rho \rceil$ is the smallest integer greater or equal than $\rho$.

3.24 Corollary. Let $P$ be a simple $n$-polytope with $m$ facets. Then

$$f_0(P) \leq 2 \binom{m}{\lfloor n/2 \rfloor} \quad \text{and} \quad \sum_{k=0}^n f_k(P) \leq 2^{n+1} \binom{m}{\lfloor n/2 \rfloor}.$$ 

Or equivalently: Let $P$ be a simplicial $n$-polytope with $m$ vertices. Then

$$f_{n-1}(P) \leq 2 \binom{m}{\lfloor n/2 \rfloor} \quad \text{and} \quad \sum_{k=0}^n f_k(P) \leq 2^{n+1} \binom{m}{\lfloor n/2 \rfloor}.$$ 

Here, for $\rho \in \mathbb{R}$ the number $\lfloor \rho \rfloor$ is the largest integer not greater than $\rho$.

3.25 Lemma. Let $P$ be an $n$-polytope.

i) There exists a simple $n$-polytope $Q$ with the same number of facets as $P$ and $f_i(P) \leq f_i(Q)$, $0 \leq i \leq n - 2$.

ii) There exists a simplicial $n$-polytope $Q^*$ with the same number of vertices as $P$ and $f_i(P) \leq f_i(Q^*)$, $1 \leq i \leq n - 1$.
3.26 Corollary. Let $P$ be an $n$-polytope with $m$ facets. Then

$$f_0(P) \leq 2 \binom{m}{n/2}.$$  

Or equivalently: Let $P$ be an $n$-polytope with $m$ vertices. Then

$$f_{n-1}(P) \leq 2 \binom{m}{n/2}.$$  

3.27 Definition [Cyclic polytopes]. The curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ given by $\gamma(t) = (t, t^2, t^3, \ldots, t^n)^T$ is called moment curve. The convex hull of $m$ points on the moment curve is called a cyclic polytope with $m$ vertices and is denoted by $C(n, m)$.

3.28 Proposition. Any $n + 1$ points on the moment curve are affinely independent. In particular, cyclic polytopes are simplicial polytopes.

3.29 Proposition [Gale’s evenness condition]. Let $t_i \in \mathbb{R}$, $1 \leq i \leq m$, $t_1 < t_2 < \cdots < t_m$, $\gamma(t_i) = (t_i, t_i^2, t_i^3, \ldots, t_i^n)^T$, $1 \leq i \leq m$, and let $S \subset \{1, \ldots, m\}$ be a subset of cardinality $n$. $F_S = \text{conv} \{\gamma(t_s) : s \in S\}$ is a facet of $C(n, m)$ if and only if $\# \{s \in S : i < s < j\}$ is even for all $i, j \in \{1, \ldots, m\} \setminus S$.

3.30 Remark. All points $\gamma(t_i)$ are vertices of $C(n, m)$ and the number of $i$-faces of $C(n, m)$ is independent of the choice of the $m$-points on the moment curve.

3.31 Proposition. The cyclic polytope $C(n, m)$ is $\lfloor n/2 \rfloor$-neighborly, i.e., the convex hull of any subset of the vertices of cardinality less than or equal $\lfloor n/2 \rfloor$ is a face.

3.32 Theorem* [McMullen’s Upper Bound Theorem, 1971]. Let $P$ be an $n$-polytope with $m$ vertices. Then

$$f_i(P) \leq f_i(C(n, m)) = \begin{cases} \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \frac{m-j}{m-2} \binom{m-j}{i+j+1} \binom{m-j}{i+j+1}, & n \text{ odd}, \\ \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{m-j}{m-j} \binom{m-j}{i+j-1} \binom{m-j}{i+j-1}, & n \text{ even}. \end{cases}$$

In particular,

$$f_{n-1}(P) \leq f_{n-1}(C(n, m)) = \begin{cases} 2^{\binom{m-\lfloor n/2 \rfloor-1}{\lfloor n/2 \rfloor}}, & n \text{ odd}, \\ \binom{m-\lfloor n/2 \rfloor}{\lfloor n/2 \rfloor} + \binom{m-\lfloor n/2 \rfloor-1}{\lfloor n/2 \rfloor-1}, & n \text{ even}. \end{cases}$$

For fixed $n$ the right hand sides are of order $m^{\lfloor n/2 \rfloor}$.

3.33 Theorem* [Barnette’s Lower Bound Theorem, 1971]. Let $P$ be a simplicial $n$-polytope with $m$ vertices. $P$ has at least as many $i$-faces as the so called stacked polytopes $P(n, m)$ with $m$ vertices for which

$$f_i(P(n, m)) = \begin{cases} \binom{n}{i} - \binom{n}{i+1}, & 0 \leq i \leq n - 2, \\ n + 1 + (m - (n + 1))(n - 1), & i = n - 1. \end{cases}$$
$P(n,n+1)$ is an $n$-simplex, and for $m \geq n + 2$ an $m$-vertex stacked $n$-polytope $P(n,m)$ is the convex hull of an $(m - 1)$-vertex stacked polytope with an additional point that is beyond exactly one facet.

3.34 Theorem [Dehn-Sommerville equations, 1905, 1927]. Let $P$ be a simple $n$-polytope. Then

$$f_i(P) = \sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} f_j(P), \quad i = 0, \ldots, n,$$

or equivalently: Let $P$ be a simplicial $n$-polytope. Then

$$f_{i-1}(P) = \sum_{j=i}^{n} (-1)^{n-j} \binom{j}{i} f_{j-1}(P), \quad i = 0, \ldots, n.$$

3.35 Definition [h-vector]. Let $P \in \mathcal{C}^n$ be a simple $n$-polytope. The vector $h(P) = (h_0(P), \ldots, h_n(P))$ with entries

$$h_i(P) = \sum_{j=0}^{n} (-1)^{i-j} \binom{j}{i} f_j(P)$$

is called $h$-vector of $P$.

3.36 Remark. In terms of the $h$-vector, the Dehn-Sommerville equations become $h_i(P) = h_{n-i}(P), i = 0, \ldots, n$.

3.37 Theorem* [McMullen’s $g$-Theorem]. McMullen’s $g$-theorem gives a complete characterization of the $f$-vectors of simple (or simplicial) polytopes in terms of its $g$-vector which is given by $g_i(P) = h_i(P) - h_{i-1}(P), i = 1, \ldots, \lfloor n/2 \rfloor$.

3.38 Remark. For any $n$-polytope $P \in \mathcal{P}^n$ we have $nf_0(P) \leq 2f_1(P)$ with equality iff $P$ simple and $n f_{n-1}(P) \leq 2f_{n-2}(P)$ with equality iff $P$ simplicial.

3.39 Theorem [Steinitz, 1906]. A non-negative integral vector $(f_0, f_1, f_2)$ is the $f$-vector of a 3-polytope if and only if i) $f_0 - f_1 + f_2 = 2$, ii) $3f_0 \leq 2f_1$, and iii) $3f_2 \leq 2f_1$. 

3.40 Theorem [Figiel, Lindenstrauss, Milman, 1977]. Let $P \in \mathcal{P}^n$ be a 0-symmetric $n$-polytope, i.e., $P = -P$. Then
\[ \ln(f_0(P)) \ln(f_{n-1}(P)) \geq \frac{1}{16} n. \]

3.41 Conjecture [Kalai, 1989]. Let $P \in \mathcal{P}^n$ be a 0-symmetric $n$-polytope. Then
\[ \sum_{i=0}^{n} f_i(P) \geq 3^n. \]
Here we have equality, for instance, for the cube $C_n$ and its polar, the cross-polytope $C_n^\star$, or, more generally, for the class of Hanner-polytopes. Recently, the conjecture has been verified for all $n \leq 4$ (see [http://front.math.ucdavis.edu/0708.3661](http://front.math.ucdavis.edu/0708.3661)).

3.42 Theorem [Balinski, 1961]. Let $P \in \mathcal{P}^n$ be an $n$-polytope. The graph of $P$ consisting of its vertices and edges is $n$-connected, i.e., the graph is still connected if $n - 1$ vertices and their adjacent edges are removed.

