

# Combinatorial Number Theory

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based on lecture notes of Terence Tao  
and the book by  
Terence Tao and Van H. Vu *Additive Combinatorics*

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## 1 Basics

**1.1 Notation.** Let  $A, B$  subsets of an Abelian group  $Z$ .

- i)  $A + B = \{a + b : a \in A, b \in B\}$ .
- ii)  $A - B = \{a - b : a \in A, b \in B\}$ .
- iii)  $kA = \underbrace{A + A + \dots + A}_{k \text{ times}}, k \in \mathbb{Z}_{\geq 1}$ .
- iv)  $t + B = \{t\} + B$  for  $t \in Z$ .

**1.2 Proposition.** Let  $A \subset \{1, \dots, N\}$  such that  $a + a' \notin A$  for all  $a, a' \in A$ , i.e.,  $A$  is a sumfree subset of  $\{1, \dots, N\}$ . Then  $|A| \leq \lfloor \frac{N+1}{2} \rfloor$ .

**1.3 Proposition.** Let  $A, B \subset \mathbb{Z}$  be finite. Then  $|A + B| \leq |A| |B|$ .

**1.4 Proposition.** Let  $A, B \subset \mathbb{Z}$  be finite. Then  $|A + B| \geq |A| + |B| - 1$ .

**1.5 Definition [Torsion-free].** An Abelian group is called torsion-free if for all  $x \in Z \setminus \{0\}$  and all  $m \in \mathbb{Z}_{\geq 1}$  it holds  $mx \neq 0$ .

**1.6 Remark.** We will need the fact that each finitely generated torsion-free group is isomorphic to a lattice.

**1.7 Definition [Freiman isomorphism].** Let  $A \subseteq Z, A' \subseteq Z'$  subsets of Abelian group  $Z, Z'$ , and let  $k \geq 2$ . A Freiman isomorphism of order  $k$  is a bijective map  $\Phi : A \rightarrow A'$  such that for all  $a_1, \dots, a_k \in A$  and  $a'_1, \dots, a'_k \in A'$

$$\sum_{i=1}^k \Phi(a_i) = \sum_{i=1}^k \Phi(a'_i) \iff \sum_{i=1}^k a_i = \sum_{i=1}^k a'_i.$$

**1.8 Remark.**

- i) If  $\Phi$  is a Freiman isomorphism of order  $k$  then also of order  $k' \leq k$ .
- ii) The composition of Freiman isomorphisms is again a Freiman isomorphism.
- iii) Let  $\Phi : A \rightarrow A'$  be a Freiman isomorphism, and let  $U_1, U_2 \subset A$ . Then  $|U_1 + U_2| = |\Phi(U_1) + \Phi(U_2)|$ .

**1.9 Lemma.** Let  $A \subseteq Z$  be a finite subset of an Abelian torsion free group. For every  $k \geq 2$  exists a Freiman isomorphism  $\Phi : A \rightarrow \Phi(A) \subseteq \mathbb{Z}$ .

**1.10 Theorem.** Let  $A, B \subseteq Z$  be finite subsets of an Abelian torsion free group  $Z$ . Then  $|A + B| \geq |A| + |B| - 1$ .

**1.11 Remark.** Let  $A, B \subseteq \mathbb{Z}/N\mathbb{Z}$ . Then  $|A + B| \geq \max\{|A|, |B|\}$  which is best possible as a (non-trivial) subgroup shows.

**1.12 Theorem [Cauchy-Davenport].** Let  $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$  for a prime  $p$ . Then  $|A + B| \geq \max\{|A| + |B| - 1, p\}$ .

**1.13 Lemma.** Let  $A, B \subseteq Z$  be finite subsets of an Abelian group  $Z$ . Then the following statements are equivalent:

- i)  $|A + B| = |A|$ .
- ii)  $|A - B| = |A|$ .
- iii)  $|A + mB - nB| = |A|$  for all  $m, n \in \mathbb{Z}_{\geq 0}$ ,  $(m, n) \neq (0, 0)$ .
- iv)  $|A + mB - nB| = |A|$  for a pair  $m, n \in \mathbb{Z}_{\geq 0}$  with  $(m, n) \neq (0, 0)$ .
- v) There exists a subgroup  $U \leq G$  such that  $B$  is a subset of a coset w.r.t.  $U$  and  $A$  is the union of cosets w.r.t.  $U$ .

**1.14 Lemma.** Let  $A, B, C \subseteq Z$  be finite subsets of an Abelian group  $Z$ . Then

$$|A - B| \leq \frac{|A + C||B + C|}{|C|}.$$

**1.15 Corollary.** Let  $A, B, A', B' \subseteq Z$  be finite subsets of an Abelian group  $Z$ .

- i) If  $|A + B| = |A|$  and  $|A' + B| = |A'|$ . Then  $|A \pm A'| \leq |A||A'|/|B|$ .
- ii) If  $|A + B| = |A| = |A + B'|$  then  $|B \pm B'| \leq |A|$ .

