Discrete Analytic Convex Geometry

Introduction

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Preface

The material presented here is stolen from different excellent sources:

- First of all: A manuscript of Ulrich Betke on convexity which is partially based on lecture notes given by Peter McMullen.

- The inspiring books by
  - Alexander Barvinok, "A course in Convexity"
  - Günter Ewald, "Combinatorial Convexity and Algebraic Geometry"
  - Peter M. Gruber, "Convex and Discrete Geometry"
  - Peter M. Gruber and Cerrit G. Lekkerkerker, "Geometry of Numbers"
  - Jiri Matousek, "Discrete Geometry"
  - Rolf Schneider, "Convex Geometry: The Brunn-Minkowski Theory"
  - Günter M. Ziegler, "Lectures on polytopes"

- and some original papers

!! and they are part of lecture notes on "Discrete and Convex Geometry" jointly written with Maria Hernandez Cifre but not finished yet.
0 Some basic and convex facts

0.1 Notation. \( \mathbb{R}^n = \{ x = (x_1, \ldots, x_n)^T : x_i \in \mathbb{R} \} \) denotes the \( n \)-dimensional Euclidean space equipped with the Euclidean inner product \( \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \), \( x, y \in \mathbb{R}^n \), and the Euclidean norm \( |x| = \sqrt{\langle x, x \rangle} \).

0.2 Definition [Linear, affine, positive and convex combination]. Let \( m \in \mathbb{N} \) and let \( x_i \in \mathbb{R}^n \), \( \lambda_i \in \mathbb{R} \), \( 1 \leq i \leq m \).

i) \( \sum_{i=1}^{m} \lambda_i x_i \) is called a linear combination of \( x_1, \ldots, x_m \).

ii) If \( \sum_{i=1}^{m} \lambda_i = 1 \) then \( \sum_{i=1}^{m} \lambda_i x_i \) is called an affine combination of \( x_1, \ldots, x_m \).

iii) If \( \lambda_i \geq 0 \) then \( \sum_{i=1}^{m} \lambda_i x_i \) is called a positive combination of \( x_1, \ldots, x_m \).

iv) If \( \lambda_i \geq 0 \) and \( \sum_{i=1}^{m} \lambda_i = 1 \) then \( \sum_{i=1}^{m} \lambda_i x_i \) is called a convex combination of \( x_1, \ldots, x_m \).

v) Let \( X \subseteq \mathbb{R}^n \). \( x \in \mathbb{R}^n \) is called linearly (affinely, positively, convexly) dependent of \( X \), if \( x \) is a linear (affine, positive, convex) combination of finitely many points of \( X \), i.e., there exist \( x_1, \ldots, x_m \in X \), \( m \in \mathbb{N} \), such that \( x \) is a linear (affine, positive, convex) combination of the points \( x_1, \ldots, x_m \).

0.3 Definition [Linearly and affinely independent points]. \( x_1, \ldots, x_m \in \mathbb{R}^n \) are called linearly (affinely) dependent, if one of the \( x_i \) is linearly (affinely) dependent of \( \{x_1, \ldots, x_m\} \setminus \{x_i\} \). Otherwise \( x_1, \ldots, x_m \) are called linearly (affinely) independent.

0.4 Proposition. Let \( x_1, \ldots, x_m \in \mathbb{R}^n \).

i) \( x_1, \ldots, x_m \) are affinely dependent if and only if \( (x_1^T, \ldots, x_m^T) \in \mathbb{R}^{n+1} \) are linearly dependent.

ii) \( x_1, \ldots, x_m \) are affinely dependent if and only if there exist \( \mu_i \in \mathbb{R} \), \( 1 \leq i \leq m \), with \( (\mu_1, \ldots, \mu_m) \neq (0, \ldots, 0) \), \( \sum_{i=1}^{m} \mu_i = 0 \) and \( \sum_{i=1}^{m} \mu_i x_i = 0 \).

iii) If \( m \geq n + 1 \) then \( x_1, \ldots, x_m \) are linearly dependent.

iv) If \( m \geq n + 2 \) then \( x_1, \ldots, x_m \) are affinely dependent.

Proof. By definition we have that \( x_1, \ldots, x_m \) are affinely dependent if and only if there exists an \( x_i \), say, and scalars \( \lambda_j \), \( 1 \leq j \neq i \leq m \), such that \( x_i = \sum_{j \neq i} \lambda_j x_j \) and \( \sum_{j \neq i} \lambda_j = 1 \). This can be reformulated as

\[
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = -\begin{pmatrix} x_i \\ 1 \end{pmatrix} + \sum_{j \neq i} \lambda_j \begin{pmatrix} x_j \\ 1 \end{pmatrix}
\]

which is equivalent to the linear dependency of the vectors \( \begin{pmatrix} x_i \\ 1 \end{pmatrix} \). For ii) we just observe that the equation above is just a reformulation of what is to show if we
set \( \mu_i = -1 \) and \( \mu_j = \lambda_j, \ j \neq i \). Finally, iv) follows from i) and iii), which is trivial.

\[ \square \]

0.5 Definition [Linear subspace, affine subspace, cone and convex set].

\( X \subseteq \mathbb{R}^n \) is called

i) linear subspace (set) if it contains all \( x \in \mathbb{R}^n \) which are linearly dependent of \( X \),

ii) affine subspace (set) if it contains all \( x \in \mathbb{R}^n \) which are affinely dependent of \( X \),

iii) (convex) cone if it contains all \( x \in \mathbb{R}^n \) which are positively dependent of \( X \),

iv) convex set if it contains all \( x \in \mathbb{R}^n \) which are convexly dependent of \( X \).

0.6 Notation. \( C^n = \{ K \subseteq \mathbb{R}^n : K \text{ convex} \} \) denotes the set of all convex sets in \( \mathbb{R}^n \). The empty set \( \emptyset \) is regarded as a convex, linear and affine set.

0.7 Theorem. \( K \subseteq \mathbb{R}^n \) is convex if and only if

\[ \lambda x + (1 - \lambda) y \in K, \quad \text{for all } x, y \in K \text{ and } 0 \leq \lambda \leq 1. \]

Proof. Of course, if \( K \) is convex and \( x, y \in K \) then for any \( \lambda \in [0, 1] \) the point \( \lambda x + (1 - \lambda) y \) is convexly dependent of \( K \) and hence it must belong to \( K \).

Conversely, let \( v \in \mathbb{R}^n \) be convexly dependent of \( K \). Then there exist \( x_1, \ldots, x_m \in K \) and scalars \( \lambda_1, \ldots, \lambda_m \geq 0 \) with \( \sum \lambda_i = 1 \) and \( v = \sum \lambda_i x_i \).

We show by induction on \( m \) that \( v \in K \). The case \( m = 1 \) is trivial, and so let \( m \geq 2 \) and \( \lambda_m < 1 \). Then

\[ v = \sum_{i=1}^{m} \lambda_i x_i = (1 - \lambda_m) \left( \sum_{i=1}^{m-1} \frac{\lambda_i}{1 - \lambda_m} x_i \right) + \lambda_m x_m = (1 - \lambda_m) x + \lambda_m x_m. \]

Since

\[ \sum_{i=1}^{m-1} \frac{\lambda_i}{1 - \lambda_m} = 1 - \sum_{i=1}^{m} \frac{\lambda_i}{1 - \lambda_m} = 1, \]

we get by our inductive argumention \( x \in K \) and thus, by our assumption \( v \in K \).

\[ \square \]

0.8 Example. The closed \( n \)-dimensional ball \( B_n(a, \rho) = \{ x \in \mathbb{R}^n : |x - a| \leq \rho \} \) with centre \( a \) and radius \( \rho > 0 \) is convex. The boundary of \( B_n(a, \rho) \), i.e., \( \{ x \in \mathbb{R}^n : |x - a| = \rho \} \) is non-convex. In the case \( a = 0 \) and \( \rho = 1 \) the ball \( B_n(0, 1) \) is abbreviated by \( B_n \) and is called \( n \)-dimensional unit ball. Its boundary is denoted by \( S^{n-1} \).
0.9 Corollary. Let $K_i \in C^n$, $i \in I$. Then $\bigcap_{i \in I} K_i \in C^n$.

Proof. Let $x, y \in \bigcap_{i \in I} K_i$ and let $\lambda \in [0, 1]$. Since $K_i$ is a convex set for all $i \in I$, we have $\lambda x + (1 - \lambda) y \in K_i$ for all $i \in I$ and hence $\lambda x + (1 - \lambda) y \in \bigcap_{i \in I} K_i$. By Theorem 0.7 we obtain the convexity of $\bigcap_{i \in I} K_i$. □

0.10 Definition [Linear, affine, positive and convex hull, dimension]. Let $X \subseteq \mathbb{R}^n$.

i) The linear hull $\text{lin} X$ of $X$ is defined by
$$\text{lin} X = \bigcap_{L \subseteq \mathbb{R}^n, L \text{ linear}, X \subseteq L} L.$$

ii) The affine hull $\text{aff} X$ of $X$ is defined by
$$\text{aff} X = \bigcap_{A \subseteq \mathbb{R}^n, A \text{ affine}, X \subseteq A} A.$$

iii) The positive (conic) hull $\text{pos} X$ of $X$ is defined by
$$\text{pos} X = \bigcap_{C \subseteq \mathbb{R}^n, C \text{ convex cone}, X \subseteq C} C.$$

iv) The convex hull $\text{conv} X$ of $X$ is defined by
$$\text{conv} X = \bigcap_{K \subseteq \mathbb{R}^n, K \text{ convex}, X \subseteq K} K.$$

v) The dimension $\text{dim} X$ of $X$ is the dimension of its affine hull, i.e., $\text{dim} \text{aff} X$.

0.11 Theorem. Let $X \subseteq \mathbb{R}^n$. Then
$$\text{conv} X = \left\{ \sum_{i=1}^{m} \lambda_i x_i : m \in \mathbb{N}, x_i \in X, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$
where the coefficients of the above combination satisfy \( \lambda \nu_1, (1 - \lambda) \mu_j \geq 0 \) and 
\[
\sum_{i=1}^{m_1} \lambda \nu_i + \sum_{j=1}^{m_2} (1 - \lambda) \mu_j = \lambda + (1 - \lambda) = 1.
\]
Hence \( \lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in M(X) \)
which shows that \( M(X) \) is convex (cf. Theorem 0.7), and so \( \text{conv} X \subseteq M(X) \).

In order to verify the reverse inclusion \( M(X) \subseteq \text{conv} X \), we observe that each \( \mathbf{x} \in M(X) \) is convexly dependent of \( X \), and hence \( \mathbf{x} \) is contained in any convex set containing \( X \), i.e., \( \mathbf{x} \in \bigcap_{K \in C^n, X \subseteq K} K = \text{conv} X \).

\[\square\]

**0.12 Remark.**

i) \( \text{conv} \{ \mathbf{x}, \mathbf{y} \} = \{ \lambda \mathbf{x} + (1 - \lambda) \mathbf{y} : \lambda \in [0, 1] \} \).

ii) \( \text{lin} X = \{ \sum_{i=1}^{m} \lambda_i \mathbf{x}_i : \lambda_i \in \mathbb{R}, \mathbf{x}_i \in X, m \in \mathbb{N} \} \).

iii) \( \text{aff} X = \{ \sum_{i=1}^{m} \lambda_i \mathbf{x}_i : \lambda_i \in \mathbb{R}, \sum_{i=1}^{m} \lambda_i = 1, \mathbf{x}_i \in X, m \in \mathbb{N} \} \).

iv) \( \text{pos} X = \{ \sum_{i=1}^{m} \lambda_i \mathbf{x}_i : \lambda_i \in \mathbb{R}, \lambda_i \geq 0, \mathbf{x}_i \in X, m \in \mathbb{N} \} \).

**0.13 Definition [(Relative) interior point and (relative) boundary point].**

Let \( X \subseteq \mathbb{R}^n \).

i) \( \mathbf{x} \in X \) is called an interior point of \( X \) if there exists a \( \rho > 0 \) such that 
\( B_n(\mathbf{x}, \rho) \subseteq X \). The set of all interior points of \( X \) is called the interior of \( X \) and is denoted by \( \text{int} X \).

ii) \( \mathbf{x} \in \mathbb{R}^n \) is called boundary point of \( X \) if for all \( \rho > 0 \), \( B_n(\mathbf{x}, \rho) \cap X \neq \emptyset \) and \( B_n(\mathbf{x}, \rho) \cap (\mathbb{R}^n \setminus X) \neq \emptyset \). The set of all boundary points of \( X \) is called the boundary of \( X \) and is denoted by \( \text{bd} X \).

iii) Let \( A = \text{aff} X \). \( \mathbf{x} \in X \) is called a relative interior point of \( X \) if there exists a \( \rho > 0 \) such that \( B_n(\mathbf{x}, \rho) \cap A \subseteq X \). The set of all relative interior points is called the relative interior of \( X \) and is denoted by \( \text{relint} X \).

iv) Let \( A = \text{aff} X \). \( \mathbf{x} \in A \) is called a relative boundary point of \( X \) if for all \( \rho > 0 \), \( B_n(\mathbf{x}, \rho) \cap X \neq \emptyset \) and \( B_n(\mathbf{x}, \rho) \cap (A \setminus X) \neq \emptyset \). The set of all relative boundary points of \( X \) is called relative boundary of \( X \) and is denoted by \( \text{relbd} X \).

**0.14 Remark.** Let \( X \subseteq \mathbb{R}^n \) be closed. Then \( X = \text{relint} X \cup \text{relbd} X \).

**0.15 Theorem.** Let \( K \in C^n, \mathbf{x} \in \text{relint} K \) and \( \mathbf{y} \in K \). Then \( -(1 - \lambda)\mathbf{x} + \lambda \mathbf{y} \in \text{relint} K \) for all \( \lambda \in [0, 1) \).

**Proof.** Let \( A = \text{aff} K \), \( \mathbf{x} \in \text{relint} K \), and for \( \lambda \in [0, 1) \) let \( \mathbf{z}_\lambda = (1 - \lambda)\mathbf{x} + \lambda \mathbf{y} \).

Since \( \mathbf{x} = \mathbf{z}_0 \in \text{relint} K \) there exists a \( \rho > 0 \) such that \( B_n(\mathbf{x}, \rho) \cap A \subseteq K \). By the theorem on intersecting lines it follows immediately that \( B_n(\mathbf{z}_\lambda, (1 - \lambda)\rho) \cap A \subseteq K \), and hence \( \mathbf{z}_\lambda \in \text{relint} K \). Or more explicitly: For \( \mathbf{a} \in B_n(\mathbf{z}_\lambda, (1 - \lambda)\rho) \cap A \) we have

\[
(1 - \lambda)\rho \geq |\mathbf{z}_\lambda - \mathbf{a}| = |(1 - \lambda)\mathbf{x} + \lambda \mathbf{y} - \mathbf{a}| = |(1 - \lambda)\mathbf{x} - (\mathbf{a} - \lambda \mathbf{y})|.
\]
Since $\lambda < 1$ we may divide both sides by $1 - \lambda$ and get $|x - (a - \lambda y)/(1 - \lambda)| \leq \rho$. Thus $(a - \lambda y)/(1 - \lambda) \in B_n(x, \rho) \cap A \subseteq K$ and by the convexity of $K$ we finally find

$$a = (1 - \lambda)\frac{a - \lambda y}{1 - \lambda} + \lambda y \in K.$$ 

\[\square\]

**0.16 Corollary.** Let $K \in C^n$ be closed. Let $x \in \text{relint } K$ and $y \in \text{aff } K \setminus K$. Then the segment $\text{conv } \{x, y\}$ intersects $\text{relbd } K$ in precisely one point.

**Proof.** $K \cap \text{conv } \{x, y\}$ is a convex, compact 1-dimensional set. Hence $K \cap \text{conv } \{x, y\} = \text{conv } \{x, \overline{y}\}$ for some $\overline{y} \in K$. Obviously, $\overline{y} \notin \text{relint } K$ and so $\overline{y} \in \text{relbd } K$. By Theorem 0.15, $\overline{y}$ is the only point of $\text{conv } \{x, \overline{y}\}$ lying on $\text{relbd } K$. \[\square\]

**0.17 Definition [Polytope and simplex].** Let $X \subset \mathbb{R}^n$ of finite cardinality, i.e., $\#X < \infty$.

1. $\text{conv } X$ is called a (convex) polytope.
2. A polytope $P \subset \mathbb{R}^n$ of dimension $k$ is called a $k$-polytope.
3. If $X$ is affinely independent and $\dim X = k$ then $\text{conv } X$ is called a $k$-simplex.

**0.18 Notation.** $\mathcal{P}_n = \{P \subset \mathbb{R}^n : P \text{ polytope}\}$ denotes the set of all polytopes in $\mathbb{R}^n$.

**0.19 Lemma.** Let $T = \text{conv } \{x_1, \ldots, x_{k+1}\} \subset \mathbb{R}^n$ be a $k$-simplex, and let $\lambda_i > 0$, $1 \leq i \leq k + 1$, with $\sum \lambda_i = 1$. Then $\sum \lambda_i x_i \in \text{relint } T$.

**Proof.** See Exercise ??.

\[\square\]

**0.20 Corollary.** Let $K \in C^n$, $K \neq \emptyset$. Then $\text{relint } K \neq \emptyset$.

**Proof.** Let $k = \dim K \geq 0$. Then there exist $x_1, \ldots, x_{k+1} \in K$ affinely independent such that $\text{aff } K = \text{aff } \{x_1, \ldots, x_{k+1}\}$. Let $T_k = \text{conv } \{x_1, \ldots, x_{k+1}\} \subseteq K$. From Lemma 0.19 we get $\text{relint } T_k \neq \emptyset$, and hence $\text{relint } K \neq \emptyset$. \[\square\]

**0.21 Theorem.** Let $P = \text{conv } \{x_1, \ldots, x_m\} \in \mathcal{P}_n$. A point $x \in \mathbb{R}^n$ belongs to $\text{relint } P$ if and only if $x$ admits a representation as $x = \sum_{i=1}^m \lambda_i x_i$ with $\lambda_i > 0$, $1 \leq i \leq m$, and $\sum_{i=1}^m \lambda_i = 1$.

**Proof.** Let $x \in \text{relint } P$ and let $y = \sum_{i=1}^m (1/m) x_i \in P$. Since $x \in \text{relint } P$ there exists a $z \in P$ such that $x = \lambda z + (1 - \lambda) y$ with $\lambda \in [0, 1)$. Let $z = \sum_{i=1}^m \mu_i x_i$, with $\mu_i \geq 0$, $1 \leq i \leq m$, and $\sum \mu_i = 1$. Then $x = \sum_{i=1}^m (\lambda \mu_i + (1 - \lambda)/m) x_i$, where all the scalars $\lambda \mu_i + (1 - \lambda)/m$ are positive and sum up to 1.

