Combinatorial Convexity

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Contents

Preface

These notes are based on a thirteen weeks' course given at the University of Magdeburg in the summer term 2011. In each week we had two clasess, i.e., two times 90 minutes of teaching, and 4 out of these 26 clasess were exercise classes. Unfortunately, there was not enough time for the last chapter on packings which is also in a very preliminary condition. Since we had prepared it already, however, we have included it in these notes..

The material presented here is stolen from different excellent sources:

- First of all: A manuscript of Ulrich Betke on convexity which is partially based on lecture notes given by Peter McMullen.
- The inspiring books by
  - Alexander Barvinok, "A course in Convexity"
  - Günter Ewald, "Combinatorial Convexity and Algebraic Geometry"
  - Peter M. Gruber, "Convex and Discrete Geometry"
  - Jiri Matousek, "Discrete Geometry"
  - Rekha R. Thomas, "Lectures in Geometric Combinatorics"
  - Günter M. Ziegler, "Lectures on polytopes"
- Lecture notes on "Algebraic Methods in Combinatorics" by Thành Nguyen `http://www.cam.cornell.edu/~thanh/paper/alg.pdf`
- Lecture notes on "Discrete and Convex Geometry" my Maria Hernandez Cifre and myself.
- And some original papers...

0 Some basic and convex facts

0.1 Notation. $\mathbb{R}^n = \{(x_1, \ldots, x_n)^\intercal : x_i \in \mathbb{R}\}$ *denotes the n-dimensional* Euclidean space *equipped with the* Euclidean inner product $\langle x, y \rangle = \sum_{i=1}^n x_i\, y_i$, $x, y \in \mathbb{R}^n$, *and the* Euclidean norm $|x| = \sqrt{\langle x, x \rangle}$.

0.2 Definition [Linear, affine, positive and convex combination]. *Let* $m \in \mathbb{N}$ *and let* $x_i \in \mathbb{R}^n$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq m$.

i) $\sum_{i=1}^m \lambda_i\, x_i$ *is called* a linear combination *of* $x_1, \ldots, x_m$.

ii) *If* $\sum_{i=1}^m \lambda_i = 1$ *then* $\sum_{i=1}^m \lambda_i\, x_i$ *is called* an affine combination *of* $x_1, \ldots, x_m$.

iii) *If* $\lambda_i \geq 0$ *then* $\sum_{i=1}^m \lambda_i\, x_i$ *is called* a positive combination *of* $x_1, \ldots, x_m$.

iv) *If* $\lambda_i \geq 0$ *and* $\sum_{i=1}^m \lambda_i = 1$ *then* $\sum_{i=1}^m \lambda_i\, x_i$ *is called* a convex combination *of* $x_1, \ldots, x_m$.

v) *Let* $X \subseteq \mathbb{R}^n$. $x \in \mathbb{R}^n$ *is called* linearly *(*affinely, positively, convexly*)* dependent *of* X, *if* x *is a linear (affine, positive, convex) combination of finitely many points of* X, *i.e., there exist* $x_1, \ldots, x_m \in X$, $m \in \mathbb{N}$, *such that* x *is a linear (affine, positive, convex) combination of the points* $x_1, \ldots, x_m$.

0.3 Definition [Linearly and affinely independent points]. $x_1, \ldots, x_m \in \mathbb{R}^n$ *are called* linearly *(*affinely*)* dependent, *if one of the* x_i *is linearly (*affinely*) dependent of* $\{x_1, \ldots, x_m\} \setminus \{x_i\}$. *Otherwise* $x_1, \ldots, x_m$ *are called* linearly *(*affinely*)* independent.

0.4 Remark. *Let* $x_1, \ldots, x_m \in \mathbb{R}^n$.

i) $x_1, \ldots, x_m$ *are affinely dependent if and only if* $\binom{x_1}{1}, \ldots, \binom{x_m}{1} \in \mathbb{R}^{n+1}$ *are linearly dependent.*

ii) $x_1, \ldots, x_m$ *are affinely dependent if and only if there exist* $\mu_i \in \mathbb{R}$, $1 \leq i \leq m$, *with* $(\mu_1, \ldots, \mu_m) \neq (0, \ldots, 0)$, $\sum_{i=1}^m \mu_i = 0$ *and* $\sum_{i=1}^m \mu_i\, x_i = 0$.

iii) *If* $m \geq n+1$ *then* $x_1, \ldots, x_m$ *are linearly dependent.*

iv) *If* $m \geq n+2$ *then* $x_1, \ldots, x_m$ *are affinely dependent.*

Proof. By definition we have that $x_1, \ldots, x_m$ are affinely dependent if and only if there exists an x_i, say, and scalars λ_j, $1 \leq j \neq i \leq m$, such that $x_i = \sum_{j \neq i} \lambda_j\, x_j$ and $\sum_{j \neq i} \lambda_j = 1$. This can be reformulated as

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} x_i \\ 1 \end{pmatrix} + \sum_{j \neq i} \lambda_j \begin{pmatrix} x_j \\ 1 \end{pmatrix}$$

which is equivalent to the linear dependency of the vectors $\binom{x_j}{1}$. For ii) we just observe that the equation above is just a reformulation of what is to show if we set $\mu_i = -1$ and $\mu_j = \lambda_j$, $j \neq i$. Fianlly, iv) follows from i) and iii), which is trivial.

□

0.5 Definition [Linear subspace, affine subspace, cone and convex set]. *$X \subseteq \mathbb{R}^n$ is called*

i) linear subspace (set) *if it contains all $x \in \mathbb{R}^n$ which are linearly dependent of X,*

ii) affine subspace (set) *if it contains all $x \in \mathbb{R}^n$ which are affinely dependent of X,*

iii) (convex) cone *if it contains all $x \in \mathbb{R}^n$ which are positively dependent of X,*

iv) convex set *if it contains all $x \in \mathbb{R}^n$ which are convexly dependent of X.*

0.6 Notation. *$\mathcal{C}^n = \{K \subseteq \mathbb{R}^n : K \text{ convex}\}$ denotes the set of all convex sets in $\mathbb{R}^n$. The empty set $\emptyset$ is regarded as a convex, linear and affine set.*

0.7 Theorem. *$K \subseteq \mathbb{R}^n$ is convex if and only if*

$$\lambda x + (1-\lambda)\, y \in K, \quad \text{for all} \quad x, y \in K \text{ and } 0 \leq \lambda \leq 1.$$

Proof. Of course, if K is convex and $x, y \in K$ then for any $\lambda \in [0,1]$ the point $\lambda\, x + (1-\lambda)\, y$ is convexly dependent of K and hence it must belong to K.

Conversely, let $v \in \mathbb{R}^n$ be convexly dependent of K, i.e., there exist $x_1, \ldots, x_m \in K$ and scalars $\lambda_1, \ldots, \lambda_m \geq 0$ with $\sum \lambda_i = 1$ and $v = \sum \lambda_i\, x_i$. We show by induction on m that $v \in K$. The case $m = 1$ is trivial and $m = 2$ is just our assumption. So let $m \geq 3$ and we may assume that $\lambda_m < 1$, say. Then

$$v = \sum_{i=1}^{m} \lambda_i\, x_i = (1-\lambda_m) \left(\sum_{i=1}^{m-1} \frac{\lambda_i}{1-\lambda_m} x_i \right) + \lambda_m\, x_m = (1-\lambda_m)\, x + \lambda_m\, x_m.$$

Since

$$\sum_{i=1}^{m-1} \frac{\lambda_i}{1-\lambda_m} = \frac{1}{1-\lambda_m} \sum_{i=1}^{m-1} \lambda_i = \frac{1-\lambda_m}{1-\lambda_m} = 1,$$

we get by our inductive argumention $x \in K$ and thus, by our assuimption $v \in K$. □

0.8 Example. *The closed n-dimensional ball $B_n(a, \rho) = \{x \in \mathbb{R}^n : |x - a| \leq \rho\}$ with centre a and radius $\rho > 0$ is convex. The boundary of $B_n(a, \rho)$, i.e., $\{x \in \mathbb{R}^n : |x - a| = \rho\}$ is non-convex. In the case $a = 0$ and $\rho = 1$ the ball $B_n(0, 1)$ is abbreviated by B_n and is called n-*dimensional unit ball*. Its boundary is denoted by S^{n-1}.*

0.9 Corollary. *Let $K_i \in \mathcal{C}^n$, $i \in I$. Then $\bigcap_{i \in I} K_i \in \mathcal{C}^n$.*

Proof. Let $x, y \in \bigcap_{i\in I} K_i$ and let $\lambda \in [0,1]$. Since K_i is a convex set for all $i \in I$, we have $\lambda\, x + (1-\lambda)\, y \in K_i$ for all $i \in I$ and hence $\lambda\, x + (1-\lambda)\, y \in \bigcap_{i\in I} K_i$. By Theorem 0.7 we obatin the convexity of $\bigcap_{i\in I} K_i$. □

0.10 Definition [Linear, affine, positive and convex hull, dimension]. *Let* $X \subseteq \mathbb{R}^n$.

i) *The* linear hull $\operatorname{lin} X$ *of* X *is defined by*

$$\operatorname{lin} X = \bigcap_{\substack{L\subseteq\mathbb{R}^n,\ L \text{ linear},\\ X\subseteq L}} L.$$

ii) *The* affine hull $\operatorname{aff} X$ *of* X *is defined by*

$$\operatorname{aff} X = \bigcap_{\substack{A\subseteq\mathbb{R}^n,\ A \text{ affine},\\ X\subseteq A}} A.$$

iii) *The* positive (conic) hull $\operatorname{pos} X$ *of* X *is defined by*

$$\operatorname{pos} X = \bigcap_{\substack{C\subseteq\mathbb{R}^n,\ C \text{ convex cone},\\ X\subseteq C}} C.$$

iv) *The* convex hull $\operatorname{conv} X$ *of* X *is defined by*

$$\operatorname{conv} X = \bigcap_{\substack{K\subseteq\mathbb{R}^n,\ K \text{ convex},\\ X\subseteq K}} K.$$

v) *The* dimension $\dim X$ *of* X *is the dimension of its affine hull, i.e.,* $\dim \operatorname{aff} X$.

0.11 Theorem. *Let* $X \subseteq \mathbb{R}^n$. *Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{m} \lambda_i\, x_i : m \in \mathbb{N}, x_i \in X, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

Proof. For short we denote by $M(X)$ the right hand side of the equation above.

First we consider the inclusion $\operatorname{conv} X \subseteq M(X)$. Since $X \subseteq M(X)$ it suffices to show that $M(X)$ is convex. Let $x, y \in M(X)$ with $x = \sum_{i=1}^{m_1} \nu_i\, x_i$ and $y = \sum_{j=1}^{m_2} \mu_j\, y_j$, $x_i, y_j \in X$, $\nu_i, \mu_j \geq 0$, and $\sum_{i=1}^{m_1} \nu_i = \sum_{j=1}^{m_2} \mu_j = 1$. Hence for $\lambda \in [0,1]$ we find

$$\lambda x + (1-\lambda) y = \sum_{i=1}^{m_1} \lambda \nu_i\, x_i + \sum_{j=1}^{m_2} (1-\lambda)\mu_j\, y_j,$$

where the coefficients of the above combination satisfy $\lambda\nu_i, (1-\lambda)\mu_j \geq 0$ and $\sum_{i=1}^{m_1} \lambda\nu_i + \sum_{j=1}^{m_2} (1-\lambda)\mu_j = \lambda + (1-\lambda) = 1$. Hence $\lambda\, x + (1-\lambda)\, y \in M(X)$ which shows that $M(X)$ is convex (cf. Theorem 0.7), and so $\operatorname{conv} X \subseteq M(X)$.

In order to verify $\operatorname{conv} X \supseteq M(X)$ let $K \in \mathcal{C}^n$ with $X \subseteq K$. Since each element of $M(X)$ is convexly dependent of X, and hence of K, we have $M(X) \subseteq K$. Thus $M(X) \subseteq \bigcap_{K\in\mathcal{C}^n,\, X\subseteq K} K = \operatorname{conv} X$. □

0.12 Remark.

i) $\operatorname{conv}\{x,y\} = \{\lambda\, x + (1-\lambda)\, y : \lambda \in [0,1]\}$.

ii) $\operatorname{lin} X = \{\sum_{i=1}^m \lambda_i x_i : m \in \mathbb{N}, x_i \in X\}$.

iii) $\operatorname{aff} X = \{\sum_{i=1}^m \lambda_i x_i : m \in \mathbb{N}, x_i \in X, \sum_{i=1}^m \lambda_i = 1\}$.

iv) $\operatorname{pos} X = \{\sum_{i=1}^m \lambda_i x_i : m \in \mathbb{N}, x_i \in X,\ \lambda_i \geq 0\}$.

0.13 Definition [(Relative) interior point and (relative) boundary point]. *Let $X \subseteq \mathbb{R}^n$.*

i) *$x \in X$ is called an* interior point *of X if there exists a $\rho > 0$ such that $B_n(x,\rho) \subseteq X$. The set of all interior points of X is called the* interior *of X and is denoted by $\operatorname{int} X$.*

ii) *$x \in \mathbb{R}^n$ is called* boundary point *of X if for all $\rho > 0$ hold $B_n(x,\rho) \cap X \neq \emptyset$ and $B_n(x,\rho) \cap (\mathbb{R}^n \backslash X) \neq \emptyset$. The set of all boundary points of X is called the* boundary *of X and is denoted by $\operatorname{bd} X$.*

iii) *Let $A = \operatorname{aff} X$. $x \in X$ is called a* relative interior point *of X if there exists a $\rho > 0$ such that $B_n(x,\rho) \cap A \subseteq X$. The set of all relative interior points is called the* relative interior *of X and is denoted by $\operatorname{relint} X$.*

iv) *Let $A = \operatorname{aff} X$. $x \in A$ is called a* relative boundary point *of X if for all $\rho > 0$ hold $B_n(x,\rho) \cap X \neq \emptyset$ and $B_n(x,\rho) \cap (A \backslash X) \neq \emptyset$. The set of all relative boundary points of X is called* relative boundary *of X and is denoted by $\operatorname{relbd} X$.*

0.14 Remark. *Let $X \subseteq \mathbb{R}^n$ be closed. Then $X = \operatorname{relint} X \cup \operatorname{relbd} X$.*

0.15 Theorem. *Let $K \in \mathcal{C}^n$, $x \in \operatorname{relint} K$ and $y \in K$. Then $(1-\lambda)x + \lambda y \in \operatorname{relint} K$ for all $\lambda \in [0,1)$.*

Proof. Let $A = \operatorname{aff} K$, $x \in \operatorname{relint} K$, and for $\lambda \in [0,1)$ let $z_\lambda = (1-\lambda)\, x + \lambda\, y$. Since $x = z_0 \in \operatorname{relint} K$ there exists a $\rho > 0$ such that $B_n(x,\rho) \cap A \subseteq K$. By the theorem on intersecting lines it follows immediately that $B_n\big(z_\lambda, (1-\lambda)\rho\big) \cap A \subseteq K$, and hence $z_\lambda \in \operatorname{relint} K$. Or more explicitly: For $a \in B_n\big(z_\lambda, (1-\lambda)\rho\big) \cap A$ we have

$$(1-\lambda)\rho \geq |z_\lambda - a| = \big|(1-\lambda)x + \lambda y - a\big| = \big|(1-\lambda)x - (a - \lambda y)\big|.$$

Since $\lambda < 1$ we may divide both sides by $1-\lambda$ and get $|x - (a - \lambda\, y)/(1-\lambda)| \leq \rho$. Thus $(a - \lambda\, y)/(1-\lambda) \in B_n(x,\rho) \cap A \subseteq K$ and by the convexity of K we finally find

$$a = (1-\lambda)\frac{a - \lambda y}{1-\lambda} + \lambda y \in K.$$

□

0.16 Corollary. *Let $K \in \mathcal{C}^n$ be closed. Let $x \in \operatorname{relint} K$ and $y \in \operatorname{aff} K \setminus K$. Then the segment $\operatorname{conv}\{x,y\}$ intersects $\operatorname{relbd} K$ in precisely one point.*

Proof. $K \cap \operatorname{conv}\{x, y\}$ is a convex, compact 1-dimensional set. Hence $K \cap \operatorname{conv}\{x, y\} = \operatorname{conv}\{x, \overline{y}\}$ for some $\overline{y} \in K$. Obviously, $\overline{y} \in \operatorname{relbd} K$ and by Theorem 0.15 we see that $\overline{y}$ is the only point of $\operatorname{conv}\{x, \overline{y}\}$ lying on $\operatorname{relbd} K$. □

0.17 Definition [Polytope and simplex]. *Let $X \subset \mathbb{R}^n$ of finite cardinality, i.e., $\#X < \infty$.*

i) $\operatorname{conv} X$ *is called a* (convex) polytope.

ii) *A polytope $P \subset \mathbb{R}^n$ of dimension k is called a k-polytope.*

iii) *If X is affinely independent and $\dim X = k$ then $\operatorname{conv} X$ is called a k-simplex.*

0.18 Notation. $\mathcal{P}^n = \{P \subset \mathbb{R}^n : P \text{ polytope}\}$ *denotes the set of all polytopes in $\mathbb{R}^n$.*

0.19 Lemma. *Let $T = \operatorname{conv}\{x_1, \dots, x_{k+1}\} \subset \mathbb{R}^n$ be a k-simplex, and let $\lambda_i > 0$, $1 \le i \le k+1$, with $\sum \lambda_i = 1$. Then $\sum \lambda_i x_i \in \operatorname{relint} T$.*

Proof. See Exercise **??**. □

0.20 Corollary. *Let $K \in \mathcal{C}^n$, $K \neq \emptyset$. Then* $\operatorname{relint} K \neq \emptyset$.

Proof. Let $k = \dim K \ge 0$. Then there exist $x_1, \dots, x_{k+1} \in K$ affinely independent such that $\operatorname{aff} K = \operatorname{aff}\{x_1, \dots, x_{k+1}\}$. Let $T_k = \operatorname{conv}\{x_1, \dots, x_{k+1}\} \subseteq K$. From Lemma 0.19 we get $\operatorname{relint} T_k \neq \emptyset$, and hence $\operatorname{relint} K \neq \emptyset$. □

0.21 Theorem. *Let $P = \operatorname{conv}\{x_1, \dots, x_m\} \in \mathcal{P}^n$. A point $x \in \mathbb{R}^n$ belongs to $\operatorname{relint} P$ if and only if x admits a representation as $x = \sum_{i=1}^m \lambda_i x_i$ with $\lambda_i > 0$, $1 \le i \le m$, and $\sum_{i=1}^m \lambda_i = 1$.*

Proof. Let $x \in \operatorname{relint} P$ and let $y = \sum_{i=1}^m (1/m)\, x_i \in P$. Since $x \in \operatorname{relint} P$ there exists a $z \in P$ such that $x = \lambda\, z + (1 - \lambda)\, y$ with $\lambda \in [0, 1)$. Let $z = \sum_{i=1}^m \mu_i x_i$, with $\mu_i \ge 0$, $1 \le i \le m$, and $\sum \mu_i = 1$. Then $x = \sum_{i=1}^m \big(\lambda \mu_i + (1 - \lambda)/m\big) x_i$, where all the scalars $\lambda \mu_i + (1 - \lambda)/m$ are positive and sum up to 1.

Next we assume that x has a representation as $x = \sum_{i=1}^m \lambda_i x_i$ with $\lambda_i > 0$, $1 \le i \le m$, and $\sum \lambda_i = 1$. Let $k = \dim P$ and without loss of generality let $x_1, \dots, x_{k+1}$ be affinely independent. Setting $\lambda = \sum_{i=1}^{k+1} \lambda_i$, Lemma 0.19 shows that

$$y = \sum_{i=1}^{k+1} \frac{\lambda_i}{\lambda} x_i \in \operatorname{relint} \operatorname{conv}\{x_1, \dots, x_{k+1}\} \subseteq \operatorname{relint} P.$$

If $\lambda = 1$ then $k + 1 = m$ and hence $x = y \in \operatorname{relint} P$. If $\lambda < 1$ let $z = 1/(1 - \lambda) \sum_{i=k+1}^m \lambda_i\, x_i \in P$ and with Theorem 0.15 we also find in this case $x = \lambda y + (1 - \lambda) z \in \operatorname{relint} P$. □

0.22 Notation.

i) *For two sets* $X, Y \subseteq \mathbb{R}^n$ *the vectorial addition*

$$X + Y = \{x + y : x \in X,\, y \in Y\}$$

is called the Minkowski sum *of* X *and* Y. *If* X *is just a singleton, i.e.,* $X = \{x\}$, *then we write* $x + Y$ *instead of* $\{x\} + Y$.

ii) *For* $\lambda \in \mathbb{R}$ *and* $X \subseteq \mathbb{R}^n$ *we denote by* $\lambda\, X$ *the set*

$$\lambda\, X = \{\lambda\, x : x \in X\}\,.$$

1 Support and separate

1.1 Notation. *Let $a \in \mathbb{R}^n$, $a \neq 0$, and $\alpha \in \mathbb{R}$. The* closed halfspaces $H^+(a,\alpha)$, $H^-(a,\alpha) \subset \mathbb{R}^n$ *are given by*

$$H^+(a,\alpha) = \big\{x \in \mathbb{R}^n : \langle a, x\rangle \geq \alpha\big\}, \quad H^-(a,\alpha) = \big\{x \in \mathbb{R}^n : \langle a, x\rangle \leq \alpha\big\}.$$

The hyperplane $H(a,\alpha)$ *is defined by*

$$H(a,\alpha) = \big\{x \in \mathbb{R}^n : \langle a, x\rangle = \alpha\big\}.$$

1.2 Definition [Supporting hyperplane]. *Let $X \subset \mathbb{R}^n$. A hyperplane $H(a,\alpha) \subset \mathbb{R}^n$ is called* supporting hyperplane *of X if:*

$$\text{i) } H(a,\alpha) \cap X \neq \emptyset \quad \textit{and} \quad \text{ii) } X \subseteq H^-(a,\alpha).$$

a is called outer normal vector *of X and if, in addition, $|a| = 1$ then it is called* outer unit normal vector *of X.*

1.3 Proposition. *Let $X \subset \mathbb{R}^n$ and let $H(a,\alpha)$ be a supporting hyperplane of X. Then*

$$H(a,\alpha) \cap \operatorname{conv} X = \operatorname{conv}\big(H(a,\alpha) \cap X\big).$$

Proof. By definition we have $\operatorname{conv} X \subset H^-(a,\alpha)$. Let $x \in H(a,\alpha) \cap \operatorname{conv} X$. Since $x \in \operatorname{conv} X$ then there exist $\lambda_i \geq 0$ and $x_i \in X$, $i = 1, \ldots, m$, with $x = \sum_{i=1}^m \lambda_i = 1$ such that $x = \sum_{i=1}^m \lambda_i x_i$. On the other hand since $x \in H(a,\alpha)$ and $X \subset H^-(a,\alpha)$ we have

$$\alpha = \langle a, x\rangle = \sum_{i=1}^m \lambda_i \langle a, x_i\rangle \leq \sum_{i=1}^m \alpha\lambda_i = \alpha.$$

Then $\langle a, x_i\rangle = \alpha$, $1 \leq i \leq m$, which ensures that $x_i \in H(a,\alpha) \cap X \subseteq \operatorname{conv}\big(H(a,\alpha) \cap X\big)$.

The reverse inclusion is trivial since $\operatorname{conv}\big(H(a,\alpha) \cap X\big) \subseteq \operatorname{conv}\big(H(a,\alpha)\big) \cap \operatorname{conv} X = H(\alpha, a) \cap \operatorname{conv} X$. □

1.4 Remark. *Let $X \subset \mathbb{R}^n$ be compact and $a \in \mathbb{R}^n \backslash \{0\}$. Then there exists a supporting hyperplane of X with outer normal vector a.*

1.5 Definition [Nearest point map (or metric projection)]. *Let $K \in \mathcal{C}^n$ be closed. The map $\Phi_K : \mathbb{R}^n \to K$, where for $x \in \mathbb{R}^n$ the point $\Phi_K(x) \in K$ is given by $|x - \Phi_K(x)| = \min\big\{|x-y| : y \in K\big\}$ is called the* nearest point map *(*metric projection*) with respect to K.*

1.6 Remark. *We prove that the nearest point map is well-defined. Notice that since K is closed, for all $x \in \mathbb{R}^n$ there exist $y_x \in K$ such that $|x - y_x| = \min\big\{|x-y| : y \in K\big\}$. We show that y_x is uniquely determined. In fact, if there exists $\overline{y} \in K$, $\overline{y} \neq y_x$, with $|x - \overline{y}| = |x - y_x|$ then we may assume that $x - y_x$ and $x - \overline{y}$ are linearly independent. Hence*

$$\left|x - \frac{y_x + \overline{y}}{2}\right| = \left|\frac{1}{2}(x - y_x) + \frac{1}{2}(x - \overline{y})\right| < \frac{1}{2}\,|x - y_x| + \frac{1}{2}\,|x - \overline{y}| = |x - y_x|\,.$$

Since $(y_x + \overline{y})/2 \in K$ by the convexity of K, it contradicts the minimality of y_x.

1.7 Theorem. *Let $K \in \mathcal{C}^n$ be closed and let $x \in \mathbb{R}^n \setminus K$. Let $a = x - \Phi_K(x)$ and $\alpha = \langle a, \Phi_K(x)\rangle$. Then $H(a,\alpha)$ is a supporting hyperplane of K with outer normal vector a.*

Proof. The choice of a and α ensures that $\Phi_K(x) \in K \cap H(a,\alpha)$ and hence $K \cap H(a,\alpha) \neq \emptyset$. We have to show that $K \subset H^-(a,\alpha)$, i.e., that $\langle a, y\rangle \leq \alpha$ for all $y \in K$.

Let $y \in K$ and $\lambda \in (0,1]$. Clearly $(1-\lambda)\Phi_K(x) + \lambda y \in K$ and then

$$\begin{aligned} |x - \Phi_K(x)|^2 &\leq \left|x - \left[(1-\lambda)\Phi_K(x) + \lambda y\right]\right|^2 = \left|x - \Phi_K(x) + \lambda\bigl(\Phi_K(x) - y\bigr)\right|^2 \\ &= |x - \Phi_K(x)|^2 + \lambda^2 |\Phi_K(x) - y|^2 + 2\lambda \langle x - \Phi_K(x), \Phi_K(x) - y\rangle, \end{aligned}$$

or equivalently

$$0 \leq \lambda^2 |\Phi_K(x) - y|^2 + 2\lambda \langle a, \Phi_K(x) - y\rangle = \lambda^2 |\Phi_K(x) - y|^2 + 2\lambda\bigl(\alpha - \langle a, y\rangle\bigr).$$

Then $\langle a, y\rangle - \alpha \leq (\lambda/2)\,|\Phi_K(x) - y|^2$ for all $\lambda \in (0,1]$, which implies $\langle a, y\rangle - \alpha \leq 0$. □

1.8 Corollary. *Let $K \in \mathcal{C}^n$, $K \neq \mathbb{R}^n$, be closed. Then*

$$K = \bigcap_{\substack{H(a,\alpha) \text{ supporting} \\ \text{hyperplane of } K}} H^-(a,\alpha),$$

i.e., K is the intersection of all its "supporting halfspaces".

Proof. Clearly $K \subseteq \bigcap_{H(a,\alpha)} H^-(a,\alpha)$. In order to prove the reverse inclusion we take $x \notin K$, and let $H(a,\alpha)$ be the supporting hyperplane defined in Theorem 1.7. Then $x \in H^+(a,\alpha)$ but $x \notin H(a,\alpha)$, i.e, $x \notin H^-(a,\alpha)$ and hence x is not contained in the intersection of the right hand side. □

1.9 Corollary. *Let $X \subset \mathbb{R}^n$ such that $\operatorname{conv} X$ is closed and $\operatorname{conv} X \neq \mathbb{R}^n$. Then*

$$\operatorname{conv} X = \bigcap_{X \subseteq H^-(a,\alpha)} H^-(a,\alpha),$$

i.e., $\operatorname{conv} X$ is the intersection of all halfspaces containing X.

Proof. If $x \notin \operatorname{conv} X$, by Corollary 1.8 there exists a supporting hyperplane $H(a,\alpha)$ of $\operatorname{conv} X$ with $x \notin H^-(a,\alpha)$. Since $X \subseteq \operatorname{conv} X \subseteq H^-(a,\alpha)$ then x is not contained in the intersection of the right hand side. The reverse inclusion is trivial. □

1.10 Lemma [Busemann-Feller Lemma]. *Let $K \in \mathcal{C}^n$ be closed. Then*

$$|\Phi_K(x) - \Phi_K(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}^n$, i.e., the nearest point map does not increase distances. In particular, it is a continuous map.

Proof. We suppose that $\Phi_K(x) \neq \Phi_K(y)$ and let $a = \Phi_K(x) - \Phi_K(y)$, $\alpha_x = \langle a, \Phi_K(x)\rangle$ and $\alpha_y = \langle -a, \Phi_K(y)\rangle$. It suffices to show that $x \in H^+(a, \alpha_x)$ and $y \in H^+(-a, \alpha_y)$, because then $\langle a, x\rangle \geq \alpha_x$ and $\langle -a, y\rangle \geq \alpha_y$, which implies

$$\langle a, x - y\rangle \geq \alpha_x + \alpha_y = \langle a, \Phi_K(x) - \Phi_K(y)\rangle = |\Phi_K(x) - \Phi_K(y)|^2 .$$

By the Cauchy-Schwarz inequality we conclude $|x - y| \geq |\Phi_K(x) - \Phi_K(y)|$. So we assume the contrary and without loss of generality let $x \notin H^+(a, \alpha_x)$, i.e., $\langle a, x\rangle < \alpha_x$. Then the ray

$$R(x) = \big\{\Phi_K(x) + \lambda\big(x - \Phi_K(x)\big) : \lambda \geq 0\big\}$$

has to intersect the hyperplane $H(-a, \alpha_y)$ in a point $\overline{x}$, say, and by the Pythagorean theorem we obtain

$$|\overline{x} - \Phi_K(y)| < |\overline{x} - \Phi_K(x)| .$$

On the other hand, by Exercise **??** we have $\Phi_K(z) = \Phi_K(x)$ for all $z \in R(x)$, and so we get the contradiction $|\overline{x} - \Phi_K(\overline{x})| = |\overline{x} - \Phi_K(x)| > |\overline{x} - \Phi_K(y)|$. □

1.11 Theorem. *Let $K \in \mathcal{C}^n$ be compact and let $\rho > 0$ such that $K \subset \operatorname{int}(\rho B_n)$. The nearest point map restricted to ρS^{n-1} is surjective, i.e., $\Phi_K : \rho\, S^{n-1} \to \operatorname{bd} K$ is surjective.*

Proof. Clearly $\Phi_K(\rho\, S^{n-1}) \subseteq \operatorname{bd} K$. So, let $x \in \operatorname{bd} K$, for which $\Phi_K(x) = x$. For $i \in \mathbb{N}$ let $x_i \in \operatorname{int}(\rho B_n)$ such that $x_i \notin K$ and $|x_i - x| < 1/i$. By Lemma 1.10 we have

$$|x - \Phi_K(x_i)| = |\Phi_K(x) - \Phi_K(x_i)| \leq |x - x_i| < \frac{1}{i}.$$

By Exercise **??**, the intersection point z_i of the ray $R(x_i) = \big\{\Phi_K(x_i) + \lambda\big(x_i - \Phi_K(x_i)\big) : \lambda \geq 0\big\}$ with $\rho\, S^{n-1}$ verifies $\Phi_K(z_i) = \Phi_K(x_i)$, and hence $|x - \Phi_K(z_i)| < 1/i$. By the compactness of $\rho\, S^{n-1}$ there exists a subsequence $(z_{i_k})_{k\in\mathbb{N}}$ of $(z_i)_{i\in\mathbb{N}}$ converging to a certain point $z \in \rho\, S^{n-1}$. Since $\lim_{i\to\infty} \Phi_K(z_i) = x$, we finally obtain by the continuity of the nearest map point that $x = \Phi_K(z) \in \Phi_K(\rho\, S^{n-1})$. □

1.12 Corollary. *Let $K \in \mathcal{C}^n$ be closed and let $x \in \operatorname{relbd} K$. Then there exists a supporting hyperplane $H(a, \alpha)$ of K with $x \in H(a, \alpha)$.*

Proof. We can suppose without loss of generality that $\dim K = n$. Let $x \in \operatorname{bd} K$ and let $\gamma > 0$ with $x \in \operatorname{int}(\gamma B_n)$. The convex set $\overline{K} = K \cap (\gamma B_n)$ is compact and $x \in \operatorname{bd} \overline{K}$. Let $\rho > \gamma$ be such that $\overline{K} \subset \operatorname{int}(\rho B_n)$. By Theorem 1.11 we can find $z \in \rho S^{n-1}$ with $x = \Phi_{\overline{K}}(z)$. Then Theorem 1.7 ensures that the hyperplane $H(a, \alpha)$, with $a = z - x$ and $\alpha = \langle a, x\rangle$, supports $\overline{K}$ at x. Finally we have to prove that $H(a, \alpha)$ is also a supporting hyperplane of K at x.