3.43 Definition [Combinatorial diameter]. Let $P \in \mathcal{P}^n$. The combinatorial distance $\delta_P(v, w)$ between two vertices $v, w \in \text{vert } P$ is the minimum length of an ”edge path” connecting $v$ and $w$, i.e., the minimum length of a sequence $[v, v_{i_1}, \ldots, v_{i_l}, w]$, with $v_{i_j} \in \text{vert } P$ and two consecutive vertices form an edge. 

$\delta(P) = \max \{ \delta_P(v, w) : v, w \in \text{vert } P \}$ is called the (combinatorial) diameter of $P$. For $n, m \in \mathbb{N}$ let
\[ \Delta(n, m) = \max \{ \delta(P) : P \in \mathcal{P}^n, \dim P = n \text{ and } f_{n-1}(P) = m \}. \]

3.44 Example. $\delta(T_n) = 1$, $\delta(C_n) = n$ and $\delta(C_n^\star) = 2$.

3.45 Conjecture [Hirsch, 1957]. $\Delta(n, m) \leq m - n$.

3.46 Remark. It is known that
i) the conjecture is true if $n \leq 3$ or $m \leq n + 5$ (Klee&Walkup, 1961/1965),
ii) the conjecture would be false for unbounded polyhedra,
iii) $\Delta(n, m) \leq \frac{1}{3} 2^{n-2}(m - n + \frac{5}{2})$ (Barnette, 1974),

iv) $\Delta(n, m) \leq 2m^{\log(n)+1}$ (Kalai, 1992),

v) it suffices to prove the conjecture for simple polytopes with $m = 2n$ (Klee&Walkup, 1961/1965)!

vi) Disproof of the Hirsch conjecture by Francisco Santos, 2010, see \url{http://front.math.ucdavis.edu/1006.2814}

3.47 Definition [0/1-polytope]. Let $[0,1]^n$ be the n-dimensional unit cube with vertices $\{0,1\}^n = \{(x_1, \ldots, x_n)^T : x_i \in \{0,1\}\}$. $P \in \mathcal{P}^n$ is called 0/1-polytope if $\text{vert } P \subset \{0,1\}^n$.

3.48 Lemma. Let $P \in \mathcal{P}^n$ be a 0/1-polytope and let $\dim P \leq n - 1$. Then there exists a 0/1-polytope $\tilde{P} \in \mathcal{P}^{n-1}$ affinely isomorphic to $P$, i.e., there exists a bijective map between $P$ and $\tilde{P}$.

3.49 Theorem [Naddef, 1989].

i) Let $P$ be a 0/1-polytope. Then $\delta(P) \leq \dim P$.

ii) Let $P \in \mathcal{P}^n$ be an n-dimensional 0/1-polytope with $m$ facets. Then $\delta(P) \leq m - n$.

3.50 Remark.

i) $f_{n-1}(P) \leq 2n!$ for a 0/1-polytope $P \in \mathcal{P}^n$.

ii) There exist 0/1-polytopes $P \in \mathcal{P}^n$ with

$$f_{n-1}(P) \geq \left(\frac{cn}{\log^2 n}\right)^{\frac{2}{3}},$$

where $c$ is a universal constant (Gatzouras, Giannopoulos, Markoulakis, 2004).
4 A bit of Gale diagrams and triangulations

4.1 Definition. Let \( a_1, \ldots, a_m \in \mathbb{R}^n \), \( m \geq n + 1 \), such that their affine hull spans \( \mathbb{R}^n \), and let \( \bar{a}_i = (a_i, 1)^T \in \mathbb{R}^{n+1}, 1 \leq i \leq m \), and let \( \bar{\mathcal{A}} = (\bar{a}_1, \ldots, \bar{a}_m) \in \mathbb{R}^{(n+1) \times m} \). Let \( \bar{B} = (b_1, \ldots, b_{m-(n+1)}) \in \mathbb{R}^{m \times (m-(n+1))} \) be a basis of the subspace \( \{ x \in \mathbb{R}^m : \bar{\mathcal{A}} x = 0 \} \), i.e., of the kernel of \( \bar{\mathcal{A}} \), and let \( \bar{B}^T = (b_1, \ldots, b_m) \in \mathbb{R}^{(m-(n+1)) \times m} \).

The vectors \( b_1, \ldots, b_m \in \mathbb{R}^{m-(n+1)} \) are called the Gale transform of the points \( a_1, \ldots, a_m \).

4.2 Remark.

i) The Gale transform is unique up to linear isomorphisms of \( \mathbb{R}^{(m-(n+1))} \).

ii) For the vectors of a Gale transform always hold \( \sum_{i=1}^m b_i = 0 \).

4.3 Proposition. Let \( P = \text{conv} \{v_1, \ldots, v_m\} \) with vertices \( v_i, 1 \leq i \leq m \), and for \( J \subseteq [m] = \{1, \ldots, m\} \) let \( V_J = \{v_j : j \in J\} \). Then \( \text{conv} V_J \) is a face of \( P \) if and only if \( \text{conv} V_{[m] \setminus J} \cap \text{aff} V_J = \emptyset \).

For \( \#J = 1 \) the statement is also true without the assumption that \( v_i \) are vertices.

4.4 Theorem. Let \( P = \text{conv} \{v_1, \ldots, v_m\}, \dim P = n \), with vertices \( v_i, 1 \leq i \leq m \), and for \( J \subseteq [m] = \{1, \ldots, m\} \) let \( V_J = \{v_j : j \in J\} \). Let \( \{b_1, \ldots, b_m\} \) be the Gale transform of \( \{v_1, \ldots, v_m\} \). Then \( \text{conv} V_J \) is a face of \( P \) if and only if \( J = [m] \) or \( 0 \in \text{relint} (\text{conv} \{b_k : k \notin J\}) \).

For \( \#J = 1 \) the statement is also true without the assumption that \( v_i \) are vertices.

4.5 Corollary. \( \{b_1, \ldots, b_m\} \in \mathbb{R}^{m-(n+1)} \) is the Gale transform of the vertex set \( \{v_1, \ldots, v_m\} \in \mathbb{R}^n \) of a polytope if and only if for each hyperplane \( H(a,0) \subset \mathbb{R}^{m-(n+1)} \) the halfspaces \( H^+(a,0) \) and \( H^-(a,0) \) contain at least two points of \( \{b_1, \ldots, b_m\} \) in its interior.

4.6 Definition [Face lattice]. For a polytope \( P \in \mathcal{P}^n \) let \( \mathcal{F}(P) \) be the set of all its faces. Together with the inclusion relation ”\( \subseteq \)” on \( \mathcal{F}(P) \) the faces form a partially ordered set (poset) denoted by \( (\mathcal{F}(P), \subseteq) \) and which is called the face lattice of \( P \).

4.7 Definition [Combinatorially isomorphic]. Two polytopes \( P, Q \in \mathcal{P}^n \) are called combinatorially isomorphic or combinatorially equivalent if the face lattices \( (\mathcal{F}(P), \subseteq) \) and \( (\mathcal{F}(Q), \subseteq) \) are isomorphic, i.e., there exists an inclusion preserving bijection between the faces of \( P \) and \( Q \).