Next we assume that $x$ has a representation as $x = \sum_{i=1}^m \lambda_i x_i$ with $\lambda_i > 0$, $1 \leq i \leq m$, and $\sum \lambda_i = 1$. Let $k = \dim P$ and without loss of generality let
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$x_1, \ldots, x_{k+1}$ be affinely independent. Setting $\lambda = \sum_{i=1}^{k+1} \lambda_i$, Lemma 0.19 shows that

$$y = \sum_{i=1}^{k+1} \frac{\lambda_i}{\lambda} x_i \in \text{relint conv} \{x_1, \ldots, x_{k+1}\} \subseteq \text{relint } P.$$

If $\lambda = 1$ then $k + 1 = m$ and hence $x = y \in \text{relint } P$. If $\lambda < 1$ let $z = 1/(1 - \lambda) \sum_{i=k+1}^{m} \lambda_i x_i \in P$ and with Theorem 0.15 we also find in this case $x = \lambda y + (1 - \lambda) z \in \text{relint } P$. □

0.22 Notation.

i) For two sets $X, Y \subseteq \mathbb{R}^n$ the vectorial addition

$$X + Y = \{x + y : x \in X, y \in Y\}$$

is called the Minkowski $^1$ sum of $X$ and $Y$. If $X$ is just a singleton, i.e., $X = \{x\}$, then we write $x + Y$ instead of $\{x\} + Y$.

ii) For $\lambda \in \mathbb{R}$ and $X \subseteq \mathbb{R}^n$ we denote by $\lambda X$ the set

$$\lambda X = \{\lambda x : x \in X\}.$$

For instance, $B_n(a, \rho) = a + \rho B_n$.

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$^1$Hermann Minkowski, 1864–1909
1 Support and separate

1.1 Notation. Let $a \in \mathbb{R}^n$, $a \neq 0$, and $\alpha \in \mathbb{R}$. The closed halfspaces $H^+(a, \alpha)$, $H^-(a, \alpha) \subset \mathbb{R}^n$ are given by

$$H^+(a, \alpha) = \{ x \in \mathbb{R}^n : \langle a, x \rangle \geq \alpha \}, \quad H^-(a, \alpha) = \{ x \in \mathbb{R}^n : \langle a, x \rangle \leq \alpha \}.$$ 

The hyperplane $H(a, \alpha)$ is defined by

$$H(a, \alpha) = \{ x \in \mathbb{R}^n : \langle a, x \rangle = \alpha \}.$$ 

1.2 Definition [Supporting hyperplane]. Let $X \subset \mathbb{R}^n$. A hyperplane $H(a, \alpha) \subset \mathbb{R}^n$ is called a supporting hyperplane of $X$ if:

i) $H(a, \alpha) \cap X \neq \emptyset$, and ii) $X \subseteq H^-(a, \alpha)$.

$a$ is called outer normal vector of $X$ and if, in addition, $|a| = 1$ then it is called outer unit normal vector of $X$.

1.3 Proposition. Let $X \subset \mathbb{R}^n$ and let $H(a, \alpha)$ be a supporting hyperplane of $X$. Then

$$H(a, \alpha) \cap \text{conv} X = \text{conv} (H(a, \alpha) \cap X).$$

Proof. Let $x \in H(a, \alpha) \cap \text{conv} X$. Since $x \in \text{conv} X$ there exist $x_i \in X$ and $\lambda_i > 0$, $i = 1, \ldots, m$, with $\sum_{i=1}^m \lambda_i = 1$ and $x = \sum_{i=1}^m \lambda_i x_i$. Since $x \in H(a, \alpha)$ we have $\langle a, x \rangle = \alpha$ and from $X \subseteq H^-(a, \alpha)$ we get $\langle a, x_i \rangle \leq \alpha$, $1 \leq i \leq m$. Hence

$$\alpha = \langle a, x \rangle = \sum_{i=1}^m \lambda_i \langle a, x_i \rangle \leq \sum_{i=1}^m \alpha \lambda_i = \alpha.$$ 

Thus $\langle a, x_i \rangle = \alpha$, $1 \leq i \leq m$, and so $x_i \in H(a, \alpha) \cap X$ which implies $x \in \text{conv} (H(a, \alpha) \cap X)$.

The reverse inclusion is trivial since $\text{conv} (H(a, \alpha) \cap X) \subseteq \text{conv} (H(a, \alpha)) \cap \text{conv} X = H(a, \alpha) \cap \text{conv} X$. \hfill $\square$

1.4 Remark. Let $X \subset \mathbb{R}^n$ be compact and $a \in \mathbb{R}^n \setminus \{0\}$. Then there exists a supporting hyperplane of $X$ with outer normal vector $a$.

1.5 Definition [Nearest point map (or metric projection)]. Let $K \subset \mathbb{C}^n$ be closed. The map $\Phi_K : \mathbb{R}^n \to K$, where for $x \in \mathbb{R}^n$ the point $\Phi_K(x) \in K$ is given by $|x - \Phi_K(x)| = \min \{ |x - y| : y \in K \}$ is called the nearest point map (metric projection) with respect to $K$.

1.6 Remark. The nearest point map is well-defined: Notice that since $K$ is closed, for all $x \in \mathbb{R}^n$ there exist $y_x \in K$ such that $|x - y_x| = \min \{ |x - y| : y \in K \}$, and $y_x$ is uniquely determined. In fact, if there exists $y \in K$, $y \neq y_x$, with $|x - y| = |x - y_x|$ then we may assume that $x - y_x$ and $x - y$ are linearly independent. Hence

$$|x - y_x + y| = \frac{1}{2} |x - y_x| + \frac{1}{2} |x - y| < \frac{1}{2} |x - y_x| + \frac{1}{2} |x - y| = |x - y_x|.$$ 

Since $K$ is convex, $(y_x + y)/2 \in K$ which contradicts the minimality of $y_x$. 

1.7 Theorem. Let $K \subseteq \mathbb{C}^n$ be closed and let $x \in \mathbb{R}^n \setminus K$. Let $a = x - \Phi_K(x)$ and $\alpha = \langle a, \Phi_K(x) \rangle$. Then $H(a, \alpha)$ is a supporting hyperplane of $K$ with outer normal vector $a$.

Proof. By the choice of $a$ and $\alpha$ it $\Phi_K(x) \in K \cap H(a, \alpha)$ and hence it remains to show $K \subseteq H^-(a, \alpha)$, i.e., that $\langle a, y \rangle \leq \alpha$ for all $y \in K$. Suppose the opposite and let $y \in K$ with $\langle a, y \rangle > \alpha$. For $\lambda \in [0, 1]$ we consider the distance of $x$ to $z(\lambda) = (1 - \lambda)\Phi_K(x) + \lambda y \in K$

$$h(\lambda) = |x - z(\lambda)|^2 = |x - \Phi_K(x) + \lambda(\Phi_K(x) - y)|^2$$

$$= |x - \Phi_K(x)|^2 + 2\lambda \langle x - \Phi_K(x), \Phi_K(x) - y \rangle + \lambda^2 |\Phi_K(x) - y|^2$$

Since $\langle a, \Phi_K(x) - y \rangle < 0$ there exists a positive $\lambda^* \in (0, 1]$ with $h(\lambda^*) < h(0)$. This, however, contradicts the fact that $h(0)$ is the minimal (squared) distance between $x$ and $K$. \qed

1.8 Corollary. Let $K \subseteq \mathbb{C}^n$, $K \neq \mathbb{R}^n$, be closed. Then

$$K = \bigcap_{H(a, \alpha) \text{ supporting hyperplane of } K} H^-(a, \alpha),$$

i.e., $K$ is the intersection of all its “supporting halfspaces”.

Proof. Clearly $K \subseteq \bigcap_{H(a, \alpha)} H^-(a, \alpha)$. In order to prove the reverse inclusion we take $x \notin K$, and let $H(a, \alpha)$ be the supporting hyperplane defined in Theorem 1.7. Then $x \in H^+(a, \alpha)$ but $x \notin H(a, \alpha)$, i.e., $x \notin H^-(a, \alpha)$ and hence $x$ is not contained in the intersection on the right hand side. \qed

1.9 Corollary. Let $X \subseteq \mathbb{R}^n$ such that $\text{conv } X$ is closed and $\text{conv } X \neq \mathbb{R}^n$. Then

$$\text{conv } X = \bigcap_{X \subseteq H^-(a, \alpha)} H^-(a, \alpha),$$

i.e., $\text{conv } X$ is the intersection of all halfspaces containing $X$.

Proof. Since each halfspace containing $X$ also contains $\text{conv } X$, we certainly have that $\text{conv } X$ is contained in the intersection of halfspaces above. On the other hand, if $x \notin \text{conv } X$, by Corollary 1.8 there exists a supporting hyperplane $H(a, \alpha)$ of $\text{conv } X$ with $x \notin H^-(a, \alpha)$. Since $X \subseteq \text{conv } X \subseteq H^-(a, \alpha)$, $x$ is also not contained in the intersection of the right hand side. \qed

1.10 Lemma [Busemann-Feller Lemma]. Let $K \subseteq \mathbb{C}^n$ be closed. Then

$$|\Phi_K(x) - \Phi_K(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}^n$, i.e., the nearest point map does not increase distances. In particular, it is a continuous map.

\footnotesize{\textsuperscript{2}Herbert Busemann, 1905–1994 \textsuperscript{3}William Feller, 1906–1970}
Corollary. Let \( a = \Phi_K(x) - \Phi_K(y) \) and \( y \in H^+(\alpha, \beta) \). It suffices to show that \( x \in H^+(a, \alpha) \)
and \( y \in H^+(-a, \beta) \), because then \( \langle a, x \rangle \geq \alpha \) and \( \langle -a, y \rangle \geq \beta \), which implies
\[
\langle a, x - y \rangle \geq \alpha + \beta = \langle a, \Phi_K(x) - \Phi_K(y) \rangle = |\Phi_K(x) - \Phi_K(y)|^2.
\]
By the Cauchy-Schwarz inequality we conclude \( x - y \geq |\Phi_K(x) - \Phi_K(y)| \). So we assume the contrary and without loss of generality let \( x \in H^+(a, \alpha) \), i.e.,
\[
\langle a, x \rangle < \alpha.
\]
Then the ray
\[
R(x) = \{ \Phi_K(x) + \lambda(x - \Phi_K(x)) : \lambda \geq 0 \}
\]
has to intersect the hyperplane \( H(-a, \alpha) \) in a point \( x \), say, and by the Pythagorean theorem we obtain
\[
|\Phi(x) - \Phi(y)| < |\Phi(x) - \Phi_K(x)|.
\]
On the other hand, by Exercise ?? we have \( \Phi_K(z) = \Phi_K(x) \) for all \( z \in R(x) \), and so we get the contradiction \( |\Phi(x) - \Phi_K(x)| = |\Phi(x) - \Phi(x)| > |\Phi(x) - \Phi(x)| \).

1.11 Theorem. Let \( K \in C^n \) be compact and let \( \rho > 0 \) such that \( K \subset \text{int}(\rho B_n) \). The nearest point map on \( \rho S^{n-1} \), i.e., \( \Phi_K : \rho S^{n-1} \to \text{bd} \ K \) is surjective.

Proof. Let \( x \in \text{bd} \ K \). Then \( \Phi_K(x) = x \) and for \( i \in \mathbb{N} \) let \( x_i \in \text{int}(\rho B_n) \) such that \( x_i \notin K \) and \( \lim_{i \to \infty} x_i = x \). By Lemma 1.10 we have
\[
|\Phi_K(x) - \Phi_K(x_i)| = |\Phi(x) - \Phi_K(x)| \leq |x - x_i|.
\]
and so \( \lim_{i \to \infty} \Phi_K(x_i) = x \) as well. By Exercise ??, the intersection point \( z_i \) of the ray \( R(x_i) = \{ \Phi_K(x_i) + \lambda(x_i - \Phi_K(x_i)) : \lambda \geq 0 \} \) with \( \rho S^{n-1} \) verifies \( \Phi_K(z_i) = \Phi_K(x_i) \), and thus \( \lim_{i \to \infty} \Phi_K(z_i) = x \). By the compactness of \( \rho S^{n-1} \) we may assume (after restricting to a convergent subsequence) that \( (z_i) \) is convergent, and so \( \lim_{i \to \infty} z_i = z \in \rho S^{n-1} \). Finally, the continuity of the nearest map point gives
\[
\Phi_K(z) = \lim_{i \to \infty} \Phi_K(z_i) = x.
\]

1.12 Corollary. Let \( K \in C^n \) be closed and let \( x \in \text{relbd} \ K \). Then there exists a supporting hyperplane \( H(a, \alpha) \) of \( K \) with \( x \in H(a, \alpha) \).

Proof. Without loss of generality let \( \dim K = n \). Let \( x \in \text{bd} K \) and let \( \gamma > 0 \) with \( x \in \text{int}(\gamma B_n) \). The convex set \( K = K \cap \gamma B_n \) is compact and \( x \in \text{bd} K \). Let \( \rho > \gamma \) be such that \( K \subset \text{int}(\rho B_n) \). By Theorem 1.11 we can find \( z \in \rho S^{n-1} \) with \( x = \Phi_K(z) \). Then Theorem 1.7 ensures that the hyperplane \( H(a, \alpha) \), with \( a = z - x \) and \( \alpha = \langle a, x \rangle \), supports \( K \) at \( x \). Finally we have to check that \( H(a, \alpha) \) is also a supporting hyperplane of \( K \) at \( x \).

By definition we have \( K \subset H^-(a, \alpha) \) and suppose that there exists \( y \in K \) with \( \langle a, y \rangle > \alpha \). Since \( \langle a, x \rangle = \alpha \) it holds \( \langle a, (1 - \lambda)(x + \lambda y) \rangle > \alpha \) for any \( \lambda \in (0, 1) \). Let \( \lambda > 0 \) be sufficiently small such that \( y = (1 - \lambda)x + \lambda y \in \gamma B_n \). Since \( y \in K \) we have \( y \in K \) with \( \langle a, y \rangle > \alpha \), a contradiction.

\( \square \)
1.13 Theorem [Separation theorem]. Let $K_1, K_2 \in \mathbb{C}^n$ with $K_1 \cap K_2 = \emptyset$. Then there exists a separating hyperplane $H(a, \alpha)$ of $K_1$ and $K_2$, i.e., $K_1 \subseteq H^+(a, \alpha)$ and $K_2 \subseteq H^-(a, \alpha)$.

If $K_1$ is closed and $K_2$ is compact, then there exists even a strictly separating hyperplane $H(a, \alpha)$ of $K_1$ and $K_2$, i.e., $K_1 \subset \text{int} H^+(a, \alpha)$ and $K_2 \subset \text{int} H^-(a, \alpha)$.

Proof. If both, $K_1$ and $K_2$ are compact the statement is certainly true by standard compactness arguments (see also Exercise ??). If $K_1$ is closed and $K_2$ is compact we take the intersection $\overline{K_1} = K_1 \cap \rho B_n$ for $\rho > 0$ sufficiently large such that the distance between $K_2$ and $K_1$ equals the distance between $K_2$ and $\overline{K_1}$. Hence we are back in the case of two compact sets, and we finally have just to observe that a strictly separating hyperplane of $K_2$ and $\overline{K_1}$ is also one for $K_2$ and $K_1$ (see also the proof of Corollary 1.12).

Next we consider the case of arbitrary disjoint convex sets $K_1$ and $K_2$. Let $x_1 \in \text{relint} K_1$ and $x_2 \in \text{relint} K_2$, and for $i \in \mathbb{N}$ let

$$K_j^i = \left[ \text{cl} \left( \left( 1 - \frac{1}{i} \right) (K_j - x_j) \right) + x_j \right] \cap (iB_n), \quad \text{for } j = 1, 2.$$  

Clearly $K_j^i \subset K_j^{i+1} \subset K_j$, for $j = 1, 2$ and any $i \in \mathbb{N}$, and for every $x \in \text{relint} K_j$ there exists an index $i_x$ such that $x \in K_j^{i_x}$ for all $i \geq i_x$. Moreover, $K_1^i$ and $K_2^i$ are compact convex sets with $K_1^i \cap K_2^i = \emptyset$ for any $i \in \mathbb{N}$. By Exercise ?? we know that there exists a separating hyperplane $H(a_i, \alpha_i)$ of $K_1^i$ and $K_2^i$ with $|a_i| = 1$, and thus

$$\langle a_i, x \rangle \leq \alpha_i \text{ for all } x \in K_1^i \quad \text{and} \quad \langle a_i, x \rangle \geq \alpha_i \text{ for all } x \in K_2^i.$$  

Since $\langle a_i, x_1 \rangle \leq \alpha_i \leq \langle a_i, x_2 \rangle$ for $i$ large and $|a_i| = 1$, the sequence $(\alpha_i)_{i \in \mathbb{N}}$ is bounded and hence $\{\alpha_i^{(n)} : i \in \mathbb{N}\} \subset \mathbb{R}^{n+1}$ is also a bounded sequence. Without loss of generality we can assume that this sequence is convergent and we write $a = \lim_{i \to \infty} a_i$ and $\alpha = \lim_{i \to \infty} \alpha_i$. In order to prove that the hyperplane $H(a, \alpha)$ separates $K_1$ and $K_2$, let $x \in K_1$. If $x \in \text{relint} K_1$ then there exists an index $i_x$ such that $\langle a_i, x \rangle \leq \alpha_i$ for all $i \geq i_x$, which implies that $\langle a, x \rangle \leq \alpha$. For $x \in \text{relbd} K_1$, we approach $x$ by the points $x_\lambda = (1 - \lambda)x + \lambda x_1 \in \text{relint} K_1$, $\lambda \in (0, 1]$. By the previous discussion we have $\langle a, x_\lambda \rangle \leq \alpha$ for all $\lambda \in (0, 1]$ and...
so also \((\mathbf{a}, \mathbf{x}_0) = (\mathbf{a}, \mathbf{x}) = \alpha\). Analogously we obtain \((\mathbf{a}, \mathbf{x}) \geq \alpha\) for all \(x \in K_2\). 

\[\square\]

1.14 Definition [Support function, breadth]. Let \(K \in \mathbb{C}^n\), \(K \neq \emptyset\). The function \(h(K, \cdot) : \mathbb{R}^n \to \mathbb{R}\) given by
\[
h(K, \mathbf{u}) = \sup \{ (\mathbf{u}, \mathbf{x}) : \mathbf{x} \in K \}
\]
is called support function of \(K\). For \(\mathbf{u} \in S^{n-1}\) the breadth of \(K\) in the direction \(\mathbf{u}\) is defined by \(h(K, \mathbf{u}) + h(K, -\mathbf{u})\).

1.15 Proposition. Let \(K \in \mathbb{C}^n\) be non-empty and compact. Then
\[
K = \bigcap_{\mathbf{u} \in S^{n-1}} \{ \mathbf{x} \in \mathbb{R}^n : (\mathbf{u}, \mathbf{x}) \leq h(K, \mathbf{u}) \}.
\]

Proof. By Corollary (1.8) it suffices to observe that the intersection is just taken over all supporting hyperplanes of \(K\). In fact, given a supporting hyperplane \(H(\mathbf{a}, \alpha)\) of \(K\) we may assume \(\mathbf{a} \in S^{n-1}\) and since \(K \subseteq H_-(\mathbf{a}, \alpha)\) but \(K \cap H(\mathbf{a}, \alpha) \neq \emptyset\), we have \(\alpha = \max_{\mathbf{x} \in K} (\mathbf{a}, \mathbf{x}) = h(K, \mathbf{a})\). 

\[\square\]

1.16 Definition [Convex function]. Let \(K \in \mathbb{C}^n\). A function \(f : K \to \mathbb{R}\) is called convex if
\[
f(\lambda \mathbf{x} + (1 - \lambda) \mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y}), \quad \text{for all } \mathbf{x}, \mathbf{y} \in K, \lambda \in (0, 1).
\]

\(f\) is called strictly convex when the above inequality holds as a strict inequality if \(\mathbf{x} \neq \mathbf{y}\). If \(-f\) is convex then \(f\) is called concave.