By definition we have $\overline{K} \subset H^-(a, \alpha)$ and thus we suppose that there exists $y \in K$ with $\langle a, y\rangle > \alpha$. Since $\langle a, x\rangle = \alpha$ it holds $\langle a, (1-\lambda)x + \lambda y\rangle > \alpha$ for any $\lambda \in (0, 1]$. Let $\overline{\lambda} > 0$ be sufficiently small such that $\overline{y} = (1 - \overline{\lambda})x + \overline{\lambda} y \in \gamma B_n$. Since $\overline{y} \in K$ we have $\overline{y} \in \overline{K}$ with $\langle a, \overline{y}\rangle > \alpha$, a contradiction. □

1.13 Theorem [Separation theorem]. *Let $K_1, K_2 \in \mathcal{C}^n$ with $K_1 \cap K_2 = \emptyset$. Then there exists a* separating hyperplane *$H(a,\alpha)$ of K_1 and K_2, i.e., $K_1 \subseteq H^+(a,\alpha)$ and $K_2 \subseteq H^-(a,\alpha)$.*

If K_1 is closed and K_2 is compact, then there exists even a strictly separating hyperplane *$H(a,\alpha)$ of K_1 and K_2, i.e., $K_1 \subset \operatorname{int} H^+(a,\alpha)$ and $K_2 \subset \operatorname{int} H^-(a,\alpha)$.*

Proof. If both, K_1 and K_2 are compact the statement is certainly true by standard compactness arguments. If K_1 is closed and K_2 is compact we take the intersection $\overline{K}_1 = K_1 \cap (\rho B_n)$ for $\rho > 0$ sufficiently large such that the distance between K_2 and K_1 equals the distance between K_2 and $\overline{K}_1$. Exercise **??** shows the result.

Then we consider the case of arbitrary disjoint sets K_1 and K_2. Let $x_1 \in \operatorname{relint} K_1$ and $x_2 \in \operatorname{relint} K_2$, and for $i \in \mathbb{N}$ let

$$K_j^i = \left[\operatorname{cl}\left(\left(1-\frac{1}{i}\right)(K_j - x_j)\right) + x_j\right] \cap (iB_n), \qquad \text{for } j = 1, 2.$$

Clearly $K_j^i \subset K_j^{i+1} \subset K_j$, for $j = 1, 2$ and any $i \in \mathbb{N}$, and for every $x \in \operatorname{relint} K_j$ there exists an index i_x such that $x \in K_j^i$ for all $i \geq i_x$. Moreover, K_1^i and K_2^i are compact convex sets with $K_1^i \cap K_2^i = \emptyset$ for any $i \in \mathbb{N}$. By Exercise **??** we know that there exists a separating hyperplane $H(a_i, \alpha_i)$ of K_1^i and K_2^i with $|a_i| = 1$, and thus

$$\langle a_i, x\rangle \leq \alpha_i \quad \text{for all } x \in K_1^i \qquad \text{and} \qquad \langle a_i, x\rangle \geq \alpha_i \quad \text{for all } x \in K_2^i.$$

Since $\langle a_i, x_1\rangle \leq \alpha_i \leq \langle a_i, x_2\rangle$ and $|a_i| = 1$, the sequence $(\alpha_i)_{i\in\mathbb{N}}$ is bounded and hence $\left\{\binom{a_i}{\alpha_i} : i \in \mathbb{N}\right\} \subset \mathbb{R}^{n+1}$ is also a bounded sequence. Without loss of generality we can assume that this sequence is convergent and we write $a = \lim_{i\to\infty} a_i$ and $\alpha = \lim_{i\to\infty} \alpha_i$. In order to prove that the hyperplane $H(a,\alpha)$ separates K_1 and K_2, let $x \in K_1$. If $x \in \operatorname{relint} K_1$ then there exists an index i_x such that $\langle a_i, x\rangle \leq \alpha_i$ for all $i \geq i_x$, which implies that $\langle a, x\rangle \leq \alpha$. For $x \in \operatorname{relbd} K_1$, we approach x by means of the points $(1-\lambda)x + \lambda x_1 \in \operatorname{relint} K_1$ when λ tends to zero, and we get the same conclusion. Analogously we obtain $\langle a, x\rangle \geq \alpha$ for all $x \in K_2$. □

1.14 Definition [Support function, breadth]. *Let $K \in \mathcal{C}^n$. The function $h(K,\cdot) : \mathbb{R}^n \to \mathbb{R}$ given by*

$$h(K,u) = \sup\{\langle u, x\rangle : x \in K\}$$

is called support function *of K. For $u \in S^{n-1}$ the* breadth of K in the direction *u is given by $h(K,u) + h(K,-u)$.*

1.15 Remark. *Let $K \in \mathcal{C}^n$ be non-empty and compact. Then*

$$K = \bigcap_{u\in S^{n-1}} \{x \in \mathbb{R}^n : \langle u, x\rangle \leq h(K,u)\}.$$

Proof. If $H(a, \alpha)$ is the supporting hyperplane of K with outer normal vector a then $\langle a, x\rangle \leq \alpha$ for all $x \in K$, and there exists $x_0 \in K$ with $\langle a, x_0\rangle = \alpha$. Hence writing $u = a/|a| \in S^{n-1}$ we have $\langle u, x\rangle \leq \alpha/|a|$ for all $x \in K$ and $\langle u, x_0\rangle = \alpha/|a|$. Thus $\alpha/|a| = h(K, u)$ and we get the result. □

1.16 Definition [Polar set]. *Let $X \subseteq \mathbb{R}^n$.*

$$X^\star = \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } x \in X\}$$

is called the polar set *of X.*

1.17 Proposition.

i) *$X^\star$ is a convex and closed set and $0 \in X^\star$.*

ii) *If $X_1 \subseteq X_2$ then $X_2^\star \subseteq X_1^\star$.*

iii) *Let M be a regular $n \times n$ matrix. Then $(MX)^\star = M^{-\intercal}X^\star$.*

iv) *Let $X_i \subseteq \mathbb{R}^n$, $i \in I$. Then $\left(\bigcup_{i\in I} X_i\right)^\star = \bigcap_{i\in I} X_i^\star$.*

v) *$X \subseteq (X^\star)^\star$.*

vi) *Let $X \subset \mathbb{R}^n$. Then $X = X^\star$ if and only if $X = B_n$.*

Proof. i) $X^\star$ is an intersection of closed and convex sets and hence $X^\star$ is closed and convex. Trivially $0 \in X^\star$.
ii) Let $y \in X_2^\star$. Then $\langle x, y\rangle \leq 1$ for all $x \in X_2$. In particular, $\langle x, y\rangle \leq 1$ for all $x \in X_1$ which proves that $y \in X_1^\star$.
iii) Let $y \in (MX)^\star$. By definition, $\langle Mx, y\rangle \leq 1$ for all $x \in X$, i.e., $\langle x, M^\intercal y\rangle \leq 1$ for all $x \in X$. This condition is equivalent to $M^\intercal y \in X^\star$, which says that $y \in M^{-\intercal}X^\star$.
iv) Just notice that

$$\begin{aligned}\left(\bigcup_{i\in I} X_i\right)^\star &= \left\{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } x \in \bigcup_{i\in I} X_i\right\} \\ &= \bigcap_{i\in I}\{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } x \in X_i\} = \bigcap_{i\in I} X_i^\star.\end{aligned}$$

v) Let $x \in X$. By definition of polar body $\langle y, x\rangle \leq 1$ for all $y \in X^\star$, i.e., $x \in (X^\star)^\star$.
vi) We suppose first that $X = B_n$. Then ${B_n}^\star = \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } x \in B_n\}$. If $y \in B_n$, it is clear that $\langle y, x\rangle \leq 1$ for all $x \in B_n$, which shows $B_n \subseteq {B_n}^\star$. If $y \in {B_n}^\star$ with $|y| > 1$ then $y/|y| \in B_n$. Since $\langle y/|y|, y\rangle = |y| > 1$ we get $y \notin {B_n}^\star$, a contradiction.

Conversely, let $x \in X = X^\star$. By the definition of the polar body we know $\langle x, x\rangle \leq 1$ for all $x \in X$, which shows $X \subseteq B_n$. Now part ii) implies $X = X^\star \supseteq {B_n}^\star = B_n$, and thus $X = B_n$.

□

1.18 Proposition.

i) *Let* $P = \operatorname{conv}\{x_1, \ldots, x_m\} \subset \mathbb{R}^n$. *Then*

$$P^\star = \{y \in \mathbb{R}^n : \langle x_i, y\rangle \le 1,\ 1 \le i \le m\}.$$

ii) *Let* $P = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \le 1,\ 1 \le i \le m\}$ *with* $a_i \in \mathbb{R}^n$. *Then*

$$P^\star = \operatorname{conv}\{0, a_1, \ldots, a_m\}.$$

Proof. i) $P^\star \subseteq \{y \in \mathbb{R}^n : \langle x_i, y\rangle \le 1,\ 1 \le i \le m\}$ holds by the definition of polar body. In order to show the reverse inclusion let $y \in \mathbb{R}^n$ with $\langle x_i, y\rangle \le 1$, $1 \le i \le m$. For any $x \in P$ there exist $\lambda_1, \ldots, \lambda_m \ge 0$ with $\sum_{i=1}^m \lambda_i = 1$ such that $x = \sum_{i=1}^m \lambda_i x_i$. Then we get $\langle x, y\rangle = \sum_{i=1}^m \lambda_i \langle x_i, y\rangle \le \sum_{i=1}^m \lambda_i = 1$, i.e., $y \in P^\star$.
ii) Clearly $P^\star \supseteq \operatorname{conv}\{0, a_1, \ldots, a_m\}$. In order to show the reverse inclusion we suppose there exists $y \in P^\star$ with $y \notin \operatorname{conv}\{0, a_1, \ldots, a_m\}$. Theorem 1.13 ensures the existence of a strictly separating hyperplane $H(a, \alpha)$ with $\langle a, x\rangle < \alpha$ for all $x \in \operatorname{conv}\{0, a_1, \ldots, a_m\}$ and $\langle a, y\rangle > \alpha$. Since $\alpha > 0$ we have, in particular, $\langle a/\alpha, a_i\rangle < 1$ for $1 \le i \le m$, which shows that $a/\alpha \in P$. It holds however $\langle a/\alpha, y\rangle > 1$, implying that $y \notin P^\star$, a contradiction. □

1.19 Lemma. *Let* $K \in \mathcal{C}^n$ *be closed with* $0 \in K$. *Then* $(K^\star)^\star = K$.

Proof. In view of Proposition 1.17, part v), it suffices to show $(K^\star)^\star \subseteq K$. We suppose there exists $y \in (K^\star)^\star$ with $y \notin K$. Then by Theorem 1.13 we can find a strictly separating hyperplane $H(a, \alpha)$ such that $\langle a, y\rangle > \alpha$ and $\langle a, x\rangle < \alpha$ for all $x \in K$. Since $\alpha > 0$ we have $\langle a/\alpha, x\rangle < 1$ for all $x \in K$ and so $a/\alpha \in K^\star$. Then from $\langle a/\alpha, y\rangle > 1$ we get $y \notin (K^\star)^\star$, a contradiction. □

2 Radon, Helly, Caratheodory and relatives

2.1 Theorem [Radon, 1921]. *Let $X \subset \mathbb{R}^n$. If $\#X \geq n+2$ then there exist $X_1, X_2 \subset X$ with $X_1 \cap X_2 = \emptyset$ and $\operatorname{conv} X_1 \cap \operatorname{conv} X_2 \neq \emptyset$.*

Proof. Since $\#X \geq n+2$, there exist $x_1, \dots, x_{n+2} \in X$ such that $\{x_1, \dots, x_{n+2}\}$ are affinely dependent. Then we can find $\lambda_i \in \mathbb{R}$, $i = 1, \dots, n+2$, not all of them zero with $\sum_{i=1}^{n+2} \lambda_i = 0$ and $\sum_{i=1}^{n+2} \lambda_i x_i = 0$. Without loss of generality we assume that $\lambda_1, \dots, \lambda_k \geq 0$ and $\lambda_{k+1}, \dots, \lambda_{n+2} \leq 0$. Then $\sum_{i=1}^{k} \lambda_i x_i = \sum_{i=k+1}^{n+2} (-\lambda_i) x_i$, and writing $\overline{\lambda} = \sum_{i=1}^{k} \lambda_i = \sum_{i=k+1}^{n+2}(-\lambda_i)$ it follows that

$$\sum_{i=1}^{k} \frac{\lambda_i}{\overline{\lambda}} x_i = \sum_{i=k+1}^{n+2} \left(-\frac{\lambda_i}{\overline{\lambda}}\right) x_i \in \operatorname{conv}\{x_1, \dots, x_k\} \cap \operatorname{conv}\{x_{k+1}, \dots, x_{n+2}\}$$

$$=: X_1 \cap X_2.$$

□

2.2 Theorem [Helly, 1913]. *Let $K_1, \dots, K_m \in \mathcal{C}^n$, $m \geq n+1$, such that for each $(n+1)$-index set $I \subseteq \{1, \dots, m\}$ we have $\bigcap_{i \in I} K_i \neq \emptyset$. Then all sets K_i have a point in common, i.e., $\bigcap_{j=1}^{m} K_i \neq \emptyset$.*

Proof. We prove it by induction on m. The case $m = n+1$ is trivially true by hypothesis. We assume the result holds for $m-1$. Let $X_j = \bigcap_{i=1, i \neq j}^{m} K_i$, for $j = 1, \dots, m$. By induction hypothesis $X_j \neq \emptyset$. Then let $x_j \in X_j$, $j = 1, \dots, m$. Since $m \geq n+2$, we can apply Theorem 2.1 to the set $\{x_1, \dots, x_m\}$ and hence, without loss of generality, we suppose that there exists $x \in \operatorname{conv}\{x_1, \dots, x_k\} \cap \operatorname{conv}\{x_{k+1}, \dots, x_m\}$, for a suitable $0 < k < m$. For $i = k+1, \dots, m$ it holds $\operatorname{conv}\{x_1, \dots, x_k\} \subseteq K_i$ whereas for $i = 1, \dots, k$ we have $\operatorname{conv}\{x_{k+1}, \dots, x_m\} \subseteq K_i$. Then $x \in \bigcap_{i=1}^{m} K_i$. □

2.3 Remark.

i) *Without any further restrictions/assumptions Helly's theorem is* not *true for infinitely many convex sets K_i. For instance, let $K_i = (0, \frac{1}{i}]$, $i \in \mathbb{N}$.*

ii) *Helly's theorem, however, can be easily generalised to infinitely many* compact *(bounded and closed) convex sets.*

2.4 Corollary. *Let $C \subset \mathcal{C}^n$ be compact. Then there exists $t \in \mathbb{R}^n$ with*

$$-C \subseteq t + n\,C.$$

Proof. For $c \in C$ we define the set $X_c = \{t \in \mathbb{R}^n : -c \in t + nC\}$. Clearly $X_c = -c - nC$, and it is convex and compact.

Let $c_1, \dots, c_{n+1} \in C$. Then $\bigcap_{i=1}^{n+1} X_{c_i} \neq \emptyset$. If we take $t = -\sum_{i=1}^{n+1} c_i$ we have, for any $j = 1, \dots, n+1$,

$$-c_j - t = \sum_{i=1, i \neq j}^{n+1} c_i = n \left(\sum_{i=1, i \neq j}^{n+1} \frac{1}{n} c_i \right) \in nC.$$

Therefore $-c_j \in t + nC$ and thus $t \in \bigcap_{i=1}^{n+1} X_{c_i}$. Applying Helly's Theorem 2.2 (cf. Remark 2.3 ii)) we get the required result. $\square$

2.5 Definition [Centerpoint]. *For a finite point set $X \subset \mathbb{R}^n$ a point $c \in \mathbb{R}^n$ is called* centerpoint *is every closed halfspace containing c containes at least $\lceil \frac{1}{n+1} \# X \rceil$ points of X.*

2.6 Theorem. *Every finite set $X \subset \mathbb{R}^n$ has a centerpoint.*

Proof. First we notice that c is a centerpoint of X if and only if it is contained in any open halfspace containing more than $\lceil \frac{n}{n+1} \# X \rceil$ points. We consider the following family of finitely many compact sets

$$\mathcal{X} = \left\{ \operatorname{conv}(H \cap X) : H \text{ is open halfspace with } \#(H \cap X) \geq \left\lceil \frac{n}{n+1} \# X \right\rceil + 1 \right\}.$$

Since each two members of this family differ by less than $\lceil \frac{1}{n+1} \# X \rceil$ points the intersection of each $n+1$ of these sets is nonempty. Hence, applying Helly's Theorem 2.2 gives the required result. $\square$

2.7 Theorem [Carathéodory, 1907]. *Let $X \subset \mathbb{R}^n$. Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{n+1} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1, x_i \in X, i = 1, \ldots, n+1 \right\}.$$

Proof. Let $x \in \operatorname{conv} X$. By Theorem 0.11 there exist $m \in \mathbb{N}$, which we suppose minimal, and $x_1, \ldots, x_m \in X$ such that $x = \sum_{i=1}^m \lambda_i x_i$ for certain $\lambda_i > 0$, $i = 1, \ldots, m$, with $\sum_{i=1}^n \lambda_i = 1$.

We have to see that $m \leq n+1$. So we assume that $m \geq n+2$, which implies that $\{x_1, \ldots, x_m\}$ are affinely dependent. Then there exist $\mu_1, \ldots, \mu_m \in \mathbb{R}$ not all of them zero with $\sum_{i=1}^m \mu_i = 0$ and $\sum_{i=1}^m \mu_i x_i = 0$. From the above two conditions we get that we can write $x = \sum_{i=1}^m (\lambda_i + \alpha \mu_i) x_i$ for any $\alpha \in \mathbb{R}$. Let k be chosen such that

$$\frac{\lambda_k}{\mu_k} = \min_{1 \leq i \leq m} \left\{ \frac{\lambda_i}{\mu_i} : \mu_i > 0 \right\}.$$

Then $\lambda_i - (\lambda_k / \mu_k) \mu_i \geq 0$ for $i = 1, \ldots, m$ and

$$x = \sum_{i=1, i \neq k}^{m} \left(\lambda_i - \frac{\lambda_k}{\mu_k} \mu_i \right) x_i,$$

which contradicts the minimality of m. The reverse inclusion is trivial by the definition of convex hull. $\square$

2.8 Remark. *Let $X \subset \mathbb{R}^n$. Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{\dim X + 1} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{\dim X + 1} \lambda_i = 1, x_i \in X \right\}.$$

As a direct consequence of Carathéodory's Theorem 2.7 we get the following result.

2.9 Corollary. *A polytope is the union of simplices.*

2.10 Corollary. *The convex hull of a compact set is compact.*

Proof. Let X be a compact set and we suppose that $X \subseteq B_n(0,\rho)$. Then it is clear that also $\operatorname{conv} X \subseteq B_n(0,\rho)$, which shows $\operatorname{conv} X$ is bounded.

In order to see $\operatorname{conv} X$ is closed let $x = \lim_{i\to\infty} x_i$, with $x_i \in \operatorname{conv} X$, $i \in \mathbb{N}$. By Carathéodory's Theorem 2.7 we can write $x_i = \sum_{j=1}^{n+1} \lambda_{ij} x_{ij}$, with $x_{ij} \in X$ and $\lambda_{ij} \geq 0$ for $j = 1, \dots, n+1$, and $\sum_{j=1}^{n+1} \lambda_{ij} = 1$.

Since $0 \leq \lambda_{ij} \leq 1$ and X is compact we know that there exist the limits $\lim_{i\to\infty} \lambda_{ij} = \lambda_j$ and $\lim_{i\to\infty} x_{ij} = x_j$ for $1 \leq j \leq n+1$. Moreover, for $j =, \dots, n+1$, $x_j \in X$, $\lambda_j \geq 0$ and $\sum_{j=1}^{n+1} \lambda_j = 1$. Finally we get

$$x = \lim_{i\to\infty} x_i = \lim_{i\to\infty} \sum_{j=1}^{n+1} \lambda_{ij} x_{ij} = \sum_{j=1}^{n+1} \lambda_j x_j,$$

which proves that $x \in \operatorname{conv} X$. $\square$

2.11 Theorem [(weak)Fractional Helly theorem]. *Let $K_1, \dots, K_m \in \mathcal{C}^n$, $m \geq n+1$, and let $\alpha \in (0,1]$ such that for at least $\alpha \binom{m}{n+1}$ of the $(n+1)$-index sets $I \subseteq \{1, \dots, m\}$ we have $\bigcap_{i\in I} K_i \neq \emptyset$. Then there exists a point in common of at least $\frac{\alpha}{n+1} \cdot m$ sets K_i.*

Proof. For $I \subseteq \{1, \dots, m\}$ let $K_I = \bigcap_{i\in I} K_i$. First, we argue that we may assume that K_i are compact convex sets. To this end let $x_I \in K_I$ for any $(n+1)$ index set I with $K_I \neq \emptyset$ and set $K_i' = \operatorname{conv}\{x_I : i \in I\} \subseteq K_i$. Then for each $(n+1)$-index set J we have $\bigcap_{j\in J} K_j \neq \emptyset$ if and only if $\bigcap_{j\in J} K_j' \neq \emptyset$. Hence let K_i, and thus K_I, be compact convex sets. The compactness of a subset $D \subset \mathbb{R}^n$, in particular, implies that is has a unique minimal point with respect to the lexicographic ordering $<_{\text{lex}}$. Next we claim:

> Let $K_I \neq \emptyset$, $\#I \geq n$, and let $y_I \in K_I$ be the unique lexicographical minumim of K_I. Then there exists an n-index subset $J \subseteq I$ such that y_I is the lexiographic minimum of K_J as well. **(Claim)**

For the proof of the claim we observe that $C = \{x \in \mathbb{R}^n : x <_{\text{lex}} y_I\}$ is a convex set with $C \cap K_I = \emptyset$. By Helly's Theorem 2.2 there must exist already an n-index set $J \subseteq I$ such that $C \cap K_J = \emptyset$ which means that $y_I \in K_J$ is the minimum of K_J as well.

To finish the proof we fix for any of the $\alpha \binom{m}{n+1}$ index sets I of cardinality $n+1$ an n-index set $J_I \subset I$ having the same lexicographic minimum as K_I. The number of distinct n tuples is $\binom{m}{n}$, hence one of them, $\overline{J}$, say, appears as J_I for at least $\alpha \binom{m}{n+1} / \binom{m}{n} = \alpha \frac{m-n}{n+1}$ distinct sets I which are of the type $\overline{J} \cup \{i\}$ for some $i \in \{1, \dots, m\}$. Hence the lexicographic minimum of $K_{\overline{J}}$ is contained in at least

$$n + \alpha \frac{m-n}{n+1} > \alpha \frac{m}{n+1}$$

sets K_i. □

2.12 Remark. *The (strong and sharp) fractional Helly theorem, which is due to Kalai, gives that* $(1-(1-\alpha)^{1/(n+1)})\cdot m$ *sets have a point in common. Obviously, for* $\alpha = 1$ *we get again the classcial Helly Theorem 2.2.*

2.13 Theorem [Colorful Carathéodory theorem]. *Let* $X_1, \dots, X_{n+1} \subset \mathbb{R}^n$ *be finite point sets such that* $0 \in \operatorname{conv} X_i$, $1 \le i \le n+1$. *There exist* $x_i \in X_i$, $1 \le i \le n+1$, *such that* $0 \in \operatorname{conv}\{x_1, \dots, x_{n+1}\}$.

Proof. For $x_i \in X_i$, $1 \le i \le n+1$, we call $\{x_1, \dots, x_{n+1}\}$ a rainbow set and $\operatorname{conv}\{x_1, \dots, x_{n+1}\}$ a rainbow simplex. Suppose there is no rainbow simplex containing 0, and let $S = \{s_1, \dots, s_{n+1}\}$, $s_i \in X_i$, be a rainbow set such that $\operatorname{conv} S$ has minimal distance to 0 attained at the point $s \in S$. Let $H = \{x \in \mathbb{R}^n : \langle s, x\rangle = \langle s, s\rangle\}$ be the hyperplane perpendicular s and containing s. Then we may assume that $S \subset H^+ = \{x \in \mathbb{R}^n : \langle s, x\rangle \ge \langle s, s\rangle\}$ which does not contain 0.

Since $\operatorname{conv} S \cap H = \operatorname{conv}(S \cap H)$ we know by Carathéodory's Theorem 2.7 that there exists an n-point set $T \subset S \cap H$ with $s \in \operatorname{conv} T$. Without loss of generality let $s_1 \notin T$. If $X_1 \subset H^+$ then $0 \notin \operatorname{conv} X_1$ and so we may assume that there exists a $z \in X_1$ with $\langle s, z\rangle < \langle s, s\rangle$. Next we consider the rainbow set $S' = S \setminus \{s_1\} \cup \{z\}$. Then $T \subset S'$ and thus $x \in \operatorname{conv} S'$. Hence $\operatorname{conv}\{s, z\} \subset \operatorname{conv} S'$, but due to the choice of z the segment $\operatorname{conv}\{s, z\}$ containes a point closer to 0 than s. □

2.14 Theorem [Tverberg's theorem]. *Let* $X \subseteq \mathbb{R}^n$ *and let* $k \in \mathbb{N}_{\ge 1}$. *If* $\#X \ge (k-1)(n+1)+1$, $k \in \mathbb{N}$, *then there exist* k *subsets* $X_1, \dots, X_k \subset X$ *with* $X_i \cap X_j = \emptyset$, $i \ne j$, *but* $\operatorname{conv} X_1 \cap \operatorname{conv} X_2 \cap \dots \cap \operatorname{conv} X_k \ne \emptyset$.

Proof. First we show that the following conic version of the theorem is equivalent to Tverberg's result:

Let $X \subseteq \mathbb{R}^{n+1}$, $0 \notin \operatorname{conv} X$, and let $k \in \mathbb{N}_{\ge 1}$. If $\#X \ge (k-1)(n+1)+1$, $k \in \mathbb{N}$, then there exist k subsets $X_1, \dots, X_k \subset X$ with $X_i \cap X_j = \emptyset$, $i \ne j$, but $\operatorname{pos} X_1 \cap \operatorname{pos} X_2 \cap \dots \cap \operatorname{pos} X_k \ne \{0\}$. (2.14.1)

To see that it implies the theorem we canonically embed our given set $X \subseteq \mathbb{R}^n$ into $\mathbb{R}^{n+1}$ by adding a 1 to all points of X as last coordinate. This embedded set $\widetilde{X}$ is contained in the hyperplane $H = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 1\}$ and hence $0 \notin \operatorname{conv} \widetilde{X}$. So, according to (2.14.1), there exist $\widetilde{X}_1, \dots, \widetilde{X}_k \subset \widetilde{X}$ with $\widetilde{X}_i \cap \widetilde{X}_j = \emptyset$, $i \ne j$, and a point $\widetilde{y} \ne 0$ with $\widetilde{y} \in \operatorname{pos} \widetilde{X}_1 \cap \operatorname{pos} \widetilde{X}_2 \cap \dots \cap \operatorname{pos} \widetilde{X}_k$. In particular, $\operatorname{pos}\{\widetilde{y}\} \subseteq \operatorname{pos} \widetilde{X}_1 \cap \operatorname{pos} \widetilde{X}_2 \cap \dots \cap \operatorname{pos} \widetilde{X}_k$ and so we may assume that $\widetilde{y} = (y, 1)^\intercal$. But then $y \in \operatorname{conv} X_1 \cap \operatorname{conv} X_2 \cap \dots \cap \operatorname{conv} X_k \ne \emptyset$, where X_i is obtained from $\widetilde{X}_i$ by removing the last coordinate from all points. The other implication is left as an exercise.

For the proof of 2.14.1 we set $N = (k-1)(n+1)$ and we consider the following k linear maps $\phi_j : \mathbb{R}^{n+1} \to \mathbb{R}^N$: The first $k-1$ maps just "writes" the coordinates of a vector $x \in \mathbb{R}^{n+1}$ at the positions $(j-1)(n+1)+1, \ldots, j(n+1)$ and all other coordinates are zero, or blockwise

$$\phi_j(x) = (0\,|\,0\,|\,\cdots\,|\,0\,|\,x\,|\,0\,|\,\cdots\,|\,0)^\intercal, \quad j = 1, \ldots, k-1.$$

The k-th map $\phi_k : \mathbb{R}^{n+1} \to \mathbb{R}^N$ just has $-x$ in each block, i.e.,

$$\phi_k(x) = (-x\,|\,-x\,|\,\cdots\,|\,-x)^\intercal.$$

We observe that for $x_1, \ldots, x_k \in \mathbb{R}^{n+1}$ holds

$$\sum_{j=1}^{k} \phi_j(x_j) = 0 \text{ if and only if } x_1 = x_2 = \cdots = x_k. \tag{2.14.2}$$

Of course, it suffices to prove 2.14.1 for a set $X = \{x_1, \ldots, x_{N+1}\} \subset \mathbb{R}^{n+1}$ of cardinality $N+1$ with the property $0 \notin \operatorname{conv} X$. Let $M = \phi_1(X) \cup \phi_2(X) \cup \cdots \cup \phi_k(X) \subset \mathbb{R}^N$ and we set

$$M_i = \{\phi_1(x_i), \phi_2(x_i), \ldots, \phi_k(x_i)\}, 1 \leq i \leq N+1.$$

In view of (2.14.2) we have $0 \in \operatorname{conv} M_i$, $1 \leq i \leq N+1$, and hence we may apply the colorful Carathéodory Theorem 2.13 and find a rainbow set $S \subseteq M$, $\#S = N+1$ with $0 \in \operatorname{conv} S$.

Let $f(i) \in \{1, \ldots, k\}$ be the index of the point of M_i in S, i.e., $S = \{\phi_{f(1)}(x_1), \phi_{f(2)}(x_2), \ldots, \phi_{f(N+1)}(x_{N+1})\}$. Since $0 \in \operatorname{conv} S$ there exist non-negative numbers α_i, $1 \leq i \leq N+1$, such that

$$0 = \sum_{i=1}^{N+1} \alpha_i\, \phi_{f(i)}(x_i) \text{ and } \sum_{i=1}^{N+1} \alpha_i = 1. \tag{2.14.3}$$

Next for $1 \leq j \leq k$ let $I_j = \{i : f(i) = j, 1 \leq i \leq N+1\}$ and let $X_j = \{x_i : i \in I_j\}$. By the linearity of the maps ϕ we can rewrite (2.14.3) as

$$0 = \sum_{i=1}^{N+1} \alpha_i\, \phi_{f(i)}(x_i) = \sum_{j=1}^{k} \sum_{i \in I_j} \alpha_i\, \phi_j(x_i) = \sum_{j=1}^{k} \phi_j \left(\sum_{i \in I_j} \alpha_i\, x_i \right).$$

Setting $u_j = \sum_{i \in I_j} \alpha_i\, x_i$, $1 \leq j \leq k$, we have $u_j \in \operatorname{pos} X_j$ and $\sum_{j=1}^{k} \phi_j(u_j) = 0$. By (2.14.2) we obtain $u_1 = u_2 = \cdots = u_k$ and so we have found a point in common of all $\operatorname{pos} X_j$.

In remains to show that $u_j \neq 0$ (and so also $I_j \neq \emptyset$). By (2.14.3) we can find a representation of $u_j = \sum_{i \in I_j} \alpha_i\, x_i$ with some $\alpha_i > 0$. Since $0 \notin \operatorname{conv} X$ we conclude $u_j \neq 0$. □

2.15 Theorem. *Let $X \subset \mathbb{R}^n$ and let $\#X = m \geq n+1$. Then there exists a point $y \in \mathbb{R}^n$ contained in at least $\gamma_n \binom{m}{n+1}$ X-simplices, i.e., simplices of the form $\operatorname{conv} S$, $S \subseteq X$, $\#S = n+1$. Here γ_n is a positive constant depending only on the dimension, and X-simplices $\operatorname{conv} S_1$, $\operatorname{conv} S_2$ are considered different if $S_1 \neq S_2$.*

Proof. We may assume that m is large, e.g., $m \geq 2n(n+1)$, otherwise we can choose γ_n so small that y is contained in a single simplex.