4.8 Theorem. There are precisely \( \lfloor \frac{1}{2} n^2 \rfloor \) combinatorial different types of \( n \)-polytopes with \( n + 2 \) vertices.
Proof. For such an \( n \)-polytope \( P = \text{conv}\{v_1, \ldots, v_{n+2}\} \) the Gale transform
\( \{b_1, \ldots, b_{n+2}\} \) is one-dimensional. Since \( 0 \in \text{int conv}\{b_1, \ldots, b_{n+2}\} \) we can classify the Gale transform by the number of points \( l, e, g \) which are \( < 0, = 0 \) and \( > 0 \), respectively. Then we have \( l + e + g = n + 2 \) and according to Corollary 4.5 we have \( l, g \geq 2 \) and by symmetry we may assume \( l \leq g \). So for a fixed \( e \in \{0, \ldots, n - 2\} \) the parameter \( l \) is bounded between 2 and \( \lfloor (n + 2 - e) / 2 \rfloor \), and so there are \( \lfloor (n - e) / 2 \rfloor \) possibilities. Altogether we have found
\[
\sum_{e=0}^{n-2} \left\lfloor \frac{n - e}{2} \right\rfloor = \left\lfloor \frac{1}{4} n^2 \right\rfloor
\]
different possibilities of Gale transforms which correspond by Corollary 4.5 to all \( n \) polytopes with \( (n + 2) \) vertices. It remains to show that these polytopes are combinatorially inequivalent. To this end let \( (l_i, e_i, g_i) \) be two different sequences with polytopes \( P_i, i = 1, 2 \), and assume that the polytopes are combinatorially equivalent. Let \( \phi \) be an isomorphism between the face lattices of \( P_i \) which induces an isomorphism on the points of the Gale transforms. First we observe that all points of the groups "\( e_i \)" have to be mapped onto each other, since removing all other points yields a face of the polytope in view of Theorem 4.4. So we may assume \( e_1 = e_2 \) and suppose \( 1 < l_1 < l_2 \leq g_2 < g_1 \). The image of the points of the first group "\( l_1 \)" cannot consists of all points of \( l_2 \) or of all points of \( g_2 \). Hence removing the image of the points yields a face of \( P_2 \) but not of \( P_1 \) according to Theorem 4.4. Hence we get a contradiction. \( \square \)

4.9 Definition [Point configuration]. A point configuration in \( \mathbb{R}^n \) is a finite set of (perhaps repeated) points with (non-repeated) labels. It is identified with its matrix \( A = (a_1, \ldots, a_m) \in \mathbb{R}^{n \times m} \).

4.10 Definition [Triangulation]. A triangulation of a point configuration \( A = (a_1, \ldots, a_m) \subset \mathbb{R}^{n \times m} \) is a collection \( \mathcal{T} \) of simplices, with vertices in \( A \), that satisfies the following properties:

i) All faces of simplices of \( \mathcal{T} \) are in \( \mathcal{T} \).

ii) The intersection of any two simplices of \( \mathcal{T} \) is a face (possible empty) of both.

iii) The union of all simplices of \( \mathcal{T} \) equals \( \text{conv} A \).

4.11 Remark.

i) All vertices of \( \text{conv} A \) are in \( \mathcal{T} \).

ii) Let \( F \) be a face of \( \text{conv} A \). Then \( \mathcal{T}_F := \{ \sigma \in \mathcal{T} : \sigma \subset F \} \) is a triangulation of \( A \cap F \).

iii) The first two properties of Definition 4.10 are the definition of a (geometric) simplicial complex.
4.12 Definition [Height function and lower convex hull]. Let $A = (a_1, \ldots, a_m) \in \mathbb{R}^{n \times m}$ be a point configuration. Let $\eta : A \to \mathbb{R}_{\geq 0}$ be a function and let $A_\eta = \left( a_1, \eta(a_1) \atop a_2, \eta(a_2) \atop \ldots \atop a_m, \eta(a_m) \right) \in \mathbb{R}^{(n+1) \times m}$ be the lifted point configuration (also called a lifting). Then $\eta$ is called a height function. The union of the faces of $\text{conv} A_\eta$ which are visible from below are called the lower convex hull of $\text{conv} A_\eta$. More precisely, $y = (y_1^T, \ldots, y_n^T, y_{n+1}) \in \text{conv} (A_\eta)$ belongs to the lower convex hull if the segment $\text{conv} \{y, (y_1^T, \ldots, y_n^T, 0)^T\}$ intersects $\text{conv} A_\eta$ only in $y$.

4.13 Definition [Regular triangulation]. A triangulation of a point configuration $A \in \mathbb{R}^{n \times m}$ is called a regular triangulation if it can be obtained by projecting the lower convex hull of a lifting of $A$.

4.14 Theorem. Every point configuration $A \in \mathbb{R}^{n \times m}$ has regular triangulations.

4.15 Definition [Delaunay triangulation]. Let $A \in \mathbb{R}^{n \times m}$ be a point configuration containing $n + 1$ affinely independent points. Let $\eta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be the height function given by $\eta(x) = |x|^2$. If the projection of the lower convex hull of $\text{conv} A_\eta$ yields a triangulation, then it is called Delaunay triangulation.

4.16 Lemma. Let $C \subset \mathbb{R}^{n+1}$ be the paraboloid $C = \{(x, |x|^2) : x \in \mathbb{R}^n\}$. Let $H(a, \alpha)$, $a \in \mathbb{R}^{n+1}$, $a_{n+1} \neq 0$, be a hyperplane, and let $\Pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection forgetting the last coordinate. Then $\Pi(C \cap H(a, \alpha))$ is either empty, a point or an $n$-dimensional sphere.

4.17 Corollary. Let $A \in \mathbb{R}^{n \times m}$ be a point configuration containing $n + 1$ affinely independent points. Let $\eta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be the height function given by $\eta(x) = |x|^2$. Let $B \subseteq A$ such that $B$ contains $n + 1$ affinely independent points. Then $B$ corresponds to the vertex set of a facet of the lower convex hull of the lifted points if and only if there is sphere passing through all points of $B$ and leaving all points of $A \setminus B$ outside.

4.18 Remark. In particular, simplices in a Delaunay triangulation $\mathcal{T}$ are characterized by the "empty sphere" property: $\sigma \in \mathcal{T}$ if and only if there is an Euclidean sphere with all vertices of $\sigma$ on the sphere and with the rest of the points outside.
A bit of Gale diagrams and triangulations
5 A glimpse of Ehrhart theory

5.1 Notation. For $S \subset \mathbb{R}^n$, $S$ bounded, we denote by $G(S)$ its lattice point enumerator, i.e.,
\[ G(S) = \#(S \cap \mathbb{Z}^n). \]

Almost all results of this section are valid for arbitrary lattices; for simplification, however, we state most of them only for the standard lattice $\mathbb{Z}^n$.

5.2 Definition [Lattice polytope]. A polytope $P = \text{conv}\{v_1, \ldots, v_m\} \subset \mathbb{R}^n$ is called a lattice polytope if $v_i \in \mathbb{Z}^n$, $1 \leq i \leq m$. The set of all lattice polytopes is denoted by $\mathcal{P}_n^\mathbb{Z}$.

5.3 Lemma. Let $a_1, \ldots, a_n \in \mathbb{Z}^n$ be linearly independent, and let $P$ be the half open parallelepiped $P = \{\rho_1 a_1 + \cdots + \rho_n a_n : 0 \leq \rho_i < 1\}$. Then
\[ G(P) = \text{vol}(P) = |\text{det}(a_1, \ldots, a_n)|. \]

5.4 Lemma. Let $T = \text{conv}\{0, v_1, v_2\} \in \mathcal{P}_2^\mathbb{Z}$ be a lattice triangle, i.e., $v_1, v_2 \in \mathbb{Z}^2$ are linearly independent. Then
\[ G(T) = \text{vol}(T) + \frac{1}{2}G(\text{bd}T) + 1. \]

5.5 Theorem [Pick]. Let $P \in \mathcal{P}_2^\mathbb{Z}$, $\dim P = 2$. Then
\[ G(P) = \text{vol}(P) + \frac{1}{2}G(\text{bd}P) + 1. \]

5.6 Corollary.

i) Let $P \in \mathcal{P}_2^\mathbb{Z}$, $\dim P = 2$, with edges $F_1, \ldots, F_m$. Then
\[ G(P) = \text{vol}(P) + \frac{1}{2} \sum_{i=1}^{m} \frac{\text{vol}(F_i)}{\text{det}(\text{aff} F_i \cap \mathbb{Z}^2)} + 1. \]

Here $\text{det}(\text{aff} F_i \cap \mathbb{Z}^2)$ is the distance of two consecutive lattice points on $\text{aff} F_i$.

ii) Let $K \in \mathcal{K}^2$. Then $G(K) \leq \text{vol}(K) + \frac{1}{2}F(K) + 1$, where $F(K)$ denotes the perimeter of $K$.