1.17 Proposition. Let \(f : K \to \mathbb{R}\) be a function differentiable on an open convex set \(K \subset \mathbb{R}^n\).

i) \(f\) is convex if and only if
\[
f(\mathbf{x}) \geq f(\mathbf{y}) + \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle, \quad \text{for all } \mathbf{x}, \mathbf{y} \in K. \quad (1.17.1)
\]

ii) Let \(f\) be twice differentiable. Then \(f\) is convex if and only if its Hessian is positive semi-definite for all points in \(K\).

Proof. i) We suppose first that \(f\) is convex. Then for any \(\mathbf{x}, \mathbf{y} \in K\) and any \(\lambda \in (0, 1)\) it holds \(f(\mathbf{y} + \lambda (\mathbf{x} - \mathbf{y})) = f((1 - \lambda) \mathbf{y} + \lambda \mathbf{x}) \leq (1 - \lambda) f(\mathbf{y}) + \lambda f(\mathbf{x})\), which implies
\[
\frac{f(\mathbf{y} + \lambda (\mathbf{x} - \mathbf{y})) - f(\mathbf{y})}{\lambda} - \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \leq f(\mathbf{x}) - f(\mathbf{y}) - \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle.
\]
The left hand side approaches zero when \(\lambda \to 0\), which proves (1.17.1).

Conversely, if (1.17.1) holds then interchanging the roles of \(\mathbf{x}\) and \(\mathbf{y}\) and adding both inequalities we get
\[
\langle \nabla f(\mathbf{x}) - \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \geq 0, \quad \text{for all } \mathbf{x}, \mathbf{y} \in K. \quad (1.17.2)
\]
Let \( x, y \in K \). We define the function \( g : [0, 1] \rightarrow \mathbb{R} \) given by \( g(\lambda) = f(y + \lambda(x - y)) \), which is differentiable. Then for \( 0 \leq \lambda_0 < \lambda_1 \leq 1 \) we get
\[
g'(\lambda_1) - g'(\lambda_0) = \langle \nabla f(y + \lambda_1(x - y)), x - y \rangle - \langle \nabla f(y + \lambda_0(x - y)), x - y \rangle \geq 0
\]
by using (1.17.2). Thus \( g' \) is an increasing function which implies that \( g \) is a convex function, and so \( g(\lambda) = \lambda g(1) + (1 - \lambda) g(0) = \lambda f(x) + (1 - \lambda) f(y) \).

ii) For \( x, y \in K \) we consider the same function \( g_{x,y}(\lambda) \) as in the previous proof. Then \( f \) is convex if and only if all \( g_{x,y} \) are convex for all \( x, y \in K \), which is equivalent to \( g''_{x,y} \geq 0 \). Hence, \( f \) is convex if and only if for all \( x, y \in K \) and \( \lambda \in [0, 1] \)
\[
(x - y)^T [(\text{Hess } f)(x + \lambda(x - y))] (x - y) \geq 0.
\]
This is certainly fulfilled if the Hessian is positive semi-definite. Otherwise, for a given \( z \) in the open set \( K \) and a given vector \( v \in \mathbb{R}^n \) we can find \( x, y \in K \) and \( \lambda \in [0, 1] \) and \( \mu \in \mathbb{R}_{>0} \) such that \( v = \mu(x - y), z = x + \lambda(x - y) \) and so \( v^T (\text{Hess } f)(z) v = \mu^2 (x - y)^T [(\text{Hess } f)(x + \lambda(x - y))] (x - y) \geq 0 \), which shows that the Hessian is positive definite for every \( z \in K \).

1.18 Theorem [Jensen’s inequality].\(^4\) Let \( K \in \mathcal{C}^n \) and let \( f : K \rightarrow \mathbb{R} \) be convex. For all \( x_1, \ldots, x_m \in K \) and \( 0 \leq \lambda_1, \ldots, \lambda_m \) with \( \sum_{i=1}^m \lambda_i = 1 \) it holds
\[
f \left( \sum_{i=1}^m \lambda_i x_i \right) \leq \sum_{i=1}^m \lambda_i f(x_i).
\]

Proof. Induction on \( m \). \( \square \)

1.19 Remark. Let \( f : K \rightarrow \mathbb{R} \) be defined on a convex set \( K \). The set \( \text{epi } f = \{ (x, x_{n+1}) \in \mathbb{R}^{n+1} : x \in K, x_{n+1} \geq f(x) \} \) is called the epigraph of \( f \). Then \( f \) is convex if and only if its epigraph \( \text{epi } f \) is convex.

1.20 Theorem*. Let \( K \in \mathcal{C}^n \) be open and let \( f : K \rightarrow \mathbb{R} \) be convex. Then \( f \) is continuous.

1.21 Theorem. Let \( K \in \mathcal{C}^n \) be bounded and let \( K \neq \emptyset \).

i) \( h(K, \cdot) \) is a convex function.

ii) \( h(K, \cdot) \) is positively homogeneous of degree 1, i.e., \( h(K, \lambda u) = \lambda h(K, u) \) for all \( \lambda \geq 0 \) and \( u \in \mathbb{R}^n \).

iii) If \( h : \mathbb{R}^n \rightarrow \mathbb{R} \) is a function satisfying i) and ii) then there exists a convex closed and bounded set \( K \in \mathcal{C}^n \) such that \( h(K, u) = h(u) \) for all \( u \in \mathbb{R}^n \).

---

\(^4\)Johan Jensen, 1859–1925
Proof. Let \( u, v \in \mathbb{R}^n \) and \( \lambda \in [0, 1] \). Then

\[
h(K, \lambda u + (1 - \lambda)v) = \sup \{ \lambda \langle u, x \rangle + (1 - \lambda) \langle v, x \rangle : x \in K \}
\]

\[
\leq \sup \{ \lambda \langle u, x \rangle : x \in K \} + \sup \{ (1 - \lambda) \langle v, x \rangle : x \in K \}
\]

\[
= \lambda h(K, u) + (1 - \lambda)h(K, v),
\]

which shows i), and the last step also gives ii).

For iii) and the given function \( h : \mathbb{R}^n \to \mathbb{R} \) we consider

\[
K = \bigcap_{v \in \mathbb{R}^n} \{ x \in \mathbb{R}^n : \langle x, v \rangle \leq h(v) \},
\]

which is a closed convex (cf. Corollary 0.9) and bounded (consider \( v = e_i \) set. If it is non-empty then clearly \( h(K, u) \leq h(u) \) for all \( u \in \mathbb{R}^n \), and so it suffices to prove that for every \( u \in \mathbb{R}^n \) there exist an \( a \in K \) with \( \langle u, a \rangle = h(u) \) and thus \( h(K, u) = h(u) \).

Since \( h \) satisfies i) and ii), its epigraph \( \text{epi } h \) is a closed (cf. Theorem 1.20) convex cone in \( \mathbb{R}^{n+1} \). Let \( u \in \mathbb{R}^n \). Since \( (u, h(u)) \in \text{bd epi } h \), we can find by Corollary 1.12 a supporting hyperplane \( H((a,t),\alpha) \) of \( \text{epi } h \) through \( (u, h(u)) \) such that \( \text{epi } h \subseteq H^-((a,t),\alpha) \). Since \( \text{epi } h \) is a cone containing \( 0 \), \( H((a,t),\alpha) \) must pass through \( 0 \), hence \( \alpha = 0 \). If \( t = 0 \) then \( (b, h(b))^T \notin H^-((a,t),0) \) for \( \langle b, a \rangle > 0 \). If \( t > 0 \), then for some \( x_{n+1} \) \( h(u) \) the point \( (u, x_{n+1})^T \) would not be contained in \( H^-((a,t),0) \). Hence \( t < 0 \) and so we may assume \( t = -1 \). Then we can write \( \text{epi } h \subseteq H^-((a,-1),0) \), and since \( \langle v, h(v) \rangle^T \in \text{epi } h \) we have \( \langle a, v \rangle \leq h(v) \) for all \( v \in \mathbb{R}^n \), which shows \( a \in K \). Moreover, by construction we have \( \langle a, u \rangle = h(u) \) and so \( h(K, u) = \langle a, u \rangle = h(u) \). \(\square\)

1.22 Definition [Polar set]. Let \( X \subseteq \mathbb{R}^n \).

\[
X^* = \{ y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in X \}
\]

is called the polar set of \( X \).

1.23 Proposition.

i) \( X^* \) is a convex and closed set and \( 0 \in X^* \).

ii) If \( X_1 \subseteq X_2 \) then \( X_2^* \subseteq X_1^* \).

iii) Let \( M \) be a regular \( n \times n \) matrix. Then \( (MX)^* = M^{-\top}X^* \).

iv) Let \( X_i \subseteq \mathbb{R}^n , i \in I \). Then \( \bigcup_{i \in I} X_i^* = \bigcap_{i \in I} X_i^* \).

v) \( X \subseteq (X^*)^* \).

vi) Let \( X \subseteq \mathbb{R}^n \). Then \( X = X^* \) if and only if \( X = B_n \).
Proof. i) $X^\ast$ is an intersection of closed and convex sets and hence $X^\ast$ is closed and convex (cf. Corollary 0.9). Obviously, $0 \in X^\ast$.

ii) Let $y \in X_2^\ast$. Then $\langle x, y \rangle \leq 1$ for all $x \in X_2$. In particular, $\langle x, y \rangle \leq 1$ for all $x \in X_1$ which proves that $y \in X_1^\ast$.

iii) Let $y \in (MX)^\ast$. By definition, $(Mx, y) \leq 1$ for all $x \in X$, i.e., $\langle x, M^\ast y \rangle \leq 1$ for all $x \in X$. This condition is equivalent to $M^\ast y \in X^\ast$, which says that $y \in M^{-1} X^\ast$.

iv) Just notice that

$$
\left( \bigcup_{i \in I} X_i \right)^\ast = \left\{ y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in \bigcup_{i \in I} X_i \right\}
= \bigcap_{i \in I} \left\{ y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in X_i \right\} = \bigcap_{i \in I} X_i^\ast.
$$

v) Let $x \in X$. By definition of polar body $\langle y, x \rangle \leq 1$ for all $y \in X^\ast$, i.e., $x \in (X^\ast)^\ast$.

vi) We suppose first that $X = B_n$. Then $B_n^\ast = \left\{ y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in B_n \right\}$. If $y \in B_n$, it is clear that $\langle y, x \rangle \leq 1$ for all $x \in B_n$, which shows $B_n \subseteq B_n^\ast$. If $y \in B_n^\ast$ with $|y| > 1$ then $y/|y| \notin B_n$. Since $\langle y/|y|, y \rangle = |y| > 1$ we get $y \notin B_n^\ast$, a contradiction.

Conversely, let $x \in X = X^\ast$. By the definition of the polar body we know $\langle x, x \rangle \leq 1$, which shows $X \subseteq B_n$. Now part ii) implies $X = X^\ast \supseteq B_n^\ast = B_n$, and thus $X = B_n$. □

1.24 Proposition.

i) Let $P = \text{conv} \{x_1, \ldots, x_m\} \subset \mathbb{R}^n$. Then

$$P^\ast = \left\{ y \in \mathbb{R}^n : \langle x_i, y \rangle \leq 1, 1 \leq i \leq m \right\}.$$

ii) Let $P = \left\{ x \in \mathbb{R}^n : \langle a_i, x \rangle \leq 1, 1 \leq i \leq m \right\}$ with $a_i \in \mathbb{R}^n$. Then

$$P^\ast = \text{conv} \{0, a_1, \ldots, a_m\}.$$

Proof. i) $P^\ast \subseteq \left\{ y \in \mathbb{R}^n : \langle x_i, y \rangle \leq 1, 1 \leq i \leq m \right\}$ holds by the definition of polar body. So let $y \in \mathbb{R}^n$ with $\langle x_i, y \rangle \leq 1, 1 \leq i \leq m$. For any $x \in P$ there exist $\lambda_1, \ldots, \lambda_m \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$ such that $x = \sum_{i=1}^m \lambda_i x_i$. Then $\langle x, y \rangle = \sum_{i=1}^m \lambda_i \langle x_i, y \rangle \leq \sum_{i=1}^m \lambda_i = 1$, i.e., $y \in P^\ast$.

ii) Clearly $P^\ast \supseteq \text{conv} \{0, a_1, \ldots, a_m\}$. So suppose there exists $y \in P^\ast$ with $y \notin \text{conv} \{0, a_1, \ldots, a_m\}$. Then $\langle a_i, y \rangle > 0$ for all $a_i$. Since $a_i$ are linearly independent, $\langle a_i, y \rangle > 0$ for all $a_i$. This shows that $y \notin P^\ast$, a contradiction. □

1.25 Lemma. Let $K \subset \mathbb{C}^n$ be closed with $0 \in K$. Then $(K^\ast)^\ast = K$. 

□
Proof. In view of Proposition 1.23, part v), it suffices to show \((K^\ast)^\ast \subseteq K\). We suppose there exists \(y \in (K^\ast)^\ast\) with \(y \notin K\). Then by Theorem 1.13 we can find a strictly separating hyperplane \(H(a, \alpha)\) such that \(\langle a, y \rangle > \alpha\) and \(\langle a, x \rangle < \alpha\) for all \(x \in K\). Since \(\alpha > 0\) we have \(\langle a/\alpha, x \rangle < 1\) for all \(x \in K\) and so \(a/\alpha \in K^\ast\). From \(\langle a/\alpha, y \rangle > 1\) we get the contradiction \(y \notin (K^\ast)^\ast\). \(\Box\)
2 Radon, Helly, Caratheodory and (a few) relatives

2.1 Theorem [Radon]. Let $X \subset \mathbb{R}^n$. If $\#X \geq n + 2$ then there exist $X_1, X_2 \subset X$ with $X_1 \cap X_2 = \emptyset$ and $\text{conv} \, X_1 \cap \text{conv} \, X_2 \neq \emptyset$.

Proof. Since $\#X \geq n + 2$, $X$ contains $n + 2$ affinely dependent points $x_1, \ldots, x_{n+2} \in X$. Thus there exist $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, n + 2$, not all zero, with $\sum_{i=1}^{n+2} \lambda_i = 0$ and $\sum_{i=1}^{n+2} \lambda_i x_i = 0$. Without loss of generality let $\lambda_1, \ldots, \lambda_k > 0$ and $\lambda_{k+1}, \ldots, \lambda_{n+2} \leq 0$. Then $\sum_{i=1}^{k} \lambda_i x_i = \sum_{i=k+1}^{n+2} (\lambda_i) x_i$, and with $\lambda = \sum_{i=1}^{k} \lambda_i = \sum_{i=k+1}^{n+2} (-\lambda_i)$ we may write

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda} x_i = \sum_{i=k+1}^{n+2} \left( -\frac{\lambda_i}{\lambda} \right) x_i \in \text{conv} \, \{x_1, \ldots, x_k\} \cap \text{conv} \, \{x_{k+1}, \ldots, x_{n+2}\}.$$

\[ \square \]

2.2 Theorem [Helly]. Let $K_1, \ldots, K_m \in C^n$, $m \geq n + 1$, such that for each $(n + 1)$-index set $I \subseteq \{1, \ldots, m\}$ we have $\bigcap_{i \in I} K_i \neq \emptyset$. Then all sets $K_i$ have a point in common, i.e., $\bigcap_{j=1}^{m} K_i \neq \emptyset$.

Proof. We use induction on $m$. The case $m = n + 1$ is certainly true by assumption, and for the proof of the induction step $m - 1 \to m$ we set $X_j = \bigcap_{i=1, i \neq j}^{m} K_i$, for $j = 1, \ldots, m$. By induction hypothesis $X_j \neq \emptyset$, and let $x_j \in X_j$, $j = 1, \ldots, m$. Hence $x_j \in K_i$ for all $i \neq j$.

Since $m \geq n + 2$, we may apply Radon’s Theorem 2.1 to $\{x_1, \ldots, x_m\}$ and hence, without loss of generality, we may suppose that there exists $x \in \text{conv} \, \{x_1, \ldots, x_k\} \cap \text{conv} \, \{x_{k+1}, \ldots, x_{m}\}$, for a suitable index $k$. Now $\text{conv} \, \{x_1, \ldots, x_k\} \subseteq K_i$ for $i > k$ and $\text{conv} \, \{x_{k+1}, \ldots, x_m\} \subseteq K_i$ for $i \leq k$, and so $x \in \bigcap_{i=1}^{m} K_i$. \[ \square \]

2.3 Remark.

i) Without any further restrictions/assumptions Helly’s theorem is not true for infinitely many convex sets $K_i$. For instance, let $K_i = (0, \frac{1}{i})$, $i \in \mathbb{N}$.

ii) Helly’s theorem, however, can be easily generalised to infinitely many compact (bounded and closed) convex sets.

2.4 Corollary. Let $C \subset C^n$ be compact. Then there exists $t \in \mathbb{R}^n$ with

$$-C \subseteq t + n \, C.$$

Proof. For $c \in C$ we set $X_c = \{t \in \mathbb{R}^n : -c \in t + n \, C\}$. Clearly, $X_c = -c - n \, C$, and so it is convex and compact. We are looking for a $t \in \bigcap_{c \in C} X_c$. According

\[ ^5 \text{Johann Karl August Radon, 1887–1956} \]

\[ ^6 \text{Eduard Helly, 1884–1943} \]
to Helly’s Theorem 2.2 (cf. also Remark 2.3 ii)) it suffices to show that for \( c_1, \ldots, c_{n+1} \in C \)
\[
\bigcap_{i=1}^{n+1} X_{c_i} \neq \emptyset.
\]
For it we consider \( t = -\sum_{i=1}^{n+1} c_i \). Then for \( j = 1, \ldots, n+1 \),
\[
-c_j - t = \sum_{i=1, i \neq j}^{n+1} c_i = n \left( \sum_{i=1, i \neq j}^{n+1} \frac{1}{n} c_i \right) \in nC,
\]
i.e., \( t \in X_{c_i} \), \( 1 \leq i \leq n+1 \). □

2.5 Definition [Centerpoint]. For a finite point set \( X \subset \mathbb{R}^n \) a point \( c \in \mathbb{R}^n \) is called centerpoint if every closed halfspace containing \( c \) contains at least \( \lfloor \frac{1}{n+1} \#X \rfloor \) points of \( X \).

2.6 Theorem. Every finite set \( X \subset \mathbb{R}^n \) has a centerpoint.

Proof. With
\[
\mathcal{M} = \{ U \subseteq X : \#U > \lfloor (n/(n+1)) \#X \rfloor \},
\]
we first notice that \( c \) is a centerpoint of \( X \) if and only if
\[
c \in \bigcap_{U \in \mathcal{M}} \text{conv } U.
\]
For if, \( c \) is not a centerpoint if and only if there exists a halfspace \( H^- \) containing \( c \) such that \( \#(H^- \cap X) < \lfloor 1/(n+1) \#X \rfloor \). Hence \( c \) is not contained in \( \text{conv } U \) with \( U = (\text{int } H^+ \cap X) \in \mathcal{M} \). On the other hand, if \( c \notin \text{conv } U \) for some \( U \in \mathcal{M} \), then Theorem 1.13 gives a halfspace \( H^- \) containing \( c \) and \( H^- \cap X = X \setminus U \).