Let $k = \lceil m/(n+1) \rceil$. Then $m \geq (k-1)(n+1)+1$ and by Tverberg's Theorem 2.14 there exists k pairwise disjoint subsets $X_i \subset X$ having a point $y \in \cap_{i=1}^{k} \operatorname{conv} X_i$. For a subset $J = \{j_0, \ldots, j_n\} \subset \{1, \ldots, k\}$ and with respect to the sets X_{j_i}, $0 \leq i \leq n$, we may apply the colorful Carathéodory theorem and find an X-simplex S_J containing y and having one vertex from each X_{j_i}. For different sets J we get different X-simplices S_J and so the number of X-simplices containing y is at least

$$\begin{aligned}\binom{k}{n+1} = \binom{\lceil m/(n+1) \rceil}{n+1} &\geq \frac{\frac{m}{n+1} \cdot \left(\frac{m}{n+1} - 1\right) \cdot \ldots \cdot \left(\frac{m}{n+1} - n\right)}{(n+1)!} \\ &= \frac{1}{(n+1)^{n+1}} \frac{m \cdot (m-(n+1)) \cdot \ldots \cdot (m - n(n+1))}{(n+1)!} \\ &\sim \frac{1}{(n+1)^{n+1}} \frac{1}{2^n} \binom{m}{n+1},\end{aligned}$$

in view of our assumption $m \geq 2n(n+1)$. □

3 Polytopes

3.1 Definition [Polyhedron]. *The intersection of finitely many closed halfspaces is called a* polyhedron.

3.2 Theorem [Minkowski, 1896, Weyl, 1935].

i) *A bounded polyhedron is a polytope.*

ii) *A polytope is a bounded polyhedron.*

Proof.

i) Let $P = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \leq \alpha_i,\ 1 \leq i \leq m\}$ be bounded. We proceed by induction on n. Every closed and bounded convex subset of $\mathbb{R}$ is an closed interval and so the case $n = 1$ is trivial. Hence, we may assume $n \geq 2$ and let $F_i = P \cap H(a_i, \alpha_i)$, $1 \leq i \leq m$. F_i is a bounded polyhedron in an affine space of dimension at most $n - 1$ and by our inductive argument there exist finite sets V_i such that $F_i = \operatorname{conv} V_i$, $1 \leq i \leq m$. It suffices to show that
$$P = \operatorname{conv}\ \{V_1 \cup V_2 \cup \cdots \cup V_m\}.$$
The inclusion "$\supseteq$" follows from $V_i \subset P$. For the reverse inclusion let $x \in P$and let l be a line passing through x. The intersection $l \cap P$ is a non-empty compact convex set of dimension at most 1. Hence we can find $y, z \in P$ such that $l \cap P = \operatorname{conv}\{y, z\}$. Since both, y and z, has to lie in the boundary of P we can find k and j with $y \in P \cap H(a_k, \alpha_k) = F_k = \operatorname{conv} V_k$ and $z \in P \cap H(a_j, \alpha_j) = F_j = \operatorname{conv} V_j$. So we have
$$x \in \operatorname{conv}\{y, z\} \subseteq \operatorname{conv}\{V_j \cup V_k\} \subseteq \operatorname{conv}\ \{V_1 \cup V_2 \cup \cdots \cup V_m\}.$$

ii) For the second statement we apply polarity to i). Let $P = \operatorname{conv}\{v_1, \ldots, v_m\}$, and here we may assume that $\dim P = n$ and $0 \in \operatorname{int} P$. By Proposition 1.17 and Lemma 1.18 i) we find that $P^\star$ is a bounded polyhedron. Applying i) to $P^\star$ we can find points $w_1, \ldots, w_l \in \mathbb{R}^n$ such that $P^\star = \operatorname{conv}\{w_1, \ldots, w_l\}$. Next we consider $(P^\star)^\star$, which can be written as (cf. Lemma 1.18 i))
$$(P^\star)^\star = \{x \in \mathbb{R}^n : \langle w_i, x\rangle \leq 1,\ 1 \leq i \leq l\}.$$
By Lemma 1.19 we also know $P = (P^\star)^\star$ and everything is shown.

□

3.3 Notation [$\mathcal{V}$-Polytope, $\mathcal{H}$-Polytope]. *A polytope given as the convex hull of finitely many points is called a* $\mathcal{V}$-polytope. *If it is given as the bounded intersection of finitely many closed halfspaces, then it is called an* $\mathcal{H}$-polytope.

3.4 Corollary. *Let $P \in \mathcal{P}^n$.*

i) *Let* $A \in \mathbb{R}^{m \times n}$ *and* $t \in \mathbb{R}^m$. *Then* $AP + t$ *is a polytope.*

ii) *Let* $U \subset \mathbb{R}^n$ *be an affine subspace. Then* $P \cap U$ *is a polytope.*

Proof. For i) we may assume that P is a $\mathcal{V}$-Polytope, i.e., $P = \operatorname{conv}\{v_1, \ldots, v_m\}$ for some points $v_i \in \mathbb{R}^n$. Hence $AP+t = t+\operatorname{conv}\{A\,v_1, \ldots, A\,v_m\} = \operatorname{conv}\{A\,v_1 + t, \ldots, A\,v_m + t\}$ is again a polytope.

For ii) we regard P as an $\mathcal{H}$-polytope, i.e., $P = \cap_{i=1}^m H^-(a_i, \alpha_i)$ for suitable hyperplanes $H(a_i, \alpha_i)$. Each affine subspace $U \subset \mathbb{R}^n$ can be written as intersection of hyperplanes and thus as intersection of halfspaces. Hence let $U = \cap_{i=1}^k H^-(u_i, \mu_i)$, $1 \le i \le k$, and so $P \cap U = \cap_{i=1}^m H^-(a_i, \alpha_i) \cap \cap_{i=1}^k H^-(u_i, \mu_i)$. Since P is bounded, $P \cap U$ is a bounded polyhedron, i.e., a polytope by Theorem 3.2 □

3.5 Definition [Faces]. *Let* $K \in \mathcal{C}^n$ *be closed and let* H *be a supporting hyperplane of* K. *If* $j = \dim(K \cap H)$, *then* $K \cap H$ *is called a* j-*face of* K. *Moreover,* K *itself is regarded as a* $(\dim K)$-*face and the empty set* $\emptyset$ *as* (-1)-*face of* K.

3.6 Notation [Vertices, edges, facets]. *A* 0-*face of* $K \in \mathcal{C}^n$, K *closed, is called* vertex, *an* 1-*face is called* edge *and a* $(\dim K - 1)$-*face is called* facet *of* K. K *itself and the empty set are called* improper faces, *whereas the remaining faces are called* proper faces *of* K.

The set of all vertices of a polytope P *is denoted by* $\operatorname{vert} P$.

3.7 Remark.

i) *Let* $K \in \mathcal{C}^n$ *be closed. Every (relative) boundary point of* K *lies in a suitable* j-*face,* $0 \le j \le \dim K - 1$.

ii) *Let* $K \in \mathcal{C}^n$, $\dim K = n$. *Let* F *be a facet of* K *and* H *a supporting hyperplane of* K *with* $F = K \cap H$. *Then* $H = \operatorname{aff} F$.

3.8 Proposition. *Each face of a polytope is a polytope, and a polytope has only finitely many faces.*

Proof. Let $P = \operatorname{conv}\{v_1, \ldots, v_m\}$ and let $F = P \cap H(a, \alpha)$ be a face of F, i.e., $H(a, \alpha)$ is a supporting hyperplane of P. By Proposition 1.3 we have $P \cap H(a, \alpha) = \operatorname{conv}\{H(a, \alpha) \cap \{v_1, \ldots, v_m\}\}$ which shows that F is a polytope.

In particular, we can write each face F of P as $\operatorname{conv} V_F$ for a suitable subset $V_F \subseteq \{v_1, \ldots, v_m\}$. Hence the number of different faces is certainly bounded by $\sum_{i=0}^m \binom{m}{i} = 2^m$. In fact, since different faces have different affine hulls and since the affine hull of a j-face, $j \in \{0, \ldots, n-1\}$, is uniquely determined by $j+1$ affinely independent points we even may bound the number of all proper faces by $\sum_{i=0}^n \binom{m}{i+1}$. □

3.9 Definition [f-vector]. *For* $P \in \mathcal{P}^n$ *let* $f_i(P)$ *be the number of* i-*faces of* P, $-1 \le i \le \dim P$. *Furthermore, let* $f_i(P) = 0$ *for* $\dim P + 1 \le i \le n$. *The vector* $f(P)$ *with entries* $f_i(P)$, $-1 \le i \le n$, *is called the* f-vector *of* P.

3.10 Remark.

i) *Let* $T_n = \operatorname{conv}\{0, e_1, \ldots, e_n\}$ *be the so called* standard simplex. *Then* $f_i(T_n) = \binom{n+1}{i+1}$, *i.e., any* $(i+1)$ *subset of the vertices are the vertices of an* i*-face.*

ii) *For any* n*-polytope* $P \in \mathcal{P}^n$ *we have* $\sum_{i=-1}^{n} f_i(P) \geq 2^{n+1}$ *with equality if and only if* P *is an* n*-simplex.*

3.11 Lemma. *Let* $P \in \mathcal{P}^n$.

i) $v \in \operatorname{vert} P$ *can not be written as a convex combination of two other points of* P, *i.e.,* $v \notin \operatorname{conv}\left(P \backslash \{v\}\right)$.

ii) *If* $P = \operatorname{conv} W$, *then* $\operatorname{vert} P \subseteq W$.

iii) $P = \operatorname{conv}\,(\operatorname{vert} P)$.

Proof. For i) let $H(a, \alpha)$ be a supporting plane of v. Then we have $\langle a, v\rangle = \alpha$ and $\langle a, x\rangle < \alpha$ for all $x \in P \setminus \{v\}$. Hence we cannot write $v = \lambda x_1 + (1-\lambda) x_2$ and $x_i \in P \setminus \{v\}$, $\lambda \in (0, 1)$.

Statement ii) is a consequence of i), because assume that there is a $v \in \operatorname{vert} P \setminus W$. Since $P = \operatorname{conv} W$ we conclude $W \subseteq P \setminus \{v\}$ which contradicts i).

For iii) let $W \subset \mathbb{R}^n$ be a minimal (w.r.t. its cardinality) set with $P = \operatorname{conv} W$. By ii) we know already $\operatorname{vert} P \subseteq W$, and so for $w \in W$ it remains to show that w is a vertex of P. By the minimality of W we have $w \notin \operatorname{conv} W \backslash \{w\}$. By Theorem 1.13 there exists a strong separation hyperplane $H(a, \alpha)$ of w and $\operatorname{conv}(W \setminus \{w\})$, and let $\langle a, w\rangle > \alpha$ and $\langle a, \overline{w}\rangle < \alpha$ for all $\overline{w} \in \operatorname{conv}(W \setminus \{w\})$. With $\alpha* = \langle a, w\rangle$ this implies firstly $P = \operatorname{conv} W \subset H^-(a, \alpha*)$ and secondly (cf. Proposition (1.3))

$$P \cap H(a, \alpha^*) = \operatorname{conv}(W \cap H(a, \alpha*)) = \{w\}.$$

Hence $w \in \operatorname{vert} P$. □

3.12 Lemma. *Let* $P \in \mathcal{P}^n$ *be an* n*-polytope with* $0 \in \operatorname{int} P$. *For a proper face* F *of* P *let*

$$F^\diamond = \{y \in P^\star : \langle x, y\rangle = 1 \text{ for all } x \in F\}.$$

Then

i) $F^\diamond$ *is a face of* $P^\star$.

ii) $F = (F^\diamond)^\diamond$.

iii) *If* G *is a face of* P *and* $F \subseteq G$, *then* $G^\diamond \subseteq F^\diamond$.

iv) $\dim F^\diamond = n - 1 - \dim F$.

Proof. First we fix the notation. Let $P = \operatorname{conv}\{v_1, \dots, v_m\}$, so that $P^\star = \{y \in \mathbb{R}^n : \langle v_i, y\rangle \leq 1,\, 1 \leq i \leq m\}$. We may assume $\dim F = k$, $F = \operatorname{conv}\{v_1, \dots, v_l\}$, $l \in \{k+1, \dots, m\}$, and that $v_1, \dots, v_{k+1}$ are affinely independent. Moreover, let $H(a, 1)$ be a supporting hyperplane of F, i.e., $F = \{x \in P : \langle a, x\rangle = 1\}$ and $\langle a, x\rangle \leq 1$ for all $x \in P$. Hence $a \in P^\star$ and , in particular, $a \in F^\diamond$.

For i) we observe that we may write

$$F^\diamond = \{y \in P^\star : \langle v_i, y\rangle = 1,\, 1 \leq i \leq l\}. \tag{3.12.1}$$

Since $\langle v_i, y\rangle \leq 1$ for all $y \in P^\star$ we conclude

$$F^\diamond = \{y \in P^\star : \langle v_1 + v_2 + \cdots + v_l, y\rangle = l\}$$

and $P^\star \subset \{y \in \mathbb{R}^n : \langle v_1 + v_2 + \cdots + v_l, y\rangle \leq l\}$. Hence $F^\diamond$ is a face of $P^\star$ which shows i).

By definition we have $(F^\diamond)^\diamond = \{x \in P : \langle y, x\rangle = 1 \text{ for all } y \in F^\diamond\}$ and so $v_i \in (F^\diamond)^\diamond$, $1 \leq i \leq l$, which implies $F \subseteq (F^\diamond)^\diamond$. For the reverse inclusion we recall that $a \in F^\diamond$ and so

$$(F^\diamond)^\diamond \subseteq \{x \in P : \langle a, x\rangle = 1\} = F.$$

iii) is obvious, and for iv) we first note that the vectors $v_1, \dots, v_{k+1}$ are, in fact, linearly independent, otherwise the system given by the right hand side of (3.12.1) could have only a solution if the vectors are affinely dependent. Hence with

$$U = \{y \in \mathbb{R}^n : \langle v_i, y\rangle = 0,\, 1 \leq i \leq l\}$$

we have $\dim U = n - (k+1)$ and so $\dim F^\diamond = \dim((a + U) \cap P^\star) \leq n - (k+1)$. Now let $z \in U$ arbitrary and for $\mu \in /$ we consider $a + \mu\, z$. Then we have

$$\langle v_i, a + \mu z\rangle = \begin{cases} 1, & 1 \leq i \leq l, \\ \langle v_i, a\rangle + \mu \langle v_i, z\rangle, & i > l. \end{cases}$$

Since $\langle v_i, a\rangle < 1$ for $i > l$ we can find and $\epsilon_z > 0$ such that $a + \mu\, z \in F^\diamond$ for all $\mu < |\epsilon_z|$. Since $z \in U$ was arbitrary we conclude $\dim F^\diamond \geq n - (k+1)$. □

3.13 Theorem. *Let $P \in \mathcal{P}^n$ be an n-polytope with $0 \in \operatorname{int} P$. Then*

$$f_{n-1-i}(P^\star) = f_i(P), \quad -1 \leq i \leq n.$$

Proof. For $i \in \{-1, n\}$ there is nothing to prove. By Lemma 3.12 we can associate to every i-face F a dual face $n-(i+1)$-face $F^\diamond$ of $P^\star$, and for different i-faces of P the associated dual faces are different as well. Hence $f_{n-i-1}(P^\star) \geq f_i(P)$. In the same way we find by Lemma 3.12 ii) that $f_i(P) \geq f_{n-i-1}(P^\star)$. □

3.14 Theorem. *Let $P \in \mathcal{P}^n$ be an n-polytope with facets $F_1, \dots, F_m$ and let $H(a_i, \alpha_i)$, $1 \leq i \leq m$, be the supporting hyperplanes of F_i, $1 \leq i \leq m$. Then*

$$P = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \leq \alpha_i,\, 1 \leq i \leq m\}.$$

Proof. We may assume that $0 \in \operatorname{int} P$. In view of Theorem 3.13, $P^\star$ has m vertices $v_1, \ldots, v_m$, say, and by Lemma 3.11 iii) we have $P^\star = \operatorname{conv}\{v_1, \ldots, v_m\}$. In view of Lemma 1.19 and Proposition 1.18 we have

$$P = P^{\star\star} = \{x \in \mathbb{R}^n : \langle v_i, x \rangle \leq 1,\, 1 \leq i \leq n\}$$

and according to Lemma 3.12 $\{v_i\}^\diamond = \{x \in P : \langle v_i, x \rangle = 1\}$ is an $(n-1)$-face of P. Finally, we observe that after an appropriate renumbering we must have $H(a_i, \alpha_i) = \operatorname{aff}\{v_i\}^\diamond$, $1 \leq i \leq m$. □

3.15 Theorem. *Let $P \in \mathcal{P}^n$ be an n-polytope.*

i) *The boundary of P is the union of all its facets.*

ii) *A k-face is the intersection of (at least) $(n-k)$ facets.*

iii) *An $(n-2)$-face is contained in exactly two facets.*

iv) *If F, G are faces of P with $F \subseteq G$, then F is a face of G.*

v) *A face of P is also a face of a facet of P.*

Proof. Again we we may assume that $0 \in \operatorname{int} P$. i) is an immediate consequence of Theorem 3.14.

For ii) let F be a k-face of P, let $P^\star = \operatorname{conv}\{v_1, \ldots, v_m\}$ with v_i vertex of $P^\star$ and let $F^\diamond = \operatorname{conv}\{v_1, \ldots, v_l\}$. By Lemma 3.12 iv) we have $\dim F^\diamond = n-(k+1)$ and so $l \geq n-k$. Moreover, with Lemma 3.12 ii) we get

$$F = (F^\diamond)^\diamond = \{x \in P : \langle v_i, x \rangle = 1,\, 1 \leq i \leq l\} = \bigcap_{i=1}^{l} \{x \in P : \langle v_i, x \rangle = 1\}.$$

Since v_i is a vertex of $P^\star$, we get by Lemma 3.12 that $\{x \in P : \langle v_i, x \rangle = 1\}$ is a facet of P. Observe, that we may assume that the first $n-k$ vertices $v_1, \ldots, v_{n-k}$ are affine independent and hence we must also have $F = \bigcap_{i=1}^{n-k} \{x \in P : \langle v_i, x \rangle = 1\}$.

With this setting we know already that we can write a $(n-2)$-face F of P as

$$F = \bigcap_{i=1}^{r} \{x \in P : \langle v_i, x \rangle = 1\}$$

for some $r \geq 2$. By Lemma 3.12 the dual face $F^\diamond = \operatorname{conv}\{v_1, \ldots, v_r\}$ is a 1-face of $P^\star$. Since v_i are vertices we conclude $r = 2$, because otherwise a vertex could be written as a convex combination of other vertices, and so contradicting Lemma 3.11 i).

For iv) let $H(a, \alpha)$ be a supporting hyperplane of F, i.e., $F = P \cap H(a, \alpha)$. Then $F = G \cap F = G \cap P \cap H(a, \alpha) = G \cap H(a, \alpha)$, which shows that F is a face of G.

Finally, we observe that ii) shows that any face is contained in a facet and thus, by iv), it is a face of that facet. □

3.16 Theorem. *Let $P \in \mathcal{P}^n$ be an n-polytope.*

i) *Let G be a face of P and let F be a face of G. Then F is a face of P.*

ii) *Let F_j be a j-face of P and let F_k be a k-face of P with $F_j \subset F_k$. There exist i-faces F_i of P, $j < i < k$, such that*

$$F_j \subset F_{j+1} \subset \cdots \subset F_{k-1} \subset F_k.$$

Proof. For i) let F and G be proper faces and let $0 \in F \subset G$. Let $H(a_F, 0)$ and $H(a_G, 0)$ be supporting hyperplanes of F and G, respectively, i.e., $F = H(a_F, 0) \cap G$, $G = H(a_G, 0) \cap P$ and $G \subset H(a_F, 0)^-$, $P \subset H(a_G, 0)^-$.

For $\mu \geq 0$ we consider the Hyperplane $H(a_F + \mu\, a_G, 0)$ and observe that

$$\langle a_F + \mu\, a_G, x\rangle = \langle a_F, x\rangle + \mu\, \langle a_G, x\rangle \quad \begin{cases} = & 0, \text{ for } x \in F \text{ and } \mu \geq 0, \\ < & 0, \text{ for } x \in G \setminus F \text{ and } \mu \geq 0. \end{cases}$$

Since for all $x \in P \setminus G$ we have $\langle a_G, x\rangle < 0$ we can easily determine a $\overline{\mu} > 0$ such that for $x \in \operatorname{vert} P \setminus \operatorname{vert} G$ we have $\langle a_F + \overline{\mu}\, a_G, x\rangle < 0$. Hence for $x \in P$ which we can write as a convex combination $x = \sum_{v \in \operatorname{vert} P \setminus \operatorname{vert} G} \lambda_v\, v + \sum_{v \in \operatorname{vert} G \setminus F} \lambda_v\, v + \sum_{v \in (\operatorname{vert} G) \cap F} \lambda_v\, v$ we find

$$\begin{aligned} \langle a_F + \overline{\mu}\, a_G, x\rangle = & \sum_{v \in \operatorname{vert} P \setminus \operatorname{vert} G} \lambda_v\, \langle a_F + \overline{\mu}\, a_G, v\rangle + \sum_{v \in (\operatorname{vert} G) \setminus F} \lambda_v\, \langle a_F + \overline{\mu}\, a_G, v\rangle \\ & + \sum_{v \in (\operatorname{vert} G) \cap F} \lambda_v\, \langle a_F + \overline{\mu}\, a_G, v\rangle \\ & \leq 0, \end{aligned}$$

with equality if and only if $x \in F$. Hence F is a face of P.

In order to verify ii) we may assume $j \leq k - 2$ and we first note that by Theorem 3.16 iv) F_j is a face of F_k. According to Theorem 3.16 it is contained in a facet G, say, of F_k, i.e., G is a $(k-1)$ face of F_K and hence it is a face of P by i). With $F_{k-1} = G$ we have shown $F_j \subset F_{k-1} \subset F_k$. In the case $j < k - 2$ we can apply the same reasoning as above to the pair F_j, F_{k-1}, and obtain so recursively the desired chain of faces. □

3.17 Remark. *Let v_0 be a vertex of an n-polytope P and let $\{v_1, \ldots, v_r\}$ be all* adjacent *vertices of v_0, i.e., $\operatorname{conv}\{v_0, v_i\}$ is an edge of P. In other words, $\{v_1, \ldots, v_r\}$ are the neighbours of v_0. Then*

i)

$$P \subset v_0 + \operatorname{pos}\{v_1 - v_0, \ldots, v_r - v_0\}.$$

ii) *Let $c \in \mathbb{R}^n$ with $\langle c, v_0\rangle \geq \langle c, v_i\rangle$, $1 \leq i \leq r$. Then*

$$\max\{\langle c, x\rangle : x \in P\} = \langle c, v_0\rangle\,.$$

3.18 Theorem [Euler-Poincaré formula]. *Let $P \in \mathcal{P}^n$. Then*

$$\sum_{i=-1}^{n} (-1)^i f_i(P) = 0. \tag{3.18.1}$$

In particular, in the 3-dimensional case, i.e., $\dim P = 3$, *it holds* $f_0 - f_1 + f_2 = 2$.

Proof. It suffices to consider n-dimensional polytopes and we proceed by induction with respct to n. In the case $n = 1$ there is nothing to prove as it is $-1 + f_0(P) - 1 = 0$. Let $n \geq 2$, $m = f_0(P)$ and let $a \in \mathbb{R}^n \setminus \{0\}$ be chosen such that for any $\alpha \in \mathbb{R}$ the hyperplane $H(a, \alpha)$ contains at most one vertex P. Such an a certainly exists because the normal vector u of a hyperplane $H(u, \mu)$ containing at least two vertices v, w, say, must be orthogonal to $v - w$.

Now let $\alpha_1 < \alpha_2 < \cdots < \alpha_{2m-1}$ such that the "odd" hyperplanes $H_{2i-1} := H(a, \alpha_{2i-1})$, $i = 1, \ldots, m$, contains a vertex of P. Obviously, we have $H_{2i} := H(a, \alpha_{2i}) \cap \operatorname{vert} P = \emptyset$, and H_1 and H_{2m-1} are supporting hyperplanes of P. Hence, for $i = 2, \ldots, 2m-2$ the polytopes $P_i := H_i \cap P$ are $(n-1)$-dimensional polytopes. For a j-face F of P, $\dim F \geq 1$, and a polytope P_i we set

$$\psi(F, P_i) = \begin{cases} 1, & P_i \cap \operatorname{relint} F \neq \emptyset, \\ 0, & P_i \cap \operatorname{relint} F = \emptyset. \end{cases}$$

Observe, that if $\psi(F, P_i) = 1$ then $F \cap H_i$ is a $(j-1)$ face of P. Now for a given j-face F the first index i_1, say, and the last index i_2, say, of a hyperplane H_i with $H_i \cap F \neq \emptyset$ has to be odd. Hence we have $\psi(F, P_i) = 1$ if and only if $i \in \{i_1 + 1, \ldots, i_2 - 1\}$ and so

$$\sum_{i=2}^{2m-2} (-1)^i \psi(F, P_i) = 1.$$

Summing over all j-faces yields

$$\sum_{j=1}^{n-1} (-1)^j \sum_{F\ j\text{-face}} \sum_{i=2}^{2m-2} (-1)^i \psi(F, P_i) = \sum_{j=1}^{n-1} (-1)^j f_j(P).$$

Changing the order of summation on the left hand side gives

$$\sum_{j=1}^{n-1} (-1)^j f_j(P) = \sum_{i=2}^{2m-2} (-1)^i \left(\sum_{j=1}^{n-1} (-1)^j \sum_{F\ j\text{-face}} \psi(F, P_i) \right), \tag{3.18.2}$$

and next we evaluate the interior sum on the right hand side.

For an even index $i \in \{2, \ldots, 2m-2\}$ or for $j > 1$ each $(j-1)$-face $\tilde{F}$ of such a P_i is the intersection of a j-face F of P with H_i. For odd i and $j = 1$ a vertex of P_i is either the intersection of an edge of P with H_i or it is the unique vertex of P lying in H_i. Hence for $j \geq 1$ we have

$$\sum_{F\ j\text{-face}} \psi(F, P_i) = \begin{cases} f_0(P_i) - 1, & j = 1 \text{ and } i \text{ odd}, \\ f_{j-1}(P_i), & \text{otherwise}, \end{cases}$$

which gives by our inductive argument

$$\begin{aligned}\sum_{j=1}^{n-1}(-1)^j \sum_{F\ j\text{-face}} \psi(F,P_i)\\ &= \begin{cases} 1+\sum_{j=1}^{n-1}(-1)^j f_{j-1}(P_i) = 1-\sum_{j=0}^{n-2}(-1)^j f_j(P_i) = (-1)^{n-1}, & i \text{ odd},\\ \sum_{j=1}^{n-1}(-1)^j f_{j-1}(P_i) = -\sum_{j=0}^{n-2}(-1)^j f_j(P_i) = (-1)^{n-1}-1, & i \text{ even.}\end{cases}\end{aligned}$$

Summing over i gives

$$\begin{aligned}\sum_{i=2}^{2\,m-2}(-1)^i\left(\sum_{j=1}^{n-1}(-1)^j \sum_{F\ j\text{-face}} \psi(F,P_i)\right) &= (-1)^{n-1}-(m-1)\\ &= -(-1)^{-1}-(-1)^n-(-1)^0\, f_0(P),\end{aligned}$$

which proves the theorem by (3.18.2). □

3.19 Proposition. *The Euler-Poincaré formula is the only linear equation satisfied by the f-vector, i.e., let $\lambda_i \in \mathbb{R}$, such that $\sum_{i=-1}^{n} \lambda_i\, f_i(P) = 0$ for all $P \in \mathcal{P}^n$. Then there exists a constant $\gamma \in \mathbb{R}$, such that $\lambda_i = \gamma\,(-1)^i$.*

Proof. Let $\sum_{i=-1}^{n} \lambda_i\, f_i(P) = 0$ be a linear equation which holds for any $P \in \mathcal{P}^n$. Taking a k-simplex T_k, $k \in \{0,\ldots,n\}$, we obtain

$$\sum_{i=-1}^{n} \lambda_i \binom{k+1}{i+1} = 0, \quad k = 0,\ldots,n+1.$$

The $(n+1)\times(n+2)$ matrix A with coefficients $a_{k,i} = \binom{k+1}{i+1}$ has rang $n+1$ and hence any solution $\lambda \in \mathbb{R}^{n+2}$ of the homogeneous system $A\,\lambda = 0$ must be a multiple of the coefficients given by the Euler-Poincare identity. □

3.20 Definition [Simple and simplicial polytopes]. *Let $P \in \mathcal{P}^n$.*

i) *P is called* simplicial *if all proper faces are simplices.*

ii) *P is called* simple *if every vertex is contained in exactly* $\dim P$ *many facets.*

3.21 Lemma. *Let $P \in \mathcal{P}^n$ be an n-polytope with $0 \in \operatorname{int} P$. The following statements are equivalent:*

i) *P is simplicial.*

ii) *All facets of P are simplices.*

iii) *$P^\star$ is simple.*

iv) *Every k-face of $P^\star$ is contained in exactly $n-k$ facets for $k = 0,\ldots,n-1$.*

Proof. We recall that by polarity we have $P = \text{conv}\{v_1, \ldots, v_m\}$ with v_i vertex of P if and only if $P^\star = \{y \in \mathbb{R}^n : \langle v_i, y \rangle \le 1, 1 \le i \le m\}$ with facets $\{v_i\}^\diamond = P^\star \cap H(v_i, 1)$, $1 \le i \le m$. Moreover, $F = \text{conv}\{v_{i_1}, \ldots, v_{i_k}\}$ is an l-face of P, $l \in \{0, \ldots, n-1\}$, with vertices $v_{i_1}, \ldots, v_{i_k}$ if and only if $F^\diamond = \{y \in P^\star : y \in H(v_{i_j}, 1), j = 1, \ldots, k\}$ is an $n - l - 1$-face of $P^\star$ contained only in the facets $\{v_{i_j}\}^\diamond$ of $P^\star$, $j = 1, \ldots, k$.
"i)⇔iv)": Let $F^\diamond$ be a k-face of $P^\star$. Then $(F^\diamond)^\diamond$ is an $n-k-1$ face of P, thus an $n-k-1$ simplex with $n-k$ vertices. Hence by the foregoing remark $F^\diamond$ is contained in exactly $n-k$ facets. On the other hand let F be a k-face of P. Then $F^\diamond$ is an $n-k-1$ of $P^\star$ contained in exactly $k+1$ facets, and thus the k-face F has exactly $k+1$ vertices and it is a simplex.
"ii)⇔iii)": Same argumentation as before where "⇒" is the case $k = 0$ and "⇐" is the case $k = n-1$.
"i)⇔ii)": Follows from the fact that every proper face of a polytope is a face of facet (see Theorem 3.15 v)). □

3.22 Theorem. *Let $P \in \mathcal{P}^n$ be a simple n-polytope. Then*

i) *Every vertex is contained in exactly $\binom{n}{k}$ k-faces of P, $k = 0, \ldots, n-1$.*

ii) *The intersection of $k \le n$ facets containing a common vertex is an $(n-k)$-face of P.*

iii) *Let $v_1, \ldots, v_n$ be the neighbours of a vertex v_0 of P. For each subset of $k < n$ neighbours $v_{i_1}, \ldots, v_{i_k}$ there exists a unique k-face F of P containing $v_0, v_{i_1}, \ldots, v_{i_k}$.*

iv) *A face of a simple polytope is simple.*

v) *Every j face of P is contained in exactly $\binom{n-j}{k-j}$ k faces of P.*

Proof. Without loss of generality we may assume that $0 \in \text{int}\, P$ and by Lemma 3.21 we know that $P^\star$ is simplicial.
i) Let v be a vertex of P and let F be a k-face of P. Then we have $\{v\} \subseteq F$ if and only if $F^\diamond \subseteq \{v\}^\diamond$ is a $n-k-1$ face of $P^\star$. Now $\{v\}^\diamond$ is facet of the simplicial polytope $P^\star$ and so it has exactly $\binom{n}{n-k}$ many $(n-k-1)$-dimensional faces.
ii) Let v be a vertex of P. Since P is simple, v is contained in n-facets $F_i = P \cap H(a_i, 1)$, $1 \le i \le n$, and we want to show that $\cap_{i=1}^k F_i$ is an $n-k$-face fo P. Since $P^\star$ is simplicial, $\{v\}^\diamond = \text{conv}\{a_1, \ldots, a_n\}$ is an $(n-1)$-simplex and so $\text{conv}\{a_1, \ldots, a_k\}$ is a $(k-1)$-face of $P^\star$. Hence $\text{conv}\{a_1, \ldots, a_k\}^\diamond$ is an $(n-k)$ face of P given by $\{x \in P : \langle a_i, x \rangle = 1,\ i = 1, \ldots, k\} = \cap_{i=1}^k F_i$.
iii) Let $\{v_0\}^\diamond = \text{conv}\{w_1, \ldots, w_n\}$ be a facet of $P^\star$. For the edges (1-faces) $\text{conv}\{v_0, v_i\}$ the associated polar face is an $(n-2)$-face and so let

$$\text{conv}\{v_0, v_i\}^\diamond = \text{conv}(\{w_1, \ldots, w_n\} \setminus \{w_i\}), \quad 1 \le i \le n.$$

Since $P^\star$ is simplicial,

$$U = \text{conv}(\{w_1, \ldots, w_n\} \setminus \{w_{i_1}, \ldots, w_{i_k}\}) = \cap_{j=1}^k \text{conv}(\{w_1, \ldots, w_n\} \setminus \{w_{i_j}\})$$

is an $(n-k-1)$-face of $P^\star$. Hence $U^\diamond$ is a k face containing the edges $\operatorname{conv}\{v_0, v_{i_j}\}$, $j = 1, \ldots, k$. For any k-face G of P containing these edges with have by polarity that $G^\diamond \subseteq \cap_{j=1}^k \operatorname{conv}(\{w_1, \ldots, w_n\} \setminus \{w_{i_j}\}) = U$. Since $\dim G^\diamond = \dim U = n-k-1$ we have $G = U^\diamond$. Thus $U^\diamond$ is uniquely determined.
iv) Let F be a k-face of P, $k \in \{1, \ldots, n-1\}$, and let v be a vertex of F. We want to show that v is contained in exactly k-facets of F, i.e., $(k-1)$-faces of P contained in F. Now $G \subset F$ is a $(k-1)$-face containing v if and only if $G^\diamond$ is a $(n-k)$-face of $P^\star$ with $F^\diamond \subset G^\diamond \subset \{v\}^\diamond$. Now $P^\star$ is simplicial and so we may assume that $\{v\}^\diamond = \operatorname{conv}\{w_1, \ldots, w_n\}$ and $F^\diamond = \{w_1, \ldots, w_{n-k}\}$. Hence there are exactly k many k-faces $G^\diamond$ with the required property.
iv) left as an exercise. □

3.23 Theorem. *Let $P \in \mathcal{P}^n$ be a simple n-polytope.*

i) $n\, f_0(P) = 2\, f_1(P)$.

ii) $\sum_{k=0}^n f_k(P) \le 2^n f_0(P)$.

iii) $f_0(P) \le 2 f_{\lceil n/2 \rceil}(P)$.