5.7 Remark.

\footnote{Georg Alexander Pick; 10.08.1859 (Vienna) – 26.07.1942 (concentration camp Theresienstadt)}
i) Inequality ii) of Corollary 5.6 cannot be generalized in a straightforward and best possible way to arbitrary lattices, since the perimeter is not an affine equivariant functional, in contrast to the area.

ii) Pick’s theorem itself, however, can be generalized in various ways. Its proof is only based on the property that a set \( S \) can be subdivided into lattice triangle such that the intersection of any two of them is a face of both. For those sets it was shown by Hadwiger&Wills that

\[
G(S) = \text{vol}(S) + \frac{1}{2} E(S) + \chi(S).
\]

Here \( \chi(S) \) is the Euler-Poincaré characteristic of \( S \), and \( E(S) \) is the number of segments between two consecutive lattice points in the boundary of \( S \), where segments are counted twice which are not bordering a 2-dimensional cell.

iii) By Pick’s theorem we immediately get the following polynomial behaviour of the lattice point enumerator of a lattice polygon \( P \in \mathcal{P}^2_\mathbb{Z} \)

\[
G(k P) = \text{vol}(P) k^2 + \frac{1}{2} G(bd\, P) k + 1,
\]

\[
G(\text{int}(k P)) = \text{vol}(P) k^2 - \frac{1}{2} G(bd\, P) k + 1.
\]

5.8 Notation. For integers \( m, n \) we denote by

\[
\binom{x + m}{n} = \frac{1}{n!} \prod_{i=0}^{n-1} (x + m - i)
\]

the polynomial of degree \( n \) with roots \( i - m, \ i = 0, \ldots, n - 1 \), and leading coefficient \( 1/n! \). In particular, the polynomials \( \binom{x+n-i}{n} \), \( i = 0, \ldots, n \), form a basis of the space of all polynomials of degree at most \( n \).

5.9 Lemma. Let \( T = \text{conv}\{0,v_1,\ldots,v_n\} \in \mathcal{P}^n_\mathbb{Z} \) be a lattice simplex, i.e., \( v_1,\ldots,v_n \in \mathbb{Z}^n \) are linearly independent, and for \( 0 \leq i \leq n \) let

\[
a_i(T) = \# \left\{ \sum_{j=1}^{n} \lambda_j v_j \in \mathbb{Z}^n : 0 \leq \lambda_j < 1, \ i - 1 < \sum_{j=1}^{n} \lambda_j \leq i \right\}.
\]

Then for all \( k \in \mathbb{N}, k \geq 1 \), we have

\[
G(kT) = \sum_{i=0}^{n} a_i(T) \binom{k + n - i}{n}.
\]

5.10 Notation [Characteristic Function]. For a set \( A \subseteq \mathbb{R}^n \) let \( \chi_A : \mathbb{R}^n \to \{0,1\} \) with

\[
\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}
\]

be its characteristic function.
5.11 Lemma [Inclusion-Exclusion Formula]. Let $A_i \subseteq \mathbb{R}^n$, $1 \leq i \leq m$, with characteristic functions $\chi(A_i)$. Then
\[
\chi(A_1 \cup A_2 \cup \cdots \cup A_m) = \sum_{I \subseteq \{1, \ldots , m\} \setminus \emptyset} (-1)^{\#I-1} \chi \left( \bigcap_{j \in I} A_j \right).
\]

5.12 Remark. By evaluating and summing both sides of Lemma 5.11 over all lattice points $\mathbb{Z}^n$ we find for bounded sets $A_i \subset \mathbb{R}^n$, $1 \leq i \leq m$
\[
G(A_1 \cup A_2 \cup \cdots \cup A_m) = \sum_{I \subseteq \{1, \ldots , m\} \setminus \emptyset} (-1)^{\#I-1} G \left( \bigcap_{j \in I} A_j \right).
\]

5.13 Theorem [Ehrhart, 1967]. Let $P \in \mathcal{P}^n_{\mathbb{Z}}$. Then there exist numbers $G_i(P) \in \mathbb{Q}$, $0 \leq i \leq n$, depending only on $P$, such that for all $k \in \mathbb{N}_{\geq 1}$
\[
G(kP) = \sum_{i=0}^n G_i(P) k^i.
\]
The right hand side is called Ehrhart-polynomial.

5.14 Definition.

i) Let $\text{GL}(n, \mathbb{Z}) = \{U \in \mathbb{Z}^{n \times n} : |\det U| = 1\}$ be the group of unimodular matrices. For $U \in \text{GL}(n, \mathbb{Z})$ and $z \in \mathbb{Z}^n$ the map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $f(x) = Ux + z$ is called an unimodular transformation.

ii) A functional $g : \mathcal{P}^n_{\mathbb{Z}} \rightarrow \mathbb{R}$ with
\[
g(P \cup Q) + g(P \cap Q) = g(P) + g(Q)
\]
for all $P, Q \in \mathcal{P}^n_{\mathbb{Z}}$ with $P \cup Q, P \cap Q \in \mathcal{P}^n_{\mathbb{Z}}$ is called additive.

5.15 Proposition. The unimodular transformations are exactly those affine transformations $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which leaves $\mathbb{Z}^n$ invariant, i.e., $f(\mathbb{Z}^n) = \mathbb{Z}^n$.

5.16 Proposition. Let $P \in \mathcal{P}^n_{\mathbb{Z}}$.

i) $G_n(P) = \text{vol} (P)$.

ii) $G_i : \mathcal{P}^n_{\mathbb{Z}} \rightarrow \mathbb{R}$ is homogeneous of degree $i$, invariant with respect to unimodular transformations and additive, $0 \leq i \leq n$.

iii) $G_i(P)$ are independent of the dimension of the space in which $P$ is embedded, i.e., let $P \in \mathcal{P}^n_{\mathbb{Z}}$ and let $\tilde{P} = \text{conv} \{(v,0)^T : v \in P\} \in \mathcal{P}^{n+1}_{\mathbb{Z}}$. Then $G_i(\tilde{P}) = G_i(P)$, $i = 0, \ldots , n$.

---

2Eugène Ehrhart, 29.04.1906 (Guebwiller (Haut-Rhin, France)) – 17.01.2000 (Strasbourg)
5.17 Theorem* [Betke, Kneser, 1985]. Every additive and unimodular invariant functional on the space of all lattice polytopes is a linear combination of the $n + 1$ functionals $G_i(\cdot)$.

5.18 Remark. Some of the coefficients $G_i(P)$ might be negative. One family of standard examples in this context are the so called Reeve-simplices: let $R_m = \text{conv} \{0, e_1, e_2, (1, 1, m)^\top\} \in \mathcal{P}_Z^3$ for $m \in \mathbb{N}$. The only lattice points contained in $R_m$ are the four vertices, the volume, however, can be arbitrarily large. Hence some $G_i(R_m)$ must be negative for large $m$. More precisely, it is $G_3(R_m) = m/6$, $G_2(R_m) = 1$, $G_1(R_m) = (12 - m)/6$ and $G_0(R_m) = 1$.

Of course, we can also rewrite the Ehrhart polynomial in terms of the binomial basis $\binom{k+n-i}{n}$ and get

5.19 Corollary. Let $P \in \mathcal{P}_Z^n$. Then there exist numbers $a_i(P) \in \mathbb{Q}$, $0 \leq i \leq n$, such that for all $k \in \mathbb{N}_{\geq 1}$

$$G(k P) = \sum_{i=0}^{n} a_i(P) \binom{k+n-i}{n}.$$ 

5.20 Example.

i) Let $T_n = \text{conv} \{0, e_1, e_2, \ldots, e_n\}$. Then

$$\#(kT_n \cap \mathbb{Z}^n) = \binom{n+k}{n},$$

and so we have $a_i(T_n) = 0$ for $1 \leq i \leq n$, and $a_0(T_n) = 1$. The $G_i(T_n)$ are $\pm 1$ – Stirling numbers of the first kind.

ii) Let $C_n = [-1, 1]^n$. Then $G(k C_n) = (2k+1)^n$ and so $G_i(C_n, \mathbb{Z}^n) = 2^i \binom{n}{i}$, $0 \leq i \leq n$. Here the $a_i(C_n)$ are some combinatorial numbers, the so called Eulerian numbers.

iii) Let $C_n^* = \text{conv} \{\pm e_i : 1 \leq i \leq n\}$. Then

$$G(k C_n^*) = \sum_{i=0}^{n} \binom{n}{i} \binom{k+n-i}{n},$$

and so $a_i(C_n^*) = \binom{n}{i}$, $i = 0, \ldots, n$. 