Hence we have to show that the intersection \( \bigcap_{U \in \mathcal{M}} \text{conv } U \), consisting of finitely many compact convex sets, is nonempty. Due to the cardinality of the sets \( U \in \mathcal{M} \) the intersection of each \( n+1 \) of these sets is nonempty, and so the intersection of their convex hulls is non-empty. Hence, Helly’s Theorem 2.2 yields \( \bigcap_{U \in \mathcal{M}} \text{conv } U \neq \emptyset \). □

2.7 Theorem [Carathéodory]. ⁷ Let \( X \subset \mathbb{R}^n \). Then
\[
\text{conv } X = \left\{ \sum_{i=1}^{n+1} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1, x_i \in X, i = 1, \ldots, n+1 \right\}.
\]

Proof. Let \( x \in \text{conv } X \). By Theorem 0.11 there exist a minimal \( m \in \mathbb{N} \) and \( x_1, \ldots, x_m \in X \) such that \( x = \sum_{i=1}^{m} \lambda_i x_i \) for certain \( \lambda_i > 0, i = 1, \ldots, m \), with \( \sum_{i=1}^{m} \lambda_i = 1 \).

We have to show \( m \leq n+1 \). So we assume that \( m \geq n+2 \). Then \( \{x_1, \ldots, x_m\} \) are affinely dependent and there exist \( \mu_1, \ldots, \mu_m \in \mathbb{R} \) not all

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⁷ Constantin Carathéodory, 1873 - 1950
of them zero with \(\sum_{i=1}^{m} \mu_i = 0\) and \(\sum_{i=1}^{m} \mu_i x_i = 0\). Hence we can write \(\mathbf{x} = \sum_{i=1}^{m} (\lambda_i - \alpha \mu_i) x_i\) for any \(\alpha \in \mathbb{R}\). Let the index \(k\) be chosen such that

\[
\frac{\lambda_k}{\mu_k} = \min_{1 \leq i \leq m} \left\{ \frac{\lambda_i}{\mu_i} : \mu_i > 0 \right\}.
\]

Then \(\lambda_i - (\lambda_k / \mu_k) \mu_i \geq 0\) for \(i = 1, \ldots, m\), \(\sum_{i=1, i \neq k}^{m} (\lambda_i - (\lambda_k / \mu_k) \mu_i) = 1\) and

\[
\mathbf{x} = \sum_{i=1, i \neq k}^{m} \left( \lambda_i - \frac{\lambda_k}{\mu_k} \mu_i \right) x_i,
\]

which contradicts the minimality of \(m\). The reverse inclusion is certainly true by Theorem 0.11. \(\square\)

2.8 Remark. Let \(X \subset \mathbb{R}^n\). Then

\[
\text{conv } X = \left\{ \sum_{i=1}^{\dim X+1} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{\dim X+1} \lambda_i = 1, x_i \in X \right\}.
\]

As a direct consequence of Carathéodory’s Theorem 2.7 we get

2.9 Corollary. A polytope is the union of simplices.

2.10 Corollary. The convex hull of a compact set is compact.

Proof. Let \(X\) be a compact set and we suppose that \(X \subset B_n(0, \rho)\). Then it is clear that also \(\text{conv } X \subset B_n(0, \rho)\), which shows \(\text{conv } X\) is bounded.

In order to see \(\text{conv } X\) is closed let \(\mathbf{x} = \lim_{i \to \infty} x_i\), with \(x_i \in \text{conv } X\), \(i \in \mathbb{N}\). By Carathéodory’s Theorem 2.7 we can write \(x_i = \sum_{j=1}^{n+1} \lambda_{ij} x_{ij}\), with \(x_{ij} \in X\) and \(\lambda_{ij} \geq 0\), \(\sum_{j=1}^{n+1} \lambda_{ij} = 1\).

Since \(0 \leq \lambda_{ij} \leq 1\) and \(X\) is bounded we may assume that for each \(j \in \{1, \ldots, n+1\}\) the limits \(\lim_{i \to \infty} \lambda_{ij} = \lambda_j\) and \(\lim_{i \to \infty} x_{ij} = x_j\) exist. By the closeness of \(X\) we have \(x_j \in X\), and moreover, \(\lambda_j \geq 0\) and \(\sum_{j=1}^{n+1} \lambda_j = 1\). So we know

\[
\mathbf{x} = \lim_{i \to \infty} \mathbf{x}_i = \lim_{i \to \infty} \sum_{j=1}^{n+1} \lambda_{ij} x_{ij} = \sum_{j=1}^{n+1} \lambda_j x_j,
\]

which proves that \(\mathbf{x} \in \text{conv } X\). \(\square\)

2.11 Theorem*. [(strong) Fractional Helly theorem]. Let \(K_1, \ldots, K_m \in \mathcal{C}^n\), \(m \geq n+1\), and let \(\alpha \in (0, 1]\) such that for at least \(\alpha \binom{m}{n+1}\) of the \((n+1)\)-index sets \(I \subseteq \{1, \ldots, m\}\) we have \(\bigcap_{i \in I} K_i \neq \emptyset\). Then there exists a point in common of at least \((1 - (1 - \alpha)^{1/(n+1)}) \cdot m\) sets \(K_i\).

2.12 Theorem [Colorful Carathéodory theorem]. Let \(X_1, \ldots, X_{n+1} \subset \mathbb{R}^n\) such that \(0 \in \text{conv } X_i\), \(1 \leq i \leq n+1\). There exist \(x_i \in X_i\), \(1 \leq i \leq n+1\), such that \(0 \in \text{conv } \{x_1, \ldots, x_{n+1}\}\).
Proof. It suffices to prove the statement for finite sets $X_i$, because otherwise we may replace $X_i$ be any finite subset $\tilde{X}_i$ containing $0$ in its convex hull. For $x_i \in X_i, 1 \leq i \leq n+1$, we call $\{x_1, \ldots, x_{n+1}\}$ a rainbow set and $\text{conv} \{x_1, \ldots, x_{n+1}\}$ a rainbow simplex. Suppose there is no rainbow simplex containing $0$, and let $S = \{s_1, \ldots, s_{n+1}\}, s_i \in X_i, be a rainbow set such that $\text{conv} S$ has minimal distance to $0$ attained at the point $s \in S$, say. Let $H = \{x \in \mathbb{R}^n : \langle s, x \rangle = \langle s, s \rangle \}$ be the hyperplane perpendicular to $s$ and containing $s$. Then we may assume that $S \subset H^+ = \{x \in \mathbb{R}^n : \langle s, x \rangle \geq \langle s, s \rangle \}$ which does not contain $0$.

Since $\text{conv} S \cap H = \text{conv} (S \cap H)$ (cf. Proposition 1.3) we know by Carathéodory’s Theorem 2.7 that there exists an $n$-point set $T \subset S \cap H$ with $s \in \text{conv} T$. Without loss of generality let $s_1 \notin T$. If $X_1 \subset H^+$ then $0 \notin \text{conv} X_1$, and so we may assume that there exists a $z \in X_1$ with $\langle s, z \rangle < \langle s, s \rangle$. The new rainbow set $S' = S \setminus \{s_1\} \cup \{z\}$ contains the segment $\text{conv} \{s, z\}$, which, by the choice of $z$, contains a point closer to $0$ than $s$ — a contradiction. □

2.13 Theorem* [Tverberg]. 8 Let $X \subseteq \mathbb{R}^n$ and let $k \in \mathbb{N}_{\geq 1}$. If $\# X \geq (k - 1)(n+1)+1, k \in \mathbb{N}$, then there exist $k$ subsets $X_1, \ldots, X_k \subset X$ with $X_i \cap X_j = \emptyset, i \neq j$, but $\text{conv} X_1 \cap \text{conv} X_2 \cap \cdots \cap \text{conv} X_k \neq \emptyset$. 

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8Helge Arnulf Tverberg, 1935–
3 The multifaceted world of polytopes

3.1 Definition [Polyhedron]. The intersection of finitely many closed half-spaces is called a polyhedron.

3.2 Theorem [Minkowski, Weyl].

i) A bounded polyhedron is a polytope.

ii) A polytope is a bounded polyhedron.

Proof. i) Let \( P = \{ x \in \mathbb{R}^n : \langle a_i, x \rangle \leq \alpha_i, 1 \leq i \leq m \} \) be bounded. We proceed by induction on \( n \). The case \( n = 1 \) is obvious. Hence, we may assume \( n \geq 2 \), and let \( F_i = P \cap H(a_i, \alpha_i), 1 \leq i \leq m \). \( F_i \) is a bounded polyhedron in an affine space of dimension at most \( n - 1 \) and by our inductive argument there exist finite sets \( V_i \) such that \( F_i = \text{conv} \ V_i, 1 \leq i \leq m \). It suffices to show that

\[
P = \text{conv} \ (V_1 \cup V_2 \cup \cdots \cup V_m).
\]

The inclusion “\( \supseteq \)" follows from \( V_i \subseteq P \) and the convexity of \( P \). For the reverse inclusion let \( x \in P \) and let \( l \) be a line passing through \( x \). The intersection \( l \cap P \) is a non-empty compact convex set of dimension at most 1. Hence we can find \( y, z \in P \) such that \( l \cap P = \text{conv} \ \{y, z\} \). Since both, \( y \) and \( z \), has to lie in the boundary of \( P \) we can find \( k \) and \( j \) with \( y \in P \cap H(a_k, \alpha_k) = F_k = \text{conv} \ V_k \) and \( z \in P \cap H(a_j, \alpha_j) = F_j = \text{conv} \ V_j \). So we have

\[
x \in \text{conv} \ \{y, z\} \subset \text{conv} \ (V_k \cup V_j) \subset \text{conv} \ (V_1 \cup V_2 \cup \cdots \cup V_m).
\]

For the second statement ii) we apply polarity to i). Let \( P = \text{conv} \ \{v_1, \ldots, v_m\} \), and here we may assume that \( \dim P = n \) and \( 0 \in \text{int} P \). By Propositions 1.23 ii) and 1.24 i) we find that \( P^* \) is a bounded polyhedron. Applying i) to \( P^* \) we can find points \( w_1, \ldots, w_l \in \mathbb{R}^n \) such that \( P^* = \text{conv} \ \{w_1, \ldots, w_l\} \). Next we consider \( (P^*)^* \), which can be written as (cf. Proposition 1.24 i))

\[
(P^*)^* = \{ x \in \mathbb{R}^n : \langle w_i, x \rangle \leq 1, 1 \leq i \leq l \}.
\]

By Lemma 1.25 we know \( P = (P^*)^* \) and we are done.

3.3 Notation [\( \mathcal{V}\)-Polytope, \( \mathcal{H}\)-Polytope]. A polytope given as the convex hull of finitely many points is called a \( \mathcal{V}\)-polytope. If it is given as the bounded intersection of finitely many closed halfspaces, then it is called an \( \mathcal{H}\)-polytope.

3.4 Corollary. Let \( P \in \mathcal{P}^n \).

i) Let \( A \in \mathbb{R}^{m \times n} \) and \( t \in \mathbb{R}^m \). Then \( AP + t \) is a polytope.

ii) Let \( U \subset \mathbb{R}^n \) be an affine subspace. Then \( P \cap U \) is a polytope.

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9Hermann Minkowski, 1864–1909
10Hermann Klaus Hugo Weyl, 1885 – 1955
Moreover,

**3.5 Definition [Faces]**. Let $K \subseteq \mathbb{C}^n$ be closed and let $H$ be a supporting hyperplane of $K$. If $j = \dim(K \cap H)$, then $K \cap H$ is called a $j$-face of $K$. Moreover, $K$ itself is regarded as a $(\dim K)$-face and the empty set $\emptyset$ as $(-1)$-face of $K$.

**3.6 Notation [Vertices, edges, facets]**. A 0-face of $K \subseteq \mathbb{C}^n$. $K$ closed, is called an exposed point or in the case of a polytope a vertex, a 1-face of a polytope is called edge and a $(\dim K - 1)$-face of $K$ is called facet of $K$. $K$ itself and the empty set are called improper faces, whereas the remaining faces are called proper faces of $K$.

The set of all vertices of a polytope $P$ is denoted by $\text{vert} P$.

**3.7 Remark.**

i) Let $K \subseteq \mathbb{C}^n$ be closed. Every (relative) boundary point of $K$ lies in a suitable $j$-face, $0 \leq j \leq \dim K - 1$ (cf. Corollary 1.12).

ii) Let $K \subseteq \mathbb{C}^n$, $\dim K = n$. Let $F$ be a facet of $K$ and $H$ a supporting hyperplane of $K$ with $F = K \cap H$. Then $H = \text{aff } F$.

**3.8 Proposition.** Every face of a polytope is a polytope, and a polytope has only finitely many faces.

**Proof.** Let $P = \text{conv} \{v_1, \ldots, v_m\}$ and let $F = P \cap H(a, \alpha)$ be a face of $F$. By Proposition 1.3 we have $P \cap H(a, \alpha) = \text{conv}(H(a, \alpha) \cap \{v_1, \ldots, v_m\})$ which shows that $F$ is a polytope.

In particular, we can write each face $F$ of $P$ as $\text{conv} V_F$ for a suitable subset $V_F \subseteq \{v_1, \ldots, v_m\}$. Since different faces have different affine hulls and since the affine hull of a $j$-face, $j \in \{0, \ldots, n - 1\}$, is uniquely determined by $j + 1$ affinely independent points we may bound the number of all proper faces by $\sum_{i=0}^{n-1} \binom{m}{i+1}$.

**3.9 Definition [f-vector].** For $P \subseteq \mathbb{P}^n$ let $f_i(P)$ be the number of $i$-faces of $P$, $-1 \leq i \leq \dim P$. Furthermore, let $f_i(P) = 0$ for $\dim P + 1 \leq i \leq n$. The vector $f(P)$ with entries $f_i(P)$, $-1 \leq i \leq n$, is called the f-vector of $P$.

**3.10 Remark.**

i) Let $T_n$ be an $n$-dimensional simplex. Then $f_i(T_n) = \binom{n+1}{i+1}$, i.e., any $(i+1)$-subset of the vertices are the vertices of an $i$-face.
3.11 Lemma. Let $P \in \mathcal{P}^n$.

i) $v \in \text{vert } P$ can not be written as a convex combination of two other points of $P$, i.e., $v \notin \text{conv } (P \setminus \{v\})$.

ii) If $P = \text{conv } W$ then $\text{vert } P \subseteq W$.

iii) $P = \text{conv } (\text{vert } P)$.

Proof. For i) let $H(a, \alpha)$ be a supporting plane of $v$. Then we have $\langle a, v \rangle = \alpha$ and $\langle a, x \rangle < \alpha$ for all $x \in P \setminus \{v\}$. Hence we cannot write $v = \lambda x_1 + (1-\lambda) x_2$ with $x_1 \in P \setminus \{v\}$, $\lambda \in [0,1]$.

Statement ii) is a direct consequence of i).

For iii) let $W \subseteq \mathbb{R}^n$ be a minimal (w.r.t. its cardinality) set with $P = \text{conv } W$. By ii) we know already $\text{vert } P \subseteq W$, and so for $w \in W$ it remains to show that $w$ is a vertex of $P$. By the minimality of $W$ we have $w \notin \text{conv } (W \setminus \{w\})$. By Theorem 1.13 there exists a strong separation hyperplane $H(a, \alpha)$ of $w$ and $\text{conv } (W \setminus \{w\})$, and let $\langle a, w \rangle > \alpha$ and $\langle a, \overline{w} \rangle < \alpha$ for all $\overline{w} \in \text{conv } (W \setminus \{w\})$. With $\alpha^* = \langle a, \overline{w} \rangle$ this implies firstly $P = \text{conv } W \subset H^-(a, \alpha^*)$ and secondly (cf. Proposition (1.3))

$$P \cap H(a, \alpha^*) = \text{conv } (W \cap H(a, \alpha^*)) = \{w\}.$$ 

Hence $w \in \text{vert } P$. \[ \square \]

3.12 Lemma. Let $P \in \mathcal{P}^n$ be an $n$-polytope with $0 \in \text{int } P$. For a proper face $F$ of $P$ let

$$F^\circ = \{ y \in P^* : \langle x, y \rangle = 1 \text{ for all } x \in F \}.$$

Then

i) $F^\circ$ is a face of $P^*$.

ii) $F = (F^\circ)^\circ$.

iii) If $G$ is a face of $P$ and $F \subseteq G$, then $G^\circ \subseteq F^\circ$.

iv) $\dim F^\circ = n - 1 - \dim F$.

Proof. First we fix some notation. Let $\text{vert } P = \{v_1, \ldots, v_m\}$. Then $P = \text{conv } (\text{vert } P)$ (Lemma 3.11) and $P^* = \{ y \in \mathbb{R}^n : \langle v_i, y \rangle \leq 1, 1 \leq i \leq m \}$ (Proposition 1.24). We may assume $\dim F = k$, $F = \text{conv } \{v_1, \ldots, v_l\}$, $l \in \{k+1, \ldots, m\}$, $v_1, \ldots, v_{k+1}$ are affinely independent. Moreover, let $H(a, 1)$ be a supporting hyperplane of $F$, i.e., $F = \{ x \in P : \langle a, x \rangle = 1 \}$ and $\langle a, x \rangle \leq 1$ for all $x \in P$. Hence $a \in P^*$ and, in particular, $a \in F^\circ$ and $\langle a, v_i \rangle < 1$ for $l+1 \leq i \leq m$. 


For i) we observe that we may write $F^o = \{ y \in P^* : \langle v_i, y \rangle = 1, 1 \leq i \leq l \}$. Since $\langle v_i, y \rangle \leq 1$ for all $y \in P^*$ we conclude

$$F^o = \{ y \in P^* : \langle v_1 + v_2 + \cdots + v_{l}, y \rangle = l \}$$

and $P^* \subset \{ y \in \mathbb{R}^n : \langle v_1 + v_2 + \cdots + v_{l}, y \rangle \leq l \}$. Hence $F^o$ is a face of $P^*$ which shows i).

By definition we have $F^o = \{ x \in P : \langle y, x \rangle = 1 \}$ for all $y \in F^o$ and so $v_i \in (F^o)^o$, $1 \leq i \leq l$, which implies $F \subseteq (F^o)^o$. For the reverse inclusion we recall that $a \in F^o$ and so $(F^o)^o \subseteq \{ x \in P : \langle a, x \rangle = 1 \} = F$.

iii) is obvious. For iv) let

$$U = \{ y \in \mathbb{R}^n : \langle v_i, y \rangle = 0, 1 \leq i \leq l \} = \{ y \in \mathbb{R}^n : \langle v_i, y \rangle = 0, 1 \leq i \leq k + 1 \}.$$ 

Then we have $F^o = (a + U) \cap P^*$. Since the vectors $v_1, \ldots, v_{k+1}$ are, in fact, linearly independent, because otherwise 0 $\in \text{aff } F$ contradicting 0 $\in \text{int } P$, we have $\dim U = n - (k+1)$ and so $\dim F^o \leq n - (k+1)$. Now let $z \in U$ arbitrary and for $\mu \in \mathbb{R}$ we consider $a + \mu z$. Then we have

$$\langle v_i, a + \mu z \rangle = \begin{cases} 1, & 1 \leq i \leq l, \\ \langle v_i, a \rangle + \mu \langle v_i, z \rangle, & i > l. \end{cases}$$

Since $\langle v_i, a \rangle < 1$ for $i > l$ we can find and $\varepsilon > 0$ such that $a + \mu z \in P^*$, and thus in $F^o$ for all $\mu < |\varepsilon|$. Since $z \in U$ was arbitrary we conclude $\dim F^o \geq n - (k + 1)$.