Here, for $\rho \in \mathbb{R}$ the number $\lceil \rho \rceil$ is the smallest integer greater or equal than ρ.

Proof. For i) we note that every edge contains exactly 2 vertices and every vertex is contained in exactly n edges.

By Theorem 3.22 i) every vertex is contained in exactly $\sum_{i=0}^n \binom{n}{k} = 2^n$ faces of P and each face has at least one vertex. This gives ii).

For iii) we assume that $P = \operatorname{conv}\{v_1, \ldots, v_m\}$ with vertices v_i and all vertices have different last coordinate. For a fixed vertex v with its n neighbors $v_1, \ldots, v_n$, say, let

$$L(v) = \{v_i i : \langle e_n, v_i \rangle < \langle e_n, v \rangle\} \text{ and } U(v) = \{v_i i : \langle e_n, v_i \rangle > \langle e_n, v \rangle\}.$$

Next we distinguish two cases depending on the cardinality of these sets.

a) $\#L(v) \ge \lceil n/2 \rceil$. On account of Theorem 3.22 iii) each $\lceil n/2 \rceil$ subset $S \subseteq L(v)$ determines an unique $\lceil n/2 \rceil$-face F of P containing the edges $\operatorname{conv}\{v, v_i\}$, $v_i \in S$. By Theorem 3.22 iv) F is simple and hence $\operatorname{conv}\{v, v_i\}$, $v_i \in S$, are the only edges of F containing v. Thus, on account of Remark 3.17, we obtain: there exists an unique $\lceil n/2 \rceil$ face F of P with

$$\langle e_n, v \rangle = \max_{x \in F} \langle e_n, x \rangle .$$

b) $\#U(v) \ge \lceil n/2 \rceil$. In the same way we conclude: there exists an unique $\lceil n/2 \rceil$ face F of P with

$$\langle e_n, v \rangle = \min_{x \in F} \langle e_n, x \rangle .$$

Hence for each vertex v there exists an $\lceil n/2 \rceil$ face F of P such that either v has the biggest or smallest last coordinate among all points of F. Since each $\lceil n/2 \rceil$ face contains a biggest as well as a smallest vertex (with respect to the last coordinate) we must have $2\, f_{\lceil n/2 \rceil}(P) \ge f_0(P)$. □

3.24 Corollary. *Let P be a simple n-polytope with m facets. Then*

$$f_0(P) \le 2\binom{m}{\lfloor n/2 \rfloor} \quad \text{and} \quad \sum_{k=0}^{n} f_k(P) \le 2^{n+1}\binom{m}{\lfloor n/2 \rfloor}.$$

Or equivalently: Let P be a simplicial n-polytope with m vertices. Then

$$f_{n-1}(P) \le 2\binom{m}{\lfloor n/2 \rfloor} \quad \text{and} \quad \sum_{k=0}^{n} f_k(P) \le 2^{n+1}\binom{m}{\lfloor n/2 \rfloor}.$$

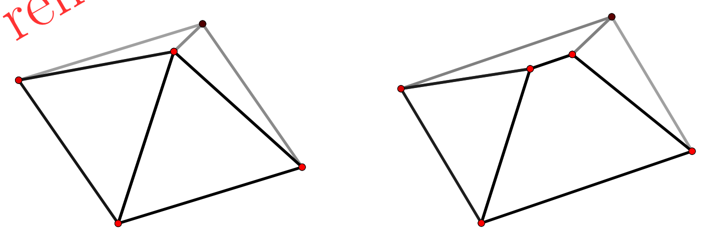
Here, for $\rho \in \mathbb{R}$ the number $\lfloor \rho \rfloor$ is the largest integer not greater than ρ.

Proof. First we note that the statement for simplical polytopes follows by "polarity". For any polytope with m facets we certainly have $f_k(P) \le \binom{m}{n-k}$, $k = 0, \dots, n-1$ (cf. Theorem 3.15 ii)), and, in particular, $f_{\lceil n/2 \rceil}(P) \le \binom{m}{\lfloor n/2 \rfloor}$. Hence with Theorem 3.23 iii) we obtain $f_0(P) \le 2\binom{m}{\lfloor n/2 \rfloor}$ which also gives the second inequlity by Theorem 3.23 ii) □

3.25 Lemma. *Let P be an n-polytope.*

i) *There exists a simple n-polytope Q with the same number of facets as P and $f_i(P) \le f_i(Q)$, $0 \le i \le n-2$.*

ii) *There exists a simplical n-polytope $Q^\star$ with the same number of vertices as P and $f_i(P) \le f_i(Q^\star)$, $1 \le i \le n-1$.*



Proof. By "polarity" is suffices to prove i), and w.l.o.g. let $P = \cap_{i=1}^{m} H^-(a_i, 1)$ be the n-Polytope with facets $P \cap H(a_i, 1)$, $i = 1, \dots, m$. The idea is to move a little bit all facets outwards, i.e., to add to the right hand sides some $\epsilon_i > 0$. For small enough values of ϵ_i the polytope $\cap_{i=1}^{m} H^-(a_i, 1+\epsilon_i)$ has still m- facets and is simple (because now the dependencies are broken by the perturbed right hand sides). It remains to check that the number of faces is not decreasing. To this end we consider for $\epsilon > 0$ the faces of the polytope

$$P(\epsilon) = \left(\bigcap_{i=2}^{m} H^-(a_i, 1)\right) \cap H(a_1, 1+\epsilon).$$

Let F be a k-face of P and let $I_F = \{i : F \subset H(a_i, 1)\}$ be the index set of all facets containing F, i.e., $F = P \cap_{i \in I_F} H(a_i, 1)$. In particular, $H(\sum_{i \in I_F} a_i, \#I_F)$

is a supporting hyperplane of F with respect to P. In the following we show

i) if $1 \notin I_F$ then $F(\epsilon) := P(\epsilon) \cap_{i \in I_F} H(a_i, 1)$ is a k-face of $P(\epsilon)$, and $I_{F(\epsilon)} = I_F$ for $\epsilon \geq 0$.

ii) if $1 \in I_F$ and a_1 is affinely dependent of $\{a_i : i \in I_F \setminus \{1\}\}$ then $F(\epsilon) := P(\epsilon) \cap_{i \in I_F \setminus \{1\}} H(a_i, 1)$ is a k-face of $P(\epsilon)$, and $I_{F(\epsilon)} = I_F \setminus \{1\}$ for $\epsilon > 0$.

iii) if $1 \in I_F$ and a_1 is affinely independent of $\{a_i : i \in I_F \setminus \{1\}\}$ then $F(\epsilon) := P(\epsilon) \cap H(a_1, 1+\epsilon) \cap_{i \in I_F \setminus \{1\}} H(a_i, 1)$ is a k-face of $P(\epsilon)$ and $I_{F(\epsilon)} = I_F$ for $0 \leq \epsilon < \epsilon_F$, *say*.

(3.25.1)

In the fist case i) we observe that $F(\epsilon)$ is certainly a face of $P(\epsilon)$ contained in the k-dimensional affine space $\cap_{i \in I_F} H(a_i, 1)$. Since $F \subseteq F(\epsilon)$ for any $\epsilon \geq 0$, we see that $F(\epsilon)$ is a k-face and $I_{F(\epsilon)} = I_F$.

For ii) we observe that in the case of affine dependency of a_1 from the remaining normal vectors $\{a_i : i \in I_F \setminus \{1\}\}$ we have $F = P \cap_{i \in I_F \setminus \{1\}} H(a_i, 1)$ and, in particular, $H(\sum_{i \in I_F \setminus \{1\}} a_i, \#I_F - 1)$ is a supporting hyperplane of F with respect to P. Hence we can argue as in case i) and observe that $F(\epsilon) \not\subset H(a_1, 1+\epsilon)$ for $\epsilon > 0$.

So we come to iii). Let $\overline{x} \in \operatorname{relint}(F)$ and let $\delta > 0$ such that $1 - \delta := \max\{\langle a_i, \overline{x}\rangle : i \notin I_F\}$. Next we observe that by the assumed affine independency there exists a $t \in \mathbb{R}^n$ such that $\langle a_i, t\rangle = 0$, $i \in I_F \setminus \{1\}$ and $\langle a_1, t\rangle = 1$. With

$$\epsilon_F := \min\{\delta / \langle a_i, t\rangle : \langle a_i, t\rangle > 0 \text{ and } i \notin I_F\}$$

we have for $0 \leq \epsilon < \epsilon_F$

$$\langle a_i, \epsilon\, t + \overline{x}\rangle \begin{cases} = 1+\epsilon, & \text{for } i = 1, \\ = 1, & \text{for } i \in I_F \setminus \{1\}, \\ < 1, & \text{if } i \notin I_F. \end{cases}$$

Hence for any $x \in F$ we can find a $\rho > 0$ such that $\epsilon\, t + \overline{x} + \rho(x - \overline{x}) \in F(\epsilon)$, which shows $\dim F(\epsilon) \geq \dim F$, but since the same normal vectors determine the affine hull of the faces $F(\epsilon)$ and F we also have $\dim F(\epsilon) \leq \dim F$. By construction we also have $I_{F(\epsilon)} = I_F$.

Since P has only finitely many faces we can find an $\overline{\epsilon_1}$ such that for all $0 < \epsilon_1 < \overline{\epsilon_1}$ and for any i-face F of P, $F(\epsilon_1)$ is an i-face of $P(\epsilon_1)$. By construction $F(\epsilon_1)$ is uniquely determined by F and so we conclude that for $0 < \epsilon_1 < \overline{\epsilon_1}$

$$f_i(P) \leq f_i(P(\epsilon_1)), \quad -1 \leq i \leq n.$$

Next we do the same with respect to second hyperplane $H(a_2, 1)$ of the polytope $P(\epsilon_1)$ and so on. In this way we can find positive functions $r_i(\epsilon_1)$, $0 < \epsilon_1 < \overline{\epsilon_1}$, such that for $0 < \epsilon_i < r_i(\epsilon_1)$, $i = 2, \ldots, m$, the polytope

$$\overline{P(\epsilon_1)} = \cap_{i=1}^m H(a_i, 1+\epsilon_i)$$

has at least as many i-faces as P, $i = -1, \ldots, n$. Obviously, for almost all possible values of $\epsilon_1, \ldots, \epsilon_m$ the polytope $\overline{P(\epsilon_1)}$ must be simple. $\square$

3.26 Corollary. *Let P be an n-polytope with m facets. Then*

$$f_0(P) \le 2\binom{m}{\lfloor n/2 \rfloor}.$$

Or equivalently: Let P be an n-polytope with m vertices. Then

$$f_{n-1}(P) \le 2\binom{m}{\lfloor n/2 \rfloor}.$$

3.27 Definition [Cyclic polytopes]. *The curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ given by $\gamma(t) = (t, t^2, t^3, \dots, t^n)^\intercal$ is called* moment curve. *The convex hull of m points on the moment curve is called a* cyclic polytope *with m vertices and is denoted by $C(n, m)$.*

3.28 Proposition. *Any $n+1$ points on the moment curve are affinely independent. In particlar, cyclic polytopes are simplicial polytopes.*

Proof. The vectors $\gamma(t_1), \dots, \gamma(t_{n+1})$, $t_i \ne t_j$, are affinely dependent if and only if the $n+1$ vectors $\binom{1}{\gamma(t_1)}, \dots, \binom{1}{\gamma(t_{n+1})} \in \mathbb{R}^{n+1}$ are linearly independent (cf. (0.4)). Hence their determinant, which is Vandermonds determinant $\prod_{i<j}(t_j - t_i)$, has to vanish, contradicting our assumption. □

3.29 Proposition [Gale's evenness condition]. *Let $t_i \in \mathbb{R}$, $1 \le i \le m$, $t_1 < t_2 < \cdots < t_m$, $\gamma(t_i) = (t_i, t_i^2, t_i^3, \dots, t_i^n)^\intercal$, $1 \le i \le m$, and let $S \subset \{1, \dots, m\}$ be a subset of cardinality n. $F_S = \operatorname{conv}\{\gamma(t_s) : s \in S\}$ is a facet of $C(n, m)$ if and only if $\#\{s \in S : i < s < j\}$ is even for all $i, j \in \{1, \dots, m\} \setminus S$.*

Proof. By Proposition 3.28 there exists an unique hyperplane $H(a, \alpha)$, $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, containing the n points $\gamma(t_s)$, $s \in S$. For $t \in \mathbb{R}$, let $f(t) = -\alpha + \sum_{k=1}^n a_k\, t^k$. Then $\langle a, v_{t_j} \rangle - \alpha = f(t_j)$, $f(t_s) = 0$ for $s \in S$, and F_S is a facet if and only if either $f(t_k) > 0$ for all $k \in \{1, \dots, m\} \setminus S$ or $f(t_k) < 0$ for all $k \in \{1, \dots, m\} \setminus S$. Since the polynomial f has n different roots, it alternates its sign at each of these roots and so the last condition is satisfied if and only if there is an even number of roots between any pair of points $t_i, t_j \in \{t_1, \dots, t_m\} \setminus \{t_s : s \in S\}$. □

3.30 Remark. *All points $\gamma(t_i)$ are vertices of $C(n, m)$ and the number of i-faces of $C(n, m)$ is independent of the choice of the m-points on the moment curve.*

3.31 Proposition. *The cyclic polytope $C(n, m)$ is $\lfloor n/2 \rfloor$-neighborly, i.e., the convex hull of any subset of the vertices of cardinality less than or equal $n/2$ is a face.*

Proof. Let $t_1 < \cdots < t_m$, $C(n, m) = \operatorname{conv}\{\gamma(t_1), \dots, \gamma(t_m)\}$, and let $S \subset \{1, \dots, m\}$ with $\#S \le \lfloor n/2 \rfloor$. We want to show that $F := \operatorname{conv}\{\gamma(t_j) : j \in S\}$ is a face of $C(n, m)$.

To this we set $g(t) = \prod_{j \in S}(t - t_j)^2$ which is a polynomial of degree at most n, and so we may write $g(t) = -\alpha + \sum_{i=1}^n a_i\, t^i$ for suitable numbers a_i, α. With

$a = (a_1, \dots, a_n)^\intercal$ we observe that $H(a, \alpha) \cap \{\gamma(t_1), \dots, \gamma(t_m)\} = \{\gamma(t_j) : j \in S\}$ and sicne $g(t)$ is a non-negative polynomial we also have $\gamma(t_i) \in H^-(a, \alpha)$, $1 \le i \le m$. Hence F is a face. □

3.32 Theorem* [McMullen's Upper Bound Theorem, 1971]. *Let P be an n-polytope with m vertices. Then*

$$f_i(P) \le f_i(C(n,m)) = \begin{cases} \sum_{j=0}^{(n-1)/2} \frac{i+2}{m-j} \binom{m-j}{j+1} \binom{j+1}{i+1-j}, & n \text{ odd}, \\ \sum_{j=1}^{n/2} \frac{m}{m-j} \binom{m-j}{j} \binom{j}{i+1-j}, & n \text{ even}. \end{cases}$$

In particular,

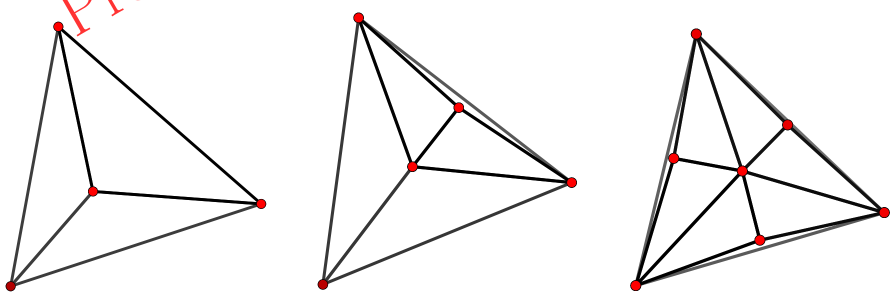
$$f_{n-1}(P) \le f_{n-1}(C(n,m)) = \begin{cases} 2\binom{m-\lfloor n/2 \rfloor -1}{\lfloor n/2 \rfloor}, & n \text{ odd}, \\ \binom{m-\lfloor n/2 \rfloor}{\lfloor n/2 \rfloor} + \binom{m-\lfloor n/2 \rfloor -1}{\lfloor n/2 \rfloor -1}, & n \text{ even}. \end{cases}$$

For fixed n the right hand sides are of order $m^{\lfloor n/2 \rfloor}$.

3.33 Theorem* [Barnette's Lower Bound Theorem, 1971]. *Let P be a simplicial n-polytope with m vertices. P has at least as many i-faces as the so called* stacked polytopes *$P(n,m)$ with m vertices for which*

$$f_i(P(n,m)) = \begin{cases} m\binom{n}{i} - i\binom{n+1}{i+1}, & 0 \le i \le n-2, \\ n+1+(m-(n+1))(n-1), & i = n-1. \end{cases}$$

$P(n, n+1)$ is an n-simplex, and for $m \ge n+2$ an m-vertex stacked n-polytope $P(n,m)$ is the convex hull of an $(m-1)$-vertex stacked polytope with an additional point that is beyond exactly one facet.



3.34 Theorem [Dehn-Sommerville equations, 1905, 1927]. *Let P be a simple n-polytope. Then*

$$f_i(P) = \sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} f_j(P), \quad i = 0, \dots, n,$$

Or equivalently: Let P be a simplicial n-polytope. Then

$$f_{i-1}(P) = \sum_{j=i}^{n} (-1)^{n-j} \binom{j}{i} f_{j-1}(P), \quad i = 0, \dots, n.$$

Proof. Let $F \subseteq P$ be an i-face. By the Euler-Poincaré formula (3.18.1) we have $\sum_{j=-1}^{i}(-1)^j f_j(F) = 0$. Hence summing over all i-faces we may write

$$0 = \sum_{F\, i-\text{face}} \sum_{j=-1}^{i} (-1)^j f_j(F) = \sum_{j=-1}^{i} (-1)^j \sum_{F\, i-\text{face}} f_j(F).$$

For $j = -1$ we just count the number of i-faces and so

$$f_i(P) = \sum_{j=0}^{i} (-1)^j \sum_{F\, i-\text{face}} f_j(F).$$

For a given j the interior sum $\sum_{F\, i-\text{face}} f_j(F)$ is the number of all pairs of faces G, F with F is an i-face and G is a j-face of F. Since P is simple, for a fixed G the number of i- faces F of P containing G is $\binom{n-j}{n-i}$ (cf. Theorem 3.22 iv)). Hence the inner sum is equal to $\binom{n-j}{n-i} f_j(P)$. □

3.35 Definition [h-**vector**]**.** *Let $P \in \mathcal{C}^n$ be a simple n-polytope. The vector $h(P) = (h_0(P), \ldots, h_n(P))$ with entries*

$$h_i(P) = \sum_{j=0}^{n} (-1)^{i-j} \binom{j}{i} f_j(P)$$

is called h-vector *of P.*

3.36 Remark. *In terms of the h-vector, the Dehn-Sommerville equations become $h_i(P) = h_{n-i}(P)$, $i = 0, \ldots, n$.*

3.37 Theorem* [**McMullen's** g-**Theorem**]**.** *McMullen's g-theorem gives a complete characterization of the f-vectors of simple (or simplicial) polytopes in terms of its g-vector which is given by $g_i(P) = h_i(P) - h_{i-1}(P)$, $i = 1, \ldots, \lfloor n/2 \rfloor$.*

3.38 Remark. *For any n-polytope $P \in \mathcal{P}^n$ we have $n f_0(P) \leq 2 f_1(P)$ with equality iff P simple and $n\, f_{n-1}(P) \leq 2\, f_{n-2}(P)$ with equality iff P simplicial.*

3.39 Theorem [**Steinitz, 1906**]**.** *A non-negative integral vector (f_0, f_1, f_2) is the f-vector of a 3-polytope if and only if* i) $f_0 - f_1 + f_2 = 2$, ii) $3\, f_0 \leq 2\, f_1$, *and* iii) $3\, f_2 \leq 2 f_1$.

Proof. Equation i) is the Euler-Poincaré formula (3.18.1) for polytopes in $\mathbb{R}^3$, ii) describes the trivial fact that every vertex is contained in at least 3 edges and each edge has exactly two vertices, and iii) is just the polar version of ii).

For the sufficiency part we have to construct a polytope with f_0 vertices and f_2 facets, where the non-negative integers f_0, f_2 satisfy

$$2\, f_0 - f_2 - 4,\;\; 2 f_2 - f_0 - 4 \geq 0.$$

These inequalities are obtained by using i) in ii) and iii). Since their difference is divisible by 3 they have the same reminder r, say, on division by 3. Thus there exist $c, s \in \mathbb{N}$ such that $2\,f_0 - f_2 - 4 = 3\,c + r$ and $2\,f_2 - f_0 - 4 = 3\,s + r$. Hence we have

$$f_0 = (4+r) + 2\,c + s \quad \text{and} \quad f_2 = (4+r) + c + 2\,s. \tag{3.39.1}$$

Let P_0 be a pyramid whose basis is a $r+3$-gon. Then $f_0(P_0) = f_2(P_0) = r+4$, and each facet containing the top vertex is a triangle, and each vertex of the basis is contained in exactly three edges, which we call a simplex vertex. Now we apply two operations to P_0, namely cutting of simple vertices and stacking over triangular faces.

If we cut off a simple vertex, then we obtain a polytope P_1, say, with $f_0(P_1) = f_0(P_0) + 2$ and $f_2(P_1) = f_2(P_0) + 1$. Moreover, the three new vertices of P_1 form a triangular face and all of them are simple. Hence, if we continue this process of cutting of simple vertices (in a proper way) we obtain after c steps a polytope P_c with $f_0(P_c) = (4+r) + 2\,c$ and $f_2(P_c) = (4+r) + c$.

On the other hand, if we replace a triangular face of P_c by a suitable simplex then we obtain a polytope P_{c+1} with $f_0(P_{c+1}) = f_0(P_c) + 1$ and $f_2(P_{c+1}) = f_2(P_0)+2$. Repeating this stacking process of triangular faces s-times, we finally arrive at a polytope P_{c+s} with $f_0(P_{c+s})$ and $f_2(P_{c+s})$ as desired in (3.39.1). $\square$

3.40 Theorem [Figiel, Lindenstrauss, Milman, 1977]. *Let $P \in \mathcal{P}^n$ be a 0-symmetric n-polytope, i.e., $P = -P$. Then*

$$\ln(f_0(P))\ln(f_{n-1}(P)) \geq \frac{1}{16}n.$$

Proof. For the proof we need two facts from convexity. First, there exits an $A \in \mathrm{GL}(n,\mathbb{R})$ such that $B_n \subset A\,P \subset \sqrt{n}\,B_n$. Hence, in our combinatorial setting we may assume that

$$B_n \subset P \subset \sqrt{n}\,B_n. \tag{3.40.1}$$

The second fact concerns *spherical caps* $C(v,\epsilon)$, which for $v \in S^{n-1}$ and $\epsilon \in [0,1]$ are defined by $C(v,\epsilon) = \{w \in S^{n-1} : \langle v, w\rangle \geq \epsilon\}$. If $\mu_{n-1}(\cdot)$ denotes the Haar probability measure on S^{n-1} then $\mu_{n-1}(C(v,\epsilon)) \in [0,1]$ measures how much of S^{n-1} is covered by $C(v,\epsilon)$. Here we need

$$\mu_{n-1}(C(v,\epsilon)) \leq \mathrm{e}^{-n\,\frac{\epsilon^2}{2}}. \tag{3.40.2}$$

For the proof we set $m_1 = f_0(P)$ and $m_2 = f_{n-1}(P)$, and we also need a $\mathcal{V}$- and an $\mathcal{H}$-representation of P

$$P = \mathrm{conv}\,\{v_i : 1 \leq i \leq m_1\} = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \leq 1,\ 1 \leq i \leq m_2\}.$$

Moreover, let $\epsilon_i = 2\,\sqrt{\ln(m_i)}/\sqrt{n}$, $i = 1, 2$.

Let $V_{\epsilon_1} = \cup_{i=1}^{m_1} C(v_i/\,|v_i|\,,\epsilon_1)$ and $F_{\epsilon_2} = \cup_{i=1}^{m_2} C(a_i/\,|a_i|\,,\epsilon_2)$. Then by (3.40.2) and the choice of ϵ_i we have $\mu_{n-1}\left(V_{\epsilon_1}\right), \mu_{n-1}\left(F_{\epsilon_2}\right) \leq 1/4$, e.g.,

$$\mu_{n-1}\left(V_{\epsilon_1}\right) \leq \sum_{i=1}^{m_1} \mu_{n-1}\left(C(v_i/\,|v_i|\,,\epsilon_1)\right) \leq m_1\,\mathrm{e}^{-n\,\frac{\epsilon_1^2}{2}} = \frac{1}{m_1} \leq \frac{1}{4}.$$

Thus we can find a $c \in S^{n-1} \setminus \{V_{\epsilon_1} \cup F_{\epsilon_2}\}$. Since $c \in S^{n-1} \setminus V_{\epsilon_1}$ we find together with (3.40.1)

$$\max_{x \in P} \langle c, x\rangle = \max_{1\leq i\leq m_1} \langle c, v_i\rangle \leq \sqrt{n} \max_{1\leq i\leq m_1} \langle c, v_i/\,|v_i|\rangle < \sqrt{n}\,\epsilon_1 = 2\sqrt{\ln(m_1)}. \tag{3.40.3}$$

In order to get a lower bound on $\max_{x\in P}\langle c, x\rangle$ we observe that $B_n \subset P$ implies $|a_i| \leq 1$, and for $c \in S^{n-1} \setminus F_{\epsilon_2}$ we obtain for $1 \leq i \leq m_2$

$$\langle c, a_i\rangle \leq \langle c, a_i/\,|a_i|\rangle \leq \epsilon_2.$$

So $(1/\epsilon_2)\, c \in P$ which gives

$$\max_{x\in P}\langle c, x\rangle \geq \frac{1}{\epsilon_2} = \frac{\sqrt{n}}{2\sqrt{\ln(m_2)}}.$$

Together with (3.40.3) the assertion is proved. □

3.41 Conjecture [Kalai, 1989]. *Let $P \in \mathcal{P}^n$ be a 0-symmetric n-polytope. Then*

$$\sum_{i=0}^{n} f_i(P) \geq 3^n.$$

Here we have equality, for instance, for the cube C_n and its polar, the cross-poyltope $C_n^\star$, or, more generally, for the class of Hanner-polytopes. Recently, the conjecture has been verified for all $n \leq 4$ (see `http://front.math.ucdavis.edu/0708.3661`).

3.42 Definition [Graph, combinatorial diameter]. *Let $P \subset \mathbb{R}^n$ be a polyhedron.*

i) *The* graph *(1-skeleton) $G(P)$ of P consists of the vertices and edges of P.*

ii) *The distances $\delta_P(v, w)$ between two vertices $v, w \in P$ (or in $G(P)$) is the minimum length of an "edge" path connecting v and w in $G(P)$.*

iii) *$\delta(P) = \max\{\delta_P(v, w) : v, w \in \operatorname{vert} P\}$ is called the* (combinatorial) diameter *of P.*

3.43 Theorem [Balinski, 1961]. *Let $P \in \mathcal{P}^n$ be an n-polytope. The graph $G(P)$ of P is n-connected, i.e., the graph is still connected if $n-1$ vertices and their incident edges are removed.*

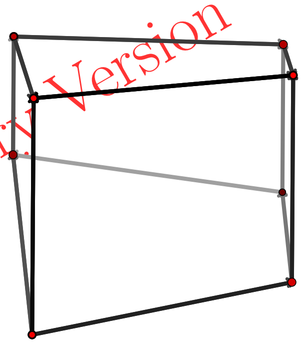
Proof. Let V be the set of vertices of P, and let $S \subset V$ with $|S| = n-1$. We have to show that for any $v_1, v_2 \in V \setminus S$ there exists a path along edges of P connecting v_1, v_2 and only using vertices of $V \setminus S$. Let $s = \frac{1}{n-1}\sum_{v\in S} v$. We distinguis two cases.
i) There exists a face F of P with $0 \leq \dim F \leq n-1$ containg s in its relative interior. Let $H(a, \alpha)$ be a supporting hyperplane of F with respect to P and let $\beta = \min\{\langle a, x\rangle : x \in P\} < \alpha$. In particular, we have $S \subset H(a, \alpha)$, and

$G = P \cap H(a, \beta)$ is a face of P. On account of Remark 3.17 any vertex $v \in V \setminus G$ has a neighbor w with $\langle a, w\rangle < \langle a, v\rangle$. Hence $w \in V \setminus S$, and we conclude that any vertex $v \in V \setminus S$ can be connected to a vertex $z \in G$ via a path only using vertices in $V \setminus S$. By induction G is connected and we are done.
ii) s in the interior of P. Then we may assume that $H = \{x \in \mathbb{R}^n : \langle a, x\rangle = 0\}$ is a hyperplane containing S and another vertex $v_0 \in V \setminus S$. Let $\beta = \min\{\langle a, x\rangle : x \in P\}$, $\alpha = \max\{\langle a, x\rangle : x \in P\}$ and we consider now the faces $F = P \cap H(a, \alpha)$ and $G = P \cap H(a, \beta)$. First we observe that by the same reasoning as before any vertex $v \in V$ with $\langle a, v\rangle \geq 0$ ($\langle a, v\rangle \leq 0$) can be connected to a vertex of F (G) using only vertices of $V \setminus S$. Since v_0 can be connected to both F and G in this way, and since F and G are connected by induction, for two vertices of $V \setminus S$ there exists an "edge path" only using vertices of $V \setminus S$. $\square$

3.44 Definition. *For integers n, m let*

$$\Delta(n, m) = \max\{\delta(P) : P \subset \mathbb{R}^n \text{ polyhedron}, \dim P = n \text{ and } f_{n-1}(P) = m\}.$$

3.45 Example. $\delta(T_n) = 1 = (n + 1) - n$, $\delta(C_n) = n \leq 2\,n - n$ *and* $\delta(C_n^\star) = 2 \leq 2^n - n$.