## 2 Plünnecke's Theorem

**2.1 Theorem [Plünnecke].** *Let  $Z$  be an Abelian group, let  $A, B \subseteq Z$  be finite, and let  $K \geq 1$  such that  $|A + B| \leq K|A|$ . Then there exists a  $A' \subseteq A$ ,  $A' \neq \emptyset$ , such that*

$$|A' + B + B| \leq K^2 |A'|$$

**2.2 Definition [Plünnecke Graph].** *Let  $Z$  be an Abelian group. A graph  $G = (V_0, V_1, V_2, E_{0 \rightarrow 1}, E_{1 \rightarrow 2})$  is called an Plünnecke graph (of depth 2) with vertices  $V_0 \cup V_1 \cup V_2 \subset Z$  and directed edges  $E_{0 \rightarrow 1} \cup E_{1 \rightarrow 2}$  if every edge  $E_{0 \rightarrow 1}$  starts in  $V_0$  and ends in  $V_1$ , every edge  $E_{1 \rightarrow 2}$  starts in  $V_1$  and ends in  $V_2$  and the following Plünnecke property is fulfilled:*

$$\begin{aligned} & \text{if } a \rightarrow a + b \in E_{0 \rightarrow 1} \text{ and } a + b \rightarrow a + b + c \in E_{1 \rightarrow 2} \\ & \text{then } a \rightarrow a + c \in E_{0 \rightarrow 1} \text{ and } a + c \rightarrow a + b + c \in E_{1 \rightarrow 2}. \end{aligned} \quad (\text{Pp})$$

**2.3 Remark.** *Let  $G = (V_0, V_1, V_2, E_{0 \rightarrow 1}, E_{1 \rightarrow 2})$  be Plünnecke graph.*

- i) *Each  $a \rightarrow a + b \in E_{0 \rightarrow 1}$  induced an injection between  $\{a + b \rightarrow a + b + c \in E_{1 \rightarrow 2}\}$  (all edges of  $E_{1 \rightarrow 2}$  starting in  $a + b$  and  $\{a \rightarrow a + c \in E_{0 \rightarrow 1}\}$  (all edges of  $E_{0 \rightarrow 1}$  emanating from  $a$ ).*
- ii) *Each  $d \rightarrow c + d \in E_{1 \rightarrow 2}$  induced an injection between  $\{d - b \rightarrow d \in E_{0 \rightarrow 1}\}$  (all edges of  $E_{0 \rightarrow 1}$  ending in  $d$  and  $\{-b + c + d \rightarrow c + d \in E_{0 \rightarrow 1}\}$  (all edges of  $E_{1 \rightarrow 2}$  ending in  $c + d$ ).*
- iii) *We may always assume that the sets  $V_i$  are pairwise disjoint.*

**2.4 Definition.** *Let  $G = (V, E)$  be a graph. For  $V' \subseteq V$  let  $G(V') = \{v \in V : \exists v' \in V' \text{ with } (v, v') \in E\}$  be all nodes which can be reached via a path of length 1 starting in  $V'$ . Recursively,  $G^k(V')$  be all nodes which can be reached via a path of length  $k$  starting in  $V'$ .*

**2.5 Remark.** *With respect to finite subsets  $A, B \subseteq Z$  of an Abelian group  $Z$  we consider the following Plünnecke graph  $G(A, B) = (V_0, V_1, V_2, E_{0 \rightarrow 1}, E_{1 \rightarrow 2})$ , where  $V_0 = A$ ,  $V_1 = A + B$ ,  $V_2 = A + B + B$  and  $E_{0 \rightarrow 1} = \{a \rightarrow a + b : a \in A, b \in B\}$  and  $E_{1 \rightarrow 2} = \{a + b \rightarrow a + b + c : a \in A, b, c \in B\}$ . In terms of the graph  $G(A, B)$ , Theorem 2.1 says:*

*Suppose  $|V_1| \leq K |V_0|$  then there exists a nonempty subset  $A' \subseteq V_0$  such that*

$$G^2(A') \leq K^2 |A'|.$$

**2.6 Lemma [Menger's theorem].** *Let  $G = (V, E)$  be a directed graph. For subsets  $V_0, V_1 \subset V$  let  $\text{maxflow}(V_0, V_1)$  be the maximal number of vertex disjoint paths from  $V_0$  to  $V_1$ , and let  $\text{mincut}(V_0, V_1)$  be the minimal number of vertices which have to be removed from  $V$  such that  $V_0$  and  $V_1$  are disconnected. Then*

$$\text{maxflow}(V_0, V_1) = \text{mincut}(V_0, V_1).$$

First, we prove Remark 2.5 for arbitrary Plünnecke graphs in the case  $K = 1$ . More precisely,

**2.7 Proposition.** *Let  $G = (V_0, V_1, V_2, E_{0 \rightarrow 1}, E_{1 \rightarrow 2})$  be a Plünnecke graph with  $|V_1| < |V_0|$ . Then there exists a nonempty subset  $A' \subseteq V_0$  such that  $G^2(A') < |A'|$ .*

**2.8 Definition [Magnification Ratio].** *Let  $G = (V_0, V_1, V_2, E_{0 \rightarrow 1}, E_{1 \rightarrow 2})$  be a Plünnecke graph with respect to a group  $Z$ .*

i)

$$D(G) = \min_{A' \subseteq V_0, A' \neq \emptyset} \frac{|G^2(A')|}{|A'|}$$

is called magnification ratio of the graph.

ii) *Let  $\bar{G} = (\bar{V}_0, \bar{V}_1, \bar{V}_2, \bar{E}_{0 \rightarrow 1}, \bar{E}_{1 \rightarrow 2})$  be another Plünnecke graph with respect to a group  $\bar{Z}$ . The Cartesian product*

$$G \times \bar{G} = (V_0 \times \bar{V}_0, V_1 \times \bar{V}_1, V_2 \times \bar{V}_2, E_{0 \rightarrow 1} \times \bar{E}_{0 \rightarrow 1}, E_{1 \rightarrow 2} \times \bar{E}_{1 \rightarrow 2})$$

is a Plünnecke graph with respect to  $Z \times \bar{Z}$ . Here the product of two edges  $a \rightarrow b$  and  $\bar{a} \rightarrow \bar{b}$  is just  $(a, \bar{a}) \rightarrow (b, \bar{b})$ .