3.13 Theorem. Let $P \in \mathcal{P}^n$ be an $n$-polytope with 0 $\in \text{int } P$. Then

$$f_{n-1-i}(P^*) = f_i(P), \quad -1 \leq i \leq n.$$  

Proof. For $i \in \{ -1, n \}$ there is nothing to prove. By Lemma 3.12 we can associate to every $i$-face $F$ injectively a dual $n - (i + 1)$-face $F^o$ of $P^*$. Hence $f_{n-1-i}(P^*) \geq f_i(P)$. Applying the same argument to the faces of $P^*$ yields the assertion.

3.14 Theorem. Let $P \in \mathcal{P}^n$ be an $n$-polytope with facets $F_1, \ldots, F_m$ and let $H(a_i, \alpha_i), 1 \leq i \leq m$, be the supporting hyperplanes of $F_i, 1 \leq i \leq m$. Then

$$P = \bigcap_{i=1}^m H^-(a_i, \alpha_i).$$

Proof. We may assume that 0 $\in \text{int } P$. In view of Theorem 3.13, $P^*$ has $m$ vertices $v_1, \ldots, v_m$, say, and by Lemma 3.11 iii) we have $P^* = \text{conv } \{ v_1, \ldots, v_m \}$. By Lemma 1.25 and Proposition 1.24 we also have

$$P = P^** = \{ x \in \mathbb{R}^n : \langle v_i, x \rangle \leq 1, 1 \leq i \leq n \} = \bigcap_{i=1}^m H^-(v_i, 1),$$

and according to Lemma 3.12, $\{ v_i \}^o = \{ x \in P : \langle v_i, x \rangle = 1 \} = P \cap H(v_i, 1)$ is an $(n - 1)$-face of $P$. Hence, after an appropriate renumbering we must have $H(a_i, \alpha_i) = H(v_i, 1), 1 \leq i \leq m$. 

3.15 Theorem. Let $P \in \mathcal{P}^n$ be an $n$-polytope.

i) The boundary of $P$ is the union of all its facets.

ii) A $k$-face is the intersection of (at least) $(n-k)$ facets.

iii) An $(n-2)$-face is contained in exactly two facets.

iv) If $F, G$ are faces of $P$ with $F \subseteq G$, then $F$ is a face of $G$.

v) A face of $P$ is also a face of a facet of $P$.

Proof. We may assume that $0 \in \text{int} P$. According to Theorem 3.14, each boundary point of $P$ is contained in a supporting hyperplane of a facet, which shows i).

For ii) let $F$ be a $k$-face of $P$, let $P^* = \text{conv} \{v_1, \ldots, v_m\}$ with $v_i$ vertex of $P^*$ and let $F^* = \text{conv} \{v_1, \ldots, v_l\}$. By Lemma 3.12 iv) we have $\dim F^* = n-(k+1)$ and so $l \geq n-k$. Moreover, with Lemma 3.12 ii) we also have

$$F = (F^*)^\circ = \{x \in P : \langle v_i, x \rangle = 1, \ 1 \leq i \leq l\} = \bigcap_{i=1}^{l} \{x \in P : \langle v_i, x \rangle = 1\}.$$

Since $v_i$ is a vertex of $P^*$, $\{v_i\}^\circ = \{x \in P : \langle v_i, x \rangle = 1\}$ is a facet of $P$, which yields ii). In particular, an $(n-2)$-face $F$ of $P$ can be written as

$$F = \bigcap_{i=1}^{r} \{x \in P : \langle v_i, x \rangle = 1\}$$

for some $r \geq 2$. By Lemma 3.12, $F^\circ = \text{conv} \{v_1, \ldots, v_r\}$ is a 1-face of $P^*$. Since $v_i$ are vertices we conclude $r = 2$, because otherwise a vertex could be written as a convex combination of other vertices, contradicting Lemma 3.11 i).

For iv) let $H(a, \alpha)$ be a supporting hyperplane of $F$, i.e., $F = P \cap H(a, \alpha)$. Then $F = G \cap F = G \cap P \cap H(a, \alpha) = G \cap H(a, \alpha)$, which shows that $F$ is a face of $G$.

Finally, we observe that ii) shows that any face is contained in a facet and thus, by iv), it is a face of that facet. \qed

3.16 Theorem. Let $P \in \mathcal{P}^n$ be an $n$-polytope.

i) Let $G$ be a face of $P$ and let $F$ be a face of $G$. Then $F$ is a face of $P$.

ii) Let $F_j$ be a $j$-face of $P$ and let $F_k$ be a $k$-face of $P$ with $F_j \subseteq F_k$. There exist $i$-faces $F_i$ of $P$, $j < i < k$, such that

$$F_j \subseteq F_{j+1} \subseteq \cdots \subseteq F_{k-1} \subseteq F_k.$$
Proof. For i) let $F$ and $G$ be proper faces and let $0 \in F \subset G$. Let $H(a_F, 0)$ and $H(a_G, 0)$ be supporting hyperplanes of $F$ and $G$, respectively, i.e., $F = H(a_F, 0) \cap G$, $G = H(a_G, 0) \cap P$ and $G \subset H(a_F, 0)^-$, $P \subset H(a_G, 0)^-$. For $\mu \geq 0$ we consider the Hyperplane $H(a_F + \mu a_G, 0)$ and observe that

$$\langle a_F + \mu a_G, x \rangle = \langle a_F, x \rangle + \mu \langle a_G, x \rangle$$

Since for all $x \in P \setminus G$ we have $\langle a_G, x \rangle < 0$ there exists a $\bar{\mu} > 0$ such that for all $v \in (\text{vert } P) \setminus G$ we have $\langle a_F + \bar{\mu} a_G, v \rangle < 0$. Hence for $x \in P$ which we can write as a convex combination $x = \sum_{v \in (\text{vert } P) \setminus G} \lambda_v v + \sum_{v \in ((\text{vert } P) \cap G)} \lambda_v v + \sum_{v \in (\text{vert } P) \cap F} \lambda_v v$ we find

$$\langle a_F + \bar{\mu} a_G, x \rangle = \sum_{v \in (\text{vert } P) \setminus G} \lambda_v \langle a_F + \bar{\mu} a_G, v \rangle + \sum_{v \in ((\text{vert } P) \cap G)} \lambda_v \langle a_F + \bar{\mu} a_G, v \rangle + \sum_{v \in (\text{vert } P) \cap F} \lambda_v \langle a_F + \bar{\mu} a_G, v \rangle \leq 0,$$

with equality if and only if $x \in F$. Hence $F$ is a face of $P$.

In order to verify ii) we may assume $j \leq k - 2$ and we first note that by Theorem 3.15 iv) $F_j$ is a face of $F_k$. According to Theorem 3.15 v) $F_j$ is contained in a facet $G$, say, of $F_k$, i.e., $G$ is a $(k - 1)$ face of $F_k$ and hence it is a face of $P$ by i). With $F_{k-1} = G$ we have shown $F_j \subset F_{k-1} \subset F_k$. In the case $j < k - 2$ we can apply the same reasoning as above to the pair $F_j, F_{k-1}$, and obtain so recursively the desired chain of faces.

3.17 Remark. Let $v_0$ be a vertex of an $n$-polytope $P$ and let $\{v_1, \ldots, v_r\}$ be all adjacent vertices of $v_0$, i.e., $\text{conv}\{v_0, v_i\}$ is an edge of $P$. In other words, $\{v_1, \ldots, v_r\}$ are the neighbours of $v_0$. Then

i) $P \subset v_0 + \text{pos}\{v_1 - v_0, \ldots, v_r - v_0\}$.

ii) Let $c \in \mathbb{R}^n$ with $\langle c, v_0 \rangle \geq \langle c, v_i \rangle$, $1 \leq i \leq r$. Then

$$\max\{\langle c, x \rangle : x \in P\} = \langle c, v_0 \rangle.$$

3.18 Theorem [Euler-Poincaré formula]. Let $P \in \mathbb{P}^n$. Then

$$\sum_{i=-1}^n (-1)^i f_i(P) = 0. \tag{3.18.1}$$

In particular, in the 3-dimensional case, i.e., $\text{dim } P = 3$, it holds $f_0 - f_1 + f_2 = 2$.

---

11Leonhard Euler, 1707–1783
12Henri Poincaré, 1854–1912
Proof. It suffices to consider $n$-dimensional polytopes and we proceed by induction with respect to $n$. In the case $n = 1$ there is nothing to prove as it is $-1 + f_0(P) - 1 = 0$. Let $n \geq 2$, $m = f_0(P)$ and let $a \in \mathbb{R}^n \setminus \{0\}$ be chosen such that for any $\alpha \in \mathbb{R}$ the hyperplane $H(a, \alpha)$ contains at most one vertex $P$.

Let $\alpha_1 < \alpha_2 < \cdots < \alpha_{2m-1}$ such that the “odd” hyperplanes $H_{2i-1} := H(a, \alpha_{2i-1})$, $i = 1, \ldots, m$, contains a vertex of $P$. The “even” hyperplanes $H_{2i} := H(a, \alpha_{2i})$, $i = 1, \ldots, m - 1$, then, do not contain vertices, and $H_1$ and $H_{2m-1}$ are supporting hyperplanes of $P$. Hence, for $i = 2, \ldots, 2m - 2$ the polytopes $P_i := H_i \cap P$ are $(n-1)$-dimensional polytopes. For a $j$-face $F$ of $P$, $j \geq 1$, and a polytope $P_i$ we set

$$\psi(F, P_i) = \begin{cases} 1, & P_i \cap \text{relint } F \neq \emptyset, \\ 0, & P_i \cap \text{relint } F = \emptyset. \end{cases}$$

Observe, since $F \not\subset H_i$, $\psi(F, P_i) = 1$ implies that $F \cap H_i$ is a $(j-1)$ face of $P$. The first index $i_1$ and the last index $i_2$ of a hyperplane $H_i$ with $H_i \cap F \neq \emptyset$ has to be odd, and $\psi(F, P_i) = 1$ if and only if $i \in \{i_1 + 1, \ldots, i_2 - 1\}$. Hence,

$$\sum_{i=2}^{2m-2} (-1)^i \psi(F, P_i) = 1.$$

Summing over all $j$-faces yields

$$\sum_{j=1}^{n-1} (-1)^j \sum_{F \text{-j-face}} \sum_{i=2}^{2m-2} (-1)^i \psi(F, P_i) = \sum_{j=1}^{n-1} (-1)^j f_j(P).$$

Changing the order of summation on the left hand side gives

$$\sum_{j=1}^{n-1} (-1)^j f_j(P) = \sum_{i=2}^{2m-2} (-1)^i \left( \sum_{j=1}^{n-1} (-1)^j \sum_{F \text{-j-face}} \psi(F, P_i) \right),$$

and next we evaluate the interior sum on the right hand side.

For an even index $i$ or for $j \geq 2$ each $(j-1)$-face $F$ of such a $P_i$ is the intersection of a $j$-face $F$ of $P$ with $H_i$. For odd $i$ and $j = 1$ a vertex of $P_i$ is either the unique vertex of $P$ lying in $H_i$ or the intersection of an edge of $P$ with $H_i$. Hence for $j \geq 1$ we have

$$\sum_{F \text{-j-face}} \psi(F, P_i) = \begin{cases} f_0(P_i) - 1, & j = 1 \text{ and } i \text{ odd}, \\ f_{j-1}(P_i), & \text{otherwise}. \end{cases}$$

By our inductive argument we get

$$\sum_{j=1}^{n-1} (-1)^j \sum_{F \text{-j-face}} \psi(F, P_i) = \begin{dcases} 1 + \sum_{j=1}^{n-1} (-1)^j f_{j-1}(P_i) = 1 - \sum_{j=0}^{n-2} (-1)^j f_j(P_i) = (-1)^{n-1}, & i \text{ odd}, \\ \sum_{j=1}^{n} (-1)^j f_{j-1}(P_i) = -\sum_{j=0}^{n-2} (-1)^j f_j(P_i) = (-1)^{n-1} - 1, & i \text{ even}. \end{dcases}$$

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and hence any solution $\lambda \in \mathbb{R}^n$ of
\[
\sum_{i=2}^{2m-2} (-1)^i \left( \sum_{j=1}^{n-1} (-1)^j \sum_{F \text{ face}} \psi(F, P_i) \right) = (-1)^{n-1} - (m - 1)
\]

which proves the theorem by (3.18.2).

\[ \square \]

### 3.19 Proposition. The Euler-Poincaré formula is the only linear equation satisfied by the $f$-vector, i.e., let $\lambda_i \in \mathbb{R}$, such that $\sum_{i=1}^{n} \lambda_i f_i(P) = 0$ for all $P \in \mathcal{P}^n$. Then there exists a constant $\gamma \in \mathbb{R}$, such that $\lambda_i = \gamma (-1)^i$.

**Proof.** Let $\sum_{i=1}^{n} \lambda_i f_i(P) = 0$ be a linear equation which holds for any $P \in \mathcal{P}^n$. Taking a $k$-simplex $T_k$, $k \in \{0, \ldots, n\}$, we obtain (cf. Remark 3.10)

\[
\sum_{i=-1}^{n} \lambda_i \binom{k+1}{i+1} = 0, \quad k = 0, \ldots, n.
\]

The $(n + 1) \times (n + 2)$ matrix $A$ with coefficients $a_{k,i} = \binom{k+1}{i+1}$ has rang $n + 1$ and hence any solution $\lambda \in \mathbb{R}^{n+2}$ of the homogeneous system $A \lambda = 0$ must be a multiple of the coefficients given by the Euler-Poincaré identity.

\[ \square \]

### 3.20 Definition [Simple and simplicial polytopes]. Let $P \in \mathcal{P}^n$.

i) $P$ is called simplicial if all proper faces are simplices.

ii) $P$ is called simple if every vertex is contained in exactly $\dim P$ many facets.

### 3.21 Lemma. Let $P \in \mathcal{P}^n$ be an $n$-polytope with $0 \in \text{int} P$. The following statements are equivalent:

i) $P$ is simplicial.

ii) All facets of $P$ are simplices.

iii) $P^*$ is simple.

iv) Every $k$-face of $P^*$ is contained in exactly $n - k$ facets for $k = 0, \ldots, n - 1$.

**Proof.** We recall that by polarity we have $P = \text{conv} \{v_1, \ldots, v_m\}$ with $v_i$ vertex of $P$ if and only if $P^* = \{y \in \mathbb{R}^n : \langle v_i, y \rangle \leq 1, 1 \leq i \leq m\}$ with facets $\{v_i\}^c = P^* \cap H(v_i, 1), 1 \leq i \leq m$. Moreover, $F = \text{conv} \{v_{i_1}, \ldots, v_{i_k}\}$ is an $l$-face of $P$ with vertices $v_{i_1}, \ldots, v_{i_k}$ if and only if $F^c = \{y \in P^* : y \in H(v_{i_j}, 1), j = 1, \ldots, k\}$ is an $n - l - 1$-face of $P^*$ contained only in the facets $\{v_{i_j}\}^c$ of $P^*$, $j = 1, \ldots, k$.

"$i) \iff iv)$": Let $F^c$ be a $k$-face of $P^*$. Then $(F^c)^c$ is an $n - k - 1$ face of $P$, thus an $n - k - 1$ simplex with $n - k$ vertices. Hence by the foregoing remark $F^c$ is
contained in exactly $n - k$ facets. On the other hand let $F$ be a $k$-face of $P$. Then $F^\circ$ is an $n - k - 1$ of $P^\ast$ contained in exactly $k + 1$ facets, and thus the $k$-face $F$ has exactly $k + 1$ vertices and it is a simplex.

"ii)$\iff$iii)" $\Rightarrow$: Same argumentation as before where "$\Rightarrow$" is the case $k = 0$ and "$\Leftarrow$" is the case $k = n - 1$.

"i)$\iff$ii)" Follows from the fact that every proper face of a polytope is a face of facet (see Theorem 3.15 v)).

3.22 Theorem. Let $P \in \mathcal{P}^n$ be a simple $n$-polytope. Then

i) Every vertex is contained in exactly $\binom{n}{k}$ $k$-faces of $P$, $k = 0, \ldots, n - 1$.

ii) The intersection of $k$ facets containing a common vertex is an $(n - k)$-face of $P$.

iii) Let $v_1, \ldots, v_n$ be the neighbours of a vertex $v_0$ of $P$. For each subset of $k$ neighbours $v_{i_1}, \ldots, v_{i_k}$ there exists a unique $k$-face $F$ of $P$ containing $v_0, v_{i_1}, \ldots, v_{i_k}$.

iv) A face of a simple polytope is simple.

v) Every $j$ face of $P$ is contained in exactly $\binom{n-j}{k-j}$ $k$ faces of $P$.

Proof. Without loss of generality we may assume that $0 \in \text{int } P$ and by Lemma 3.21 we know that $P^\ast$ is simplicial.

i) Let $v$ be a vertex of $P$ and let $F$ be a $k$-face of $P$. Then we have $\{v\} \subseteq F$ if and only if $F^\circ \subseteq \{v\}^\circ$ is a $(n - k - 1)$-face of $P^\ast$. Now $\{v\}^\circ$ is facet of the simplicial polytope $P^\ast$ and so it has exactly $\binom{n}{n-k}$ many $(n - k - 1)$-dimensional faces.

ii) Let $v$ be a vertex of $P$. Since $P$ is simple, $v$ is contained in $n$-facets $F_i = P \cap H(a_i, 1)$, $1 \leq i \leq n$, and we want to show that $\cap_{i=1}^n F_i$ is an $n - k$-face fo $P$. Since $P^\ast$ is simplicial, $\{v\}^\circ = \text{conv } \{a_1, \ldots, a_n\}$ is an $(n - 1)$-simplex and so $\text{conv } \{a_1, \ldots, a_k\}$ is a $(k - 1)$-face of $P^\ast$. Hence $\text{conv } \{a_1, \ldots, a_k\}^\circ$ is an $(n - k)$ face of $P$ given by $\{x \in P : \langle a_i, x \rangle = 1, i = 1, \ldots, k\} = \cap_{i=1}^n F_i$.

iii) Let $\{v_0\}^\circ = \text{conv } \{w_1, \ldots, w_n\}$ be a facet of $P^\ast$. For the edges (1-faces) $\text{conv } \{v_0, v_i\}$ the associated polar face is an $(n - 2)$-face and so let

$$\text{conv } \{v_0, v_i\}^\circ = \text{conv } \{\{w_1, \ldots, w_n\} \setminus \{w_i\}\}, \quad 1 \leq i \leq n.$$ 

Since $P^\ast$ is simplicial,

$$U = \text{conv } \{\{w_1, \ldots, w_n\} \setminus \{w_i, \ldots, w_k\}\} = \cap_{j=1}^k \text{conv } \{\{w_1, \ldots, w_n\} \setminus \{w_j\}\}$$

is an $(n - k - 1)$-face of $P^\ast$. Hence $U^\circ$ is a $k$ face containing the edges $\text{conv } \{v_0, v_j\}$, $j = 1, \ldots, k$. For any $k$-face $G$ of $P$ containing these edges with have by polarity that $G^\circ \subseteq \cap_{j=1}^k \text{conv } \{\{w_1, \ldots, w_n\} \setminus \{w_j\}\} = U$. Since $\dim G^\circ = \dim U = n - k - 1$ we have $G = U^\circ$. Thus $U^\circ$ is uniquely determined.

iv) Let $F$ be a $k$-face of $P$, $k \in \{1, \ldots, n - 1\}$, and let $v$ be a vertex of $F$. We want to show that $v$ is contained in exactly $k$-faces of $F$, i.e., $(k - 1)$-faces
of \( P \) contained in \( F \). Now \( G \subset F \) is a \((k-1)\)-face containing \( v \) if and only if \( G^\circ \) is a \((n-k)\)-face of \( P^* \) with \( F^\circ \subset G^\circ \subset \{v\}^\circ \). Now \( P^* \) is simplicial and so we may assume that \( \{v\}^\circ = \text{conv} \{w_1, \ldots, w_n\} \) and \( F^\circ = \{w_1, \ldots, w_{n-k}\} \). Hence there are exactly \( k \) many \( k \)-faces \( G^\circ \) with the required property, namely \( F^\circ \cup \{w_j\}, j = n - k + 1 \) dots, \( n \).

iv) left as an exercise. \( \square \)

### 3.23 Theorem
Let \( P \in \mathcal{P}^n \) be a simple \( n \)-polytope.

i) \( n f_0(P) = 2 f_1(P) \).

ii) \( \sum_{k=0}^n f_k(P) \leq 2^n f_0(P) \).

iii) \( f_0(P) \leq 2 f_{\lfloor n/2 \rfloor}(P) \).