3.46 Conjecture [Hirsch, 1957]. $\Delta(n, m) \leq m - n$.

3.47 Remark. *It is known that*

i) *the conjecture is true if $n \leq 3$ or $m \leq n+5$, but, in general, for unbounded polyhedra the conjecture is false; for $m \geq 2\,n$ it is $\Delta(n, m) \geq m-n+\lfloor n/4\rfloor$. (Klee&Walkup, 1961/1965),*

ii) $\Delta(n, m) \leq m\,2^{n-3}$, *(Barnette, 1969; Larman, 1970),*

iii) *Disproof of the Hirsch conjecture for polytopes by Francisco Santos, 2010, see* `http://front.math.ucdavis.edu/1006.2814`

3.48 Theorem [Kalai, 1992; Kalai&Kleitman, 1992].

$$\Delta(n, m) \leq m^{\log n + 2}$$

Proof. First we will establish the recurrence

$$\Delta(n,m) \le \Delta(n-1,m-1) + 2\Delta(n,\lfloor m/2\rfloor) + 2 \tag{3.48.1}$$

Let P be an n-dimensional polytope with m facets. For an "edge" path ω of the polytope let $F(\omega)$ be the set of facets of P which are incident with one of the vertices of P, i.e., all facets which are visited on the path ω. The length of a path ω is denoted $|\omega|$.

For a vertex w of P let

$$k_w = \max\left\{p : \#\left(\bigcup_{\omega \text{ path starting in } w,\, |\omega|\le p} \mathrm{F}(\omega)\right) \le \lfloor m/2\rfloor\right\}.$$

Now let v, u be two vertices of P attaining the diameter of P. By the definition of k_v (and k_u) we have $\#(\cup_{\omega \text{ starting in v},|\omega|=k_v+1}\mathrm{F}(\omega)) > m/2$ and hence there exists a facet F of P which can be reached from v by a path of length at most k_v+1 and from u by a path of length at most k_u+1. Thus we conclude

$$\delta(P) \le \Delta(n-1,m-1) + k_v + k_u + 2,$$

and it remains to show $k_v, k_u \le \Delta(n,\lfloor m/2\rfloor)$.

To this end let Q given by all facet defining inequalities corresponding to facets of $\cup_{\omega \text{ starting in v},|\omega|\le k_v}\mathrm{F}(\omega)$. Then we have $P \subset Q$ and Q has at most $q \le \lfloor m/2\rfloor$ facets. Let w be a vertex of P with $\delta(v,w) = k_v$. Then we have $v, w \in Q$ and let ω_Q be a shortest "edge" path in Q joining v, w. Next we claim that

$$|\omega_Q| = k_v. \tag{3.48.2}$$

By definition we have $|\omega_Q| \le k_v$ and so suppose that $|\omega_Q| < k_v$. Then ω_Q uses an edge which is not an edge of P. Let e_Q be the first such edge on ω_Q. Then this edge must be intersected by one of the facet defining hyperplanes of P which are not in $\cup_{\omega \text{ starting in v},|\omega|\le k_v}\mathrm{F}(\omega)$. Hence this facet can be reached by a path in P of length $\le k_v$ which contradicts the choice of k_v.

This shows (3.48.2) and so $k_v = |\omega_Q| \le \Delta(n,q) \le \Delta(n,\lfloor m/2\rfloor)$.

Finally, in order to get the desired bound $\Delta(n,m) \le m^{\log n+2}$ from the recursion (3.48.1) we refer to ... □

3.49 Definition [**0/1-polytope**]. *Let* $[0,1]^n$ *be the* n*-dimensional unit cube with vertices* $\{0,1\}^n = \{(x_1,\ldots,x_n)^\intercal : x_i \in \{0,1\}\}$. $P \in \mathcal{P}^n$ *is called* 0/1-polytope *if* $\operatorname{vert} P \subset \{0,1\}^n$.

3.50 Lemma. *Let* $P \in \mathcal{P}^n$ *be a* 0/1*-polytope and let* $\dim P \le n-1$. *Then there exists a* 0/1*-polytope* $\widetilde{P} \in \mathcal{P}^{n-1}$ affinely isomorphic *to* P, *i.e., there exists a bijective map between* P *and* $\widetilde{P}$.

Proof. Let $P \subset H(a,\alpha)$, and we may assume that $a_n \ne 0$. Let $\Pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection forgetting the last coordinate. It suffices to show that $\Pi|_{H(a,\alpha)}$ is injective. Let $x, y \in H(a,\alpha)$ with $\Pi(x) = \Pi(y)$. Thus $x_i = y_i$, $1 \le i \le n-1$, and so $0 = \langle a, x\rangle - \langle a, y\rangle = a_n\, x_n - a_n\, y_n$. Hence we conclude $x = y$. □

3.51 Theorem [Naddef, 1989].

i) *Let P be a 0/1-polytope. Then $\delta(P) \leq \dim P$.*

ii) *Let $P \in \mathcal{P}^n$ be an n-dimensional 0/1-polytope with m facets. Then $\delta(P) \leq m - n$.*

Proof.
i) Let $P \in \mathcal{P}^n$ be a 0/1-polytope. We prove the statemnet via induction on n. Since $n = 1$ is trivial we assume $n > 1$. Let $v, w \in P$ be two vertices of P. If there exists a facet F of $[0,1]^n$ containg both of them, then induction gives

$$\delta_P(v, w) \leq \delta(P \cap F) \leq \dim(P \cap F) \leq \dim P.$$

Here, of course, we have identified F with $[0,1]^{n-1}$. If they are not contained in a common facet F of $[0,1]^n$, they must differ in all coordinates. But then we can find a neighbor $\overline{v}$, say, of v which is contained in a common facet with w. Hence by the same reasoning as above we obtain

$$\delta_P(v, w) \leq 1 + \delta_P(\overline{v}, w) \leq 1 + \delta(P \cap F) \leq 1 + \dim(P \cap F) \leq \dim P,$$

where we just have to observe that $\dim(P \cap F) \leq \dim P - 1$ since $v \notin P \cap F$.
ii) Again we use induction with respect to n. If $n = 1$ there is nothing to prove and so let $n > 1$. The case $m \geq 2n$ is covered by statement i). On the other hand, if $m < 2n$ then any two vertices of P are conatined in a common facet F of P. Since F has at most $m - 1$ facets we obtain by Lemma 3.50 and indcution

$$\delta(P) \leq \delta(F) \leq (m-1) - (n-1) = m - n.$$

□

3.52 Remark.

i) *$f_{n-1}(P) \leq 2\,n!$ for a 0/1-polytope $P \in \mathcal{P}^n$.*

ii) *There exist 0/1-polytopes $P \in \mathcal{P}^n$ with*

$$f_{n-1}(P) \geq \left(\frac{c\,n}{\log^2 n}\right)^{\frac{n}{2}},$$

where c is a universal constant (Gatzouras, Giannopoulos, Markoulakis, 2004).

4 A bit of Gale diagrams and triangulations

4.1 Definition. *Let $a_1, \dots, a_m \in \mathbb{R}^n$, $m \geq n+1$, such that their affine hull spans $\mathbb{R}^n$, and let $\overline{a}_i = (a_i, 1)^\intercal \in \mathbb{R}^{n+1}$, $1 \leq i \leq m$, and let $\overline{A} = (\overline{a}_1, \dots, \overline{a}_m) \in \mathbb{R}^{(n+1)\times m}$. Let $\overline{B} = (\overline{b}_1, \dots, \overline{b}_{m-(n+1)}) \in \mathbb{R}^{m\times(m-(n+1))}$ be a basis of the subspace $\{x \in \mathbb{R}^m : \overline{A}\, x = 0\}$, i.e., of the kernel of $\overline{A}$, and let $\overline{B}^\intercal = (b_1, \dots, b_m) \in \mathbb{R}^{(m-(n+1))\times m}$.*

The vectors $b_1, \dots, b_m \in \mathbb{R}^{m-(n+1)}$ are called the Gale transform *of the points $a_1, \dots, a_m$.*

4.2 Remark.

i) *The Gale transform is unique up to linear isomorphisms of $\mathbb{R}^{(m-(n+1))}$.*

ii) *For the vectors of a Gale transform always hold $\sum_{i=1}^m b_i = 0$.*

4.3 Proposition. *Let $P = \operatorname{conv}\{v_1, \dots, v_m\}$ with vertices v_i, $1 \leq i \leq m$, and for $J \subseteq [m] = \{1, \dots, m\}$ let $V_J = \{v_j : j \in J\}$. Then $\operatorname{conv} V_J$ is a face of P if and only if $\operatorname{conv} V_{[m]\setminus J} \cap \operatorname{aff} V_J = \emptyset$.*

For $\#J = 1$ the statement is also true without the assumption that v_i are vertices.

Proof. In the case $\#J = 1$ we may just apply Lemma 3.11. So let us assume that all v_i are vertices, and let $\operatorname{conv} V_J$ be a face of P with supporting hyperplane $H(a, \alpha)$. Then we have $H(a, \alpha) \cap P = \operatorname{conv} V_J$. So $v_i \in H(a, \alpha)$ if and only if $v_i \in \operatorname{conv} V_J$ which is equivalent to $i \in J$ since all v_i are vertices of P. In other words, $\operatorname{conv} V_{[m]\setminus J} \cap \operatorname{aff} V_J \subseteq \operatorname{conv} V_{[m]\setminus J} \cap H(a, \alpha) = \emptyset$.

For the reverse direction, let L be the orthogonal complement of $\operatorname{aff} V_J$ and for $x \in \mathbb{R}^n$ let $x|L$ be the orthogonal projection of x onto L. In view of our assumption, we have $\operatorname{conv}(V_{[m]\setminus J}|L) \cap (\operatorname{aff} V_J)|L = \emptyset$. Observe, that $(\operatorname{aff} V_J)|L$ is just a point and by Lemma 3.11 it is a vertex of the set $\operatorname{conv}(V_{[m]\setminus J}|L \cup (\operatorname{aff} V_J)|L)$. Hence there exists a supporting plane $H(a, \alpha)$ in L at the vertex $(\operatorname{aff} V_J)|L)$ and so the hyperplane $\tilde{H} = H(a, \alpha) + L$ supports P in $P \cap \tilde{H} = \operatorname{conv}(\tilde{H} \cap \{v_{!}, \dots, v_m\}) = \operatorname{conv} V_J$. □

4.4 Theorem. *Let $P = \operatorname{conv}\{v_1, \dots, v_m\}$, $\dim P = n$, with vertices v_i, $1 \leq i \leq m$, and for $J \subseteq [m] = \{1, \dots, m\}$ let $V_J = \{v_j : j \in J\}$. Let $\{b_1, \dots, b_m\}$ be the Gale transform of $\{v_1, \dots, v_m\}$. Then $\operatorname{conv} V_J$ is a face of P if and only if $J = [m]$ or $0 \in \operatorname{relint}(\operatorname{conv}\{b_k : k \notin J\})$.*

For $\#J = 1$ the statement is also true without the assumption that v_i are vertices.

Proof. In the case $\#J = 1$ the statement follows from Lemma 3.11, so let us assume that all v_i are vertices. If $\operatorname{conv} V_J$ is not a face, we get from Proposition 4.3 that there exists a $z \in \operatorname{aff} V_J \cap \operatorname{conv} V_{[m]\setminus J}$. Hence we may write

$$z = \sum_{j\in J} \mu_j\, v_j = \sum_{i\in[m]\setminus J} \lambda_i\, v_i \text{ with } \sum_{j\in J} \mu_j = \sum_{i\in[m]\setminus J} \lambda_i = 1, \quad \lambda_i \geq 0,$$

which is equivalent to the existence of scalars ρ_i, $i \in [m]$, with the property

$$\text{i)} \sum_{i\in[m]} \rho_i\, v_i = 0, \quad \text{ii)} \sum_{i\in[m]} \rho_i = 0, \quad \text{iii)} \sum_{i\in[m]\setminus J} \rho_i = 1, \quad \text{iv)}\ \rho_i \geq 0,\ i \in [m]\setminus J.$$

i) and ii) are equivalent to the property that

$$\begin{pmatrix} v_1 & \dots & v_m \\ 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_m \end{pmatrix} = 0.$$

According to the Gale transform this is equivalent to the statement, that there exists a $t \in \mathbb{R}^{m-(n+1)}$ with $\rho_i = \langle b_i, t\rangle$, $1 \leq i \leq m$. In view of iii) and iv) we further know that $\langle b_i, t\rangle \geq 0$, $i \in [m] \setminus J$, and for at least one index it is strictly positive. Hence, $0 \notin \operatorname{relint}(\operatorname{conv}\{b_k : k \notin J\})$.

On the other hand, if $0 \notin \operatorname{relint}(\operatorname{conv}\{b_k : k \notin J\})$ then there exists a separating hyperplane of 0 and $\operatorname{conv}\{b_k : k \notin J\}$ which does not contain all points b_k, $k \notin J$. Hence there exists a $t \in \mathbb{R}^{m-(n+1)}$ with $\langle b_i, t\rangle \geq 0$, $i \in [m] \setminus J$, and for at least one index it is strictly positive. Setting $\rho_i = \langle b_i, t\rangle$, $1 \leq i \leq m$, we may assume that iii) and iv) holds. In view of Remark 4.2 ii) we also know that ii) is true, and b the definition of the Gale transform i) holds as well. Hence, as pointed out before, this is equivalent to $\operatorname{aff} V_J \cap \operatorname{conv} V_{[m]\setminus J} \neq \emptyset$. □

4.5 Corollary. *$\{b_1, \dots, b_m\} \in \mathbb{R}^{m-(n+1)}$ is the Gale transform of the vertex set $\{v_1, \dots, v_m\} \in \mathbb{R}^n$ of a polytope if and only if for each hyperplane $H(a, 0) \subset \mathbb{R}^{m-(n+1)}$ the halfspaces $H^+(a, 0)$ and $H^-(a, 0)$ contain at least two points of $\{b_1, \dots, b_m\}$ in its interior.*

Proof. Let $B = \{b_1, \dots, b_m\}$. First, we note that on account of Remark 4.2 ii) we know $0 \in \operatorname{int}\operatorname{conv} B$ and hence for each hyperplane $H(a, 0) \subset \mathbb{R}^{m-(n+1)}$ we have $\#(\operatorname{int} H^-(a, 0) \cap B), \#(\operatorname{int} H^-(a, 0) \cap B) \geq 1$. Moreover, in view of Theorem 4.4 we also know that v_i vertex if and only if $0 \in \operatorname{relint}(\operatorname{conv}\{b_k : k \neq i\}) = \operatorname{int}(\operatorname{conv}\{b_k : k \neq i\})$.

Hence, if $\#(\operatorname{int} H^-(a, 0) \cap B) = 1$, say, and if $b_i = \operatorname{int} H^-(a, 0) \cap B$ then v_i is a not vertex. Conversely, if $0 \notin \operatorname{int}(\operatorname{conv}\{b_k : k \neq i\})$ then there exists a separating hyperplane of 0 and $\operatorname{conv}\{b_k : k \neq i\})$ and so a vector $a \in \mathbb{R}^{m-(n+1)}$ with $\langle a, b_k\rangle > 0$, $k \neq i$, violating the assumption $\#(\operatorname{int} H^-(a, 0) \cap B) \geq 2$. □

4.6 Definition [Face lattice]. *For a polytope $P \in \mathcal{P}^n$ let $\mathcal{F}(P)$ be the set of all its faces. Together with the inclusion relation "$\subseteq$" on $\mathcal{F}(P)$ the faces form a* partially ordered set (poset) *denoted by $(\mathcal{F}(P), \subseteq)$ and which is called the* face lattice *of P.*

4.7 Definition [Combinatorially isomorphic]. *Two polytopes $P, Q \in \mathcal{P}^n$ are called* combinatorially isomorphic *or* combinatorially equivalent *if the face lattices $(\mathcal{F}(P), \subseteq)$ and $(\mathcal{F}(Q), \subseteq)$ are isomorphic, i.e., there exists an inclusion preserving bijection between the faces of P and Q.*

4.8 Theorem. *There are precisely $\lfloor \frac{1}{4} n^2 \rfloor$ combinatorial different types of n-polytopes with $n+2$ vertices.*

Proof. For such an n-polytope $P = \operatorname{conv}\{v_1, \ldots, v_{n+2}\}$ the Gale transform $\{b_1, \ldots, b_{n+2}\}$ is one-dimensional. Since $0 \in \operatorname{int}\operatorname{conv}\{b_1, \ldots, b_{n+2}\}$ we can classify the Gale transform by the number of points l, e, g which are < 0, $= 0$ and > 0, respectively. Then we have $l + e + g = n + 2$ and according to Corollary 4.5 we have $l, g \geq 2$ and by symmetry we may assume $l \leq g$. So for a fixed $e \in \{0, \ldots, n-2\}$ the parameter l is bounded between 2 and $\lfloor (n+2-e)/2 \rfloor$, and so there are $\lfloor (n-e)/2 \rfloor$ possibilities. Altogether we have found

$$\sum_{e=0}^{n-2} \left\lfloor \frac{n-e}{2} \right\rfloor = \left\lfloor \frac{1}{4} n^2 \right\rfloor$$

different possibilities of Gale transforms which correspond by Corollary 4.5 to all n polytopes with $(n+2)$ vertices. It remains to show that these polytopes are combinatorially inequivalent. To this end let (l_i, e_i, g_i) be two different sequences with polytopes P_i, $i = 1, 2$, and assume that the polytopes are combinatorially equivalent. Let ϕ be an isomorphism between the face lattices of P_i which induces an isomorphism on the points of the Gale transforms. First we observe that all points of the groups "e_i" have to be mapped onto each other, since removing all other points yields a face of the polytope in view of Theorem 4.4. So we may assume $e_1 = e_2$ and suppose $1 < l_1 < l_2 \leq g_2 < g_1$. The image of the points of the first group "l_1" cannot consists of all points of l_2 or of all points of g_2. Hence removing the image of the points yields a face of P_2 but not of P_1 according to Theorem 4.4. Hence we get a contradiction. □

4.9 Definition [Point configuration]. *A point configuration in $\mathbb{R}^n$ is a finite set of (perhaps repeated) points with (non-repeated) labels. It is identified with its matrix $A = (a_1, \ldots, a_m) \in \mathbb{R}^{n \times m}$.*

4.10 Definition [Triangulation]. *A* triangulation *of a point configuration $A = (a_1, \ldots, a_m) \subset \mathbb{R}^{n \times m}$ is a collection $\mathcal{T}$ of simplices, with vertices in A, that satisfies the following properties:*

i) *All faces of simplices of $\mathcal{T}$ are in $\mathcal{T}$.*

ii) *The intersection of any two simplices of $\mathcal{T}$ is a face (possible empty) of both.*

iii) *The union of all simplices of $\mathcal{T}$ equals* conv *A.*

4.11 Remark.

i) *All vertices of* conv *A are in $\mathcal{T}$.*

ii) *Let F be a face of* conv *A. Then $\mathcal{T}_F := \{\sigma \in \mathcal{T} : \sigma \subset F\}$ is a triangulation of $A \cap F$.*

iii) *The frist two properties of Definition 4.10 are the definition of a (geometric)* simplicial complex.

4.12 Definition [Height function and lower convex hull]. *Let* $A = (a_1, \dots, a_m) \in \mathbb{R}^{n \times m}$ *be a point configuration. Let* $\eta : A \to \mathbb{R}_{\geq 0}$ *be a function and let*

$$A_\eta = \begin{pmatrix} a_1 & a_2 & \dots & a_m \\ \eta(a_1) & \eta(a_2) & \dots & \eta(a_m) \end{pmatrix} \in \mathbb{R}^{(n+1) \times m}$$

be the lifted point configuration *(also called a lifting). Then* η *is called a height function . The union of the faces of* conv A_η *which are visible form below are called the* lower convex hull *of* conv A_η. *More precisely,* $y = (y_1, \dots, y_n, y_{n+1})^\intercal \in \operatorname{conv}(A_\eta)$ *belongs to the lower convex hull if the segment* $\operatorname{conv}\{y, (y_1, \dots, y_n, 0)^\intercal\}$ *intersects* conv A_η *only in* y.

4.13 Definition [Regular triangulation]. *A triangulation of a point configuration* $A \in \mathbb{R}^{n \times m}$ *is called a* regular triangulation *if it can be obtained by projecting the lower convex hull of a lifting of* A.

4.14 Theorem. *Every point configuration* $A \in \mathbb{R}^{n \times m}$ *has regular triangulations.*

Proof. It suffices to find a height function $\eta : A \to \mathbb{R}_{\geq 0}$, or more precisely, non-negative numbers $\eta_1, \dots, \eta_m$ such that for any choice of $n+1$ affinely independent points $a_{i_1}, \dots, a_{i_{n+1}} \in A$ the unique hyperplane containing the lifted points $(a_{i_j}, \eta_{i_j})^\intercal$, $1 \leq j \leq n+1$, contains no other lifted points. Then all faces of the lifted configuration are simplices and the projected faces of the lower convex hull gives the desired regular triangulation. Now for a given set $a_{i_1}, \dots, a_{i_{n+1}} \in A$ this is equivalent to say that

$$\det \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ a_{i_1} & a_{i_2} & \dots & a_{i_{n+1}} & a_k \\ \eta_{i_1} & \eta_{i_2} & \dots & \eta_{i_{n+1}} & \eta_k \end{pmatrix} \tag{4.14.1}$$

is non zero for all $k \in [m] \setminus \{i_1, \dots, i_{n+1}\}$. Evaluating this determinant with respect to the last row shows that the determinant is zero if and only if $(\eta_{i_1}, \eta_{i_2}, \dots, \eta_{i_{n+1}}, \eta_k)^\intercal$ satisfies a non-trivial linear equation. Hence except points lying in a hyperplane of the form $\langle w, (\eta_1, \dots, \eta_m)^\intercal \rangle = 0$ for a certain $w \in \mathbb{R}^m$ yielding a non-zero determinant in (4.14.1).

Hence for almost any choice of $(\eta_1, \dots, \eta_m)^\intercal \in \mathbb{R}^m_{\geq 0}$ all the determinants of type (4.14.1) for any choice of affinely independent points $a_{i_1}, \dots, a_{i_{n+1}} \in A$ and $a_k \in A \setminus \{a_{i_1}, \dots, a_{i_{n+1}}\}$ are non zero. □

4.15 Definition [Delaunay triangulation]. *Let* $A \in \mathbb{R}^{n \times m}$ *be a point configuration containing* $n+1$ *affinely independent points. Let* $\eta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *be the height function given by* $\eta(x) = |x|^2$. *If the projection of the lower convex hull of* conv A_η *yields a triangulation, then it is called* Delaunay triangulation.

4.16 Lemma. *Let* $C \subset \mathbb{R}^{n+1}$ *be the paraboloid* $C = \{(x, |x|^2)^\intercal : x \in \mathbb{R}^n\}$. *Let* $H(a, \alpha)$, $a \in \mathbb{R}^{n+1}$, $a_{n+1} \neq 0$, *be a hyperplane, and let* $\Pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ *be the projection forgetting the last coordinate. Then* $\Pi(C \cap H(a, \alpha))$ *is either empty, a point or an* n*-dimensional sphere.*

Proof. Since $a_{n+1} \neq 0$ we may assume $a_{n+1} = 1$ and may write $H(a, \alpha) = \{x \in \mathbb{R}^{n+1} : x_{n+1} = \alpha - \sum_{i=1}^n a_i x_i\}$. The intersection $C \cap H(a, \alpha)$ is given by

$$\begin{aligned} C \cap H(a,\alpha) &= \left\{ (x, |x|^2)^\intercal \in \mathbb{R}^{n+1} : \sum_{i=1}^n x_i^2 = \alpha - \sum_{i=1}^n a_i x_i \right\} \\ &= \left\{ (x, |x|^2)^\intercal \in \mathbb{R}^{n+1} : \sum_{i=1}^n \left(x_i + \frac{a_i}{2} \right)^2 = \alpha + \sum_{i=1}^n \left(\frac{a_i}{2} \right)^2 \right\}. \end{aligned} \tag{4.16.1}$$

Since the set on the right hand side does not involve the last coordinate x_{n+1} it describes the projection which is depending on the sign of the right side either empty, a point or an n-dimensional sphere □

4.17 Corollary. *Let* $A \in \mathbb{R}^{n \times m}$ *be a point configuration containing* $n + 1$ *affinely independent points. Let* $\eta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *be the height function given by* $\eta(x) = |x|^2$. *Let* $B \subseteq A$ *such that* B *contains* $n+1$ *affinely independent points. Then* B *corresponds to the vertex set of a facet of the lower convex hull of the lifted points if and only if there is sphere passing through all points of* B *and leaving all points of* $A \setminus B$ *outside.*

Proof. Since $\dim B = n$ there is at most one sphere passing through all points of B. If there is no such sphere, Lemma 4.16 shows that the lifted points are not contained in a hyperplane. On the other hand, if there is such a sphere S there is a hyperplane H_B containing the lifted points (see (4.16.1)). Due to the convexity of the paraboloid, points in the interior, on the surface, and in the exterior of S are lifted to points below H_B, on H_B or above H_B, respectively. Hence, B is lifted to the vertex set of a facet of the lower convex hull of the lifted points if and only if there is no point in S except the points of B on the boundary. □

4.18 Remark. *In particular, simplices in a Delaunay triangulation* $\mathcal{T}$ *are characterized by the "empty sphere" property:* $\sigma \in \mathcal{T}$ *if and only if there is an Euclidean sphere with all vertices of* σ *on the sphere and with the rest of the points outside.*

5 A glimpse of Ehrhart theory

5.1 Notation. *For* $S \subset \mathbb{R}^n$, *S bounded, we denote by* $\mathrm{G}(S)$ *its* lattice point enumerator, *i.e.,*

$$\mathrm{G}(S) = \#(S \cap \mathbb{Z}^n).$$

Almost all results of this section are valid for arbitrary lattices; for simplification, however, we state most of them only for the standard lattice $\mathbb{Z}^n$.

5.2 Definition [Lattice polytope]. *A polytope* $P = \mathrm{conv}\,\{v_1, \ldots, v_m\} \subset \mathbb{R}^n$ *is called a* lattice polytope *if* $v_i \in \mathbb{Z}^n$, $1 \le i \le m$. *The set of all lattice polytopes is denoted by* $\mathcal{P}^n_{\mathbb{Z}}$.

5.3 Lemma. *Let* $a_1, \ldots, a_n \in \mathbb{Z}^n$ *be linearly independent, and let P be the half open parallelepiped* $P = \{\rho_1 a_1 + \cdots + \rho_n a_n : 0 \le \rho_i < 1\}$. *Then*

$$\mathrm{G}(P) = \mathrm{vol}\,(P) = |\det(a_1, \ldots, a_n)|.$$

Proof. Let $\Lambda = \{z_1 a_1 + \cdots + z_n a_n : z_i \in \mathbb{Z}\}$. First we observe that for $g \ne h \in \Lambda$ we have $(g + P) \cap (h + P) = \emptyset$ and that

$$m\,P = \dot\bigcup_{0 \le m_i < m} ((m_1 a_1 + \cdots + m_n a_n) + P)\,,$$

where m_i, $m \in \mathbb{N}$. Moreover, for every $a \in \mathbb{Z}^n$ we have $\mathrm{G}(a + P) = \mathrm{G}(P)$ and so $\mathrm{G}(m\,P) = m^n \mathrm{G}(P)$. Finally, since P is Riemann-integrable we find

$$\mathrm{vol}\,(P) = \lim_{m \to \infty} \# \left(P \cap \frac{1}{m} \mathbb{Z}^n \right) \frac{1}{m^n} = \lim_{m \to \infty} \frac{\mathrm{G}(m\,P)}{m^n} = \mathrm{G}(P).$$

□

5.4 Lemma. *Let* $T = \mathrm{conv}\,\{0, v_1, v_2\} \in \mathcal{P}^2_{\mathbb{Z}}$ *be a lattice triangle, i.e.,* $v_1, v_2 \in \mathbb{Z}^2$ *are linearly independent. Then*

$$\mathrm{G}(T) = \mathrm{vol}\,(T) + \frac{1}{2}\mathrm{G}(\mathrm{bd}\,T) + 1.$$

Proof. Let $U = \{\lambda_1 v_1 + \lambda_2 v_2 : 0 \le \lambda_i < 1\}$. By Corollary 5.3 we have

$$\mathrm{G}(U) = |\det(v_1, v_2)| = \mathrm{vol}\,(U) = 2\mathrm{vol}\,(T).$$

Next we subdivide the lattice points in U into three parts: $U_1 = \mathrm{int}\,T \cap \mathbb{Z}^2$, $U_2 = \mathrm{bd}\,T \cap \mathbb{Z}^2$ and $U_3 = U \setminus T \cap \mathbb{Z}^2$. Since $v_1, v_2 \notin U$ we have $\mathrm{G}(U) = \#U_1 + \#U_2 + \#U_3 - 2$. Furthermore, we observe that $U \setminus T = \{\lambda_1 v_1 + \lambda_2 v_2 : 0 \le \lambda_i < 1, \lambda_1 + \lambda_2 > 1\} = (v_1 + v_2) - \mathrm{int}\,T$, which shows that we have a bijection between the sets U_3 and U_1. Hence $\#U_3 = \#U_1$ and so

$$2\mathrm{vol}\,(T) = \mathrm{G}(U) = 2\#U_1 + \#U_2 - 2 = 2\mathrm{G}(T) - \mathrm{G}(\mathrm{bd}\,T) - 2,$$

which proves the result. □

5.5 Theorem [Pick]. [1] *Let* $P \in \mathcal{P}^2_{\mathbb{Z}}$, $\dim P = 2$. *Then*

$$\mathrm{G}(P) = \mathrm{vol}\,(P) + \frac{1}{2}\mathrm{G}(\mathrm{bd}\,P) + 1.$$

Proof. Let $P = \mathrm{conv}\,\{v_1, \ldots, v_m\}$, where we may assume that $v_1, \ldots, v_m \in \mathbb{Z}^2$ are the vertices of P in cyclic order. We use induction on m. The case $m = 3$ is covered by Lemma 5.4. For $m > 3$ let $P_1 = \mathrm{conv}\,\{v_1, v_2, v_m\}$ and $P_2 = \mathrm{conv}\,\{v_2, \ldots, v_m\}$. Both lattice polygons have less vertices than P and with $S = \mathrm{conv}\,\{v_2, v_m\}$ we get by induction

$$\begin{aligned}
\mathrm{G}(P) &= \mathrm{G}(P_1) + \mathrm{G}(P_2) - \mathrm{G}(S) \\
&= \mathrm{vol}\,(P_1) + \frac{1}{2}\mathrm{G}(\mathrm{bd}\,P_1) + 1 + \mathrm{vol}\,(P_2) + \frac{1}{2}\mathrm{G}(\mathrm{bd}\,P_2) + 1 - \mathrm{G}(S) \\
&= \mathrm{vol}\,(P) + \frac{1}{2}\left(\mathrm{G}(\mathrm{bd}\,P) + 2\,\mathrm{G}(S) - 2\right) + 2 - \mathrm{G}(S) \\
&= \mathrm{vol}\,(P) + \frac{1}{2}\mathrm{G}(\mathrm{bd}\,P) + 1.
\end{aligned}$$

□

5.6 Corollary.

i) *Let* $P \in \mathcal{P}^2_{\mathbb{Z}}$, $\dim P = 2$, *with edges* $F_1, \ldots, F_m$. *Then*

$$\mathrm{G}(P) = \mathrm{vol}\,(P) + \frac{1}{2}\sum_{i=1}^{m} \frac{\mathrm{vol}_1(F_i)}{\det(\mathrm{aff}\,F_i \cap \mathbb{Z}^2)} + 1.$$

Here $\det(\mathrm{aff}\,F_i \cap \mathbb{Z}^2)$ *is the distance of two consecutive lattice points on* $\mathrm{aff}\,F_i$.

ii) *Let* $K \in \mathcal{K}^2$. *Then* $\mathrm{G}(K) \leq \mathrm{vol}\,(K) + \frac{1}{2}\mathrm{F}(K) + 1$, *where* $F(K)$ *denotes the perimeter of* K.