**2.9 Remark.**

i) *Proposition 2.7 says: if  $|V_1| < |V_0|$  then  $D(G) < 1$ .*

ii) *Let  $Z = \mathbb{Z}^k$ ,  $A = \{0\}$ ,  $B = \{e_1, \dots, e_k\}$ , where  $e_i$  denotes the  $i$ -th unit vector. The Plünnecke graph  $G(A, B)$  is denoted by  $G_k$  and it is*

$$D(G_k) = \frac{k(k+1)}{2}.$$

*For the associated reflected graph  $G_k^\top$  (all edges are reversed) we find  $D(G_k^\top) = \frac{2}{k(k+1)}$ .*

**2.10 Lemma.** *Let  $G, \bar{G}$  be Plünnecke graphs. Then*

$$D(G \times \bar{G}) = D(G) \cdot D(\bar{G}).$$

**2.11 Proposition.** *Let  $G = (V_0, V_1, V_2, E_{0 \rightarrow 1}, E_{1 \rightarrow 2})$  be a Plünnecke graph with  $|V_1| < K|V_0|$ ,  $K \geq 1$ . Then  $D(G) \leq K^2$ , i.e., there exists a nonempty subset  $A' \subseteq V_0$  such that  $G^2(A') \leq K^2|A'|$ .*

Observe, Proposition 2.11 implies Theorem 2.1.

**2.12 Corollary.** *Let  $A, B$  finite subsets of an Abelian group  $Z$  and let  $|A+B| \leq K|A|$  for some  $K \geq 1$ . For any  $n \in \mathbb{N}$  there exists  $A_n \subseteq A$ ,  $A_n \neq \emptyset$ , such that*

$$|A_n + nB| \leq K^{c_n} |A_n|,$$

where  $c_n$  depends only on  $n$  (in fact, it can be shown that  $c_n = n$  works).

**2.13 Corollary [Sumset Estimate].** *Let  $A, B$  finite subsets of an Abelian group  $Z$  and let  $|A + B| \leq K|A|$  for some  $K \geq 1$ . For any  $n, m \in \mathbb{N}$  we have*

$$|nB - mB| \leq K^{c_{n,m}} |A|,$$

where  $c_{n,m}$  depends only on  $n$  and  $m$ .

**2.14 Corollary.** *Let  $A, B$  finite subsets of an Abelian group  $Z$  and let  $|A + B| \leq K|A|$  for some  $K \geq 1$ . Let  $\delta \in (0, 1)$ . Then there exists  $A' \subseteq A$  with  $|A'| \geq (1 - \delta)|A|$  and*

$$|A' + B + B| \leq \frac{2K^2}{\delta} |A| \quad \left( \leq \frac{2K^2}{\delta(1-\delta)} |A'| \right).$$

**2.15 Proposition.** *There exist subsets  $A, B \subset \mathbb{Z}^2$  with  $|A| \sim n^2$ ,  $|B| \sim n$ , such that  $|A + B| \sim n^2$ , but  $|A + B + B| \sim n^3$ .*





### 3 Covering

**3.1 Lemma [Ruzsa's quotient lemma].** *Let  $A, B$  finite subsets of an Abelian group  $Z$ . Then there exists a set  $X \subseteq Z$  with  $|X| \leq \frac{|A+B|}{|A|}$  such that  $B \subseteq X + A - A$ , i.e.,  $B$  is covered by at most  $\frac{|A+B|}{|A|}$  translates of  $A - A$ .*

**3.2 Corollary.** *If  $|A+A| \sim |A|$  then  $nA - mA$  can be covered by  $O(1)$  translates of  $A - A$ .*

**3.3 Lemma.** *Let  $Z$  be a finite Abelian group and let  $A \subseteq Z$ ,  $A \neq \emptyset$ . Then  $G$  can be covered by  $O(\frac{|Z|}{|A|} \log |Z|)$  translates of  $A$ , i.e., there exists  $X \subseteq Z$ ,  $|X| \in O(\frac{|Z|}{|A|} \log |Z|)$  such that  $G = X + A$ .*

**3.4 Lemma [Improved quotient lemma].** *Let  $A, B$  finite subsets of an Abelian group  $Z$ . Then there exists a set  $X \subseteq Z$  with  $|X| \leq 2 \frac{|A+B|}{|A|}$  such that  $B$  is covered by  $X + A - A$  at least  $\frac{|A|}{2}$  times, i.e., for every  $y \in B$  there are at least  $\frac{|A|}{2}$  triplets  $(x, a, a') \in X + A - A$  such that  $y = x + a - a'$ .*

**3.5 Theorem.** *Let  $A$  be a finite subset of an Abelian group  $Z$  such that  $|A + A| \sim |A|$ . For any fixed  $n, m \geq 0$  there exists a set  $X_{n,m} \subseteq Z$  with  $|X_{n,m}| \in O(\log |A|)$  such that  $mA - nA \subseteq X_{n,m} + A$ .*