**Proof.** For i) we note that every edge contains exactly 2 vertices and every vertex is contained in exactly \( n \) edges.

By Theorem 3.22 i) every vertex is contained in exactly \( \sum_{i=0}^n \binom{n}{i} = 2^n \) faces of \( P \) and each face has at least one vertex. This gives ii).

For iii) we assume that \( P = \text{conv} \{v_1, \ldots, v_m\} \) with vertices \( v_i \) and all vertices have different last coordinate. For a fixed vertex \( v \) with its \( n \) neighbors \( v_1, \ldots, v_n \), say, let

\[
L(v) = \{v_i : \langle e_n, v_i \rangle < \langle e_n, v \rangle\} \quad \text{and} \quad U(v) = \{v_i : \langle e_n, v_i \rangle > \langle e_n, v \rangle\}.
\]

Next we distinguish two cases depending on the cardinality of these sets.

a) \( \#L(v) \geq \lfloor n/2 \rfloor \). On account of Theorem 3.22 iii) each \( \lfloor n/2 \rfloor \) subset \( S \subseteq L(v) \) determines an unique \( \lfloor n/2 \rfloor \)-face \( F \) of \( P \) containing the edges \( \text{conv} \{v, v_i\}, v_i \in S \). By Theorem 3.22 iv), \( F \) is simple and so \( \text{conv} \{v, v_i\}, v_i \in S \), are the only edges of \( F \) containing \( v \). Thus, on account of Remark 3.17, there exists an \( \lfloor n/2 \rfloor \) face \( F \) of \( P \) with

\[
\langle e_n, v \rangle = \max_{x \in F} \langle e_n, x \rangle.
\]

b) \( \#U(v) \geq \lfloor n/2 \rfloor \). In the same way we conclude that there exists an \( \lfloor n/2 \rfloor \)

face \( F \) of \( P \) with \( \langle e_n, v \rangle = \min_{x \in F} \langle e_n, x \rangle \).

Hence for each vertex \( v \) there exists an \( \lfloor n/2 \rfloor \) face \( F \) of \( P \) such that \( v \) has either the largest or smallest last coordinate among all points of \( F \). Since each \( \lfloor n/2 \rfloor \) face contains a larest as well as a smallest vertex (with respect to the last coordinate) we must have \( 2 f_{\lfloor n/2 \rfloor}(P) \geq f_0(P) \). \( \square \)

### 3.24 Corollary
Let \( P \) be a simple \( n \)-polytope with \( m \) facets. Then

\[
f_0(P) \leq 2^m\binom{m}{\lfloor n/2 \rfloor} \quad \text{and} \quad \sum_{k=0}^n f_k(P) \leq 2^{n+1}\binom{m}{\lfloor n/2 \rfloor}.
\]

Or equivalently: Let \( P \) be a simplicial \( n \)-polytope with \( m \) vertices. Then

\[
f_{m-1}(P) \leq 2^m\binom{m}{\lfloor n/2 \rfloor} \quad \text{and} \quad \sum_{k=0}^n f_k(P) \leq 2^{n+1}\binom{m}{\lfloor n/2 \rfloor}.
\]
Proof. First we note that the statement for simplical polytopes follows by "polarity". For any polytope with \( m \) facets we certainly have 
\[ f_k(P) \leq \binom{m}{n-k}, \]
\( k = 0, \ldots, n - 1 \) (cf. Theorem 3.15 ii)), and, in particular, 
\[ f_{\lfloor n/2 \rfloor}(P) \leq \binom{m}{\lfloor n/2 \rfloor}. \]
Hence with Theorem 3.23 iii) we obtain 
\[ f_0(P) \leq 2 \binom{m}{\lfloor n/2 \rfloor} \] which also gives the second inequality by Theorem 3.23 ii) □

3.25 Lemma*. Let \( P \) be an \( n \)-polytope.

i) There exists a simple \( n \)-polytope \( Q \) with the same number of facets as \( P \) and 
\[ f_i(P) \leq f_i(Q), \quad 0 \leq i \leq n - 2. \]

ii) There exists a simplical \( n \)-polytope \( Q^* \) with the same number of vertices as \( P \) and 
\[ f_i(P) \leq f_i(Q^*), \quad 1 \leq i \leq n - 1. \]

3.26 Corollary. Let \( P \) be an \( n \)-polytope with \( m \) facets. Then 
\[ f_0(P) \leq 2 \binom{m}{\lfloor n/2 \rfloor}. \]
Or equivalently: Let \( P \) be an \( n \)-polytope with \( m \) vertices. Then 
\[ f_{n-1}(P) \leq 2 \binom{m}{\lfloor n/2 \rfloor}. \]

3.27 Definition [Cyclic polytopes]. The curve \( \gamma : \mathbb{R} \to \mathbb{R}^n \) given by \( \gamma(t) = (t, t^2, t^3, \ldots, t^n)^\top \) is called moment curve. The convex hull of \( m \) points on the moment curve is called a cyclic polytope with \( m \) vertices and is denoted by \( C(n, m) \).

3.28 Proposition. Any \( n + 1 \) points on the moment curve are affinely independent. In particular, cyclic polytopes are simplicial polytopes.

Proof. Suppose \( \gamma(t_1), \ldots, \gamma(t_{n+1}), t_i \neq t_j \), are affinely dependent. Then they are contained in a hyperplane \( H(\mathbf{a}, a_0), \mathbf{a} \neq 0 \), say. Hence \( \langle \mathbf{a}, \gamma(t_i) \rangle - a_0 = 0 \), or in coordinates
\[ \sum_{j=1}^{n} a_j(t_i)^j - a_0 = 0, \quad 1 \leq i \leq n + 1, \]
contradicting the fact that a polynomial has at most degree many roots. □
3.29 Theorem* [McMullen’s Upper Bound Theorem, 1971]. Let $P$ be an $n$-polytope with $m$ vertices. Then

$$f_i(P) \leq f_i(C(n, m)) = \begin{cases} \sum_{j=0}^{(n-1)/2} \frac{i+2}{m-j} \binom{m-j}{j+1} \binom{m-j}{i+1-j}, & n \text{ odd,} \\ \sum_{j=1}^{n/2} \frac{m-j}{m-j} \binom{m-j}{j+1}, & n \text{ even.} \end{cases}$$

In particular,

$$f_{n-1}(P) \leq f_{n-1}(C(n, m)) = \begin{cases} 2 \left( \frac{m-\lfloor n/2 \rfloor - 1}{\lfloor n/2 \rfloor} \right) + \binom{m-\lfloor n/2 \rfloor - 1}{\lfloor n/2 \rfloor}, & n \text{ even.} \end{cases}$$

For fixed $n$ the right hand sides are of order $m^{\lfloor n/2 \rfloor}$.

3.30 Theorem* [Barnette’s Lower Bound Theorem, 1973]. Let $P$ be a simplicial $n$-polytope with $m$ vertices. $P$ has at least as many $i$-faces as the so called stacked polytopes $P(n, m)$ with $m$ vertices for which

$$f_i(P(n, m)) = \begin{cases} \binom{m}{n} - i\binom{n+1}{i+1}, & 0 \leq i \leq n-2, \\ n+1 + (m-(n+1))(n-1), & i = n-1. \end{cases}$$

$P(n, n+1)$ is an $n$-simplex, and for $m \geq n+2$ an $m$-vertex stacked $n$-polytope $P(n, m)$ is the convex hull of an $(m-1)$-vertex stacked polytope with an additional point that is beyond exactly one facet.

3.31 Remark. For any $n$-polytope $P \in \mathcal{P}^n$ we have $nf_0(P) \leq 2f_1(P)$ with equality iff $P$ simple and $nf_{n-1}(P) \leq 2f_{n-2}(P)$ with equality iff $P$ simplicial.

3.32 Theorem [Steinitz, 1906]. A non-negative integral vector $(f_0, f_1, f_2)$ is the $f$-vector of a 3-polytope if and only if i) $f_0 - f_1 + f_2 = 2$, ii) $3f_0 \leq 2f_1$, and iii) $3f_2 \leq 2f_1$.

Proof. Equation i) is the Euler-Poincaré formula (3.18.1) for polytopes in $\mathbb{R}^3$, ii) describes the fact that every vertex is contained in at least 3 edges and each edge has exactly two vertices, and iii) is just the polar version of ii).

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13Peter McMullen, born 1942
14David W. Barnette
15Ernst Steinitz, 1871 – 1928
For the sufficiency part we have to construct a polytope with \( f_0 \) vertices and \( f_2 \) facets, where the non-negative integers \( f_0, f_2 \) satisfy

\[
2f_0 - f_2 - 4, \quad 2f_2 - f_0 - 4 \geq 0.
\]

These inequalities are obtained by using i) in ii) and iii). Since their difference is divisible by 3 they have the same remainder \( r \), say, on division by 3. Thus there exist \( c, s \in \mathbb{N} \) such that

\[
2f_0 - f_2 - 4 = 3c + r \quad \text{and} \quad 2f_2 - f_0 - 4 = 3s + r.
\]

Hence we have

\[
f_0 = (4 + r) + 2c + s \quad \text{and} \quad f_2 = (4 + r) + c + 2s. \tag{3.32.1}
\]

Let \( P_0 \) be a pyramid whose basis is a \( r + 3 \)-gon. Then \( f_0(P_0) = f_2(P_0) = r + 4 \), and each facet containing the top vertex is a triangle, and each vertex of the basis is contained in exactly three edges, which we call a simplex vertex. Now we apply two operations to \( P_0 \), namely cutting of simple vertices and stacking over triangular faces.

If we cut off a simple vertex, then we obtain a polytope \( P_1 \), say, with \( f_0(P_1) = f_0(P_0) + 2 \) and \( f_2(P_1) = f_2(P_0) + 1 \). Moreover, the three new vertices of \( P_1 \) form a triangular face and all of them are simple. Hence, if we continue this process of cutting of simple vertices (in a proper way) we obtain after \( c \) steps a polytope \( P_c \) with \( f_0(P_c) = (4 + r) + 2c \) and \( f_2(P_c) = (4 + r) + c \).

On the other hand, if we replace a triangular face of \( P_c \) by a suitable simplex then we obtain a polytope \( P_{c+1} \) with \( f_0(P_{c+1}) = f_0(P_c) + 1 \) and \( f_2(P_{c+1}) = f_2(P_c) + 2 \). Repeating this stacking process of triangular faces \( s \)-times, we finally arrive at a polytope \( P_{c+s} \) with \( f_0(P_{c+s}) \) and \( f_2(P_{c+s}) \) as desired in (3.32.1). \( \square \)

3.33 Conjecture [Kalai, 1989]. \(^{16}\) Let \( P \in \mathcal{P}^n \) be a 0-symmetric \( n \)-polytope. Then

\[
\sum_{i=0}^{n} f_i(P) \geq 3^n.
\]

Here we have equality, for instance, for the cube \( C_n \) and its polar, the cross-polytope \( C_n^* \), or, more generally, for the class of Hanner-polytopes. In 2007 the conjecture has been verified for all \( n \leq 4 \) (see http://front.math.ucdavis.edu/0708.3661).

3.34 Theorem [Figiel, Lindenstrauss, Milman, 1977]. \(^{17}^{18}^{19}\) Let \( P \in \mathcal{P}^n \) be a 0-symmetric \( n \)-polytope, i.e., \( P = -P \). Then

\[
\ln(f_0(P)) \ln(f_{n-1}(P)) \geq \frac{1}{16} n.
\]

Proof. For the proof we need two facts from convexity. First, there exists an \( A \in \text{GL}(n, \mathbb{R}) \) such that \( B_n \subset A P \subset \sqrt{n} B_n \). Hence, in our combinatorial setting we may assume that

\[
B_n \subset P \subset \sqrt{n} B_n. \tag{3.34.1}
\]

\(^{16}\)Gil Kalai, born 1955
\(^{17}\)Tadeusz Figiel
\(^{18}\)Joram Lindenstrauss, 1936–2012
\(^{19}\)Vitali Milman, born 1939
The second fact concerns spherical caps \( C(v, \epsilon) \), which for \( v \in S^{n-1} \) and \( \epsilon \in [0, 1] \) are defined by \( C(v, \epsilon) = \{ w \in S^{n-1} : \langle v, w \rangle \geq \epsilon \} \). If \( \mu_{n-1}(\cdot) \) denotes the Haar probability measure on \( S^{n-1} \) then \( \mu_{n-1}(C(v, \epsilon)) \in [0, 1] \) measures how much of \( S^{n-1} \) is covered by \( C(v, \epsilon) \). Here we need

\[
\mu_{n-1}(C(v, \epsilon)) \leq e^{-n^2 \frac{\epsilon^2}{2}}. \tag{3.34.2}
\]

For the proof we set \( m_1 = f_0(P) \) and \( m_2 = f_{n-1}(P) \), and we also need a \( \mathcal{V} \)- and an \( \mathcal{H} \)-representation of \( P \)

\[
P = \text{conv} \{ v_i : 1 \leq i \leq m_1 \} = \{ x \in \mathbb{R}^n : \langle a_i, x \rangle \leq 1, 1 \leq i \leq m_2 \}.
\]

Moreover, let \( \epsilon_i = 2 \sqrt{\ln(m_i)} / \sqrt{n} \), \( i = 1, 2 \).

Let \( V_{\epsilon_1} = \bigcup_{i=1}^{m_1} C(v_i, |v_i|, \epsilon_1) \) and \( F_{\epsilon_2} = \bigcup_{i=1}^{m_2} C(a_i, |a_i|, \epsilon_2) \). Then by (3.34.2) and the choice of \( \epsilon_i \) we have \( \mu_{n-1}(V_{\epsilon_1}), \mu_{n-1}(F_{\epsilon_2}) \leq 1/4 \), e.g.,

\[
\mu_{n-1}(V_{\epsilon_1}) \leq \sum_{i=1}^{m_1} \mu_{n-1}(C(v_i, |v_i|, \epsilon_1)) \leq m_1 e^{-n^2 \frac{\epsilon_1^2}{2}} = \frac{1}{m_1} \leq \frac{1}{4}.
\]

Thus we can find a \( c \in S^{n-1} \setminus (V_{\epsilon_1} \cup F_{\epsilon_2}) \). Since \( c \in S^{n-1} \setminus V_{\epsilon_1} \) we get with (3.34.1)

\[
\max_{x \in P} |\langle c, x \rangle| = \max_{1 \leq i \leq m_1} |\langle c, v_i \rangle| \leq \sqrt{n} \max_{1 \leq i \leq m_1} |\langle c, v_i / |v_i| \rangle| < \sqrt{n} \epsilon_1 = 2 \sqrt{\ln(m_1)}.
\tag{3.34.3}
\]

In order to get a lower bound on \( \max_{x \in P} |\langle c, x \rangle| \) we observe that \( B_n \subset P \) implies \( |a_i| \leq 1 \), and since \( c \in S^{n-1} \setminus F_{\epsilon_2} \) we obtain for \( 1 \leq i \leq m_2 \)

\[
|\langle c, a_i \rangle| \leq |\langle c, a_i / |a_i| \rangle| \leq \epsilon_2.
\]

Hence \( (1/\epsilon_2)c \in P \) which gives

\[
\max_{x \in P} |\langle c, x \rangle| \geq \frac{1}{\epsilon_2} = \frac{\sqrt{n}}{2 \sqrt{\ln(m_2)}}.
\]

Together with (3.34.3) the assertion is proved.

\[ \square \]

3.35 Definition [Graph, combinatorial diameter]. Let \( P \subset \mathbb{R}^n \) be a polyhedron.

i) The distances \( \delta_P(v, w) \) between two vertices \( v, w \in P \) (or in \( G(P) \)) is the minimum length of an "edge" path connecting \( v \) and \( w \) in \( G(P) \).

ii) \( \delta(P) = \max\{ \delta_P(v, w) : v, w \in \text{vert } P \} \) is called the (combinatorial) diameter of \( P \).

3.36 Example. \( \delta(T_n) = 1 = (n + 1) - n, \delta(C_n) = n = 2n - n \) and \( \delta(C_n^*) = 2 \leq 2^n - n \).
3.37 Definition. For integers $n,m$ let

$$\Delta(n,m) = \max \{ \delta(P) : P \subset \mathbb{R}^n \text{ polyhedron, } \dim P = n \text{ and } f_{n-1}(P) = m \}.$$

3.38 Remark. In 1957 Hirsch\textsuperscript{20} conjectured $\Delta(n,m) \leq m - n$. It is known that

i) the conjecture is true if $n \leq 3$ or $m \leq n + 5$. For unbounded polyhedra the conjecture is false, namely, for $m \geq 2n$ it is $\Delta(n,m) \geq m - n + \lfloor n/4 \rfloor.$ (Klee&Walkup, 1961/1965),

ii) $\Delta(n,m) \leq m 2^{n-3},$ (Barnette, 1969; Larman, 1970),

iii) Disproof of the Hirsch conjecture for polytopes by Francisco Santos, 2010, see http://front.math.ucdavis.edu/1006.2814

3.39 Theorem [Kalai, 1992; Kalai&Kleitman, 1992].\textsuperscript{21}

$$\Delta(n,m) \leq m \log n + 2.$$

Proof. First we will establish the recurrence

$$\Delta(n,m) \leq \Delta(n-1,m-1) + 2\Delta(n,\lfloor m/2 \rfloor) + 2 \quad (3.39.1)$$

Let $P$ be an $n$-dimensional polytope with $m$ facets. For an edge-path $\omega$ of the polytope let $F(\omega)$ be the set of facets of $P$ which are incident with one of the vertices of $F(\omega)$, i.e., all facets which are visited on the path $\omega$. The length of a path $\omega$ is denoted $|\omega|$. For a vertex $w$ of $P$ let

$$k_w = \max \left\{ \# \left( \bigcup_{\omega \text{ path starting in } w, |\omega| \leq p} F(\omega) \right) \leq \lfloor m/2 \rfloor \right\}.$$

Now let $v, u$ be two vertices of $P$ attaining the diameter of $P$. By the definition of $k_v$ (and $k_u$) we have $\#(\bigcup_{\omega \text{ starting in } v, |\omega| = k_v+1} F(\omega)) > m/2$ and hence there exists a facet $F$ of $P$ which can be reached from $v$ by a path of length at most $k_v + 1$ and from $u$ by a path of length at most $k_u + 1$. Thus we conclude

$$\delta(P) \leq \Delta(n-1,m-1) + k_v + k_u + 2,$$

\textsuperscript{20}Warren M. Hirsch
\textsuperscript{21}Daniel J. Kleitman, born 1934
and it remains to show $k_v, k_u \leq \Delta(n, |m/2|)$.