Proof. For $v \in \mathbb{Z}^2$ it is $\mathrm{G}\big(\{\mathrm{conv}\,\{0, v\}\big) = |v| \,/\, \det(\mathrm{lin}\,\{v\} \cap \mathbb{Z}^2) + 1$, and so

$$\mathrm{G}(\mathrm{bd}\,P) = \sum_{i=1}^{m}\left(\frac{\mathrm{vol}_1(F_i)}{\det(\mathrm{aff}\,F_i \cap \mathbb{Z}^2)} + 1\right) - m.$$

Thus the first statement is just a reformulation of Pick's Theorem 5.5.

For ii), let $P = \mathrm{conv}\,(K \cap \mathbb{Z}^2)$. If $\dim P = 1$ then all lattice points are contained in a segment $\mathrm{conv}\,\{v, w\}$, say, and so

$$\mathrm{G}(K) = \mathrm{G}(P) = \frac{|v - w|}{\det\big(\mathrm{lin}\,\{v - w\} \cap \mathbb{Z}^2\big)} + 1 \leq |v - w| + 1 \leq \frac{1}{2}\mathrm{F}(K) + 1,$$

[1] Georg Alexander Pick; 10.08.1859 (Vienna) – 26.07.1942 (concentration camp Theresienstadt)

by the monotonicity of the perimeter. For $\dim P = 2$ we may use this monotonicity with respect to the identity in i) and get

$$\begin{aligned} \mathrm{G}(K) = \mathrm{G}(P) &= \mathrm{vol}\,(P) + \frac{1}{2}\sum_{i=1}^{m} \frac{\mathrm{vol}_1(F_i)}{\det(\mathrm{aff}\, F_i \cap \mathbb{Z}^2)} + 1 \\ &\le \mathrm{vol}\,(P) + \frac{1}{2}\sum_{i=1}^{m} \mathrm{vol}_1(F_i) + 1 \le \mathrm{vol}\,(K) + \frac{1}{2}\mathrm{F}(K) + 1. \end{aligned}$$

□

5.7 Remark.

i) *Inequality ii) of Corollary 5.6 can not be generalized in a straightforward and best possible way to arbitrary lattices, since the perimeter is not an affine equivariant functional, in contrast to the area.*

ii) *Pick's theorem itself, however, can be generalized in various ways. Its proof is only based on the property that a set S can be subdivided into lattice triangle such that the intersection of any two of them is a face of both. For those sets it was shown by Hadwiger&Wills that*

$$\mathrm{G}(S) = \mathrm{vol}\,(S) + \frac{1}{2}\mathrm{E}(S) + \chi(S).$$

Here $\chi(S)$ is the Euler-Poincaré characteristic of S, and $\mathrm{E}(S)$ is the number of segments between two consecutive lattice points in the boundary of S, where segments are counted twice which are not bordering a 2-dimensional cell.

iii) *By Pick's theorem we immediately get the following polynomial behaviour of the lattice point enumerator of a lattice polygon $P \in \mathcal{P}^2_{\mathbb{Z}}$*

$$\begin{aligned} \mathrm{G}(k\,P) &= \mathrm{vol}\,(P)\,k^2 + \frac{1}{2}\,\mathrm{G}(\mathrm{bd}\,P)\,k + 1, \\ \mathrm{G}\big(\mathrm{int}\,(k\,P)\big) &= \mathrm{vol}\,(P)\,k^2 - \frac{1}{2}\,\mathrm{G}(\mathrm{bd}\,P)\,k + 1. \end{aligned}$$

5.8 Notation. *For integers m, n we denote by*

$$\binom{x+m}{n} = \frac{1}{n!}\prod_{i=0}^{n-1}(x+m-i)$$

the polynomial of degree n with roots $i-m$, $i = 0, \dots, n-1$, and leading coefficient $1/n!$. In particular, the polynomials $\binom{x+n-i}{n}$, $i = 0, \dots, n$, form a basis of the space of all polynomials of degree at most n.

5.9 Lemma. *Let $T = \mathrm{conv}\,\{0, v_1, \dots, v_n\} \in \mathcal{P}^n_{\mathbb{Z}}$ be a lattice simplex, i.e., $v_1, \dots, v_n \in \mathbb{Z}^n$ are linearly independent, and for $0 \le i \le n$ let*

$$\mathrm{a}_i(T) = \#\left\{\sum_{j=1}^{n} \lambda_j v_j \in \mathbb{Z}^n : 0 \le \lambda_j < 1,\ i-1 < \sum_{j=1}^{n} \lambda_j \le i\right\}.$$

Then for all $k \in \mathbb{N}$, $k \geq 1$, we have

$$\mathrm{G}(k\,T) = \sum_{i=0}^{n} \mathrm{a}_i(T) \binom{k+n-i}{n}.$$

Proof. For $0 \leq i \leq n$ we denote by U_i the set whose cardinality is denoted by $\mathrm{a}_i(T)$, i.e., $U_i = \{\sum_{j=1}^{n} \lambda_j v_j \in \mathbb{Z}^n : 0 \leq \lambda_j < 1,\ i-1 < \sum_{j=1}^{n} \lambda_j \leq i\}$, and let

$$Q_{k-i} = \left\{ \sum_{i=1}^{n} q_i\, v_i : q_i \in \mathbb{N} \text{ and } \sum_{i=1}^{n} q_i \leq k-i \right\},$$

where $Q_{k-i} = \{0\}$ if $i > k$. Furthermore, we set $U = \{\sum_{i=1}^{n} \lambda_i v_i \in \mathbb{Z}^n : 0 \leq \lambda_i < 1\}$. Then U is the disjoint union of the U_i's, and next we claim

$$k\,T \cap \mathbb{Z}^n = \dot{\bigcup}_{i=0}^{n} (U_i + Q_{k-i})\,. \tag{5.9.1}$$

First we observe that each $z \in \mathbb{Z}^n$ admits a unique representation as

$$z = u_z + \sum_{i=1}^{n} q_i\, v_i, \tag{5.9.2}$$

where $u_z \in U$ and $q_i \in \mathbb{Z}$. To see this we write $z = \sum_{i=1}^{n} \rho_i\, v_i$ with $\rho_i \in \mathbb{R}$, which can be rewritten as

$$z = \sum_{i=1}^{n} \rho_i\, v_i = \sum_{i=1}^{n} (\rho_i - \lfloor \rho_i \rfloor)\, v_i + \sum_{i=1}^{n} \lfloor \rho_i \rfloor\, v_i.$$

Since $\sum_{i=1}^{n} \lfloor \rho_i \rfloor\, v_i \in \mathbb{Z}^n$ we have $u_z = \sum_{i=1}^{n} (\rho_i - \lfloor \rho_i \rfloor)\, v_i \in U$. Such a repesentation is also unique. For if, let $z = u_z + \sum_{i=1}^{n} q_i\, v_i$ and $z = \tilde{u}_z + \sum_{i=1}^{n} \tilde{q}_i\, v_i$ with $\tilde{u}_z \in U$. Then we find $u_z - \tilde{u}_z = \sum_{i=1}^{n} (q_i - \tilde{q}_i)\, v_i$. On the right hand side we have an integral combiantion of the v_i whereas on the left hand side we have an linear combination with coefficients in $(-1, 1)$. Hence we conclude $q_i = \tilde{q}_i$, $1 \leq i \leq n$, and $u_z = \tilde{u}_z$.

For $z \in k\,T \cap \mathbb{Z}^n$ and with respect to the representation (5.9.2) we additionally know that $q_i \in \mathbb{N}$. With $z = \sum_{i=1}^{n} \rho_i\, v_i$ and $u_z = \sum_{i=1}^{n} \lambda_i\, v_i$ we have $q_i + \lambda_i = \rho_i$, $0 \leq i \leq n$. Hence, assuming $u_z \in U_m$ gives $\sum_{i=1}^{n} q_i = \sum_{i=1}^{n} \rho_i - \sum_{i=1}^{n} \lambda_i < k - (m-1)$, and so $\sum_{i=1}^{n} q_i \leq k - m$, which shows $z \in U_m + Q_{k-m}$.

On the other hand, let $u = \sum_{i=1}^{n} \lambda_i\, v_i \in U_m$ and let $q_i \in \mathbb{N}$ with $\sum_{i=1}^{n} q_i \leq k - m$. Then for $z = u + \sum_{i=1}^{n} q_i\, v_i$ we get $z \in \mathbb{Z}^n$ and $z = \sum_{i=1}^{n} (\lambda_i + q_i) v_i$ with $\sum_{i=1}^{n} (\lambda_i + q_i) = \sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} q_i \leq m + k - m = k$. Hence, $z \in k\,T \cap \mathbb{Z}^n$.

On account of the unique representation (5.9.2) the sets $U_i + Q_{k-i}$ are pairwise disjoint and so we have verified (5.9.1). Moreover, by (5.9.2) we also know $\#(U_i + Q_{k-i}) = \#U_i\, \#Q_{k-i}$, so that may we conclude

$$\#(k\,T \cap \mathbb{Z}^n) = \sum_{i=0}^{n} \#U_i\, \#Q_{k-i} = \sum_{i=0}^{n} \mathrm{a}_i(T)\, \#Q_{k-i}.$$

It finally remains to show that for an integer m

$$\#\left\{q \in \mathbb{N}^n : \sum_{i=1}^{n} q_i \leq m\right\} = \binom{n+m}{n},$$

which follows easily by induction on m, n. $\square$

5.10 Notation [Characteristic Function]. *For a set* $A \subseteq \mathbb{R}^n$ *let* $\chi_A : \mathbb{R}^n \to \{0,1\}$ *with*

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

be its characteristic function.

5.11 Lemma [Inclusion-Exclusion Formula]. *Let* $A_i \subseteq \mathbb{R}^n$, $1 \leq i \leq m$, *with characteristic functions* $\chi(A_i)$. *Then*

$$\chi(A_1 \cup A_2 \cup \cdots \cup A_m) = \sum_{\substack{I \subseteq \{1,\ldots,m\} \\ I \neq \emptyset}} (-1)^{\#I-1} \chi\left(\bigcap_{j \in I} A_j\right).$$

Proof. First we observe that $\chi(A) \cdot \chi(B) = \chi(A \cap B)$ for any two subsets $A, B \subseteq \mathbb{R}^n$. Hence the right hand side can be rewritten as

$$\overline{\mathbf{1}} - \prod_{i=1}^{n} (\overline{\mathbf{1}} - \chi(A_i)), \tag{5.11.1}$$

where $\overline{\mathbf{1}}$ is the 1-constant function. Now the function in (5.11.1) takes the value 1 exactly for those $x \in \mathbb{R}^n$ for which one of the functions $\overline{\mathbf{1}} - \chi(A_i)$ takes the value 0, i.e., if and only if $x \in A_1 \cup A_2 \cup \cdots \cup A_m$. $\square$

5.12 Remark. *By evaluating and summing both sides of Lemma 5.11 over all lattice points* $\mathbb{Z}^n$ *we find for bounded sets* $A_i \subset \mathbb{R}^n$, $1 \leq i \leq m$

$$\mathrm{G}(A_1 \cup A_2 \cup \cdots \cup A_m) = \sum_{\substack{I \subseteq \{1,\ldots,m\} \\ I \neq \emptyset}} (-1)^{\#I-1} \mathrm{G}\left(\bigcap_{j \in I} A_j\right).$$

5.13 Theorem [Ehrhart, 1967]. [2] *Let* $P \in \mathcal{P}_{\mathbb{Z}}^n$. *Then there exist numbers* $\mathrm{G}_i(P) \in \mathbb{Q}$, $0 \leq i \leq n$, *depending only on* P, *such that for all* $k \in \mathbb{N}_{\geq 1}$

$$\mathrm{G}(k\,P) = \sum_{i=0}^{n} \mathrm{G}_i(P)\,k^i.$$

The right hand side is called Ehrhart-polynomial.

[2] Eugéne Ehrhart, 29.04.1906 (Guebwiller (Haut-Rhin, France)) – 17.01.2000 (Strasbourg)

Proof. Without loss of generality let $\dim P = n$. On account of Theorem 4.14 there exists a triangulation $T = \{\tau_1, \dots, \tau_m\}$ of P where the vertices of each τ_i are vertices of P. Then each τ_i as well as the intersection of two τ_i's are m-dimensional lattice simplices or empty. Hence with Remark 5.12 and Lemma 5.9 we get

$$\mathrm{G}(k\,P) = \mathrm{G}\left(\bigcup_{i=1}^{m} k\,\tau_i\right) = \sum_{\substack{I\subseteq\{1,\dots,m\}\\ I\neq\emptyset}} (-1)^{\#I-1}\mathrm{G}\left(k\bigcap_{j\in I}\tau_j\right)$$
$$= \sum_{\substack{I\subseteq\{1,\dots,m\}\\ I\neq\emptyset}} (-1)^{\#I-1} \sum_{i=0}^{\dim\left(\bigcap_{j\in I}\tau_j\right)} \mathrm{G}_i\left(\bigcap_{j\in I}\tau_j\right) k^i,$$

which also implies $\mathrm{G}_i(P) \in \mathbb{Q}$. □

5.14 Definition.

i) *Let* $\mathrm{GL}(n,\mathbb{Z}) = \{U \in \mathbb{Z}^{n\times n} : |\det U| = 1\}$ *be the group of* unimodular matrices . *For* $U \in \mathrm{GL}(n,\mathbb{Z})$ *and* $z \in \mathbb{Z}^n$ *the map* $f : \mathbb{R}^n \to \mathbb{R}^n$ *with* $f(x) = U\,x + z$ *is called an* unimodular transformation.

ii) *A functional* $g : \mathcal{P}^n_{\mathbb{Z}} \to \mathbb{R}$ *with*

$$g(P \cup Q) + g(P \cap Q) = g(P) + g(Q)$$

for all $P, Q \in \mathcal{P}^n_{\mathbb{Z}}$ *with* $P \cup Q, P \cap Q \in \mathcal{P}^n_{\mathbb{Z}}$ *is called* additive.

5.15 Proposition. *The unimodular transformations are exactly those affine transformations* $f : \mathbb{R}^n \to \mathbb{R}^n$ *which leaves* $\mathbb{Z}^n$ *invariant, i.e.,* $f(\mathbb{Z}^n) = \mathbb{Z}^n$.

Proof. First we observe that for $U \in \mathrm{GL}(n,\mathbb{Z})$ we certainly have $U\mathbb{Z}^n \subseteq \mathbb{Z}^n$ and since $\det U = \pm 1$ we also have $U^{-1} \in \mathrm{GL}(n,\mathbb{Z})$ and so in particular $U^{-1}\mathbb{Z}^n \subseteq \mathbb{Z}^n$. Altogether we have $U\mathbb{Z}^n = \mathbb{Z}^n$ and so for any unimodular transformation f we know $f(\mathbb{Z}^n) = \mathbb{Z}^n$.

If, on the other hand, $f(x) = A\,x + t$, $A \in \mathbb{R}^{n\times n}$, $t \in \mathbb{R}^n$, is an affine transformation with $f(\mathbb{Z}^n) = \mathbb{Z}^n$ then we have $t \in \mathbb{Z}^n$ and $A \in \mathbb{Z}^{n\times n}$. In particular, we must have $A\mathbb{Z}^n = \mathbb{Z}^n$ and so A must be regular. But then we also know $\mathbb{Z}^n = A^{-1}\mathbb{Z}^n$ which shows $A^{-1} \in \mathbb{Z}^{n\times n}$. Hence $\det A$ as well as $\det A^{-1}$ are integral and so $\det A = \pm 1$, i.e., $A \in \mathrm{GL}(n,\mathbb{Z})$. □

5.16 Proposition. *Let* $P \in \mathcal{P}^n_{\mathbb{Z}}$.

i) $\mathrm{G}_n(P) = \mathrm{vol}\,(P)$.

ii) $\mathrm{G}_i : \mathcal{P}^n_{\mathbb{Z}} \to \mathbb{R}$ *is homogeneous of degree* i*, invariant with respect to unimodular transformations and additive,* $0 \le i \le n$.

iii) $\mathrm{G}_i(P)$ *are independent of the dimension of the space in which* P *is embedded, i.e., let* $P \in \mathcal{P}^n_{\mathbb{Z}}$ *and let* $\tilde{P} = \mathrm{conv}\,\{(v,0)^\intercal : v \in P\} \in \mathcal{P}^{n+1}_{\mathbb{Z}}$. *Then* $\mathrm{G}_i(P) = \mathrm{G}_i(\tilde{P})$, $i = 0, \dots, n$.

Proof. By the Riemann integrability of the characteristic function of P we get in view of Theorem 5.13

$$\begin{aligned}\mathrm{vol}\,(P) &= \lim_{m\to\infty} \frac{\#(P\cap \frac{1}{m}\mathbb{Z}^n)}{m^n} = \lim_{m\to\infty} \frac{\mathrm{G}(m\,P)}{m^n}\\ &= \lim_{m\to\infty} \sum_{i=0}^{n} \mathrm{G}_i(P)\,m^{i-n} = \mathrm{G}_n(P).\end{aligned}$$

For ii) we observe that for $k, m \in \mathbb{N}_{\geq 1}$, $U \in \mathrm{GL}(n,\mathbb{Z})$, $t\in\mathbb{Z}^n$ and $P, Q \in \mathcal{P}^n_{\mathbb{Z}}$

$$\sum_{i=0}^{n} \left(\mathrm{G}_i(P)m^i\right) k^i = \mathrm{G}((km)P) = \mathrm{G}(k(mP)) = \sum_{i=0}^{n} \left(\mathrm{G}_i(m\,P)\right) k^i,$$

$$\sum_{i=0}^{n} \mathrm{G}_i(U\,P+t)k^i = \mathrm{G}(k(U\,P+t)) = \mathrm{G}(k\,P) = \sum_{i=0}^{n} \mathrm{G}_i(P)k^i,$$

$$\begin{aligned}\sum_{i=0}^{n} \left(\mathrm{G}_i(P)+\mathrm{G}_i(Q)\right) k^i &= \mathrm{G}(k\,P)+\mathrm{G}(k\,Q) = \mathrm{G}(k(P\cup Q)) + \mathrm{G}(k(P\cap Q))\\ &= \sum_{i=0}^{n} \left(\mathrm{G}_i(P\cup Q)+\mathrm{G}_i(P\cap Q)\right) k^i,\end{aligned}$$

where for the last we also have to assume that $P\cup Q, P\cap Q \in \mathcal{P}^n_{\mathbb{Z}}$. Comparing the coefficients in all three equations shows the required properties of $\mathrm{G}_i(P)$.

Obviously, $\mathrm{G}(k\,P) = \mathrm{G}(k\,\tilde{P})$ for $k\in\mathbb{N}_{\geq 1}$, and this shows iii). □

5.17 Theorem* [Betke, Kneser, 1985]. [3] [4] *Every additive and unimodular invariant functional on the space of all lattice polytopes is a linear combination of the $n+1$ functionals $\mathrm{G}_i(\cdot)$.*

5.18 Remark. *Some of the coefficients $\mathrm{G}_i(P)$ might be negative. One family of standard examples in this context are the so called* Reeve-simplices*: let $R_m = \mathrm{conv}\left\{0, e_1, e_2, (1,1,m)^\intercal\right\} \in \mathcal{P}^3_{\mathbb{Z}}$ for $m\in\mathbb{N}$. The only lattice points contained in R_m are the four vertices, the volume, however, can be arbitrarily large. Hence some $\mathrm{G}_i(R_m)$ must be negative for large m. More precisely, it is $\mathrm{G}_3(R_m) = m/6$, $\mathrm{G}_2(R_m) = 1$, $\mathrm{G}_1(R_m) = (12-m)/6$ and $\mathrm{G}_0(R_m) = 1$.*

Of course, we can also rewrite the Ehrhart polynomial in terms of the binomial basis $\binom{k+n-i}{n}$ and get

5.19 Corollary. *Let $P\in\mathcal{P}^n_{\mathbb{Z}}$. Then there exist numbers $\mathrm{a}_i(P)\in\mathbb{Q}$, $0\leq i\leq n$, such that for all $k\in\mathbb{N}_{\geq 1}$*

$$\mathrm{G}(k\,P) = \sum_{i=0}^{n} \mathrm{a}_i(P)\binom{k+n-i}{n}.$$

[3] Martin Kneser, 21.01.1928 (Greifswald) – 16.02.2004 (Göttingen)
[4] Ulrich Betke, 13.01.1948 (Bremen) – 24.05.2008 (Siegen)

5.20 Example.

i) *Let* $T_n = \operatorname{conv}\{0, e_1, e_2, \ldots, e_n\}$. *Then*

$$\#(k\,T_n \cap \mathbb{Z}^n) = \binom{n+k}{n},$$

and so we have $\mathrm{a}_i(T_n) = 0$ *for* $1 \le i \le n$, *and* $\mathrm{a}_0(T_n) = 1$. *The* $\mathrm{G}_i(T_n)$ *are – up to* ± 1 *– Stirling numbers of the first kind.*

ii) *Let* $C_n = [-1,1]^n$. *Then* $\mathrm{G}(k\,C_n) = (2\,k+1)^n$ *and so* $\mathrm{G}_i(C_n, \mathbb{Z}^n) = 2^i\binom{n}{i}$, $0 \le i \le n$. *Here the* $\mathrm{a}_i(C_n)$ *are some combinatorial numbers, the so called Eulerian numbers.*

iii) *Let* $C_n^\star = \operatorname{conv}\{\pm e_i : 1 \le i \le n\}$. *Then*

$$\mathrm{G}(k\,C_n^\star) = \sum_{i=0}^{n} \binom{n}{i}\binom{k+n-i}{n},$$

and so $\mathrm{a}_i(C_n^\star) = \binom{n}{i}$, $i = 0, \ldots, n$.

iv) *Let* $\widehat{T_2} = \operatorname{conv}\{0, e_1, e_2\} \subset \mathbb{R}^3$ *be the 2-dimensional standard triangle embedded in* $\mathbb{R}^3$. *Then by Pick's Theorem 5.5*

$$\mathrm{G}(k\,\widehat{T_2}) = \frac{1}{2}\,k^2 + \frac{3}{2}\,k + 1 = (-1)\binom{k}{3} + \frac{1}{2}\binom{k+1}{3} - \frac{1}{2}\binom{k+2}{3} + \binom{k+3}{3}.$$

Hence, in this case we have $\mathrm{a}_3(\widehat{T_2}) = -1$, $\mathrm{a}_2(\widehat{T_2}) = 1/2$, $\mathrm{a}_1(\widehat{T_2}) = -1/2$, $\mathrm{a}_0(\widehat{T_2}) = 1$.

Finally, we state without proof some highlights of Ehrhart theory (and which are left to the lecture in the next term).

5.21 Remark.

i) Stanley's Non-Negativity Theorem: *Let* $P \in \mathcal{P}_{\mathbb{Z}}^n$, $\dim P = n$. *Then* $\mathrm{a}_i(P) \in \mathbb{N}_{\ge 0}$, $0 \le i \le n$.

ii) Stanley's Monotonicity Theorem: *Let* $P, Q \in \mathcal{P}_{\mathbb{Z}}^n$, $\dim P = \dim Q = n$ *with* $P \subseteq Q$. *Then* $\mathrm{a}_i(P, \mathbb{Z}^n) \le \mathrm{a}_i(Q, \mathbb{Z}^n)$, $0 \le i \le n$.

iii) Ehrhart-Macdonald Reciprocity: *Let* $P \in \mathcal{P}_{\mathbb{Z}}^n$, $\dim P = n$. *Then*

$$\mathrm{G}(\operatorname{int} k\,P) = (-1)^n \sum_{i=0}^{n} \mathrm{G}_i(P)(-k)^i.$$

iv) *Let* $P \in \mathcal{P}_{\mathbb{Z}}^n$, $\dim P = n$. *Then for* $i \not\equiv n \bmod 2$

$$\mathrm{G}_i(P) = \frac{1}{2}\sum_{j=i}^{n-1}(-1)^{i+j} \sum_{F\, j\text{-face of } P} \mathrm{G}_i(F).$$

In particular,

$$\mathrm{G}_{n-1}(P) = \frac{1}{2} \sum_{F \text{ facet of } P} \frac{\operatorname{vol}_{n-1}(F)}{\det(\operatorname{aff} F \cap \mathbb{Z}^n)}.$$

6 Borsuk, Hadwiger and planks

6.1 Definition [Diameter]. *Let $X \subset \mathbb{R}^n$ be bounded.*

$$\mathrm{D}(X) = \sup\{|x - y| : x, y \in X\}$$

is called diameter *of X.*

6.2 Remark. *Let $K \in \mathcal{C}^n$ be a convex body with support function $h(K, \cdot)$. Let $x, y \in K$ such that $\mathrm{D}(K) = |x - y|$, and let $u = (x - y)/|x - y|$. The hyperplanes $H(u, \langle u, x\rangle)$ and $H(-u, \langle -u, y\rangle)$ are supporting hyperplanes in x, y, respectively, and it is*

$$\mathrm{D}(K) = \sup_{u \in S^{n-1}} h(K, u) + h(K, -u).$$

6.3 Definition [Convex body, smooth convex body]. *A* convex body *is a convex compact set, $K \in \mathcal{C}^n$. It is called* smooth *if there exists an unique supporting hyperplane for any boundary point of K. The set of all convex bodies is denoted by $\mathcal{K}^n$.*

6.4 Definition [Borsuk number]. *Let $\mathfrak{B}(n)$ be the smallest number $k \in \mathbb{N}$ such that any bounded set $X \subset \mathbb{R}^n$ can be partitioned into k sets having smaller diameter than X.*

6.5 Remark. *A regular simplex shows that $\mathfrak{B}(n) \geq n + 1$ and in 1932 Karol Borsuk*[5] *asked the question whether $\mathfrak{B}(n) = n + 1$ which led to the so called Borsuk conjecture.*

6.6 Theorem [Borsuk-Ulam]. *Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the boundary of the n-dimensional unit ball B_n. Let $f : S^{n-1} \to \mathbb{R}^{n-1}$ be a continuous mapping. Then there exists a $x \in S^{n-1}$ with $f(x) = f(-x)$.*

6.7 Proposition. *B_n can be partitioned into $n + 1$ closed sets of smaller diameter, but not less.*

Proof. Let $R_n = \mathrm{conv}\{v_1, \dots, v_{n+1}\}$ be a regular simplex with $|v_i| = 1$, $1 \leq i \leq n + 1$. Then we have $\langle v_i, v_j\rangle = -1/n$, for $i \neq j$. For $J \subset [n + 1]$, $\#J = n$, let $T_J = B_n \cap \mathrm{pos}\{v_j : j \in J\}$. Each point of B_n is contained in one of these $n+1$ sets T_J, and none of the T_J contains antipodal points, i.e., points $x, -x$. Hence $\mathrm{D}(T_J) < \mathrm{D}(B_n)$.

Suppose we could partition B_n with n (compact) sets $X_1, \dots, X_n \subset B_n$ of smaller diameter. For $x \in S^{n-1}$ let $f(x) = (\mathrm{dist}(x, X_1), \dots, \mathrm{dist}(x, X_{n-1}))^\intercal$. Here $\mathrm{dist}(x, X_i)$ denotes the Euclidean distance (defined as an infimum) between x and the set X_i. Since $\mathrm{dist}(x, X_i)$ is a continuous map, $f : S^{n-1} \to \mathbb{R}^n$ is continuous as well. In view of Theorem 6.6 there exists a $y \in S^{n-1}$ with

[5] Karol Borsuk (8. Mai 1905 in Warschau – 24. Januar 1982 in Warschau)

$f(y) = f(-y)$. If all coordinates of $f(y)$ are non zero then we have $y \in X_n$ and so $-y \in X_n$ as well contradicting $\mathrm{D}(X_n) < \mathrm{D}_(B_n) = 2$.

Hence without loss of generality we may assume that the first coordinate of $f(y)$ – and thus of $f(-y)$ – is zero. But then again we obtain the contradiction that the antipodal points $y, -y$ are contained in the same set, namely X_1. □

6.8 Theorem. *Any smooth convex body $K \in \mathcal{C}^n$ can be partitioned into $n+1$ sets of smaller diameter.*

Proof. We consider the surjective and continuous map $f : \mathrm{bd}\, K \to S^{n-1}$ which maps each boundary point $x \in \mathrm{bd}\, K$ onto the unit vector which is the outward normal vector of the supporting hyperplane in x. In view of Proposition 6.7 we can partitioned S^{n-1} into $n+1$ closed sets $S_1, \ldots, S_{n+1}$ with $\mathrm{D}(S_i) < 1$. Let $B_i = f^{-1}(S_i) \subset \mathrm{bd}\,(K)$, $1 \leq i \leq n+1$. Then $\mathrm{bd}\, K = \cup B_i$ and B_i are compact sets. By Remark 6.5 we have for any pair of points $x, y \in \mathrm{bd}\, K$ with $\mathrm{D}(K) = |x - y|$ that $|f(x) - f(y)| = \mathrm{D}(S^{n-1}) = 2$ and so we conclude that $\mathrm{D}(B_i) < \mathrm{D}(K)$, $1 \leq i \leq n+1$.

Next we define subsets $K_i \subset K$, $1 \leq i \leq n+1$, by $x \in K_i$ if the distance of x to B_i is not bigger than to any other set B_j. It remains to show $\mathrm{D}(K_i) < \mathrm{D}(K)$. Suppose the contrary and let $\mathrm{D}(K_1) = \mathrm{D}(K)$. Since K_1 is compact (observe that B_1 is compact) there exists points $s, t \in K_1$ with $|s - t| = \mathrm{D}(K_1) = \mathrm{D}(K)$. But then $s, t \in \mathrm{bd}\, K \cap K_1 = B_1$ which contradicts $\mathrm{D}(B_1) < \mathrm{D}(K)$. □

6.9 Proposition. $\mathfrak{B}(n) \leq 2^n$.

Proof. Let $X \subset \mathbb{R}^n$ be a bounded set of diameter 1, say. In order to show this upper bound we need the so called theorem of Jung, claiming that such a set is contained in a ball of radius $r = (1/\sqrt{2})\, \sqrt{n/(n+1)} < 1/\sqrt{2}$. Without loss of generality let this be the ball $r\, B_n$. Next we divide this ball via the orthants into 2^n congruent pieces of the type $T = r\, B_n \cap \{x \in \mathbb{R}^n : x_i \geq 0,\ 1 \leq i \leq n\}$. Since all points in T have the same sign pattern the distance of two points $x, y \in T$ is at most $\sqrt{|x|^2 + |y|^2} \leq \sqrt{2}\, r < 1$. □

6.10 Theorem [Kahn&Kalai]. *For large n it is $\mathfrak{B}(n) \geq (1.1)^{\sqrt{n}}$.*

Proof. For an even integer d let

$$X_d = \{v \in \{-1, 1\}^d : v_1 = 1 \text{ and the number of } 1's \text{ in } v \text{ is even}\}.$$

Then we have $\#X_d = \sum_{i \text{ odd}}^{d-1} \binom{d}{i} = 2^{d-2}$ and on this set we define a metric $\delta : X_d \times X_d$ by $\delta(v, w) = \arccos(|\langle v, w\rangle|/d) \in [0, \pi/2]$, i.e., $\delta(v, w)$ is the angle between the "lines" generated by v, w.

With respect to this distance δ we find for the diameter $\mathrm{D}_\delta(X_d)$ of X_d that $\mathrm{D}_\delta(X_d) = \max\{\delta(v, w) : v, w \in X_d\} = \pi/2$. The key ingredient is now a result of Frankl&Wilson (which we are not going to prove) saying that

> Let $p \geq 3$ be a prime and $d = 4p$. Let $A \subset X_d$ with $\mathrm{D}_\delta(A) < \pi/2$. Then $\#(A) \leq \sum_{j=0}^{p-1} \binom{d}{j}$.