**3.6 Proposition.** *There exist finite subsets  $A, B$  of an Abelian group  $Z$  such that  $|A + B| \sim |A|$ , but  $|A - B| \geq |A|^{2 - \log 6 / \log 7}$ .*

**3.7 Proposition.** *Let  $A, B$  finite subsets of an Abelian group  $Z$  such that  $|A + B| \sim |A|$ . Then for every  $\epsilon > 0$  there exists a subset  $A' \subseteq A$  such that  $|A'| \sim |A|$  and  $|A' - B| \leq C_\epsilon |A|^{1+\epsilon}$ .*

**3.8 Proposition.** *For any integer  $n \geq 1$  there exist finite subsets  $A, B$  of an Abelian group  $Z$  such that  $|A + B| \sim |A| \sim C^n$ ,  $|B| \sim n$ , but  $|A' - B| \geq n|A'|$  for all non-empty subsets of  $A$ .*



## 4 Freiman's theorem

**4.1 Definition.** Let  $A, A'$  be subsets of an Abelian group  $Z$ .

- i)  $A' \subseteq A$  is called a refinement of  $A$  if  $|A'| \sim |A|$ .
- ii)  $A'$  is called a small convolution of  $A$  if there exists an  $X \subseteq Z$  with  $|X| = O(1)$  and  $A' = X + A$ .

**4.2 Remark.** Let  $A$  be a finite subset of an Abelian group  $Z$  which is essentially closed under addition, i.e.,  $|A + A| \sim |A|$ .

- i) If  $A'$  is a refinement of  $A$  then  $|A' + A'| \sim |A'|$ .
- ii) If  $A'$  is a small convolution of  $A$  then  $|A' + A'| \sim |A'|$ .
- iii) If  $\Phi$  is a Freiman Isomorphism of order  $k \geq 2$  then  $|\Phi(A) + \Phi(A)| \sim |\Phi(A)|$ .

**4.3 Theorem [Finite Torsion].** Let  $Z$  be an Abelian group of Torsion  $r < \infty$ . Let  $A \subseteq Z$  with  $|A + A| \sim |A|$ . Then  $A$  is a refinement of a subgroup  $U \leq Z$ .

**4.4 Definition [Generalized arithmetic progression].**

- i) Let  $N = (N_1, \dots, N_d) \in \mathbb{N}^d$  and let  $[0, N] = \{n \in \mathbb{Z}^d : 0 \leq n_i \leq N_i, 1 \leq i \leq d\}$ .  $d$  is called the rank of the box  $[0, N]$ , and  $|[0, N]| = \prod_{i=1}^d (N_i + 1)$  is called the (discrete) volume of the box  $[0, N]$ .
- ii) Let  $Z$  be an Abelian group, let  $\Phi : \mathbb{Z}^d \rightarrow Z$  be an affine homomorphism, and let  $[0, N] \subset \mathbb{Z}^d$  be a box of rank  $d$ . The image  $\Phi([0, N])$  is called a (generalized) arithmetic progression of dimension  $d$ , length  $N$  and volume  $|[0, N]|$ . If  $\Phi|_{[0, N]}$  is injective the progression is called proper.

**4.5 Remark.** With the notation as above let  $\Phi(0) = a$  and  $\Phi(e_i) = v_i$ ,  $1 \leq i \leq d$ , where  $e_i$  are the unit vectors of  $\mathbb{Z}^d$ . Then

$$\Phi([0, N]) = \left\{ a + \sum_{i=1}^d n_i v_i : 0 \leq n_i \leq N_i, 1 \leq i \leq d \right\},$$

and  $a$  is called base point and  $v_i$  are called basis vectors of the arithmetic progression  $\Phi([0, N])$ .

**4.6 Proposition.** Let  $Z$  be an Abelian group,  $\Phi : \mathbb{Z}^d \rightarrow Z$  be a Freiman isomorphism of order (at least) 2, and let  $[0, N] \subset \mathbb{Z}^d$  be a box. Then  $\Phi([0, N])$  is essentially closed under addition, i.e.,  $|\Phi([0, N]) + \Phi([0, N])| \sim |\Phi([0, N])|$ .

**4.7 Theorem [Freiman].** Let  $Z$  be a torsion-free Abelian group, and let  $A \subset Z$  be essentially closed under addition, i.e.,  $|A + A| \sim |A|$ . Then  $A$  is a refinement of a small convolution of a proper generalized arithmetic progression  $P$  of bounded rank, i.e., there exists a set  $X \subset Z$ ,  $|X| = O(1)$ , with  $A \subseteq X + P$ , and  $|A| \sim |X + P| \sim |P|$ .

**4.8 Lemma.** Let  $A$  be a finite subset of a torsion-free Abelian group  $Z$ . Let  $n, N \in \mathbb{N}$  such that  $2|nA - nA| < N$ . Then there exists a subset  $A' \subseteq A$  with  $|A'| \geq |A|/n$  and a Freiman isomorphism  $\Phi : A' \rightarrow B \subset \mathbb{Z}/N\mathbb{Z}$  of order  $n$ .

**4.9 Lemma.** Let  $N \in \mathbb{N}$ ,  $A \subseteq \mathbb{Z}/N\mathbb{Z}$  with  $|A| \sim N$  and  $|A + A| \sim |A|$ . Then the set  $2A - 2A$  contains a generalized proper arithmetic progression  $P$  of bounded rank with  $|P| \sim N$ .