To this end let $Q$ be given by all facet defining inequalities corresponding to facets of $\bigcup_\omega$ starting in $v, |\omega| \leq k_v F(\omega)$. Then we have $P \subset Q$ and $Q$ has at most $q \leq |m/2|$ facets. Let $w$ be a vertex of $P$ with $\delta(v, w) = k_v$. Then $v, w$ are vertices of $Q$ as well, and let $Q_\omega$ be a shortest edge-path in $Q$ joining $v, w$. Next we claim that

$$|\omega_Q| = k_v.$$  \hspace{1cm} (3.39.2)

By definition we have $|\omega_Q| \leq k_v$ and so suppose that $|\omega_Q| < k_v$. Then $\omega_Q$ uses an edge which is not an edge of $P$. Let $e_Q$ be the first such edge on $\omega_Q$. Then this edge must be intersected by one of the facet defining hyperplanes of $P$ which are not in $\bigcup_\omega$ starting in $v, |\omega| \leq k_v F(\omega)$. Hence this facet can be reached by a path in $P$ of length $\leq k_v$ which contradicts the choice of $k_v$.

This shows (3.39.2) and so $k_v = |\omega_Q| \leq \Delta(n, q) \leq \Delta(n, |m/2|)$.

Finally, in order to get the desired bound $\Delta(n, m) \leq m^{\log n + 2}$ from the recursion (3.39.1) we refer to Lecture 3 of Günter Ziegler’s book. \hspace{1cm} $\square$
4 The space of convex bodies

4.1 Notation. In the following let $\mathcal{K}^n$ be the set of all convex bodies in $\mathbb{R}^n$, i.e., the set of all non-empty convex compact sets in $\mathbb{R}^n$.

4.2 Remark. Let $K_i \in \mathcal{K}^n$ and $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, m$. Then $\sum_{i=1}^n \lambda_i K_i \in \mathcal{K}^n$.

4.3 Definition [Outer parallel body, Hausdorff distance].

i) Let $M \subset \mathbb{R}^n$ be a non-empty compact set and $\rho \in \mathbb{R}_{\geq 0}$. The set $M + \rho B_n$ is called the outer parallel body of $M$ at distance $\rho$.

ii) Let $M_1, M_2 \subset \mathbb{R}^n$ be non-empty compact subsets. The Hausdorff distance of $M_1$ and $M_2$ is defined as

$$\delta(M_1, M_2) = \min \{ \rho \geq 0 : M_1 \subseteq M_2 + \rho B_n \text{ and } M_2 \subseteq M_1 + \rho B_n \}.$$ 

4.4 Remark. The Hausdorff-distance $\delta(\cdot, \cdot)$ defines a metric on the space of non-empty compact subsets of $\mathbb{R}^n$.

4.5 Definition [Convergent sequences of convex bodies]. A sequence of convex bodies $K_i \in \mathcal{K}^n$, $i \in \mathbb{N}$, is called convergent (to $K \in \mathcal{K}^n$) iff there exists a $K \in \mathcal{K}^n$ such that $\lim_{i \to \infty} \delta(K_i, K) = 0$; in this case we also write $K_i \to K$ or $\lim_{i \to \infty} K_i = K$.

4.6 Lemma. Let $K_i \in \mathcal{K}^n$, $i \in \mathbb{N}$, with $K_{i+1} \subseteq K_i$ and let $K = \bigcap_{i=1}^{\infty} K_i$. Then $K \in \mathcal{K}^n$ and $K_i \to K$.

Proof. First we show that $K \neq \emptyset$. To this end let $x_i \in K_i$, $i \in \mathbb{N}$. Since $K_i \subseteq K_1$ for all $i$ we may assume that the sequence $(x_i)_{i \in \mathbb{N}}$ converges to a point $x^* \in K_1$, say. Since for any $j \in \mathbb{N}$ the subsequence $(x_i)_{i \geq j}$ is contained in $K_j$ we also have $x^* \in K_j$. Hence $x^* \in K$.

In order to verify $K_i \to K$ we have to show that for any $\rho > 0$ there exists an $i_\rho \in \mathbb{N}$ such that $\delta(K_i, K) \leq \rho$ for $i \geq i_\rho$. Obviously, the inclusion $K \subset K_i + \rho B_n$ holds for any $\rho > 0$. So suppose that for a given $\rho > 0$ there does not exist an $i_\rho$ with $K_i \subset K + \rho B_n$ for $i \geq i_\rho$. Then $K_i \not\subseteq K + \rho B_n$ for all $i \in \mathbb{N}$, and therefore there exist points $x_i \in K_i \setminus (K + \rho B_n)$. As in the first part we can assume that $x_i \to x^* \in K$. Hence, $|x_i - x^*| \leq \rho$ for $i$ sufficiently large, and so $x_i \in x^* + \rho B_n \subseteq K + \rho B_n$, which contradicts the choice of $x_i$. \hfill $\square$

4.7 Lemma. Let $K_i \in \mathcal{K}^n$, $i \in \mathbb{N}$, be a Cauchy\footnote{Felix Hausdorff, 1868–1942} sequence of convex bodies, i.e., $\forall \epsilon > 0 \exists m_\epsilon \in \mathbb{N}$ such that $\delta(K_i, K_j) < \epsilon$, $\forall i, j \geq m_\epsilon$. Then there exists a $K \in \mathcal{K}^n$ with $K_i \to K$. In other words, the space $\mathcal{K}^n$ is complete.

\footnote{Augustin-Louis Cauchy, 1789-1857}
Proof. Let \( L_j = \text{cl conv } (\cup_{i \geq j} K_i) \). Since \((K_i)_{i \in \mathbb{N}}\) is a Cauchy-sequence, the set \( \text{cl } \cup_{i \geq j} K_i \) is bounded and hence compact. Thus \( L_j \in K^n \). Since \( L_{j+1} \subseteq L_j \) we may apply Lemma 4.6, and get \( L_j \to K = \cap_{j \geq 1} L_j \). In view of the definition of \( L_j \) we conclude that for any \( \rho > 0 \) and \( j \geq j_\rho \), say,

\[
K_j \subseteq L_j \subset K + \rho B_n.
\]

On the other hand, since \((K_i)_{i \in \mathbb{N}}\) is a Cauchy-sequence we have \( K_i \subset K_j + \rho B_n \) for all \( i \geq j \geq m_\rho \), and so

\[
K \subseteq L_j \subset K_j + \rho B_n.
\]

Thus \( \delta(K_j, K) \leq \rho \) for all \( j \geq \max\{j_\rho, m_\rho\} \), which means \( K_j \to K \).

4.8 Definition [Bounded sequences]. A sequence of convex bodies \( K_i \in K^n \), \( i \in \mathbb{N} \), is called bounded, if there exists an \( \gamma \in \mathbb{R}_{>0} \) such that \( K_i \subseteq \gamma B_n \) for all \( i \in \mathbb{N} \).

4.9 Theorem [Selection theorem of Blaschke, 1916]. A bounded sequence of convex bodies \( K_i \in K^n \), \( i \in \mathbb{N} \), contains a convergent subsequence.

Proof. On account of Lemma 4.7 it suffices to find a Cauchy-subsequence within the sequence \((K_i)_{i \in \mathbb{N}}\). After a suitable scaling and translation we may assume \((K_i)_{i \in \mathbb{N}} \subseteq [0,1]^n\). For \( j \in \mathbb{N} \), the cube \([0,1]^n\) can be subdivided into \( 2^n \) cubes of size \([0,2^{-j}]^n\), which we denote by \( W_k^{(j)}, k = 1, \ldots, 2^n \). Now we construct for each \( j \in \mathbb{N} \) a subsequence \((K_i^{(j)})_{i \in \mathbb{N}}\) in the following recursive way: Let \( K_i^{(0)} = K_i \), \( i \in \mathbb{N} \). For \( K_i^{(j)}, j \geq 1 \), we consider for each \( K_i^{(j-1)} \) the cubes \( W_k^{(j)} \) having a non-empty intersection with \( K_i^{(j-1)} \), i.e.,

\[
U_i^{(j)} = \left\{ W_k^{(j)} : W_k^{(j)} \cap K_i^{(j-1)} \neq \emptyset \right\}.
\]

Since there are only finitely many possible sets \( U_i^{(j)} \), but infinitely many convex bodies \( K_i^{(j-1)} \), there exists a infinite family of convex bodies \( K_i^{(j)} \) having the same set \( U_i^{(j)} \), say. This family forms our new sequence \( K_i^{(j)} \), \( i \in \mathbb{N} \). Since all bodies \( K_i^{(j)} \) intersect the same cubes \( W_k^{(j)} \), and since the maximum distance between two points in \( W_k^{(j)} \) is \( \sqrt{n} 2^{-j} \) we conclude

\[
\delta(K_k^{(j)}, K_k^{(s)}) \leq \sqrt{n} 2^{-j},
\]

for all \( r, s \in \mathbb{N} \). Finally, we consider the diagonal sequence of these sequences, i.e., \((K_i^{(i)})_{i \in \mathbb{N}}\). For \( m \geq j \) we have \( K_k^{(m)} \subseteq (K_i^{(j)})_{i \in \mathbb{N}} \) and so

\[
\delta(K_k^{(m)}, K_k^{(j)}) \leq \sqrt{n} 2^{-j},
\]

which shows that \((K_i^{(i)})_{i \in \mathbb{N}}\) is a Cauchy-sequence. \( \square \)

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24 Wilhelm Blaschke, 1885–1962
4.10 Theorem. Let $K \in \mathcal{K}^n$ and $\rho > 0$. Then there exists a polytope $P \in \mathcal{P}^n$ such that $P \subseteq K \subseteq P + \rho B_n$, in particular we have $\delta(K, P) \leq \rho$.

Proof. Since $\bigcup_{x \in K} (x + \rho \text{int}(B_n))$ is an open covering of the compact set $K$, there exist finitely many $x_1, \ldots, x_m \in K$, say, such that

$$K \subseteq \bigcup_{i=1}^{m} (x_i + \rho B_n) \subseteq \overline{\text{conv}} \left\{x_1, \ldots, x_m\right\} + \rho B_n.$$ 

Since $P \subseteq K$ the assertion is proved. $\Box$

4.11 Corollary. Let $K \in \mathcal{K}^n$ with $0 \in \text{relint} K$ and let $\lambda > 1$. Then there exists a $P \in \mathcal{P}^n$ such that $P \subset K \subset \lambda P$.

Proof. Without loss of generality we may assume $\dim K = n$, and let $r > 0$ such that $r B_n \subseteq K$. By Theorem 4.10 we can find for any $\rho = r (\lambda - 1)/\lambda$ a polytope $P$ such that $P \subseteq K \subseteq P + \rho B_n$. Since $\rho < r$, the last inclusion implies $(r - \rho) B_n \subseteq P$ and thus

$$P \subseteq K \subseteq P + \frac{\rho}{r - \rho} (r - \rho) B_n \subseteq \frac{r}{r - \rho} P = \lambda P.$$ 

$\Box$
5 A glimpse on mixed volumes

5.1 Definition [Volume, characteristic function]. Let $M \subseteq \mathbb{R}^n$. The function $\chi_M : \mathbb{R}^n \to \{0, 1\}$ given by

$$\chi_M(x) = \begin{cases} 
1, & x \in M, \\
0, & x \notin M,
\end{cases}$$

is called characteristic function of $M$. If $M$ is bounded and $\chi_M$ is Riemann integrable then

$$\text{vol}(M) = \int_{\mathbb{R}^n} \chi_M(x) \, dx = \int_M 1 \, dx$$

is called the (n-dimensional) volume (Jordan measure) of $M$. $M$ is also called Riemann or Jordan-measurable.

5.2 Remark.

i) The volume of a (rectangular) box $B = [0, \alpha_1] \times [0, \alpha_2] \times \cdots \times [0, \alpha_n] \subset \mathbb{R}^n$ is given by $\text{vol}(B) = \prod_{i=1}^n \alpha_i$.

ii) Compact convex sets are Jordan-measurable. If $M \subset \mathbb{R}^n$ is Jordan-measurable then $\text{vol}(M) = \lim_{k \to \infty} \frac{1}{k^n} \# (M \cap \frac{1}{k} \mathbb{Z}^n)$.

iii) Let $M, M_1, M_2 \subset \mathbb{R}^n$, $M_1 \subseteq M_2$, be Riemann-integrable. Then

a) $\text{vol}(A \cdot M + t) = |\det A| \text{vol}(M)$ for any $A \in \mathbb{R}^{n \times n}$ with $\det A \neq 0$, and $t \in \mathbb{R}^n$.

b) $\text{vol}(\lambda \cdot M) = |\lambda|^n \text{vol}(M)$ for any $\lambda \in \mathbb{R}$.

c) $\text{vol}(M_1) \leq \text{vol}(M_2)$.

d) If $\dim M \leq n - 1$ then $\text{vol} M = 0$.

5.3 Notation. Let $K \subset \mathbb{R}^n$ be compact and contained in a $j$-dimensional affine plane $A$. The $j$-dimensional volume of $K$ with respect to $A$ is denoted by $\text{vol}_j(K)$, i.e., $\text{vol}_j(K) = \int_A \chi_K(x) \, dx$, where here $dx$ is the $j$-dimensional volume element with respect to the space $A$.

5.4 Lemma [Cavalieri’s principle]. 27 Let $K \subset \mathbb{R}^n$ be compact and for $t \in \mathbb{R}$ let $K_t = K \cap \{x \in \mathbb{R}^n : x_n = t\}$. Then

$$\text{vol}(K) = \int_{\mathbb{R}} \text{vol}_{n-1}(K_t) \, dt.$$
Proof. By Fubini’s theorem we may write
\[
\operatorname{vol} (K) = \int_{\mathbb{R}} \left( \int_{\mathbb{R}^{n-1}} \chi_K((\overline{x}, t)^T) d\overline{x} \right) dt = \int_{\mathbb{R}} \left( \int_{\{x \in \mathbb{R}^n : x_n = t\}} \chi_K(x) d\overline{x} \right) dt = \int_{\mathbb{R}} \operatorname{vol}_{n-1}(K_t) dt.
\]

5.5 Lemma [Pyramid, Prism]. Let \( Q \subset \mathbb{R}^n \) be an \((n-1)\)-dimensional compact convex set.

i) Let \( P = \operatorname{conv} \{Q, s\} \) be a pyramid, and let \( h \) be the distance between the apex \( s \) and \( \operatorname{aff} Q \). Then
\[
\operatorname{vol} (K) = \frac{h}{n} \operatorname{vol}_{n-1}(Q).
\]

ii) Let \( P = \operatorname{conv} \{x + Q, y + Q\} \) be a prism (i.e., \( \operatorname{conv} \{x, y\} \) is not parallel to \( \operatorname{aff} Q \)), and let \( h \) be the distance between \( x + \operatorname{aff} Q \) and \( y + \operatorname{aff} Q \). Then
\[
\operatorname{vol} (P) = h \operatorname{vol}_{n-1}(Q).
\]

Proof. We may assume that \( Q \subset \{x \in \mathbb{R}^n : x_n = 0\} \), \( s = (\overline{s}, h)^T \) with \( h > 0 \) and \( \overline{s} \in \mathbb{R}^{n-1} \). For \( \mu \in \mathbb{R} \) let \( P_\mu = P \cap \{x \in \mathbb{R}^n : x_n = \mu\} \). Clearly \( P_\mu = \emptyset \) for \( \mu \not\in [0, h] \), and for \( \mu \in [0, h] \) it holds
\[
P_\mu = \left(1 - \frac{\mu}{h}\right) Q + \frac{\mu}{h} \overline{s},
\]
With Lemma 5.4 we get
\[
\operatorname{vol} (P) = \int_0^h \operatorname{vol}_{n-1}(P_\mu) \, d\mu = \int_0^h \operatorname{vol}_{n-1} \left( \left(1 - \frac{\mu}{h}\right) Q + \frac{\mu}{h} \overline{s} \right) \, d\mu = \int_0^h \operatorname{vol}_{n-1}(Q) \int_0^h \left(1 - \frac{\mu}{h}\right)^{n-1} \, d\mu = \operatorname{vol}_{n-1}(Q) \left[ \frac{h}{n} \left(1 - \frac{\mu}{h}\right)^n \right]_0^h = \frac{h}{n} \operatorname{vol}_{n-1}(Q).
\]
Statement ii) for prisms can be proven analogously, where here \( P_\mu \) is always a translate of \( Q \) or empty. □

5.6 Proposition.

i) If \( \operatorname{int} K \neq \emptyset \) then \( \operatorname{vol} (K) > 0 \).
A glimpse on mixed volumes

ii) Let \( K_1, K_2 \subset \mathbb{R}^n \) be compact. Then
\[
\text{vol}(K_1 \cup K_2) = \text{vol}(K_1) + \text{vol}(K_2) - \text{vol}(K_1 \cap K_2).
\]

iii) Let \( K, M \in \mathbb{K}^n \) with \( \langle x, y \rangle = 0 \) for all \( x \in K, y \in M \) and let \( k = \dim K \) and \( m = \dim M \). Then \( \dim(K + M) = k + m \) and
\[
\text{vol}_{k+m}(K + M) = \text{vol}_k(K)\text{vol}_m(M).
\]

iv) Let \( W \) be an \( n \)-cube of edge length \( \lambda \). Then \( \text{vol}(W) = \lambda^n \) and, in particular, \( \text{vol}([-1, 1]^n) = 2^n \).

v) Let \( T = \text{conv}\{v_0, v_1, \ldots, v_n\} \) be an \( n \)-simplex. Then
\[
\text{vol}(T) = \frac{1}{n!} |\det(v_1 - v_0, \ldots, v_n - v_0)|.
\]

Proof.

i) If \( \text{int } K \neq \emptyset \) then there exist \( \varepsilon > 0 \) and \( t \in \mathbb{R}^n \) such that \( t + [0, \varepsilon] \subset K \). Hence, \( \text{vol}(K) \geq \text{vol}(t + [0, \varepsilon]) = \varepsilon^n \).

ii) It is a direct consequence of the corresponding identity for the characteristic functions \( \chi_{K_1 \cup K_2} = \chi_{K_1} + \chi_{K_2} - \chi_{K_1 \cap K_2} \).

iv) We may assume that \( n = k + m \) and that
\[
K \subset \{(x_1, \ldots, x_n)^\top \in \mathbb{R}^n : x_{k+1} = \cdots = x_n = 0\}
\]
\[
M \subset \{(x_1, \ldots, x_n)^\top \in \mathbb{R}^n : x_1 = \cdots = x_k = 0\}.
\]

Then \( K + M = \{(x', x'')^\top : (x', 0)^\top \in K, x' \in \mathbb{R}^k, (0, x'')^\top \in M, x'' \in \mathbb{R}^{n-k}\} \) and \( \dim(K + M) = n \).