---

3Martin Kneser, 21.01.1928 (Greifswald) – 16.02.2004 (Göttingen)
4Ulrich Betke, ???.?? (???) – 24.05.2008 (Siegen)
iv) Let $\tilde{T}_2 = \text{conv}\{0, e_1, e_2\} \subseteq \mathbb{R}^3$ be the 2-dimensional standard triangle embedded in $\mathbb{R}^3$. Then by Pick’s Theorem 5.5

$$G(k\tilde{T}_2) = \frac{1}{2} k^2 + \frac{3}{2} k + 1 = (-1) \left(\binom{k}{3}\right) + \frac{1}{2} \left(\binom{k + 1}{3}\right) - \frac{1}{2} \left(\binom{k + 2}{3}\right) + \left(\binom{k + 3}{3}\right).$$

Hence, in this case we have $a_3(\tilde{T}_2) = -1$, $a_2(\tilde{T}_2) = 1/2$, $a_1(\tilde{T}_2) = -1/2$, $a_0(\tilde{T}_2) = 1$.

Finally, we state without proof some highlights of Ehrhart theory (and which are left to the lecture in the next term).

5.21 Remark.

i) Stanley’s Non-Negativity Theorem: Let $P \in \mathcal{P}_n^\mathbb{Z}$, $\dim P = n$. Then $a_i(P) \in \mathbb{N}_{\geq 0}$, $0 \leq i \leq n$.

ii) Stanley’s Monotonicity Theorem: Let $P, Q \in \mathcal{P}_n^\mathbb{Z}$, $\dim P = \dim Q = n$ with $P \subseteq Q$. Then $a_i(P, \mathbb{Z}^n) \leq a_i(Q, \mathbb{Z}^n)$, $0 \leq i \leq n$.

iii) Ehrhart-Macdonald Reciprocity: Let $P \in \mathcal{P}_n^\mathbb{Z}$, $\dim P = n$. Then

$$G(\text{int } kP) = (-1)^n \sum_{i=0}^{n} G_i(P)(-k)^i.$$ 

iv) Let $P \in \mathcal{P}_n^\mathbb{Z}$, $\dim P = n$. Then for $i \neq n \text{ mod } 2$

$$G_i(P) = \frac{1}{2} \sum_{j=i}^{n-1} (-1)^{i+j} \sum_{\text{face of } P} G_i(F).$$

In particular,

$$G_{n-1}(P) = \frac{1}{2} \sum_{\text{facet of } P} \frac{\text{vol}_{n-1}(F)}{\det(\text{aff } F \cap \mathbb{Z}^n)}.$$
A glimpse of Ehrhart theory
6 Borsuk, Hadwiger and planks

6.1 Definition [Diameter]. Let $X \subset \mathbb{R}^n$ be bounded.

$$D(X) = \sup \{|x - y| : x, y \in X\}$$

is called diameter of $X$.

6.2 Remark. Let $K \in \mathcal{C}^n$ be a convex body with support function $h(K, \cdot)$. Let $x, y \in K$ such that $D(K) = |x - y|$, and let $u = (x - y)/|x - y|$. The hyperplanes $H(u, \langle u, x \rangle)$ and $H(-u, \langle -u, y \rangle)$ are supporting hyperplanes in $x, y$, respectively, and it is

$$D(K) = \sup_{u \in S^{n-1}} h(K, u) + h(K, -u).$$

6.3 Definition [Convex body, smooth convex body]. A convex body is a convex compact set, $K \in \mathcal{C}^n$. It is called smooth if there exists an unique supporting hyperplane for any boundary point of $K$. The set of all convex bodies is denoted by $\mathcal{K}^n$.

6.4 Definition [Borsuk number]. Let $\mathfrak{B}(n)$ be the smallest number $k \in \mathbb{N}$ such that any bounded set $X \subset \mathbb{R}^n$ can be partitioned into $k$ sets having smaller diameter than $X$.

6.5 Remark. A regular simplex shows that $\mathfrak{B}(n) \geq n + 1$ and in 1932 Karol Borsuk asked the question whether $\mathfrak{B}(n) = n + 1$ which led to the so called Borsuk conjecture.

6.6 Theorem [Borsuk-Ulam]. Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the boundary of the $n$-dimensional unit ball $B_n$. Let $f : S^{n-1} \to \mathbb{R}^n$ be a continuous mapping. Then there exists a $x \in S^n$ with $f(x) = f(-x)$.

6.7 Proposition. $B_n$ can be partitioned into $n + 1$ sets of smaller diameter, but not less.

6.8 Theorem. Any smooth convex body $K \in \mathcal{C}^n$ can be partitioned into $n + 1$ sets of smaller diameter.

6.9 Proposition. $\mathfrak{B}(n) \leq 2^n$.

6.10 Theorem [Kahn&Kalai]. For large $n$ it is $\mathfrak{B}(n) \geq (1.1)^{\sqrt{n}}$.

6.11 Remark. The smallest known dimension with $\mathfrak{B}(n) > n + 1$ is 298. More precisely, Hinrichs&Richter proved $\mathfrak{B}(n) > n + 11$ for all $n \geq 298$.  

6.12 Definition [Gohberg-Markus-Hadwiger number]. Let $\mathcal{H}(n)$ be smallest number $m \in \mathbb{N}$ such that any convex body $K \subset \mathbb{R}^n$ can be covered by $m$ smaller homothetic copies of $K$. $\mathcal{H}(n)$ is called the Gohberg-Markus-Hadwiger number.

6.13 Remark.

i) The cube $C_n$ shows that $\mathcal{H}(n) \geq 2^n$, and it is conjectured that this lower bound is extremal, i.e., $\mathcal{H}(n) = 2^n$. The problem or conjecture is known as the Gohberg-Markus-Hadwiger problem/conjecture.

ii) So far the conjecture has only be confirmed for $n \leq 2$ (and possible for $n \leq 3$) and for some special classes of convex bodies. In particular for smooth bodis, where always $n + 1$ smaller homothetic copies suffice.

iii) A boundary point $x$ of a convex body $K \in \mathcal{K}^n$ is illuminated by a direction $v \in \mathbb{R}^n$, if the line $x + \mathbb{R}v$ intersects the interior of $K$. The minimal number $m$ needed to illuminate any convex body in $\mathbb{R}^n$ by $m$ directions is called the illumination number and it is denoted by $I(n)$. It was shown by Boltjanski that $\mathcal{H}(n) = I(n)$.

6.14 Definition [Minimal Width, plank].

i) Let $K \in \mathcal{K}^n$ be convex body with support function $h(K, \cdot)$. The minimal breadth of $K$, i.e.,
$$\min_{u \in S^{n-1}} h(K, u) + h(K, -u)$$
is called the minimal width of $K$ and will be denoted by $\Delta(K)$.

ii) For a vector $u \in \mathbb{R}^n \setminus \{0\}$ and $\alpha, \beta \in \mathbb{R}$ the set
$$S(u, \alpha, \beta) = \{x \in \mathbb{R}^n : \beta \leq \langle u, x \rangle \leq \alpha\}$$
is called a (parallel) strip or plank of width $(\alpha - \beta)/|u|$.