Hence in order to partition the set X_d into sets having smaller diameter than X_d we need at least $2^{d-2}/\sum_{j=0}^{p-1}\binom{d}{j}$ sets. Since $\mathrm{e}(m/e)^m < m! < me(m/e)^m$ we find

$$\sum_{j=0}^{p-1}\binom{d}{j} < p\binom{4p}{p} = p\frac{(4p)!}{p!(3p)!} < \frac{4p^2}{e}\left(\frac{256}{27}\right)^p$$

and so

$$2^{d-2}/\sum_{j=0}^{p-1}\binom{d}{j} > \frac{16^p}{4}\frac{e}{4p^2}\left(\frac{27}{256}\right)^p = \frac{e}{d^2}\left(\frac{27}{16}\right)^{d/4} > \frac{e}{d^2}(1.13)^d > (1.1)^d.$$

Next we define a mapping from $g : X_d \to \mathbb{R}^{d^2}$ by $g(v) = v\,v^\intercal$, where we interpret the entries of the $(d \times d)$-matrix $v\,v^\intercal$ as coordinates in $\mathbb{R}^{d^2}$. Observe that in this notation we have for $v, w \in X_d$

$$\begin{aligned}\langle g(v), g(w)\rangle = \langle v\,v^\intercal, w\,w^\intercal\rangle &= \sum_{i=1}^{d}\sum_{j=1}^{d}(v_i v_j)(w_i w_j) = \sum_{i=1}^{d}(v_i w_i)\sum_{j=1}^{d}(v_j w_j)\\ &= (\langle v, w\rangle)^2.\end{aligned}$$

Hence we have for $v, w \in X_d$

$$\begin{aligned}|g(v) - g(w)|^2 &= \langle g(v), g(v)\rangle + \langle g(w), g(w)\rangle - 2\,\langle g(v), g(w)\rangle\\ &= |v|^4 + |w|^4 - 2(\langle v, w\rangle)^2 = 2(d^2 - (\langle v, w\rangle)^2).\end{aligned}$$

Hence $\mathrm{D}(g(X_d)) = 2\,d^2$ and $|g(v) - g(w)| = \mathrm{D}(g(X_d))$ if and only if $\delta(v, w) = \mathrm{D}_\delta(X_d)$. Thus if $g(X_d)$ can be partitioned into k, say, subsets of smaller diameter then the inverse map g^{-1} applied to these subsets gives a partition of X_d into k subsets of smaller diameter. Hence for such a k we have a lower bound of $(1.1)^{\sqrt{d^2}}$. □

6.11 Remark. *The smallest known dimension with $\mathfrak{B}(n) > n+1$ is 298. More precisely, Hinrichs&Richter proved $\mathfrak{B}(n) > n + 11$ for all $n \geq 298$.*

6.12 Definition [Gohberg-Markus-Hadwiger number]. *Let $\mathfrak{H}(n)$ be the smallest number $m \in \mathbb{N}$ such that any convex body $K \subset \mathbb{R}^n$ can be covered by m smaller homothetic copies of K. $\mathfrak{H}(n)$ is called the* Gohberg[6]-Markus-Hadwiger[7] number.

6.13 Remark.

i) *The cube C_n shows that $\mathfrak{H}(n) \geq 2^n$, and it is conjectured that this lower bound is extremal, i.e., $\mathfrak{H}(n) = 2^n$. The problem or conjecture is known as the Gohberg-Markus-Hadwiger problem/conjecture.*

ii) *So far the conjecture has only be confirmed for $n \leq 2$ (and possible for $n \leq 3$) and for some special classes of convex bodies. In particular for smooth bodies, where always $n + 1$ smaller homothetic copies suffice.*

[6] Israel Gohberg, 23. August 1928 – 12. Oktober 2009
[7] Hugo Hadwiger, 23. Dezember 1908 – 29. Oktober 1981

iii) *A boundary point x of a convex body $K \in \mathcal{K}^n$ is illuminated by a direction $v \in \mathbb{R}^n$, if the line $x + \mathbb{R}\, v$ intersects the interior of K. The minimal number m needed to illuminate any convex body in $\mathbb{R}^n$ by m directions is called the* illumination number *and it is denoted by $\mathfrak{I}(n)$. It was shown by Boltjanski that $\mathfrak{H}(n) = \mathfrak{I}(n)$.*

6.14 Definition [Minimal Width, plank].

i) *Let $K \in \mathcal{K}^n$ be a convex body with support function $h(K, \cdot)$. The minimal breadth of K, i.e.,*

$$\min_{u \in S^{n-1}} h(K, u) + h(K, -u)$$

is called the minimal width *of K and will be denoted by $\Delta(K)$.*

ii) *For a vector $u \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \geq \beta \in \mathbb{R}$ the set*

$$S(u, \alpha, \beta) = \{x \in \mathbb{R}^n : \beta \leq \langle u, x \rangle \leq \alpha\}$$

is called a (parallel) strip *or* plank *of width $(\alpha - \beta)/|u|$.*

6.15 Remark [Plank problem]. *Tarski[8] asked the following: Let $K \in \mathcal{K}^n$ be a convex body of minimal width $\Delta(K)$ which is entirely covered by m planks of width μ_i, say. Is it true that*

$$\sum_{i=1}^{m} \mu_i \geq \Delta(K)?$$

6.16 Lemma. *Let $K \in \mathcal{K}^n$ be a convex body and let $y \in \mathbb{R}^n \setminus \{0\}$. Then there exists a chord in direction y with parallel supporting hyperplanes at its ends, i.e., there exists a $t \in \mathbb{R}^n$ such that the two boundary points $\{v, w\} = \{t + \mathbb{R}\, y\} \cap \mathrm{bd}\, K$ have parallel supporting hyperplanes.*

Proof. Let $\bar{t} \in \mathbb{R}^n$ be such that the length of $\{\bar{t} + \mathbb{R}\, y\} \cap K$ is maximal, and let $\mathrm{conv}\,\{v, w\} = \{t + \mathbb{R}\, y\} \cap \mathrm{bd}\, K$. Then $\mathrm{int}\,(K) \cap \mathrm{int}\,((v - w) + K) = \emptyset$ and so there exists a supporting hypeprlane $H(a, \alpha)$ of K and $(v - w) + K$ containing v. Hence for all $x \in K$ we may assume that $\langle a, x \rangle \leq \alpha = \langle a, v \rangle \leq \langle a, (v - w) + x \rangle$, and so $\langle a, w \rangle \leq \langle a, x \rangle \leq \langle a, v \rangle$ for all $x \in K$. □

6.17 Lemma. *Let $K \in \mathcal{K}^n$ be a convex body of minimal width $\Delta(K)$ and let $y \in \mathbb{R}^n$ with $|y| < \Delta(K)/2$. Then $L_y = (K - y) \cap (K + y) \in \mathcal{K}^n$ is non-empty with $\Delta(L_y) \geq \Delta(K) - 2\,|y|$.*

Proof. With respect to the direction y let $t \in \mathbb{R}^n$ be choosen accroding to Lemma 6.16, and let α be the length of $\{t + \mathbb{R}\, y\} \cap K$. Since there exists parallel supporting hyperplanes at the endpoints $\{v, w\}$ of this intersection, we

[8] Alfred Tarski bzw. ursprünglich Alfred Tajtelbaum oder Teitelbaum, 14. Januar 1901 – 26. Oktober 1983

certainly have $\alpha \geq \Delta(K)$. We assume that $w = v + \alpha\, y/|y|$ and observe that $v + y, w - y \in L_y$. Moreover with $\lambda = (\alpha - 2|y|)/\alpha \in (0,1)$ we have

$$\begin{aligned}\lambda(K+y) + (1-\lambda)(v+y) &= \lambda K + (1-\lambda)v + y \\ &= \lambda K + (1-\lambda)w - y = \lambda(K-y) + (1-\lambda)(w-y).\end{aligned}$$

Hence $\lambda(K+y) + (1-\lambda)(v+y) \in L_y$ which shows that

$$\Delta(L_y) \geq \lambda\,\Delta(K) = \frac{\alpha - 2|y|}{\alpha}\Delta(K) \geq \Delta(K) - 2|y|\,.$$

□

6.18 Notation.

i) *In the following we will assume that for a plank $S(u,\alpha,\beta)$ the length of u is $1/2$ of its width, i.e., $2|u|^2 = \alpha - \beta$ and so we may write $\alpha = |u|^2 - \gamma$ and $\beta = -|u|^2 - \gamma$ for a certain constant γ. In this way, a plank is uniquely determined by u and γ and therefore we may write $S(u,\gamma)$ for a plank.*

ii) *For a set of planks $S(u_i,\gamma_i)$ of widths $2|u_i|$, $1 \leq i \leq m$, and for a sign vector $\epsilon \in \{-1,1\}^m$ let*

$$\begin{aligned}P_\epsilon &= \left\{x \in \mathbb{R}^n : \langle u_i, x\rangle \begin{cases}\geq |u_i|^2 - \gamma_i, & \epsilon_i = 1\\ \leq -|u_i|^2 - \gamma_i, & \epsilon_i = -1\end{cases}, 1 \leq i \leq m\right\}\\ &= \left\{x \in \mathbb{R}^n : \langle \epsilon_i\, u_i, x\rangle \geq |u_i|^2 - \epsilon_i\gamma_i,\, 1 \leq i \leq m\right\}\end{aligned}$$

be a polyhedron "outside the planks".

6.19 Lemma. *Let $S(u_i,\gamma_i) \subset \mathbb{R}^n$ be planks of widths $2|u_i|$, $1 \leq i \leq m$. Then*

$$\bigcup_{\epsilon\in\{-1,1\}^m}\left(P_\epsilon - \sum_{i=1}^m \epsilon_i\, u_i\right) = \mathbb{R}^n.$$

Proof. For abbreviation we will also use the following notation: Let $\gamma = (\gamma_1,\ldots,\gamma_m)^\intercal$, and let $U = (u_1,\ldots,u_m) \in \mathbb{R}^{n\times m}$ be the matrix with columns u_i. For a vector $\delta \in \mathbb{R}^m$ let $H_\delta = \{x \in \mathbb{R}^n : \langle U\,\delta, x\rangle \geq |U\delta|^2 - \langle\gamma,\delta\rangle\}$.

Now let $\epsilon = (\epsilon_1,\ldots,\epsilon_m)^\intercal$ be given and with $\delta^{(j)} = (0,\ldots,0,\epsilon_j,0,\ldots,0)^\intercal \in \mathbb{R}^m$ we have $P_\epsilon = \cap_{i=1}^m H_{\delta^{(j)}}$. From this we see that

$$Q_\epsilon = \cap_{\epsilon'\in\{-1,1\}^m} H_{(\epsilon-\epsilon')/2} \subseteq P_\epsilon,$$

and it suffices to show that

$$\bigcup_{\epsilon\in\{-1,1\}^m}(Q_\epsilon - U\epsilon) = \mathbb{R}^n.$$

Now $\overline{x} \in \mathbb{R}^n$ belongs to $Q_\epsilon - U\epsilon$ if and only if $\overline{x} + U\epsilon \in H_{(\epsilon-\epsilon')/2}$ for all $\epsilon' \in \{-1,1\}^m$, i.e.,

$$\left\langle U(\epsilon-\epsilon')/2, \overline{x} + U\epsilon\right\rangle \geq \left|U(\epsilon-\epsilon')/2\right|^2 - \left\langle \gamma, (\epsilon-\epsilon')/2\right\rangle.$$

This can be rewritten in the form

$$2\langle U\epsilon, \overline{x}\rangle + 2\langle \gamma, \epsilon\rangle + \langle U\epsilon, U\epsilon\rangle \geq 2\left\langle U\epsilon', \overline{x}\right\rangle + 2\left\langle \gamma, \epsilon'\right\rangle + \left\langle U\epsilon', U\epsilon'\right\rangle,$$

for $\epsilon' \in \{-1,1\}^m$. Hence $\overline{x}$ belongs to those $Q_\epsilon - U\epsilon$ for which the linear function on the left hand side is maximal. □

6.20 Theorem [Bang]. *The answer to the Plank problem is "Yes", i.e., let $K \in \mathcal{K}^n$ be a convex body of minimal width $\Delta(K)$ which is entirely covered by m planks $S(u_i, \alpha_i, \beta_i)$ of width μ_i, say. Then $\sum_{i=1}^m \mu_i \geq \Delta(K)$.*

Proof. Let the planks be given in the form $S(u_i, \gamma_i) \subset \mathbb{R}^n$ so that they have width $2|u_i|$, $1 \leq i \leq m$. Suppose $\sum_{i=1}^m 2|u_i| < \Delta(K)$ and we want to argue that K is not entirely covered by the planks. To this end we consider

$$\begin{aligned}\bigcup_{\epsilon\in\{-1,1\}^m} ((K \cap P_\epsilon) - U\epsilon) &= \bigcup_\epsilon ((K - U\epsilon) \cap (P_\epsilon - U\epsilon)) \\ &\supseteq \bigcup_\epsilon \left(\left(\bigcap_{\epsilon'} (K - U\epsilon')\right) \cap (P_\epsilon - U\epsilon)\right) \\ &= \left(\bigcap_{\epsilon'} (K - U\epsilon')\right) \cap \left(\bigcup_\epsilon (P_\epsilon - U\epsilon)\right) \\ &= \bigcap_{\epsilon'} (K - U\epsilon'),\end{aligned}$$

where the last equality comes from Lemma 6.19. In view of Lemma 6.17 and its notation we have

$$\bigcap_{\epsilon'} (K - U\epsilon') = \left(\cdots\left(((K_{u_1})_{u_2})_{u_3}\right)\cdots\right)_{u_m},$$

and the right hand side is a convex body of width at least $\Delta(K) - \sum_{i=1}^m 2|u_i| > 0$. Hence for some ϵ the set $K \cap P_\epsilon$ is a convex body with interior points, which shows that the planks do not cover K. □

6.21 Remark. *Bang also raised the following stronger question: Let $K \in \mathcal{K}^n$ be a convex body with support function $h(K,\cdot)$, and suppose K is covered by m planks $S(u_i, \alpha_i, \beta_i)$ with $u_i \in S^{n-1}$ and width $\alpha_i - \beta_i$, $1 \leq i \leq m$. Is it true that*

$$\sum_{i=1}^m \frac{\alpha_i - \beta_i}{h(K,u_i) + h(K,-u_i)} \geq 1?$$

For centrally symmetric bodies it was proven by Keith Ball in 1991.

7 Packings

7.1 Definition [Packing sets]. *A subset $D \subset \mathbb{R}^n$ is called a* packing set *of $K \in \mathcal{K}^n$ if for all $x, y \in D$, $x \neq y$,*

$$(x + \operatorname{int} K) \cap (y + \operatorname{int} K) = \emptyset.$$

The family of all packing sets of K is denoted by $\mathcal{P}(K)$.

7.2 Definition [Density of a Packing]. *Let $K \in \mathcal{K}^n$ and $D \in \mathcal{P}(K)$.*

$$\delta(K, D) = \limsup_{\lambda \to \infty} \frac{\operatorname{vol}(K)\#\{x \in D : x + K \subset \lambda C_n\}}{\operatorname{vol}(\lambda C_n)}$$

is called the density of the packing $D + K$ *(with respect to the gauge body $C_n = [-1, 1]^n$).*

7.3 Remark. *$\delta(K, D)$ depends on the gauge body C_n. For instance, let $D = \{z \in \mathbb{Z}^n : z \geq 0\} \in \mathcal{P}([0, 1]^n)$. Then $\delta([0, 1]^n, D) = \frac{1}{2^n}$. Now let $H_n = C_n \cap \{x \in \mathbb{R}^n : |x_1 - x_2 + x_3 + \cdots + x_n| \leq n - 1\}$. Then $[0, 1]^n \subset H_n$, $\operatorname{vol}(H_n) = 2^n - 2/n!$ and we get*

$$\limsup_{\lambda \to \infty} \frac{\operatorname{vol}([0, 1]^n)\#\{x \in D : x + [0, 1]^n \subset \lambda H_n\}}{\operatorname{vol}(\lambda H_n)} = \frac{1}{2^n - \frac{2}{n!}}.$$

7.4 Theorem*. *Let $K \in \mathcal{K}^n$. The supremum $\sup\{\delta(K, D) : D \in \mathcal{P}(K)\}$ is independent of the chosen gauge body, and there exists a packing set $D_K \in \mathcal{P}(K)$ such that*

$$\sup\{\delta(K, D) : D \in \mathcal{P}(K)\} = \delta(K, D_K).$$

7.5 Definition [Density of a Densest Packing]. *Let $K \in \mathcal{K}^n$.*

$$\delta(K) = \sup\{\delta(K, D) : D \in \mathcal{P}(K)\}$$

is called the density of a densest packing of K *and a set $D_K \in \mathcal{P}(K)$ with $\delta(K) = \delta(K, D_K)$ is called a* densest packing set of K.

7.6 Proposition. *Let $K \in \mathcal{K}^n$. The following properties hold:*

i) $0 < \delta(K) \leq 1$.

ii) *$\delta(t + A\,K) = \delta(K)$ for all $A \in \mathrm{GL}(n, \mathbb{R})$ and $t \in \mathbb{R}^n$.*

iii) *Let $K \in \mathcal{K}_0^n$. Then $D \in \mathcal{P}(K)$ if and only if $|x - y|_K \geq 2$ for all $x, y \in D$, $x \neq y$.*

iv) *Let $K \in \mathcal{K}^n$ and $D \in \mathcal{P}(K)$. Then*

$$\delta(K, D) = \limsup_{\lambda \to \infty} \frac{\operatorname{vol}(K)\,\#\{D \cap \lambda C_n\}}{\operatorname{vol}(\lambda C_n)}.$$

v) *$\mathcal{P}(K) = \mathcal{P}\left(\frac{1}{2}(K-K)\right)$ and for $D \in \mathcal{P}(K)$ we have*

$$\delta(K,D) = \delta\left(\tfrac{1}{2}(K-K), D\right)\,\frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,\left(\frac{1}{2}(K-K)\right)},$$

and consequently

$$\delta(K) = \delta\left(\tfrac{1}{2}(K-K)\right)\,\frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,\left(\frac{1}{2}(K-K)\right)}.$$

Proof. Items i)-iii) follow immediately from the definition. For iv) let $\gamma > 0$ with $K, K-K \subset \gamma\, C_n$ and for a given $\lambda > \gamma$ let

$$m_1(\lambda) = \#\{x \in D : x + K \subset \lambda\, C_n\} \quad \text{and} \quad m_2(\lambda) = \#\{D \cap \lambda\, C_n\}.$$

In order to estimate the difference between $m_1(\lambda), m_2(\lambda)$ let $x \in D$ such that $x + K \subset \lambda\, C_n$, but $x \notin \lambda\, C_n$. Then we get $x + K \subset \lambda\, C_n \setminus (\lambda-\gamma)\, C_n$ and so

$$\begin{aligned} \mathrm{vol}\,(K)\, m_1(\lambda) &\le \mathrm{vol}\,(K)\, m_2(\lambda) + [\lambda^n - (\lambda-\gamma)^n]\,\mathrm{vol}\,(C_n) \\ &\le \mathrm{vol}\,(K)\, m_2(\lambda) + \lambda^{n-1}\mathrm{vol}\,(C_n)\, \mathrm{c}(n,\gamma), \end{aligned} \tag{7.6.1}$$

where $\mathrm{c}(n,\gamma)$ is a constant only depending on γ and n. On the other hand if $x \in D \cap \lambda\, C_n$ with $x + K \not\subset \lambda\, C_n$. Then $x + K \subset (\lambda+\gamma)\, C_n \setminus (\lambda-\gamma)\, C_n$ and hence

$$\begin{aligned} \mathrm{vol}\,(K)\, m_2(\lambda) &\le \mathrm{vol}\,(K)\, m_1(\lambda) + \left[(\lambda+\gamma)^n - (\lambda-\gamma)^n\right]\mathrm{vol}\,(C_n) \\ &\le \mathrm{vol}\,(K)\, m_1(\lambda) + \lambda^{n-1}\mathrm{vol}\,(C_n)\, \bar{\mathrm{c}}(n,\gamma), \end{aligned} \tag{7.6.2}$$

for another constant $\bar{\mathrm{c}}(n,\gamma)$. From (7.6.1) and (7.6.2) we obtain

$$-\frac{\bar{\mathrm{c}}(n,\gamma)}{\lambda} \le \frac{\mathrm{vol}\,(K)\,\#\{x \in D : x + K \subset \lambda\, C_n\}}{\mathrm{vol}\,(\lambda\, C_n)} - \frac{\mathrm{vol}\,(K)\,\#\{D \cap \lambda\, C_n\}}{\mathrm{vol}\,(\lambda\, C_n)} \le \frac{\mathrm{c}(n,\gamma)}{\lambda},$$

which shows iv). For v) we note that

$$\begin{aligned} &(x + \mathrm{int}\, K) \cap (y + \mathrm{int}\, K) \neq \emptyset \\ &\quad\Leftrightarrow x - y \in \mathrm{int}\, K - \mathrm{int}\, K = \mathrm{int}\,(K-K) = \frac{1}{2}\mathrm{int}\,(K-K) - \frac{1}{2}\mathrm{int}\,(K-K) \\ &\quad\Leftrightarrow \left(x + \frac{1}{2}\,\mathrm{int}\,(K-K)\right) \cap \left(y + \frac{1}{2}\mathrm{int}\,(K-K)\right) \neq \emptyset. \end{aligned}$$

and thus $\mathcal{P}(K) = \mathcal{P}\left(\frac{1}{2}(K-K)\right)$. The relation between the densities is a direct consequence of iv). □

7.7 Lemma. *Let $S \subset \mathbb{R}^n$ be a bounded and measurable set with $\mathrm{vol}\,(S) > 0$ and let $D \in \mathcal{P}(K)$. Then there exist $v, w \in \mathbb{R}^n$ such that*

$$\frac{\mathrm{vol}\,(K)\,\#\{(w+S) \cap D\}}{\mathrm{vol}\,(S)} \le \delta(K,D) \le \frac{\mathrm{vol}\,(K)\,\#\{(v+S) \cap D\}}{\mathrm{vol}\,(S)}.$$

Proof. First we consider the upper bound. Let $\gamma > 0$ such that $S \subset \gamma\, C_n$ and let $\varepsilon(\lambda) \in \mathbb{R}$ with $\varepsilon(\lambda) \to 0$ as λ tends infinity and (cf. Proposition 7.6 iv))

$$\varepsilon(\lambda) + \frac{\delta(K,D)}{\mathrm{vol}\,(K)} = \frac{\#\{D \cap \lambda\, C_n\}}{\mathrm{vol}\,(\lambda\, C_n)}. \tag{7.7.1}$$

Now let $x \in D \cap \lambda\, C_n$. Since $\{v \in \mathbb{R}^n : x \in v + S\} = x - S \subset (\lambda + \gamma)\, C_n$ we get

$$\int_{(\lambda+\gamma)\, C_n} \#\left\{(v + S) \cap \left(D \cap \lambda\, C_n\right)\right\} \,\mathrm{d}v = \mathrm{vol}\,(S)\, \#\{D \cap \lambda\, C_n\}.$$

Hence there exist $v_\lambda \in \mathbb{R}^n$ such that

$$\begin{aligned}\#\left\{(v_\lambda + S) \cap \left(D \cap \lambda\, C_n\right)\right\} &\geq \mathrm{vol}\,(S)\, \frac{\#\{D \cap \lambda\, C_n\}}{\mathrm{vol}\,((\lambda + \gamma)\, C_n)} \\ &= \mathrm{vol}\,(S)\, \frac{\#\{D \cap \lambda\, C_n\}}{\mathrm{vol}\,(\lambda\, C_n)} \left(\frac{\lambda}{\lambda + \gamma}\right)^n,\end{aligned}$$

and with (7.7.1) we obtain

$$\begin{aligned}\#\left\{(v_\lambda + S) \cap \left(D \cap \lambda\, C_n\right)\right\} &\geq \left(\frac{\delta(K,D)\mathrm{vol}\,(S)}{\mathrm{vol}\,(K)} + \varepsilon(\lambda)\, \mathrm{vol}\,(S)\right)\left(1 - \frac{\gamma}{\lambda + \gamma}\right)^n \\ &\geq \frac{\delta(K,D)\mathrm{vol}\,(S)}{\mathrm{vol}\,(K)} + \tilde{\varepsilon}(\lambda),\end{aligned}$$

for suitable numbers $\tilde{\varepsilon}(\lambda)$ satisfying $\lim_{\lambda\to\infty} \tilde{\varepsilon}(\lambda) = 0$. Since the left hand side is an integer we can find $\tilde{\lambda} \in \mathbb{R}_{>0}$ such that $\#\{(v_{\tilde{\lambda}} + S) \cap \left(D \cap \tilde{\lambda}\, C_n\right)\} \geq \delta(K,D)\mathrm{vol}\,(S)/\mathrm{vol}\,(K)$, which gives the upper bound. The lower bound can be proven in the same way. □

7.8 Remark. *Let* $K \in \mathcal{K}^n$.

$$\mathrm{R}(K) = \min\{R > 0 : \exists x \in \mathbb{R}^n \text{ with } K \subseteq x + R\, B_n\}$$

is called the circumradius *of* K. *The point* $t_c \in \mathbb{R}^n$ *with* $K \subseteq t_c + \mathrm{R}(K)\, B_n$ *is called* circumcenter *and it is uniquely determined. Moreover, for some* $k \in \{1, \ldots, n\}$, *there exist* $k+1$ *points* $x_0, \ldots, x_k \in \mathrm{bd}\, K \cap \mathrm{bd}\left(t_c + \mathrm{R}(K)\, B_n\right)$ *and* $\lambda_i > 0$, $0 \leq i \leq k$, *with* $\sum_{i=0}^k \lambda_i = 1$ *such that* $t_c = \sum_{i=0}^k \lambda_i\, x_i$.

7.9 Corollary. *Let* $K \in \mathcal{K}_0^n$. *Then*

i) $\delta(K) \geq 2^{-n}$,

ii) $\delta(B_n) \leq (n+1)\sqrt{2}^{\,-n}$.

Proof. i) Let $D_s \in \mathcal{P}(K)$ such that for every $x \in \mathbb{R}^n$ we have $(x + K) \cap (D_s + K) \neq \emptyset$. Thus $(x + 2K) \cap D_s \neq \emptyset$ and now we use the lower bound of Lemma 7.7 with respect to the set $S = 2K$ and the packing set D_s.

ii) For $r \in \mathbb{R}_{>0}$ let $f(r,n) = \max\Big\{\#\{D \cap \operatorname{int}(r\,B_n)\} : D \in \mathcal{P}(B_n)\Big\}$. In view of the upper bound given in Lemma 7.7 it suffices to show (with $S = \sqrt{2}\,\operatorname{int} B_n$)

$$f\big(\sqrt{2}, n\big) \leq n+1. \tag{7.9.1}$$

Let $l = f\big(\sqrt{2}, n\big)$ and let $x_i \in \operatorname{int}\big(\sqrt{2}\,B_n\big)$, $1 \leq i \leq l$, and $D = \{x_1, \ldots, x_l\} \in \mathcal{P}(B_n)$. Let $\mathrm{R} < \sqrt{2}$ be the circumradius of D and without loss of generality we may assume $D \subset \mathrm{R}B_n$ (with circumcenter 0). Therefore we can find $k+1$ points $x_1, \ldots, x_{k+1}$ with $k \in \{1, \ldots, n\}$, say, and $\lambda_i > 0$, $0 \leq i \leq k$, such that

$$\sum_{i=1}^{k+1} \lambda_i\, x_i = 0. \tag{7.9.2}$$

We further have for $1 \leq i \neq j \leq l$,

$$4 \leq |x_i - x_j|^2 \leq 2\,\mathrm{R}^2 - 2\,\langle x_i, x_j\rangle < 4 - 2\,\langle x_i, x_j\rangle\,.$$

In particular, for $x_j \in D \setminus \{x_1, \ldots, x_{k+1}\}$ we would have $\langle x_i, x_j\rangle < 0$, $1 \leq i \leq k+1$, which is impossible on account of (7.9.2). Hence $D = \{x_1, \ldots, x_{k+1}\}$ and so $f(\sqrt{2}, n) \leq n+1$. On the other hand, for instance, the circumradius of a regular simplex of edge length 2 is equal to $\sqrt{2}\sqrt{n/(n+1)}$, and so we also know $f(\sqrt{2}, n) \geq n+1$. □

7.10 Remark. *Chronologically, the first upper bound is due to Blichfeldt (1928), who proved*

$$\delta(B_n) \leq \frac{n+2}{2}\sqrt{2}^{\,-n}.$$

This was slightly improved by Rogers (1958) by a factor of 2/e, roughly speaking. In 1973/74 Sidelnikov showed that

$$\delta(B_n) \leq 2^{-(0.509+o(1))n}, \quad n \text{ large}.$$

Subsequently, this bound was improved by Levenštein (1975), Kabatjanskiĭ and Levenštein (1978) to

$$\delta(B_n) \leq 2^{-(0.599+o(1))n}, \quad n \text{ large},$$

which is still the best known bound.

7.11 Definition [Lattice]. *Let $b_1, \ldots, b_n \in \mathbb{R}^n$ be linearly independent. The set*

$$\Lambda = \{z_1 b_1 + z_2 b_2 + \cdots + z_n b_n : z_i \in \mathbb{Z},\ 1 \leq i \leq n\}$$

is called lattice. *The set of generating vectors $\{b_1, \ldots, b_n\}$ or the matrix $B = (b_1, \ldots, b_n)$ with columns b_i is called* basis *of Λ. An element $b \in \Lambda$ is called* lattice point *of Λ. The set of all lattices in $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$.*

7.12 Definition [Unimodular matrix]. *An integral matrix $U \in \mathbb{Z}^{n\times n}$ is called* unimodular *iff $|\det U| = 1$. The group of all unimodular matrices is denoted by $\mathrm{GL}(n, \mathbb{Z})$.*

Oberserve that a matrix is unimodular if and only the matrix and its inverse are integral matrices.

7.13 Proposition. $\mathrm{GL}(n,\mathbb{Z}) = \{U \in \mathbb{R}^{n\times n} : U\mathbb{Z}^n = \mathbb{Z}^n\}$.

Proof. $U \in \mathrm{GL}(n,\mathbb{Z})$ if and only if $U, U^{-1} \in \mathbb{Z}^{n\times n}$ wich is equivalent to $U\,\mathbb{Z}^n \subseteq \mathbb{Z}^n$ and $U^{-1}\,\mathbb{Z}^n \subseteq \mathbb{Z}^n$. Since the last inclusion is equivalent to $\mathbb{Z}^n \subseteq U\,\mathbb{Z}^n$ we are done. □

7.14 Lemma. *Let* $\Lambda = B\mathbb{Z}^n \in \mathcal{L}^n$. $A = (a_1, \dots, a_n)$ *is a basis of* Λ *iff there exists a* $U \in \mathrm{GL}(n,\mathbb{Z})$ *such that* $A = B\,U$.

Proof. A is a basis of Λ if and only if $A\,\mathbb{Z}^n = \Lambda = B\,\mathbb{Z}^n$ which is equivalent to $B^{-1}A \in \mathrm{GL}(n,\mathbb{Z})$ by Proposition 7.13. □

7.15 Definition [Determinant, fundamental cell]. *Let* $\Lambda \in \mathcal{L}^n$ *with basis* $B = (b_1, \dots, b_n)$.

i) $\det \Lambda = |\det B|$ *is called* determinant *of* Λ.

ii) $P_B = \{\rho_1 b_1 + \dots + \rho_n b_n : 0 \le \rho_i < 1,\ 1 \le i \le n\} = B\,[0,1)^n$ *is called* fundamental cell *or* fundamental parallelepiped *of* Λ *(w.r.t. the basis* B*).*

7.16 Remark.

i) $\det \Lambda$ *is independent of the choice of the basis (cf. Lemma 7.14).*

ii) $\det \Lambda = \mathrm{vol}\,(P_B)$ *and* $\det(\mu\Lambda) = |\mu|^n \det \Lambda$, $\mu \in \mathbb{R}$.

iii) $\det \Lambda \le |b_1|\,|b_2| \cdot \ldots \cdot |b_n|$, *with equality if and only if the vectors* b_i *are pairwise orthogonal (Hadamard inequality).*

iv) $P_B \cap \Lambda = \{0\}$. *Since* $(P_B - P_B) = B\,(-1,1)^n$ *we even have* $(P_B - P_B) \cap \Lambda = \{0\}$.