**4.10 Definition.** Let  $Z$  be an Abelian (additive) group. Let  $e : Z \times Z \rightarrow S^1 = \{z \in \mathbb{C} : |z| = 1\}$  be a map such that

- i)  $e(x + x', y) = e(x, y)e(x', y)$  and  $e(x, y + y') = e(x, y)e(x, y')$
- ii) for all  $x$  (or  $y$ )  $\in Z \setminus \{0\}$  exists  $y$  (or  $x$ )  $\in Z \setminus \{0\}$  such that  $e(x, y) \neq 1$ .

$e$  is called a bi-character.

**4.11 Remark.**

- i)  $e(0, y) = e(x, 0) = 1$ ,  $e(x, -y) = e(-x, y) = \overline{e(x, y)}$ .
- ii) For  $\mathbb{Z}/N\mathbb{Z}$  the function  $e(x, y) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \rightarrow S^1$  with  $e(x, y) = e^{2\pi i xy/N}$  is a bi-character.
- iii) Let  $e_1, e_2$  be bi-characters with respect to groups  $Z_1, Z_2$ . Then  $e_1 \otimes e_2 : (Z_1 \times Z_2) \times (Z_1 \times Z_2)$  with  $(e_1 \otimes e_2)(x_1 \times x_2, y_1 \times y_2) = e_1(x_1, y_1) \cdot e_2(x_2, y_2)$ ,  $x_1, y_1 \in Z_1$ ,  $x_2, y_2 \in Z_2$ , is a bi-character on  $Z_1 \times Z_2$ .
- iv) Every finite Abelian group has a bi-character.

**4.12 Lemma.** Let  $Z$  be a finite Abelian group, and let  $e : Z \times Z \rightarrow S^1$  be a bi-character. Then

$$\frac{1}{|Z|} \sum_{x \in Z} e(x, y) \overline{e(x, y')} = \delta_{y, y'} \quad \text{and} \quad \frac{1}{|Z|} \sum_{y \in Z} e(x, y) \overline{e(x', y)} = \delta_{x, x'}$$

where  $\delta_{x, y} = 1$  if  $x = y$  and 0 otherwise.

**4.13 Remark.** Let  $\mathbb{C}^Z$  be the set of all complex-valued functions on  $Z$ . For  $f, g \in \mathbb{C}^Z$  let

$$\langle f, g \rangle_Z = \frac{1}{|Z|} \sum_{x \in Z} f(x) \overline{g(x)}.$$

- i)  $\langle \cdot, \cdot \rangle$  is an inner product on  $\mathbb{C}^Z$ .
- ii)  $e(\cdot, y)$ ,  $y \in Z$ , form an orthonormal system.
- iii) Let  $f \in \mathbb{C}^Z$ . Then

$$f = \sum_{y \in Z} \langle f, e(\cdot, y) \rangle_Z e(\cdot, y).$$

**4.14 Definition [Fourier transform].** Let  $f \in \mathbb{C}^Z$ , where  $Z$  is a finite Abelian group, and let  $e : Z \times Z \rightarrow S^1$  be a bi-character. Then  $\widehat{f} : Z \rightarrow \mathbb{C}$  with

$$\widehat{f}(y) = \langle f, e(\cdot, y) \rangle_Z = \frac{1}{|Z|} \sum_{x \in Z} f(x) \overline{e(x, y)}$$

is called the Fourier transform of  $f$ .

**4.15 Theorem.** Let  $Z$  be a finite Abelian group, and let  $e : Z \times Z \rightarrow S^1$  be a bi-character. Let  $f \in \mathbb{C}^Z$ .

- i)  $f(x) = \sum_{y \in Z} \widehat{f}(y) e(x, y)$  (Fourier inversion formula)
- ii)  $\frac{1}{|Z|} \sum_{x \in Z} f(x) \overline{g(x)} = \sum_{y \in Z} \widehat{f}(y) \overline{\widehat{g}(y)}$  (Plancherel formula)
- iii)  $\frac{1}{|Z|} \sum_{x \in Z} |f(x)|^2 = \sum_{y \in Z} |\widehat{f}(y)|^2$  (Parseval relation)

**4.16 Definition [Convolution].** Let  $f, g \in \mathbb{C}^Z$ . Then  $f \star g \in \mathbb{C}^Z$  with

$$f \star g(x) = \frac{1}{|Z|} \sum_{y \in Z} f(y) g(x - y)$$

is called convolution of  $f$  and  $g$ .

**4.17 Lemma.**

- i)  $\widehat{f \star g}(y) = \widehat{f}(y) \widehat{g}(y)$ .
- ii) Let  $\widetilde{f}(x) = \overline{f(-x)}$ . Then  $\widehat{\widetilde{f}}(y) = \overline{\widehat{f}(y)}$ .

**4.18 Proposition.** Let  $A \subseteq Z$  with characteristic function  $\chi_A$ , and let  $c = |A|/|Z|$ . Then

- i)  $\sum_{y \in Z} |\widehat{\chi_A}(y)|^2 = c$ .
- ii)  $|\widehat{\chi_A}(y)| \leq c$ .
- iii) Let  $\epsilon \in (0, 1]$ . Then  $|\{y \in Z : |\widehat{\chi_A}(y)| \geq \epsilon c\}| \leq \epsilon^{-2} \frac{1}{c}$ .
- iv)  $\sum_{y \in Z} |\widehat{\chi_A}(y)|^4 \leq c^3$ .