6.15 Remark [Plank problem]. Tarski asked the following: Let $K \in \mathcal{K}^n$ be a convex body of minimal width $\Delta(K)$ which is entirely covered by $m$ planks of width $\mu_i$, say. Is it true that
$$\sum_{i=1}^{m} \mu_i \geq \Delta(K)?$$

6.16 Lemma. Let $K \in \mathcal{K}^n$ be a convex body and let $v \in \mathbb{R}^n \setminus \{0\}$. Then there exists a chord in direction $v$ with parallel supporting hyperplanes at its ends, i.e., there exists a $t \in \mathbb{R}^n$ such that the two boundary points $\{v, w\} = \{t + \mathbb{R}v\} \cap \text{bd} K$ have parallel supporting hyperplanes.
6.17 Lemma. Let \( K \in \mathbb{K}^n \) be a convex body of minimal width \( \Delta(K) \) and let \( y \in \mathbb{R}^n \) with \( |y| < \Delta(K)/2 \). Then \( L_y = (K - y) \cap (K + y) \in \mathbb{K}^n \) is non-empty with \( \Delta(L_y) \geq \Delta(K) - 2|y| \).

6.18 Notation.

i) In the following we will assume that for a plank \( S(u, \alpha, \beta) \) the length of \( u \) is \( 1/2 \) of its width, i.e., \( 2|u|^2 = \alpha - \beta \) and so we may write \( \alpha = |u|^2 - \gamma \) and \( \beta = -|u|^2 - \gamma \) for a certain constant \( \gamma \). In this way, a plank is uniquely determined by \( u \) and \( \gamma \) and therefore we may write \( S(u, \gamma) \) for a plank.

ii) For a set of planks \( S(u_i, \gamma_i) \subset \mathbb{R}^n \) of widths \( 2|u_i| \), \( 1 \leq i \leq m \), and for a sign vector \( \epsilon \in \{-1, 1\}^m \) let

\[
P_\epsilon = \left\{ x \in \mathbb{R}^n : \langle u_i, x \rangle \begin{cases} \geq |u_i|^2 - \gamma_i, & \epsilon_i = 1 \\ \leq -|u_i|^2 - \gamma_i, & \epsilon_i = -1 \end{cases}, 1 \leq i \leq m \right\}
\]

be a polyhedron "outside the planks".

6.19 Lemma. Let \( S(u_i, \gamma_i) \subset \mathbb{R}^n \) be planks of widths \( 2|u_i| \), \( 1 \leq i \leq m \). Then

\[
\bigcup_{\epsilon \in \{-1, 1\}^m} \left( P_\epsilon - \sum_{i=1}^m \epsilon_i u_i \right) = \mathbb{R}^n.
\]

6.20 Theorem [Bang]. The answer to the Plank problem is “Yes”, i.e., let \( K \in \mathbb{K}^n \) be a convex body of minimal width \( \Delta(K) \) which is entirely covered by \( m \) planks \( S(u_i, \alpha_i, \beta_i) \) of width \( \mu_i \), say. Then \( \sum_{i=1}^m \mu_i \geq \Delta(K) \).

6.21 Remark. Bang also raised the following stronger question: Let \( K \in \mathbb{K}^n \) be a convex body with support function \( h(K, \cdot) \), and suppose \( K \) is covered by \( m \) planks \( S(u_i, \alpha_i, \beta_i) \) with \( u_i \in S^{n-1} \) and width \( \alpha_i - \beta_i \), \( 1 \leq i \leq m \). Is it true that

\[
\sum_{i=1}^m \frac{\alpha_i - \beta_i}{h(K, u_i) + h(K, -u_i)} \geq 1?
\]

For centrally symmetric bodies it was proven by Keith Ball in 1991.
7 Packings

7.1 Definition [Packing sets]. A subset \( D \subset \mathbb{R}^n \) is called a packing set of \( K \in K^n \) if for all \( x, y \in D \), \( x \neq y \),

\[
(x + \text{int } K) \cap (y + \text{int } K) = \emptyset.
\]

The family of all packing sets of \( K \) is denoted by \( \mathcal{P}(K) \).

7.2 Definition [Density of a Packing]. Let \( K \in K^n \) and \( D \in \mathcal{P}(K) \).

\[
\delta(K, D) = \limsup_{\lambda \to \infty} \frac{\text{vol}(K)\#\{x \in D : x + K \subset \lambda C_n\}}{\text{vol}(\lambda C_n)}
\]

is called the density of the packing \( D + K \) (with respect to the gauge body \( C_n = [-1, 1]^n \)).

7.3 Remark. \( \delta(K, D) \) depends on the gauge body \( C_n \). For instance, let \( D = \{z \in \mathbb{Z}^n : z \geq 0\} \in \mathcal{P}([0,1]^n) \). Then \( \delta([0,1]^n, D) = \frac{1}{2^n} \). Now let \( H_n = C_n \cap \{x \in \mathbb{R}^n : \|x_1 - x_2 + x_3 + \cdots + x_n\| \leq n - 1\} \). Then \( [0,1]^n \subset H_n \), \( \text{vol}(H_n) = 2^n - 2/n! \) and we get

\[
\limsup_{\lambda \to \infty} \frac{\text{vol}([0,1]^n)\#\{x \in D : [0,1]^n \subset \lambda H_n\}}{\text{vol}(\lambda H_n)} = \frac{1}{2^n - \frac{2}{n!}}.
\]

7.4 Theorem. Let \( K \in K^n \). The supremum \( \sup\{\delta(K, D) : D \in \mathcal{P}(K)\} \) is independent of the chosen gauge body, and there exists a packing set \( D_K \in \mathcal{P}(K) \) such that

\[
\sup\{\delta(K, D) : D \in \mathcal{P}(K)\} = \delta(K, D_K).
\]

7.5 Definition [Density of a Densest Packing]. Let \( K \in K^n \).

\[
\delta(K) = \sup\{\delta(K, D) : D \in \mathcal{P}(K)\}
\]

is called the density of a densest packing of \( K \) and a set \( D_K \in \mathcal{P}(K) \) with \( \delta(K) = \delta(K, D_K) \) is called a densest packing set of \( K \).

7.6 Proposition. Let \( K \in K^n \). The following properties hold:

i) \( 0 < \delta(K) \leq 1 \).

ii) \( \delta(t + AK) = \delta(K) \) for all \( A \in \text{GL}(n, \mathbb{R}) \) and \( t \in \mathbb{R}^n \).

iii) Let \( K \in \mathcal{K}_{0}^n \). Then \( D \in \mathcal{P}(K) \) if and only if \( |x - y|_{K} \geq 2 \) for all \( x, y \in D \), \( x \neq y \).
iv) Let $K \in \mathcal{K}^n$ and $D \in \mathcal{P}(K)$. Then

$$\delta(K, D) = \limsup_{\lambda \to \infty} \frac{\text{vol}(K) \#\{(D \cap \lambda C_n)\}}{\text{vol}(\lambda C_n)}.$$

v) $\mathcal{P}(K) = \mathcal{P}\left(\frac{1}{2}(K - K)\right)$ and for $D \in \mathcal{P}(K)$ we have

$$\delta(K, D) = \delta\left(\frac{1}{2}(K - K), D\right) \frac{\text{vol}(K)}{\text{vol}\left(\frac{1}{2}(K - K)\right)},$$

and consequently

$$\delta(K) = \delta\left(\frac{1}{2}(K - K)\right) \frac{\text{vol}(K)}{\text{vol}\left(\frac{1}{2}(K - K)\right)}.$$

7.7 Lemma. Let $S \subset \mathbb{R}^n$ be a bounded and measurable set with $\text{vol}(S) > 0$ and let $D \in \mathcal{P}(K)$. Then there exist $v, w \in \mathbb{R}^n$ such that

$$\frac{\text{vol}(K) \#\{(w + S) \cap D\}}{\text{vol}(S)} \leq \delta(K, D) \leq \frac{\text{vol}(K) \#\{(v + S) \cap D\}}{\text{vol}(S)}.$$

7.8 Remark. Let $K \in \mathcal{K}^n$.

$$R(K) = \min\{R > 0 : \exists x \in \mathbb{R}^n \text{ with } K \subseteq x + R B_n\}$$

is called the circumradius of $K$. The point $t_c \in \mathbb{R}^n$ with $K \subseteq t_c + R(K) B_n$ is called circumcenter and it is uniquely determined. Moreover, for some $k \in \{1, \ldots, n\}$, there exist $k + 1$ points $x_0, \ldots, x_k \in \text{bd} K \cap \text{bd} (t_c + R(K) B_n)$ and $\lambda_i > 0$, $0 \leq i \leq k$, with $\sum_{i=0}^{k} \lambda_i = 1$ such that $t_c = \sum_{i=0}^{k} \lambda_i x_i$.