7.17 Proposition. *Let* $\Lambda \in \mathcal{L}^2$ *and let* $a_1, a_2 \in \Lambda$ *be linearly independent. Then*

$$a_1, a_2 \text{ basis of } \Lambda \Leftrightarrow \mathrm{conv}\,\{0, a_1, a_2\} \cap \Lambda = \{0, a_1, a_2\}.$$

Proof. If a_1 and a_2 are basis then every point of Λ has an unique representation as an integral linear combination of a_1 and a_2. Hence $\mathrm{conv}\,\{0, a_1, a_2\} \cap \Lambda = \{0, a_1, a_2\}$. In order to show the reverse implication we set $T_A = \mathrm{conv}\,\{0, a_1, a_2\}$ and $\overline{P_A} = \{\rho_1 a_1 + \rho_2 a_2 : 0 \le \rho_1, \rho_2 \le 1\}$. It suffices to verify that $\overline{P_A} \cap \Lambda = \{0, a_1, a_2, a_1 + a_2\}$. Let $b \in \overline{P_A} \cap \Lambda$. If $b \in T_A \cap \Lambda$ then we have $b \in \{0, a_1, a_2\}$. Hence we may assume $b \notin T_A$ and thus $b = \rho_1 a_1 + \rho_2 a_2$ with $0 \le \rho_1, \rho_2 \le 1$, but $\rho_1 + \rho_2 > 1$. So we have $(1 - \rho_1) + (1 - \rho_2) \le 1$ and thus

$$(a_1 + a_2) - b = (1 - \rho_1)a_1 + (1 - \rho_2)a_2 \in T_A,$$

which shows $b \in \{a_1, a_2, a_1 + a_2\}$. □

7.18 Remark. *An analogous statement to Lemma 7.17 does not exist in dimension* ≥ 3. *For* $n \geq 3$ *and* $m \in \mathbb{N}$ *let* $b(m) = (1, \ldots, 1, m)^\intercal \in \mathbb{R}^n$ *and* $T^n(m) = \mathrm{conv}\,\{0, e_1, \ldots, e_{n-1}, b(m)\}$. *Then*

$$T^n(m) \cap \mathbb{Z}^n = \{0, e_1, \ldots, e_{n-1}, b(m)\},$$

but the determinant of the lattice with basis $\{e_1, \ldots, e_{n-1}, b(m)\}$ *is* m. $T^n(m)$ *are called* Reeve simplices.

7.19 Proposition. *Let* $\Lambda = B\mathbb{Z}^n \in \mathcal{L}^n$. *Then*

$$\mathbb{R}^n = \dot{\bigcup}_{b \in \Lambda} (b + P_B)\,,$$

i.e., $\mathbb{R}^n$ *is the pairwise disjunct union of the lattice translates* $b + P_B$.

Proof. Each $x \in \mathbb{R}^n$ can be decomposed as $x = (x - \lfloor x \rfloor_B) + \lfloor x \rfloor_B$. The first summand is in P_B and second is a lattice point of Λ. To show that the union is disjunct we observe that the intersection of two lattice translates $b + P_B, \bar{b} + P_B$ of P_B, $b, \bar{b} \in \Lambda$, is non-empty, if and only if $b - \bar{b} \in (P_B - P_B) \cap \Lambda$. By Remark 7.16 iv) this is equivalent to $b = \bar{b}$. □

7.20 Corollary. *Let* $K \in \mathcal{K}^n$ *and* $\Lambda \in \mathcal{L}^n \cap \mathcal{P}(K)$. *Then*

$$\delta(K, \Lambda) = \frac{\mathrm{vol}\,(K)}{\det \Lambda}.$$

Proof. Let P_B be a fundamental cell of Λ. By Proposition 7.19, $\mathbb{R}^n$ is the pairwise disjoint union of the translates $b + P_B$, $b \in \Lambda$. Hence $\#\{(x + P_B) \cap \Lambda\} \leq 1$ for all $x \in \mathbb{R}^n$, whereas the property that $\mathbb{R}^n$ is covered by all translates is equivalent to the lower bound $\#\{(x + P_B) \cap \Lambda\} \geq 1$ for all $x \in \mathbb{R}^n$. Hence $\#\{(x + P_B) \cap \Lambda\} = 1$ for all $x \in \mathbb{R}^n$. Together with $\mathrm{vol}\,(P_B) = \det \Lambda$ the identity follows from Lemma 7.7.

□

7.21 Definition [Density of a densest Lattice Packing]. *For* $K \in \mathcal{K}^n$ *the set* $\mathcal{P}_{\mathcal{L}}(K) = \mathcal{L}^n \cap \mathcal{P}(K)$ *is called the family of all* packing lattices *of* K. *For* $\Lambda \in \mathcal{P}_{\mathcal{L}}(K)$ *the arrangement* $\Lambda + K$ *is called a* lattice packing *of* K *and*

$$\delta_{\mathcal{L}}(K) = \sup\big\{\delta(K, \Lambda) : \Lambda \in \mathcal{P}_{\mathcal{L}}(K)\big\}$$

is called the density of a densest lattice packing of K.

7.22 Definition [Critical determinant and admissible lattices]. *Let* $K \in \mathcal{K}^n_0$. *A lattice* Λ *is called* admissible for K *(or* K-admissible*) if* $\mathrm{int}\, K \cap \Lambda = \{0\}$.

$$\Delta(K) = \inf\{\det \Lambda : \Lambda \textit{ admissible for } K\}$$

is called the critical determinant of K.

7.23 Remark. *Let* $K \in \mathcal{K}_0^n$ *and* $\Lambda \in \mathcal{L}^n$. *Then* $\big(1/\lambda_1(K,\Lambda)\big)\,\Lambda$ *is admissible for* K.

7.24 Proposition* [Critical lattice]. *For* $K \in \mathcal{K}_0^n$ *there exists a* K*-admissible lattice* Λ_K *with* $\det \Lambda_K = \Delta(K)$. *Such a lattice will be called* a critical lattice of K.

7.25 Proposition. *Let* $K \in \mathcal{K}^n$. *Then*

$$\delta_{\mathcal{L}}(K) = \frac{\mathrm{vol}\,(K)}{\Delta(K-K)}.$$

Proof. By Proposition 7.6 v) and the definition of admissible lattices (cf. Definition 7.22) the family of packing sets $\mathcal{P}_{\mathcal{L}}(K) = \mathcal{P}_{\mathcal{L}}\left(\frac{1}{2}(K-K)\right)$ coincides with the set of all admissible lattices for $K-K$. From Corollary 7.20 and Proposition 7.24 we get the desired identity. □

7.26 Remark.

i) $0 < \delta_{\mathcal{L}}(K) \leq \delta(K) \leq 1$.

ii) $\delta_{\mathcal{L}}(A\,K + t) = \delta_{\mathcal{L}}(K)$ *for all* $A \in \mathrm{GL}(n,\mathbb{R})$ *and* $t \in \mathbb{R}^n$.

iii) *For* $K \in \mathcal{K}_0^n$ *we have* $\delta_{\mathcal{L}}(K) = 2^{-n}\mathrm{vol}\,(K)/\Delta(K)$.

7.27 Theorem [Minkowski-Hlawka, 1943]. *Let* $S \subset \mathbb{R}^n$ *be a bounded Jordan-measurable set with* $\mathrm{vol}\,(S) < 1$. *Then there exists a lattice* $\Lambda \in \mathcal{L}^n$ *with*

$$\det \Lambda = 1 \quad \textit{and} \quad S \cap \Lambda \setminus \{0\} = \emptyset.$$

Proof. Since S is bounded and Jordan-measurable of $\mathrm{vol}\,(S) < 1$, there exists a prime p such that

$$\begin{aligned} &\text{i)}\ \frac{1}{p^{n-1}}\#\left(S \cap \frac{1}{p^{(n-1)/n}}\mathbb{Z}^n\right) < 1, \\ &\text{ii)}\ S \subset \left\{x \in \mathbb{R}^n : |x_i| < p^{1/n},\ 1 \leq i \leq n\right\}. \end{aligned} \tag{7.27.1}$$

If we suppose that we can find p^{n-1} sublattices of $\mathbb{Z}^n$ of determinant p^{n-1}, such that $\{0\}$ is the only common point in each two of them, then (7.27.1) i) would immediately imply the assertion. However, although we can not find those sublattices, we can find sublattices whose common points different from 0 are "far away", which is good by (7.27.1) ii).

Let $U_p = \{u \in \mathbb{Z}^n : u_1 = 1,\ 0 \leq u_i < p,\ i = 2,\ldots,n\}$. For $u \in U_p$ let $\Lambda(u)$ be the lattice with basis $u, p\,e_2, \ldots, p\,e_n$. Obviously, we have $\det \Lambda(u) = p^{n-1}$ and there are p^{n-1} sublattices of that type. Since $u_1 = 1$ we observe that

$$z \in \Lambda(u) \Leftrightarrow z_i \equiv (z_1\,u_i) \bmod p,\ i = 2,\ldots,n, \tag{7.27.2}$$

and next we claim for $u \neq \bar{u} \in U_p$

$$\Lambda(u) \cap \Lambda(\overline{u}) \subset \{0\} \cup \{x \in \mathbb{R}^n : \exists x_i \text{ with } |x_i| \geq p\}. \tag{7.27.3}$$

To see this let $z \in \Lambda(u) \cap \Lambda(\overline{u})$. By (7.27.2) we have $z_1(u_i - \bar{u}_i) \equiv 0 \bmod p$ and since $-(p-1) \leq u_i - \bar{u}_i \leq p-1$, we conclude $z_1 \equiv 0 \bmod p$. If $z_1 \neq 0$ we are done, and if $z_1 = 0$ we get from (7.27.2) that $z_i \equiv 0 \bmod p$, $i = 2, \ldots, n$. Thus, either $z = 0$ or at least one coordinate is not less than p in absolute value.

In view of (7.27.1) ii) the inclusion (7.27.3) shows that the p^{n-1} sets

$$S \cap \frac{1}{p^{(n-1)/n}} \Lambda(u) \setminus \{0\}, \quad u \in U_p,$$

are pairwise disjoint, and so (cf. (7.27.1) i))

$$\sum_{u \in U_p} \# \left(S \cap \frac{1}{p^{(n-1)/n}} \Lambda(u) \setminus \{0\} \right) \leq \# \left(S \cap \frac{1}{p^{(n-1)/n}} \mathbb{Z}^n \right) < p^{n-1}.$$

Thus, at least one of the lattices $\frac{1}{p^{(n-1)/n}} \Lambda(u)$ has the desired properties. □

7.28 Remark. *Theorem 7.27 remains true for Jordan-measurable, unbounded, closed sets.*

7.29 Corollary. *Let $S \subset \mathbb{R}^n$ be a bounded Jordan-measurable set with $S = -S$ and with $\mathrm{vol}\,(S) < 2$. Then there exists a lattice $\Lambda \in \mathcal{L}^n$ with*

$$\det \Lambda = 1 \quad \text{and} \quad S \cap \Lambda \setminus \{0\} = \emptyset.$$

7.30 Corollary. *Let $K \in \mathcal{K}_0^n$. Then*

$$\delta_{\mathcal{L}}(K) \geq 2^{-(n-1)} \qquad \left[\Leftrightarrow \Delta(K) \leq \frac{\mathrm{vol}\,(K)}{2} \right].$$

Proof. Without loss of generality let $\mathrm{vol}\,(K) = 2 - \varepsilon$ for some $\varepsilon > 0$. Corollary 7.29 shows that there exist a lattice Λ_ε, with $\det \Lambda_\varepsilon = 1$, which is admissible for K and thus $2\Lambda_\varepsilon \in \mathcal{P}_{\mathcal{L}}(K)$. Hence we get

$$\delta_{\mathcal{L}}(K) \geq \frac{\mathrm{vol}\,(K)}{\det(2\Lambda_\varepsilon)} = 2^{-(n-1)} - \frac{\varepsilon}{2^n}.$$

Since this is true for every $\varepsilon > 0$ the statement is shown. □

7.31 Theorem*. *Let $S \subset \mathbb{R}^n$, $n \geq 2$, be a bounded ray set (i.e., if $x \in S$ then $\lambda x \in S$ for all $\lambda \in [0,1]$) with $\mathrm{vol}\,(S) < \zeta(n)$. Then there exists a lattice $\Lambda \in \mathcal{L}^n$ with $\det \Lambda = 1$ and $S \cap \Lambda \setminus \{0\} = \emptyset$.*
Here $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ denotes the (Riemann) zeta function (ζ-function).

7.32 Theorem* [K. Ball, 1992]. $\delta(B_n) \geq (n-1)2^{-(n-1)}\zeta(n)$.

7.33 Remark. *The densest lattice packings of balls are only known in dimensions $n \leq 8$ (Lagrange 1773, Gauß 1840, Korkine-Zolotareff 1872/73, Blichfeldt, 1934) and $n = 24$ (Cohn& Kumar, 2004).*

7.34 Theorem [Swinnerton-Deyer, 1953]. *Let $K \in \mathcal{K}_0^n$ and let $\Lambda_K \in \mathcal{L}^n$ be a critical lattice of K. Then*

$$\#\big(K \cap \Lambda_K \setminus \{0\}\big) \geq n(n+1).$$

Proof. Let $B = (b_1, \ldots, b_n)$ be a basis of Λ_K and let $\{\pm a_1, \ldots, \pm a_k\} = K \cap \Lambda_K \setminus \{0\}$. We suppose that $k < n(n+1)/2$ and let $a_i = B\, z_i$ for some $z_i \in \mathbb{Z}^n$. For $T \in \mathbb{R}^{n \times n}$ with entries $t_{l,m}$ and for $\rho \in \mathbb{R}$ let $B_{\rho,T} = B(I_n + \rho T)$. In the following we show that we can choose parameters $t_{l,m}$ and ρ such that the corresponding lattice $\Lambda_{\rho,T} = B_{\rho,T}\mathbb{Z}^n$ is still admissible for K, but $\det \Lambda_{\rho,T} < \det \Lambda_K$.

Let $H_i = \big\{x \in \mathbb{R}^n : \langle u_i, x\rangle = 1\big\}$ be a supporting hyperplane of K at the point a_i and let $a_{i,\rho,T} = B_{\rho,T}\, z_i$. Then we have

$$a_{i,\rho,T} \in H_i \Leftrightarrow \langle u_i, B_{\rho,T}\, z_i\rangle = 1 \Leftrightarrow \langle u_i, BT\, z_i\rangle = 0 \Leftrightarrow \sum_{l,m=1}^{n} c_{l,m,i}\, t_{l,m} = 0,$$

where $c_{l,m,i}$ are certain numbers, depending on B, z_i, u_i. Since $k < n\,(n+1)/2$ we can find non-trivial scalars $\overline{t}_{l,m} \in \mathbb{R}$, say, such that

$$\text{i) } \overline{t}_{l,m} = \overline{t}_{m,l}, \qquad \text{ii) } a_{i,\rho,\overline{T}} \in H_i,\ 1 \leq i \leq k, \text{ and } \rho \in \mathbb{R}. \tag{7.34.1}$$

Next we argue that we can find $\overline{\rho} \in \mathbb{R}$ such that $\Lambda_{\rho,\overline{T}}$ is a K-admissible lattice for all $|\rho| \leq \overline{\rho}$. First we notice that for sufficiently small ρ, $\Lambda_{\rho,\overline{T}} \in \mathcal{L}^n$. Now we suppose that there exists a sequence $\rho_i \to 0$ such that there exists $u_{\rho_i,\overline{T}} \in \Lambda_{\rho_i,\overline{T}} \setminus \{0\} \cap \operatorname{int} K$ for $i \in \mathbb{N}$. Since $\Lambda_{\rho_i,\overline{T}} \to \Lambda_K$ we may assume that $u_{\rho_i,\overline{T}} \to a_1$, say. We also know that $a_{1,\rho_i,\overline{T}} \to a_1$ and so we have $a_{1,\rho_i,\overline{T}} - u_{\rho_i,\overline{T}} \to 0$, which implies $a_{1,\rho_i,\overline{T}} = u_{\rho_i,\overline{T}}$ for sufficiently large i. Since $a_{1,\rho_i,\overline{T}} \in H_1$ we have $u_{\rho_i,\overline{T}} \notin \operatorname{int} K$, a contradiction.

Now since $\Lambda_{\rho,\overline{T}}$ is K-admissible for all $|\rho| \leq \overline{\rho}$ we must have $\det \Lambda_{\rho,\overline{T}} \geq \det \Lambda_K$ for all $|\rho| \leq \overline{\rho}$. Thus

$$\begin{aligned} &\det(I_n + \rho\overline{T}) \geq 1 \Leftrightarrow 1 + \tau_1\,\rho + \tau_2\,\rho^2 + \cdots + \tau_n\,\rho^n \geq 1, \quad \text{where} \\ &\tau_1 = \sum_{j=1}^{n} \overline{t}_{j,j}, \quad \tau_2 = \sum_{1 \leq i < j \leq n} (\overline{t}_{i,i}\,\overline{t}_{j,j} - \overline{t}_{i,j}\,\overline{t}_{j,i}), \quad \tau_3 = \ldots \end{aligned} \tag{7.34.2}$$

Since the inequality in (7.34.2) must hold for all $|\rho| \leq \overline{\rho}$ we first get $\tau_1 = 0$ and then $\tau_2 \geq 0$, which gives on account of (7.34.1) i)

$$\begin{aligned} 0 &\leq 2\tau_2 - (\tau_1)^2 = 2\sum_{i<j} \overline{t}_{i,i}\,\overline{t}_{j,j} - 2\sum_{i<j}(\overline{t}_{i,j})^2 - \sum_{i=1}^{n}(\overline{t}_{i,i})^2 - 2\sum_{i<j}\overline{t}_{i,i}\,\overline{t}_{j,j} \\ &= -\sum_{i=1}^{n}(\overline{t}_{i,i})^2 - 2\sum_{i<j}(\overline{t}_{i,j})^2. \end{aligned}$$

Hence $\overline{T} = 0$, which contradicts the choice of $\overline{T}$. □

7.35 Theorem. *Let $K \in \mathcal{K}_0^2$.*

i) *Let $b_1, b_2 \in \operatorname{bd} K$ such that $b_2 - b_1 \in \operatorname{bd} K$. Then the lattice $\Lambda = (b_1, b_2)\mathbb{Z}^2$ is admissible for K.*

ii) *Let $\Lambda \in \mathcal{L}^2$ be a critical lattice of K. Then there exists a basis b_1, b_2 of Λ such that $b_1, b_2, b_2 - b_1 \in \operatorname{bd} K$.*

Proof. For i) we first notice that, since $b_1, b_2, b_2 - b_1 \in \operatorname{bd} K$ then

$$\operatorname{int} K \cap \big\{ z_1 b_1 + z_2 b_2 : z_2 \in \{0, \pm 1\},\ z_1 \in \mathbb{Z} \big\} = \{0\}.$$

Now assume that there exists $b = z_1 b_1 + z_2 b_2 \in \operatorname{int} K$ and without loss of generality let $z_2 \geq 2$. Further let $\varepsilon > 0$ such that $b \pm \varepsilon b_1 \in \operatorname{int} K$ and let $P = \operatorname{conv}\{\pm b_1, b \pm \varepsilon b_1\}$. Since

$$\operatorname{vol}_1\big(P \cap \operatorname{lin}\{b_1\}\big) = 2\,|b_1|, \quad \operatorname{vol}_1\big(P \cap (\operatorname{lin}\{b_1\} + z_2 b_2)\big) \geq 2\varepsilon\,|b_1|$$

we have $\operatorname{vol}_1\big(P \cap (\operatorname{lin}\{b_1\} + b_2)\big) > |b_1|$. This shows, however, that either b_2 or $b_2 - b_1$ belong to the interior of K.

Now we come to ii). Since $\det \Lambda = \Delta(K)$ we can find two linearly independent points $b_1, b_2 \in \Lambda \cap \operatorname{bd} K$. On account of Lemma 7.17 we may assume that b_1, b_2 build a basis of Λ. Now let $\alpha_1, \alpha_2 \in \mathbb{R}_{\geq 0}$ be maximal such that $b_2 - \alpha_1 b_1 \in \operatorname{bd} K$ and $b_2 + \alpha_2 b_1 \in \operatorname{bd} K$.

If $\alpha_1 \geq 1$ then we must have $b_2 - b_1 \in \operatorname{bd} K$ and we are done. Similarly, if $\alpha_2 \geq 1$ we have $b_2 + b_1 \in \operatorname{bd} K$ and the basis $b_1, b_1 + b_2$ of Λ has the required properties. So we may assume $\alpha_i < 1$. If $\alpha_1, \alpha_2 > 0$ then $\pm b_2 + \operatorname{lin}\{b_1\}$ are supporting lines of K, and from this we conclude that $K \cap \Lambda \setminus \{0\} = \{\pm b_1, \pm b_2\}$, which contradicts Theorem 7.34. Hence we may assume that $\alpha_2 = 0$ and since $\alpha_1 < 1$ we can find $\lambda \in (0,1)$ such that $\operatorname{vol}_1\big(K \cap (\lambda b_2 + \operatorname{lin}\{b_1\})\big) = |b_1|$. Let u and v be the corresponding points in the boundary, such that $v = u - b_1$. By i) we know that the lattice with basis b_1, u is admissible for K, but $\big|\det(b_1, u)\big| = \big|\det(b_1, \lambda b_2)\big| < \Delta(K)$, which contradicts the definition of $\Delta(K)$. □

7.36 Corollary. *Let $K \in \mathcal{K}_0^2$ and H_K be an affinely regular hexagon of minimal volume with vertices on $\operatorname{bd} K$. Then*

$$\delta_{\mathcal{L}}(K) = \frac{3}{4}\frac{\operatorname{vol}(K)}{\operatorname{vol}(H_K)} \qquad \left[\Leftrightarrow \Delta(K) = \frac{1}{3}\operatorname{vol}(H_K)\right].$$

7.37 Theorem* [Fejes Tóth, 1950; Rogers, 1951]. *Let $K \in \mathcal{K}^2$. Then*

$$\delta(K) = \delta_{\mathcal{L}}(K).$$

7.38 Theorem* [Hales, 1998/2005].

$$\delta(B_3) = \delta_{\mathcal{L}}(B_3) = \frac{\pi}{3\sqrt{2}}.$$

Exercises

1.1 Exercise. *Prove or disprove:*

i) $\text{conv}\{X+Y\} = \text{conv}\,X + \text{conv}\,Y$ *for all* $X, Y \subseteq \mathbb{R}^n$.

ii) *If* $X \subseteq \mathbb{R}^n$ *closed, then* $\text{conv}\,X$ *is also closed.*

iii) *If* $X \subseteq \mathbb{R}^n$ *open, then* $\text{conv}\,X$ *is also open.*

iv) *If* $X \subseteq \mathbb{R}^n$ *is compact, then* $\text{conv}\,X$ *is also compact.*

v) *For* $K \in \mathcal{C}^n$ *and* $\lambda, \mu \in \mathbb{R}$ *we have* $(\lambda+\mu)\,K = \lambda\,K + \mu\,K$.

1.2 Exercise. *Let* $T = \text{conv}\{x_1, \ldots, x_{k+1}\} \subset \mathbb{R}^n$ *be a* k*-simplex, and let* $\lambda_i > 0$, $1 \leq i \leq k+1$, *with* $\sum \lambda_i = 1$. *Then* $\sum \lambda_i\,x_i \in \text{relint}\,T$.

1.3 Exercise. *Let* $K \in \mathcal{C}^n$ *be closed and let* $x \notin K$. *Then we have*

$$\Phi_K(z) = \Phi_K(x) \quad \text{fr alle} \quad z \in R(x) = \big\{\Phi_K(x) + \lambda\big(x - \Phi_K(x)\big) : \lambda \geq 0\big\}.$$

1.4 Exercise. *Let* $K \in \mathcal{C}^n$ *be closed and let* $x \in \text{relbd}\,K$. *Then there exists a supporting hyperplane* $H(a,\alpha)$ *of* K *with* $x \in H(a,\alpha)$.

1.5 Exercise. *Use Theorem 1.13 in order to show Farkas lemma:*
Let $A \in \mathbb{R}^{m\times n}$ *and* $b \in \mathbb{R}^m$. *Then there exists a non-negative* $x \in \mathbb{R}^n$, *i.e.,* $x \geq 0$, *with* $A\,x = b$ *if and only if* $\langle b, y\rangle \geq 0$ *for all* $y \in \mathbb{R}^m$ *with* $A^{\intercal}y \geq 0$.

1.6 Exercise. *Let* $K_1, \ldots, K_m, C \in \mathcal{C}^n$. *For each choice of* $n+1$ *sets* $K_{i_1}, \ldots, K_{i_{n+1}}$ *there exists an* $u \in \mathbb{R}^n$, *such that* $u + C$ *intersects the sets* $K_{i_1}, \ldots, K_{i_{n+1}}$. *Show that there exists an* $u \in \mathbb{R}^n$, *such that* $u + C$ *intersects all sets* $K_1, \ldots, K_m$.

1.7 Exercise. *Let* $X, Y \subset \mathbb{R}^n$ *finite. Show that there exist a strictly separating hyperplane of* X *and* Y *if and only if each pair of subsets* $X' \subseteq X$, $Y' \subseteq Y$ *with* $\#X' + \#Y' \leq n+2$ *can be strictly separated by a hyperplane.*

2.1 Exercise.

i) *Show that each subset* $X \subset \mathbb{R}^2$ *with diameter* 1 *is contained in a circle of radius* $1/\sqrt{3}$. *Here the diameter is the maximum Euclidean distance between two points of* X *Furthermore show, that equality holds if and only if* X *contains three points spanning a regular triangle of edge length* 1. *(Hint: Helly and so...)*

ii) *What could be an analogous result* n*-space?*

2.2 Exercise. *Show that any set* $X \subset \mathbb{R}^2$ *with diameter* 1 *can be partitioned into three sets with smaller diameter.*

2.3 Exercise. *Let $K \in \mathcal{C}^n$ compact and $0 \in \operatorname{int} K$. Given an i-dimensional linear subspace $L \subseteq \mathbb{R}^n$, $0 \leq i \leq n$, let $K|L$ be the orthogonal projection of K onto L. show that*

$$(K|L)^\star \cap L = K^\star \cap L \text{ und/bzw. } (K \cap L)^\star \cap L = K^\star|L.$$

2.4 Exercise.

i) *Let $T_n = \operatorname{conv}\{v_0, \ldots, v_n\}$ be an n-simplex. Show that F is an i-face of T_n if and only if $F = \operatorname{conv}\{v_{i_j} : j \in I\}$ for any $i+1$ subset I of $\{0, \ldots, n\}$.*

ii) *Let $P \in \mathcal{P}^n$ be an n-dimensional polytop. Show that $\sum_{i=-1}^{n} f_i(P) \geq 2^{n+1}$ with equality if and only P is an n-simplex.*

2.5 Exercise. *Let $Q \in \mathcal{P}^n$ with $\dim Q = n - 1$.*

i) *Let $v \in \mathbb{R}^n \setminus \operatorname{aff} Q$. $P = \operatorname{conv}\{Q, \{v\}\}$ is called a* pyramid with basis Q *and apex v. Show that $f_k(P) = f_k(Q) + f_{k-1}(Q)$, $0 \leq k \leq n-1$.*

ii) *Let $\operatorname{conv}\{x, y\} \not\subset \operatorname{aff} Q$ be a segment with* $\operatorname{relint} Q \cap \operatorname{relint} \operatorname{conv}\{x, y\} \neq \emptyset$. *The set $P = \operatorname{conv}(Q \cup \operatorname{conv}\{x, y\})$ is called a* bipyramid with basis Q. *Show that:*

$$f_k(P) = \begin{cases} f_k(Q) + 2f_{k-1}(Q), & 0 \leq k \leq n-2, \\ 2f_{n-2}(Q), & k = n-1. \end{cases}$$

iii) *Let $x \notin \operatorname{aff} Q$. $P = Q + \operatorname{conv}\{-x, x\}$ is called a* prism with basis Q. *Show that:*

$$f_k(P) = \begin{cases} 2f_0(Q), & k = 0, \\ 2f_k(Q) + f_{k-1}(Q), & 1 \leq k \leq n-1. \end{cases}$$

2.6 Exercise. *Show that any polytope is the projection of a simplex $S \subset \mathbb{R}^N$ ist.*

3.1 Exercise. *Let $P \in \mathcal{C}^n$ be a simple n-polytop. Show that any j-face of P is contained in exactly $\binom{n-j}{n-i}$ i-faces of P, $i \geq j$.*

3.2 Exercise. *Let $C(n, m)$ be a cyclic polytope with respect to the points $\gamma(t_i)$, $1 \leq i \leq m$.*

i) *Show that $\gamma(t_i)$ is a vertex of $C(n, m)$.*

ii) *Let $\tilde{C}(n, m) = \operatorname{conv}\{\gamma(s_i) : 1 \leq i \leq m\}$ for pairwise distinct s_i. Show that there exists a bijection between the k-faces of $C(n, m)$ and $\tilde{C}(n, m)$.*

3.3 Exercise. *A polytope P is called k-*neighborly, *$k \in \{1, \ldots, n\}$, if the convex hull of each k-subset of $|P$ is a face. Show that.*

i) *If P is a k-neighborly polytope, then each subset of k vertices are affinely independent.*

ii) *Each l face of a k-neighborly polytope is a simplex for $l \leq k-1$.*

3.4 Exercise. *Let P be a k-neighborly n-polytop with $k \geq \lfloor n/2 \rfloor + 1$. Then $f_0(P) = n+1$, i.e., P is an n-simplex. (*Hint: Radon helps...*)*

4.1 Exercise. *Let P be a 4-dimensional cyclic polytope with seven vertices.*

i) *Determine a Gale transform of P.*

ii) *Determine the face lattice of P.*

4.2 Exercise. *An n-polytop $P = \operatorname{conv}\{v_1, \dots, v_m\}$ with vertices v_i, $1 \leq i \leq m$, is simplicial if and only if the Gale transform $\{b_1, \dots, b_m\}$, $b_i \in \mathbb{R}^{m-(n+1)}$ satisfies that $0 \notin \operatorname{relint} \operatorname{conv}(\{b_1, \dots, b_m\} \cap H(a, 0))$ for any hyperplane $H(a, 0) \subset \mathbb{R}^{m-(n+1)}$ (containing 0).*

4.3 Exercise. *Determine the number of combinatorial different types of simplicial n-polytopes with $n+2$ vertices.*

4.4 Exercise. *Let $W = \{w_1, \dots, w_m\} \subset \mathbb{R}^n$ be a finite point set.*

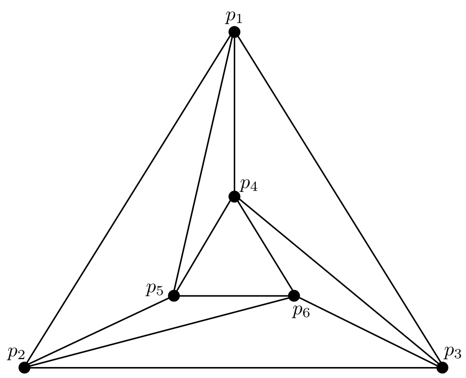
$$\operatorname{vor}(w_i) = \{x \in \mathbb{R}^n : |w_i - x| \leq |w_j - x| \text{ fr alle } 1 \leq j \leq m\}$$

is called the (Dirichlet)-Voronoi-cell *of W at w_i, and $\cup_{i=1}^m \operatorname{vor}(w_i)$ is called the* Voronoi-diagram. *($|\cdot|$ is the Euclidean norm.)*

i) *Show that the Voronoi-cells are convex n-dimensional polyeder.*

ii) *Let v be a vertex of $\operatorname{vor}(w_i)$. Show that there exists a ball with center v which does not contain in its interior any point of W, but at least $n+1$ points of W are on the boundary.*

4.5 Exercise. *Show that the following triangulation is not regular. (Hint: We may assume that the possible heights of p_4, p_5, p_6 are all equal to 0 (why?). What can then be said about the heights of p_1, p_2, p_3.)*



4.6 Exercise. *Let $a_i \in \mathbb{Z}^n$, $1 \leq i \leq n$, be linearly independent, and let $P = \{\sum_{i=1}^n \rho_i \, a_i : 0 \leq \rho_i < 1, \, 1 \leq i \leq n\}$. Prove or disprove that $\#((t+P) \cap \mathbb{Z}^n) = \#(P \cap \mathbb{Z}^n)$ for all $t \in \mathbb{R}^n$.*

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