**4.19 Lemma.** Let  $A \subseteq Z$  with characteristic function  $\chi_A$ , let  $c = |A|/|Z|$  and  $|A + A| \leq K|A|$ . Then

- i)  $\chi_A \star \chi_A(x) \neq 0$  if and only if  $x \in A + A$ .
- ii)  $\sum_{y \in Z} |\widehat{\chi_A}(y)|^4 \geq c^3 K^{-1}$ .

iii) Let

$$\Lambda = \left\{ y \in Z : |\widehat{\chi}_A(y)| \geq \frac{c}{2\sqrt{K}} \right\}.$$

Then  $|\Lambda| \leq 4K c^{-1}$  and

$$\sum_{y \in \Lambda} |\widehat{\chi}_A(y)|^4 \geq \frac{3}{4} \sum_{y \in Z} |\widehat{\chi}_A(y)|^4.$$

**4.20 Remark.** Let  $f(x) = \chi_A * \chi_A * \chi_{-A} * \chi_{-A}$ . Then  $f(x) \neq 0$  if and only if  $x \in 2A - 2A$ , and it holds  $f(y) = |\widehat{\chi}_A(y)|^4$ .

**4.21 Lemma.** Let  $A \subseteq Z$ ,  $|A|/|Z| = c$ , and let  $|A + A| \leq K|A|$ . Let

$$\Lambda = \left\{ y \in Z : |\widehat{\chi}_A(y)| \geq \frac{c}{2\sqrt{K}} \right\} \text{ and}$$

$$X = \left\{ x \in Z : |e(x, y) - 1| < \frac{1}{4} \text{ for all } y \in \Lambda \right\}.$$

Then  $X \subseteq 2A - 2A$ .

**4.22 Theorem [Gowers-Walters].** Let  $Z$  be a torsion-free Abelian group. Then every generalized arithmetic progression in  $Z$  with bounded rank is the refinement of a proper generalized arithmetic progression bounded rank.

**4.23 Corollary [Freiman's theorem without small convolutions].** Let  $Z$  be a torsion-free Abelian group, and let  $A \subset Z$  with  $|A + A| \sim |A|$ . Then  $A$  is a refinement of a proper generalized arithmetic progression  $P$  of bounded rank, i.e.,  $A \subseteq P$  and  $|A| \sim |P|$ .

**4.24 Lemma.** Let  $\Lambda \subset \mathbb{R}^d$  be a lattice (discrete subgroup) containing  $d$  linearly independent vectors  $s_1, \dots, s_d$ . Then there exists  $b_1, \dots, b_d \in \Lambda$  such that for  $1 \leq k \leq d$

$$\text{lin}\{s_1, \dots, s_k\} \cap \Lambda = \{z_1 b_1 + \dots + z_k b_k : z_i \in \mathbb{Z}\} = (b_1, \dots, b_k)\mathbb{Z}^k.$$

**4.25 Corollary.** Let  $\Lambda \subset \mathbb{R}^d$  be a lattice of rank  $k$ . Then there exists  $b_1, \dots, b_k \in \Lambda$  such that  $\Lambda = (b_1, \dots, b_k)\mathbb{Z}^k$ .

**4.26 Definition.** Let  $\Lambda \subset \mathbb{R}^d$  be a lattice.  $v \in \Lambda$  is called irreducible or primitive if  $\lambda v \notin \Lambda$  for all  $\lambda \in (0, 1)$ .

**4.27 Corollary.** Let  $\Lambda \subset \mathbb{R}^d$  be a lattice of rank  $k$  and let  $v \in \Lambda$  be irreducible. Then there exists a lattice  $\Lambda'$  of rank  $k - 1$  such that  $\Lambda = \mathbb{Z}v \oplus \Lambda'$ .

## 5 Partial Sums

**5.1 Lemma.** *Let  $X, Y$  be finite sets and let  $f : X \rightarrow Y$ . Then*

$$|\{(x, x') \in X \times X : f(x) = f(x')\}| \geq |X|^2/|Y|.$$

**5.2 Lemma.** *Let  $X, Y$  be finite sets and let  $f : X \rightarrow Y$ . Let*

$$Y_p = \left\{ y \in Y : |\{x \in X : f(x) = y\}| \geq \frac{1}{2} \frac{|X|}{|Y|} \right\}.$$

*Then*

$$|\{x \in X : f(x) \in Y_p\}| \geq \frac{1}{2}|X|.$$

**5.3 Definition.** *Let  $A, B \subseteq Z$  subsets of a group  $Z$ , and let  $G \subseteq A \times B$ . The set*

$$A \overset{G}{+} B = \{a + b : (a, b) \in G\}$$

*is called partial sum set and*

$$A \overset{G}{-} B = \{a - b : (a, b) \in G\}$$

*partial difference set.*

**5.4 Proposition.** *Let  $A, B \subseteq Z$  with  $|A| \gg |B|$  and  $|A + B| \sim |A|$ . Then there exists a subset  $G \subseteq A \times B$  with  $|G| \sim |A \times B|$  and  $|A \overset{G}{+} B| \sim |A|$ .*