7.9 Corollary. Let $K \in \mathcal{K}_0^n$. Then

i) $\delta(K) \geq 2^{-n}$,

ii) $\delta(B_n) \leq (n + 1) \sqrt{2}^{-n}$.

7.10 Remark. Chronologically, the first upper bound is due to Blichfeldt (1928), who proved

$$\delta(B_n) \leq \frac{n + 2}{2} \sqrt{2}^{-n}.$$ 

This was slightly improved by Rogers (1958) by a factor of $2/e$, roughly speaking. In 1973/74 Sidelnikov showed that

$$\delta(B_n) \leq 2^{-(0.509 + o(1))n}, \quad n \text{ large}.$$ 

Subsequently, this bound was improved by Levenštei (1975), Kabatjanskiĭ and Levenštei (1978) to

$$\delta(B_n) \leq 2^{-(0.599 + o(1))n}, \quad n \text{ large},$$

which is still the best known bound.
7.11 Definition [Lattice]. Let \( b_1, \ldots, b_n \in \mathbb{R}^n \) be linearly independent. The set
\[
\Lambda = \{ z_1 b_1 + z_2 b_2 + \cdots + z_n b_n : z_i \in \mathbb{Z}, 1 \leq i \leq n \}
\]
is called lattice. The set of generating vectors \( \{b_1, \ldots, b_n\} \) or the matrix \( B = (b_1, \ldots, b_n) \) with columns \( b_i \) is called basis of \( \Lambda \). An element \( b \in \Lambda \) is called lattice point of \( \Lambda \). The set of all lattices in \( \mathbb{R}^n \) is denoted by \( L^n \).

7.12 Definition [Unimodular matrix]. An integral matrix \( U \in \mathbb{Z}^{n \times n} \) is called unimodular iff \( |\det U| = 1 \). The group of all unimodular matrices is denoted by \( GL(n, \mathbb{Z}) \).

Observe that a matrix is unimodular if and only the matrix and its inverse are integral matrices.

7.13 Proposition. \( GL(n, \mathbb{Z}) = \{ U \in \mathbb{R}^{n \times n} : U \mathbb{Z}^n = \mathbb{Z}^n \} \).

7.14 Lemma. Let \( \Lambda = B \mathbb{Z}^n \in L^n \). \( A = (a_1, \ldots, a_n) \) is a basis of \( \Lambda \) iff there exists a \( U \in GL(n, \mathbb{Z}) \) such that \( A = BU \).

7.15 Definition [Determinant, fundamental cell]. Let \( \Lambda \in L^n \) with basis \( B = (b_1, \ldots, b_n) \).
   
   i) \( \det \Lambda = |\det B| \) is called determinant of \( \Lambda \).
   
   ii) \( P_B = \{ \rho_1 b_1 + \cdots + \rho_n b_n : 0 \leq \rho_i < 1, 1 \leq i \leq n \} = B [0,1)^n \) is called fundamental cell or fundamental parallelepiped of \( \Lambda \) (w.r.t. the basis \( B \)).

7.16 Remark.
   
   i) \( \det \Lambda \) is independent of the choice of the basis (cf. Lemma 7.14).
   
   ii) \( \det \Lambda = \text{vol}(P_B) \) and \( \det(\mu \Lambda) = |\mu|^n \det \Lambda, \mu \in \mathbb{R} \).
   
   iii) \( \det \Lambda \leq |b_1| \cdot |b_2| \cdots |b_n| \), with equality if and only if the vectors \( b_i \) are pairwise orthogonal (Hadamard inequality).
   
   iv) \( P_B \cap \Lambda = \{0\} \). Since \( (P_B - P_B) = B (-1,1)^n \) we even have \( (P_B - P_B) \cap \Lambda = \{0\} \).

7.17 Proposition. Let \( \Lambda \in L^2 \) and let \( a_1, a_2 \in \Lambda \) be linearly independent. Then
\[
a_1, a_2 \text{ basis of } \Lambda \iff \text{conv} \{0, a_1, a_2\} \cap \Lambda = \{0, a_1, a_2\}.
\]

7.18 Remark. An analogous statement to Lemma 7.17 does not exist in dimension \( \geq 3 \). For \( n \geq 3 \) and \( m \in \mathbb{N} \) let \( b(m) = (1, \ldots, 1, m)^\top \in \mathbb{R}^n \) and \( T^n(m) = \text{conv} \{0, e_1, \ldots, e_{n-1}, b(m)\} \). Then
\[
T^n(m) \cap \mathbb{Z}^n = \{0, e_1, \ldots, e_{n-1}, b(m)\},
\]
but the determinant of the lattice with basis \( \{e_1, \ldots, e_{n-1}, b(m)\} \) is \( m \). \( T^n(m) \) are called Reeve simplices.
7.19 Proposition. Let \( \Lambda = BZ^n \in \mathcal{L}^n \). Then
\[ \mathbb{R}^n = \bigcup_{b \in \Lambda} (b + P_B), \]
i.e., \( \mathbb{R}^n \) is the pairwise disjunct union of the lattice translates \( b + P_B \).

7.20 Corollary. Let \( K \in \mathcal{K}^n \) and \( \Lambda \in \mathcal{L}^n \cap \mathcal{P}(K) \). Then
\[ \delta(K, \Lambda) = \frac{\text{vol}(K)}{\det \Lambda}. \]

7.21 Definition [Density of a densest Lattice Packing]. For \( K \in \mathcal{K}^n \) the set \( \mathcal{P}_\mathcal{L}(K) = \mathcal{L}^n \cap \mathcal{P}(K) \) is called the family of all packing lattices of \( K \). For \( \Lambda \in \mathcal{P}_\mathcal{L}(K) \) the arrangement \( \Lambda + K \) is called a lattice packing of \( K \) and
\[ \delta_\mathcal{L}(K) = \sup \{ \delta(K, \Lambda) : \Lambda \in \mathcal{P}_\mathcal{L}(K) \} \]
is called the density of a densest lattice packing of \( K \).

7.22 Definition [Critical determinant and admissible lattices]. Let \( K \in \mathcal{K}_0^n \). A lattice \( \Lambda \) is called admissible for \( K \) (or \( K \)-admissible) if \( \text{int} K \cap \Lambda = \{0\} \).
\[ \Delta(K) = \inf \{ \det \Lambda : \Lambda \text{ admissible for } K \} \]
is called the critical determinant of \( K \).

7.23 Remark. Let \( K \in \mathcal{K}_0^n \) and \( \Lambda \in \mathcal{L}^n \). Then \( (1/\lambda_1(K, \Lambda)) \Lambda \) is admissible for \( K \).

7.24 Proposition [Critical lattice]. For \( K \in \mathcal{K}_0^n \) there exists a \( K \)-admissible lattice \( \Lambda_K \) with \( \det \Lambda_K = \Delta(K) \). Such a lattice will be called a critical lattice of \( K \).

7.25 Proposition. Let \( K \in \mathcal{K}^n \). Then
\[ \delta_\mathcal{L}(K) = \frac{\text{vol}(K)}{\Delta(K - K)}. \]

7.26 Remark. 

i) \( 0 < \delta_\mathcal{L}(K) \leq \delta(K) \leq 1 \).

ii) \( \delta_\mathcal{L}(AK + t) = \delta_\mathcal{L}(K) \) for all \( A \in \text{GL}(n, \mathbb{R}) \) and \( t \in \mathbb{R}^n \).

iii) For \( K \in \mathcal{K}_0^n \) we have \( \delta_\mathcal{L}(K) = 2^{-n} \text{vol}(K)/\Delta(K) \).
7.27 Theorem [Minkowski-Hlawka, 1943]. Let $S \subset \mathbb{R}^n$ be a bounded Jordan-measurable set with $\text{vol}(S) < 1$. Then there exists a lattice $\Lambda \in \mathcal{L}^n$ with
\[
\det \Lambda = 1 \quad \text{and} \quad S \cap \Lambda \setminus \{0\} = \emptyset.
\]

7.28 Remark. Theorem 7.27 remains true for Jordan-measurable, unbounded, closed sets.

7.29 Corollary. Let $S \subset \mathbb{R}^n$ be a bounded Jordan-measurable set with $S = -S$ and with $\text{vol}(S) < 2$. Then there exists a lattice $\Lambda \in \mathcal{L}^n$ with
\[
\det \Lambda = 1 \quad \text{and} \quad S \cap \Lambda \setminus \{0\} = \emptyset.
\]

7.30 Corollary. Let $K \in \mathcal{K}_0^n$. Then
\[
\delta_{\mathcal{L}}(K) \geq 2^{-(n-1)} \left[ \leftrightarrow \Delta(K) \leq \frac{\text{vol}(K)}{2} \right].
\]

7.31 Theorem. Let $S \subset \mathbb{R}^n$, $n \geq 2$, be a bounded ray set (i.e., if $x \in S$ then $\lambda x \in S$ for all $\lambda \in [0,1]$) with $\text{vol}(S) < \zeta(n)$. Then there exists a lattice $\Lambda \in \mathcal{L}^n$ with $\det \Lambda = 1$ and $S \cap \Lambda \setminus \{0\} = \emptyset$. Here $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ denotes the (Riemann) zeta function (ζ-function).

7.32 Theorem [K. Ball, 1992]. $\delta(B_n) \geq (n - 1)2^{-(n-1)}\zeta(n)$.

7.33 Remark. The densest lattice packings of balls are only known in dimensions $n \leq 8$ (Lagrange 1773, Gauss 1840, Korkine-Zolotareff 1872/73, Blichfeldt, 1934) and $n = 24$ (Cohn & Kumar, 2004).

7.34 Theorem [Swinnerton-Dyer, 1953]. Let $K \in \mathcal{K}_0^n$ and let $\Lambda_K \in \mathcal{L}^n$ be a critical lattice of $K$. Then
\[
\#(K \cap \Lambda_K \setminus \{0\}) \geq n(n + 1).
\]

7.35 Theorem. Let $K \in \mathcal{K}_0^n$.
\begin{enumerate}
\item[i)] Let $b_1, b_2 \in \text{bd } K$ such that $b_2 - b_1 \in \text{bd } K$. Then the lattice $\Lambda = (b_1, b_2)\mathbb{Z}^d$ is admissible for $K$.
\item[ii)] Let $\Lambda \in \mathcal{L}^d$ be a critical lattice of $K$. Then there exists a basis $b_1, b_2$ of $\Lambda$ such that $b_1, b_2, b_2 - b_1 \in \text{bd } K$.
\end{enumerate}
7.36 Corollary. Let $K \in \mathcal{K}_0^2$ and $H_K$ be an affinely regular hexagon of minimal volume with vertices on $\text{bd} \ K$. Then

$$
\delta_L(K) = \frac{3}{4} \frac{\text{vol}(K)}{\text{vol}(H_K)} \quad \iff \quad \Delta(K) = \frac{1}{3} \frac{\text{vol}(H_K)}{\text{vol}(K)}.
$$

7.37 Theorem [Fejes Tóth, 1950; Rogers, 1951]. Let $K \in \mathcal{K}^2$. Then

$$
\delta(K) = \delta_L(K).
$$

7.38 Theorem [Hales, 1998/2005].

$$
\delta(B_3) = \delta_L(B_3) = \frac{\pi}{3 \sqrt{2}}.
$$
Index

0/1-polytope, 37
C(n, m), 31
F^3, 21
H(a, α), 7
H^+(a, α), H^-(a, α), 7
S^{n-1}, 2
C^n, 2
Δ(K), 56
GL(n, Z), 63
K^n, 53
G(·), 45
L^n, 62
P^n, 5
P^Z, 45
R^n, 1
aff X, 3
B_n, 2
B_n(a, ρ), 2
bd X, 4
conv X, 3
dim X, 3
|x|, 1
int X, 1
lin X, 3
H-polytope, 19
V-polytope, 19
pos X, 3
reld X, 4
relint X, 3
vert P, 20
ζ-function, 67
f-vector, 20
h-vector, 33
x'y, 1

dependent, 1
independent, 1
ball, 2
Ball, K., 67
Barnette’s Lower Bound Theorem, 32
Betke, Ulrich, 61
Blichfeldt, 62
Boltjanski, 55
Borsuk, 53
boundary, 4
point, 4
breadth, 10

Carathéodory, 14
characteristic function, 48
circumradius, 61
combinatorial diameter, 36
combinatorial distance, 36
combinatorially equivalent, 40
combinatorially isomorphic, 40
compact set, 13
cone, 2
convex
body, 55
combination, 1
hull, 3
set, 2
convex body, 33
smooth, 53
convexly dependent, 1
critical determinant, 65
critical lattice, 65
cyclic polytope, 31
Dehn-Sommerville equations, 33
Delaunay triangulation, 42
density of a densest lattice packing of K, 61
density of a densest packing, 59
density of a packing, 59
dimension, 3
edge, 20
Ehrhart, 49
INDEX

Ehrhart polynomial, 49
Ehrhart, Eugène, 49
Ehrhart-Polynomial, 49
Euclidean
    inner product, 1
    norm, 1
    space, 1
Euler-Poincaré formula, 25
face lattice, 40
faces, 20
facet, 20
family of convex sets, 2
family of polytopes, 5
fundamental cell, 63
fundamental parallelepiped, 63
Gale
    transform, 39
Gale transform, 39
Gohberg, 55
Gohberg-Markus-Hadwiger number, 55
Gohberg-Markus-Hadwiger number, 55
Hadamard, 47
halfspace, 7
height function, 42
Helly, 13
Hinrichs, 55
Hirsch conjecture, 36
hyperplane, 7
    Separating, 10
    Supporting, 7
illumination number, 55
improper faces, 20
Inclusion-Exclusion Formula, 49
interior, 4
    point, 4
Kalai, 35
Kneser, Martin, 51
lattice
    basis, 62
    determinant, 63
    point, 62
lattice packing, 64
lattice point enumerator, 45
lattice polytope, 45
Lemma
    of Busemann-Feller, 8
Levenštejn & Kabatjanskii, 62
linear
    combination, 1
    hull, 3
    subspace, 2
linearly
    dependent, 1
    independent, 1
Markus, 55
McMullen’s g-theorem, 33
McMullen’s Upper Bound Theorem, 32
metric projection, 7
minimal width, 56
Minkowski sum, 6
Minkowski-Hlawka, 65
moment curve, 31
nearest point map, 7
neighbour, 24
outer normal vector, 7
outer unit normal vector, 7
packing set, 59
Pick’s theorem, 46
Pick, Alexander, 46
plank, 56
Plank problem, 56
polar set, 11
polyhedron, 19
polytope
    k-polytope, 5
    poset, 10
positive
    combination, 1
    hull, 3
positively dependent, 1
proper faces, 20
Radon, 13
Reeve simplices, 64
Reeve-simplices, 51
relative
    boundary, 4
INDEX

boundary point, 4
interior, 4
interior point, 4
Richter, 55
Rogers, 62
separating hyperplane, 10
Sidelnikov, 62
simple polytopes, 26
simplex
  k-simplex, 5
simplicial complex, 42
simplicial polytopes, 26
smooth, 53
stacked polytopes, 32
Steinitz, 34
strip, 56
support function, 10
supporting hyperplane, 7
Swinerton-Dyer, 67

Theorem
  of Carathéodory, 14
  of Helly, 13
  of Radon, 13
theorem
  of Separation, 10
triangulation, 41
  Delaunay, 42
    regular, 42
unimodular matrix, 50, 63
unimodular transformation, 50
unit ball, 2
unit cube, 37

vector
  h, 33
vertex, 20

Wills, 47