**5.5 Theorem [Balog-Szemerédi-Gowers].** *Let  $A, B \subseteq Z$  be finite sets, and let  $G \subseteq A \times B$  such that*

$$|G| \geq |A||B|/K \text{ and } |A \overset{G}{+} B| \leq K'|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}$$

*for some constants  $K \geq 1, K' > 0$ . Then there exists subsets  $A' \subseteq A, B' \subseteq B$  such that*

$$|A'| \geq \frac{1}{4\sqrt{2}K}|A|, \quad |B'| \geq \frac{1}{4K}|B| \quad \text{and} \quad |A' + B'| \leq 2^{12}K^5(K')^3|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}.$$

**5.6 Lemma.** *Let  $G(A, B, E)$  be a bipartite Graph with edges  $|E| \geq |A||B|/K$  for some  $K \geq 1$ . For any  $\epsilon \in (0, 1)$  there exists a subset  $A' \subseteq A$  such that*

$$|A'| \geq \frac{1}{\sqrt{2}K}|A|,$$

*and such that at least  $(1 - \epsilon)$  of the pairs  $a, a'$  of  $A'$  are connected by at least  $\frac{\epsilon}{2K^2}|B|$  paths of length 2 in  $G(A, B, E)$ .*

**5.7 Corollary.** *Let  $G(A, B, E)$  be a bipartite Graph with edges  $|E| \geq |A||B|/K$  for some  $K \geq 1$ . Then there exists  $A' \subseteq A, B' \subseteq B$  with  $|A'| \geq \frac{1}{4\sqrt{2}K}|A|, |B'| \geq \frac{1}{4K}|B|$  such that all  $(a', b') \in A' \times B'$  are connected by at least  $\frac{1}{2^{12}K^5}|A||B|$  paths of length 3 in  $G(A, B, E)$ .*

**5.8 Theorem\* [Bourgain].** Let  $A, B \subseteq Z$  finite with  $|A|, |B| \leq N$ , and let  $G \subseteq A \times B$  with  $|G| \geq N^2/K$  such that  $|A \overset{G}{+} B| \leq N$ . Then there exist  $A' \subseteq A$ ,  $B' \subseteq B$  with  $|G \cap (A' \times B')| \geq K^{-9}N^2$  and  $|A' - B'| \leq (K^{13}/N)|G \cap (A' \times B')|$ .

**5.9 Definition [Kakeya Problem].**

- i) A Kakeya set in  $\mathbb{R}^n$  is a compact set  $E \subset \mathbb{R}^n$ , such that  $E$  contains for every direction  $v \in \mathbb{R}^n$  a segment of length 1 in this direction, i.e., for every  $v \in \mathbb{R}^n$ ,  $|v|_2 = 1$ , there exists an  $a \in \mathbb{R}^n$  such that  $\{a + tv : t \in [0, 1]\} \subset E$ . It is conjectured that these sets have Hausdorff and Minkowski dimension  $n$ .
- ii) Finite Fields: Let  $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ ,  $p$  prime and large, and let  $n \geq 2$ . For a given  $a \in \mathbb{F}^n$ ,  $v \in \mathbb{F}^n \setminus \{0\}$  the set  $L(a, v) = \{a + tv : t \in \mathbb{F}\}$  is called a line in direction  $v$ .  $E \subseteq \mathbb{F}^n$  is called a Kakeya set in  $\mathbb{F}^n$ , if for every  $v \in \mathbb{F}^n \setminus \{0\}$  there exists a  $a \in \mathbb{F}^n$  such that  $L(a, v) \subset E$ . It is (was) conjectured that  $|E| \geq c(n, \epsilon) |\mathbb{F}|^{n-\epsilon}$  for any  $\epsilon > 0$ .

**5.10 Proposition.** Let  $E \subseteq \mathbb{F}^n$  be a Kakeya set. Then  $|E| \geq \frac{1}{2} |\mathbb{F}|^{\frac{n+1}{2}}$ .

**5.11 Proposition.** Let  $p \geq 3$ , and let  $E \subseteq \mathbb{F}^n$  be a Kakeya set. Then  $|E| \geq \frac{1}{100} |\mathbb{F}|^{\frac{13n+12}{25}}$ .

**5.12 Lemma.** Let  $g \in \mathbb{F}[x_1, \dots, x_n]$  with  $\text{grad}(g) \leq |\mathbb{F}| - 1$  and  $g \not\equiv 0$ . Then there exists a  $y \in \mathbb{F}^n$  with  $g(y) \neq 0$ .

**5.13 Lemma [Schwartz-Zippel].** Let  $g \in \mathbb{F}[x_1, \dots, x_n]$  with  $\text{grad}(g) = d$  and  $g \not\equiv 0$ . Then

$$|\{x \in \mathbb{F}^n : g(x) = 0\}| \leq d |\mathbb{F}|^{n-1}.$$

**5.14 Lemma.** Let  $E \subseteq \mathbb{F}^n$  with  $|E| < \binom{n+d}{d}$ . Then there exists a  $P \in \mathbb{F}[x_1, \dots, x_n]$ ,  $\text{grad}P \leq d$ ,  $P \not\equiv 0$  and  $P(x) = 0$  for all  $x \in E$ .

**5.15 Lemma.** Let  $E \subseteq \mathbb{F}^n$  be a Kakeya set and let  $P \in \mathbb{F}[x_1, \dots, x_n]$  with  $\text{grad}P \leq |\mathbb{F}| - 1$  and  $P(x) = 0$  for all  $x \in E$ . Then  $P \equiv 0$ .

**5.16 Theorem [Divr].** Let  $E \subseteq \mathbb{F}^n$  be a Kakeya set. Then

$$|E| \geq \binom{|\mathbb{F}| + n - 1}{n} = \frac{|\mathbb{F}|^n}{n!} + O(|\mathbb{F}|^{n-1}).$$

**5.17 Definition.**

- i) A graph  $G = (V, E)$  is called complete if  $(v, v') \in E$  for all  $v, v' \in V$ ,  $v \neq v'$ .
- ii) An edge  $k$ -coloring of a graph  $G = (V, E)$  is a partitioning of  $E$  into  $k$  different (color-) classes  $E_1, \dots, E_k$ .



iii) Let  $E_1, \dots, E_k$  be a  $k$ -coloring of  $G = (V, E)$ . A subgraph  $G' = (V', E')$  is called  $E_j$ -monochrom if  $E' \subseteq E_j$ .

**5.18 Definition [Ramsey numbers].** Let  $m, n \geq 1$ . The Ramsey number  $R(m, n)$  is the smallest number such for any 2-coloring  $E_{\text{blue}}, E_{\text{red}}$  of a complete graph with at least  $R(m, n)$  vertices there exist either a complete  $E_{\text{blue}}$ -monochrom subgraph with  $n$  vertices or a complete  $E_{\text{red}}$ -monochrom subgraph with  $m$  vertices.

**5.19 Theorem [Ramsey].**  $R(n, m) \leq \binom{n+m-2}{n-1}$ .

**5.20 Remark.** The bound is sharp for some small values of  $n, m$ . For instance,  $R(3, 3) = 6$ .

**5.21 Corollary.** Let  $m \geq 2$  and let  $n_1, \dots, n_m \in \mathbb{N}_{\geq 1}$ . There exists a number  $R(n_1, \dots, n_m; m)$ , such that for any  $m$ -coloring  $E_1, \dots, E_m$  of a complete graph with at least  $R(n_1, \dots, n_m; m)$  vertices there exists a  $j_* \in \{1, \dots, m\}$  and a  $E_{j_*}$ -monochrom complete subgraph with  $n_{j_*}$  vertices.

**5.22 Theorem [Schur].** Let  $m, k \in \mathbb{N}_{\geq 1}$ . There exists a number  $N = N(m, k)$  such that for a any partition  $[1, N] = A_1 \cup A_2 \cup \dots \cup A_m$  into  $m$  subsets at least one of the  $A_j$  contains a subset of the form

$$\{x_1, \dots, x_k, x_1 + \dots + x_k\}.$$

In fact, we can choose  $N(m, k) = R(k+1, \dots, k+1; m)$  (cf. Corollary 5.21).

**5.23 Corollary.** Let  $n \geq 2$ . There exists a  $s(n) \in \mathbb{N}$  such that for all primes  $p > s(n)$  the congruence

$$x^n + y^n \equiv z^n \pmod{p}$$

has as non-trivial solution ( $p$  is not a divisor of  $xyz$ ).

**5.24 Proposition.** Let  $m, n \in \mathbb{N}_{\geq 1}$  and let  $1 \leq s \leq m$ . There exists an integer  $\bar{d} = \bar{d}(n, m, s) \geq 1$  such that for any partition

$$[0, n-1]^{\bar{d}} = E_1 \cup \dots \cup E_m$$

into  $m$  non-empty subsets  $E_i$  there exist either  $j_* \in \{1, \dots, m\}$ ,  $a \in [0, n-1]^{\bar{d}}$  and  $v \in [0, 1]^{\bar{d}} \setminus \{0\}$  with

$$a + [0, n-1] \cdot v \subseteq E_{j_*},$$

or there exist  $s$  distinct classes  $E_{j_1}, \dots, E_{j_s}$ ,  $a \in [0, n-1]^{\bar{d}}$  and  $v_i \in [0, 1]^{\bar{d}} \setminus \{0\}$ ,  $1 \leq i \leq s$  such that

$$a + [1, n-1] \cdot v_i \subseteq E_{j_i}, \quad 1 \leq i \leq s.$$

**5.25 Theorem [Hales-Jewett].** *Let  $m, n \in \mathbb{N}_{\geq 1}$ . There exists an integer  $\bar{d} = \bar{d}(n, m) \geq 1$  such that for any partition  $[0, n-1]^{\bar{d}} = E_1 \cup \dots \cup E_m$  into  $m$  non-empty subsets  $E_i$  there exist a  $j_* \in \{1, \dots, m\}$ ,  $a \in [0, n-1]^{\bar{d}}$  and  $v \in [0, 1]^{\bar{d}} \setminus \{0\}$  with*

$$a + [0, n-1] \cdot v \subseteq E_{j_*}.$$

**5.26 Theorem [van der Waerden].** *Let  $Z$  be an arbitrary additive group, and let  $m, k \in \mathbb{N}_{\geq 1}$ . There exists a number  $N = N(m, k)$  such that for a any given proper arithmetic progression  $P$  of length  $N$  and any partition  $P = A_1 \cup A_2 \cup \dots \cup A_m$  of  $P$  into  $m$  color classes, at least one of the  $A_j$  contains a monochromatic proper arithmetic sub-progression  $P' \subseteq P$  of length  $k$ .*