Now for \( x = (x_1, \ldots, x_n)^\top \) we write \( x' = (x_1, \ldots, x_k)^\top \) and \( x'' = (x_{k+1}, \ldots, x_n)^\top \).

Then \( \chi_{K+M}(x) = \chi_K((x', 0)^\top)\chi_M((0, x'')^\top) \). By Fubini’s theorem we get
\[
\text{vol}(K + M) = \int_{\mathbb{R}^n} \chi_{K+M}(x) \, dx
\]
\[
= \int_{\mathbb{R}^{n-k}} \left( \int_{\mathbb{R}^k} \chi_K((x', 0)^\top) \chi_M((0, x'')^\top) \, dx' \right) \, dx''
\]
\[
= \int_{\mathbb{R}^m} \chi_M((0, x'')^\top) \left( \int_{\mathbb{R}^k} \chi_K((x', 0)^\top) \, dx' \right) \, dx''
\]
\[
= \text{vol}_k(K)\text{vol}_m(M).
\]

v) Let \( A = (v_1 - v_0, \ldots, v_n - v_0) \) and let \( T = \text{conv}\{0, e_1, \ldots, e_n\} \). Then \( T = A T + v_0 \). Hence \( \text{vol}(T) = |\det A|\text{vol}(T) \), and since \( T \) may be regarded as a pyramid with basis \( \text{conv}\{0, e_1, \ldots, e_n-1\} \) and apex \( e_n \), we recursively find by Lemma 5.5
\[
\text{vol}(T) = \frac{1}{n!}.
\]

Hence \( \text{vol}(T) = |\det A|/n! \). \qed
5.7 Lemma. Let $P \in \mathcal{P}^n$ with $0 \in \text{int } P$ and let $F_1, \ldots, F_m$ be the facets of $P$ with outer normal vectors $u_i \in \mathbb{R}^n$, $|u_i| = 1$. Then
\[
\text{vol} (P) = \frac{1}{n} \sum_{i=1}^m \text{vol}_{n-1}(F_i) h(P, u_i).
\]

Proof. Let $V_i$ be the pyramids over the facets with apex $0$, i.e., $V_i = \text{conv} \{0, F_i\}$, $1 \leq i \leq m$. Then $P = \bigcup_{i=1}^m V_i$. Moreover, since $V_i \cap V_j = \text{conv} \{F_i \cap F_j, 0\}$ for $1 \leq i, j \leq m$, we have $\dim (V_i \cap V_j) \leq n - 1$ for $i \neq j$. Hence, in view Proposition 5.6 ii) we get
\[
\text{vol} (P) = \sum_{i=1}^m \text{vol} (V_i) = \sum_{i=1}^m \frac{h(P, u_i)}{n} \text{vol}_{n-1}(F_i).
\]

5.8 Corollary. Let $P \in \mathcal{P}^n$, $\dim P = n$, and let $F_1, \ldots, F_m$ be the facets of $P$ with outer normal vectors $u_i \in \mathbb{R}^n$, $|u_i| = 1$. Then
\[
\sum_{i=1}^m \text{vol}_{n-1}(F_i) u_i = 0.
\]

Proof. We may assume that $0 \in \text{int } P$. Let $t = \sum_{i=1}^m \text{vol}_{n-1}(F_i) u_i$ and let $\lambda > 0$ such that $0 \in \text{int } (\lambda t + P)$. By Lemma 5.7 we get
\[
\text{vol} (P) = \text{vol} (\lambda, t + P) = \frac{1}{n} \sum_{i=1}^m \text{vol}_{n-1}(F_i) h(\lambda t + P, u_i)
\]
\[
= \frac{1}{n} \sum_{i=1}^m \text{vol}_{n-1}(F_i) h(P, u_i) + \frac{1}{n} \sum_{i=1}^m \text{vol}_{n-1}(F_i) \langle u_i, \lambda t \rangle
\]
\[
= \text{vol} (P) + \frac{\lambda}{n} \langle t, t \rangle.
\]
Hence $t = 0$. □

5.9 Corollary. Let $P \in \mathcal{P}^n$, $\dim P = n$, and let $F_1, \ldots, F_m$ be the facets of $P$ with outer normal vectors $u_i \in \mathbb{R}^n$, $|u_i| = 1$. Then
\[
\text{vol} (P) = \frac{1}{n} \sum_{i=1}^m \text{vol}_{n-1}(F_i) h(P, u_i).
\]

Proof. Let $t \in \mathbb{R}^n$ such that $0 \in \text{int } (t + P)$. As in the proof of Corollary 5.8 we find that
\[
\text{vol} (P) = \text{vol} (t + P) = \frac{1}{n} \sum_{i=1}^m \text{vol}_{n-1}(F_i) h(P, u_i) + \frac{1}{n} \left( \sum_{i=1}^m \text{vol}_{n-1}(F_i) u_i, t \right).
\]
According to Corollary 5.8 the last summand is 0. □
5.10 Theorem* [Minkowski]. Let $u_1, \ldots, u_m \in \mathbb{R}^n$, $|u_i| = 1$, such that $0 \in \text{int conv } \{u_1, \ldots, u_m\}$. Let $f_1, \ldots, f_m \in \mathbb{R}_{>0}$ be positive numbers such that $\sum_{i=1}^m f_i u_i = 0$. Then there exists an unique (up to translations) $n$-polytope $P \in \mathcal{P}^n$ with $m$ facets $F_1, \ldots, F_m$ such that $\text{vol}_{n-1}(F_i) = f_i$ and $u_1, \ldots, u_m$ are the outer unit normal vectors of the facets.

5.11 Definition [Valuations on $\mathcal{K}^n$]. A function $f : \mathcal{K}^n \to \mathbb{R}$ is called

i) monotonous, if $K_1 \subseteq K_2$ implies $f(K_1) \leq f(K_2)$,

ii) continuous, if $K_i \to K$ implies $f(K_i) \to f(K)$,

iii) homogeneous of degree $r$ if for $\lambda > 0$ it holds $f(\lambda K) = \lambda^r f(K)$,

iv) translation invariant, if for $t \in \mathbb{R}^n$ it holds $f(t + K) = f(K)$,

v) rotation invariant, if for any rotation $g : \mathbb{R}^n \to \mathbb{R}^n$ it holds $f(g(K)) = f(K)$,

vi) rigid motion invariant, if it is translation and rotation invariant,

vii) additive, if for $K_1, K_2 \in \mathcal{K}^n$ with $K_1 \cup K_2 \in \mathcal{K}^n$ it holds

\[ f(K_1 \cup K_2) = f(K_1) + f(K_2) - f(K_1 \cap K_2), \]

viii) simple, if $f(K) = 0$ for $\dim K \leq n - 1$.

An additive functional is also called a valuation.

5.12 Lemma. $\text{vol} : \mathcal{K}^n \to \mathbb{R}_{\geq 0}$ is continuous.

Proof. Let $(K_i)_{i \in \mathbb{N}} \subseteq \mathcal{K}^n$ with $\lim_{i \to \infty} K_i = K$, i.e., for each $\rho > 0$ there exists an $i_\rho$ with

\[ K_i \subseteq K + \rho B_n \quad \text{and} \quad K \subseteq K_i + \rho B_n \quad \text{for} \quad i \geq i_\rho. \]

We have to show $\lim_{i \to \infty} \text{vol}(K_i) = \text{vol}(K)$. First we treat the case when $\dim K \leq n - 1$, and w.l.o.g. let $K$ be contained in the hyperplane $H = \{x \in \mathbb{R}^n : x_n = 0\}$. Then we can find an $(n - 1)$-dimensional polytope $Q \subset H$, say, such that $(K + B_n) \cap H \subset Q$. For a positive number $\rho$ let $P_\rho$ be the prism given by $P_\rho = \text{conv } \{Q + \rho e_n, Q - \rho e_n\}$. The for $\rho \leq 1$ and $i \geq i_\rho$ we have $K_i \subseteq K + \rho B_n \subseteq P_\rho$, and thus (cf. Lemma 5.5 ii))

\[ 0 \leq \text{vol}(K_i) \leq \text{vol}(P_\rho) = 2 \rho \text{vol}_{n-1}(Q). \]

Hence $\lim_{i \to \infty} \text{vol}(K_i) = \text{vol}(K)$. In the case $\dim K = n$ we may assume that $r B_n \subseteq K$ for some positive number $r$, and by Exercise ?? we obtain for $i \geq i_\rho$

\[ \left(1 - \frac{\rho}{r}\right)^n \leq \frac{\text{vol}(K_i)}{\text{vol}(K)} \leq \left(1 + \frac{\rho}{r}\right)^n. \]

Thus $\text{vol}(K_i) \to \text{vol}(K)$.

5.13 Theorem. $\text{vol} : \mathcal{K}^n \to \mathbb{R}_{\geq 0}$ is a monotonous, continuous, rigid motion invariant, simple valuation, which is homogeneous of degree $n$.

5.14 Theorem* [Hadwiger’s characterization of the volume]. \(^{28}\) Let $\phi : \mathcal{K}^n \to \mathbb{R}$ such that

\[ \phi(K) = \phi(\lambda K) = \lambda^r \phi(K) \quad \text{for} \quad \lambda \in \mathbb{R}_{\geq 0}, \quad \phi(K_1 \cup K_2) = \phi(K_1) + \phi(K_2) - \phi(K_1 \cap K_2) \quad \text{for} \quad K_1, K_2 \in \mathcal{K}^n, \quad \phi(K) = 0 \quad \text{for} \quad \dim K \leq n - 1. \]

Then there exists a unique $r$ such that $\phi = \text{vol}_r$, and $r$ is called Hadwiger’s characteristic of the volume.

\(^{28}\)Hugo Hadwiger, 1908–1981
\( \mathbb{R} \) be a continuous, rigid motion invariant and simple valuation. Then there exists a constant \( c \in \mathbb{R} \) such that \( \phi(K) = c \text{vol}(K) \).

**5.15 Definition [Homothetic bodies].** Two convex bodies \( K, L \in \mathcal{K}^n \) are called homothetic iff there exist \( \mu \geq 0 \) and \( t \in \mathbb{R}^n \) such that \( K = t + \mu L \).

**5.16 Theorem [Brunn-Minkowski’s Theorem].** \(^{29}\) Let \( K, L \subset \mathbb{R}^n \) be compact, and let \( 0 \leq \lambda \leq 1 \). Then

\[
\text{vol} \left( \lambda K + (1 - \lambda) L \right)^{1/n} \geq \lambda \text{vol}(K)^{1/n} + (1 - \lambda) \text{vol}(L)^{1/n},
\]

i.e., the \( n \)-th root of the volume is a concave function.

Moreover, if \( K \) and \( L \) are convex then equality holds if and only if \( K \) and \( L \) are homothetic or \( K \) and \( L \) lie in parallel hyperplanes.

**Proof.** [(without the equality case)] By the homogeneity of the volume the inequality is equivalent to

\[
\text{vol}(K + L)^{1/n} \geq \text{vol}(K)^{1/n} + \text{vol}(L)^{1/n}.
\]

First we prove this inequality for the family of *polyboxes*, i.e., the union of a finite number of boxes with edges parallel to the coordinate axes and disjoint interiors, which we then use in order to approximate \( K \) and \( L \).

Let \( P \) and \( Q \) be two polyboxes consisting of \( m_P \) and \( m_Q \) boxes, respectively. We proceed by induction on \( m_P + m_Q \).

For \( m_P + m_Q = 2 \) we may assume \( P = a + ([0, \alpha_1] \times \cdots \times [0, \alpha_n]) \), i.e., \( P \) has edge length \( \alpha_i > 0 \) in direction \( e_i \), and let \( Q = b + ([0, \beta_1] \times \cdots \times [0, \beta_n]) \) for suitable \( a, b \in \mathbb{R}^n \). Then \( P + Q = a + b + ([0, \alpha_1 + \beta_1] \times \cdots \times [0, \alpha_n + \beta_n]) \), and for (5.16.2) in this particular case we have to show

\[
\text{vol}(P + Q)^{1/n} = \left( \prod_{i=1}^{n} (\alpha_i + \beta_i) \right)^{1/n} \geq \text{vol}(P)^{1/n} + \text{vol}(Q)^{1/n},
\]

which is equivalent to

\[
\left( \prod_{i=1}^{n} \frac{\alpha_i}{\alpha_i + \beta_i} \right)^{1/n} + \left( \prod_{i=1}^{n} \frac{\beta_i}{\alpha_i + \beta_i} \right)^{1/n} \leq 1.
\]

By the arithmetic-geometric mean inequality this is true.

So let \( m_P + m_Q > 2 \) and we assume \( m_p \geq 2 \). We translate the boxes of \( P \) such that two boxes \( P_1, P_2 \) of \( P \) lie on different sides of a coordinate hyperplane \( H \), i.e., \( P_1 \subset H^+ \) and \( P_2 \subset H^- \) and we may assume \( H = \{ x \in \mathbb{R}^n : x_n = 0 \} \).

Let \( P^+ = P \cap H^+ \), \( P^- = P \cap H^- \). Then both, \( P^+ \) and \( P^- \) have less than \( m_p \) boxes. Next we translate the polybox \( Q \) along the direction \( e_n \) such that

\( ^{29} \)Hermann Brunn, 1862–1939
\[ \frac{\text{vol}(Q^+)}{\text{vol}(Q)} = \frac{\text{vol}(P^+)}{\text{vol}(P)}, \text{ where } Q^+ = Q \cap H^+. \] Then with \( Q^- = Q \cap H^- \) we also have \( \frac{\text{vol}(P)}{\text{vol}(Q)} = \frac{\text{vol}(P^+)}{\text{vol}(Q^+)} = \frac{\text{vol}(P^-)}{\text{vol}(Q^-)} \).

\( P^+ + Q^+ \) and \( P^- + Q^- \) have no common interior points and both of them have less than \( m_P + m_Q \) boxes. Hence in view of our inductive argumentation we get

\[
\text{vol}(P + Q) = \text{vol}\left((P + Q) \cap H^+\right) + \text{vol}\left((P + Q) \cap H^-\right) \\
\geq \text{vol}(P^+ + Q^+) + \text{vol}(P^- + Q^-) \\
\geq \left(\text{vol}(P^+)^{1/n} + \text{vol}(Q^+)^{1/n}\right)^n + \left(\text{vol}(P^-)^{1/n} + \text{vol}(Q^-)^{1/n}\right)^n \\
= \text{vol}(P^+) \left(1 + \frac{\text{vol}(Q^+)^{1/n}}{\text{vol}(P^+)^{1/n}}\right)^n + \text{vol}(P^-) \left(1 + \frac{\text{vol}(Q^-)^{1/n}}{\text{vol}(P^-)^{1/n}}\right)^n \\
= \text{vol}(P) \left(1 + \frac{\text{vol}(Q)^{1/n}}{\text{vol}(P)^{1/n}}\right)^n = \left(\text{vol}(P)^{1/n} + \text{vol}(Q)^{1/n}\right)^n.
\]

Now given two Jordan measurable compact sets \( K \) and \( L \) we can approximate their volume by the volume of two sequences of polyboxes \( P_m \subset K \) and \( Q_m \subset L \), \( m \in \mathbb{N} \) consisting fo small cubes, i.e., \( \lim_{m \to \infty} \text{vol}(P_m) = \text{vol}(K) \) and \( \lim_{m \to \infty} \text{vol}(Q_m) = \text{vol}(L) \). Since \( P_m + Q_m \subset K + L \) we get

\[
\text{vol}(K + L)^{1/n} \geq \limsup_{m \to \infty} \text{vol}(P_m + Q_m)^{1/n} \\
\geq \lim_{m \to \infty} \text{vol}(P_m)^{1/n} + \lim_{m \to \infty} \text{vol}(Q_m)^{1/n} \\
= \text{vol}(K)^{1/n} + \text{vol}(L)^{1/n}.
\]

\[ \square \]

**5.17 Corollary.** Let \( K \subset \mathbb{R}^n \) with \( K = -K \). For a \( k \)-dimensinal linear subspace \( L_k \subset \mathbb{R}^n \), \( k \in \{1, \ldots, n-1\} \), and \( x \in \mathbb{R}^n \) let \( v(L_k, x) = \text{vol}_k(K \cap (x + L_k)) \). Then

\[ v(L_k, 0) \geq v(L_k, x) \]

for all \( x \in \mathbb{R}^n \), i.e., the volume maximal section of a o-symmetric convex body with a plane contains the origin.

**Proof.** Let \( K(x) = K \cap (x + L_k) \). Since \( K = -K \) it holds \( K(x) = -K(-x) \), and so \( v(L_k, x) = v(L_k, -x) \). By the convexity we have \( (1/2)K(x) + (1/2)K(-x) \subset K(0) \) and together with Theorem 5.16 we get

\[
v(L_k, 0) = \text{vol}_k(K(0)) \geq \text{vol}_k\left(\frac{1}{2}K(x) + \frac{1}{2}K(-x)\right) \\
\geq \left(\frac{1}{2}\text{vol}_k(K(x))^\frac{1}{n} + \frac{1}{2}\text{vol}_k(K(-x))^\frac{1}{n}\right)^k = \text{vol}_k(K(x)) = v(L_k, x).
\]

\[ \square \]
5.18 Proposition [Multiplicative version of the B-M inequality (5.16.1)].
Let $K, L \subset \mathbb{R}^n$ be compact. Inequality (5.16.1) is equivalent to
\[
\text{vol} \left( \lambda K + (1 - \lambda) L \right) \geq \text{vol} (K)^\lambda \text{vol} (L)^{1-\lambda} \quad \text{for all } \lambda \in [0, 1].
\] (5.18.1)

Proof. By the Brunn-Minkowski inequality (5.16.1) and the weighted arithmetic-geometric mean inequality we get for $\lambda \in [0, 1]$\[
\text{vol} \left( \lambda K + (1 - \lambda) L \right)^{1/n} \geq \lambda \text{vol} (K)^{1/n} + (1 - \lambda) \text{vol} (L)^{1/n} \geq \text{vol} (K)^{\lambda/n} \text{vol} (L)^{(1-\lambda)/n}.
\]

For the reverse implication we may assume that $\text{vol} (K), \text{vol} (L) \neq 0$ and we set
\[
\lambda = \frac{\text{vol} (K)^{1/n}}{\text{vol} (K)^{1/n} + \text{vol} (L)^{1/n}} \in (0, 1).
\]
Then $\text{vol} (K)/\lambda^n = \text{vol} (L)/(1-\lambda)^n = (\text{vol} (K)^{1/n} + \text{vol} (L)^{1/n})^{n}$ and with (5.18.1) we get\[
\text{vol} (K + L) = \text{vol} \left( \lambda \frac{K}{\lambda} + (1 - \lambda) \frac{L}{1 - \lambda} \right) \geq \left( \frac{\text{vol} (K)}{\lambda^n} \right)^\lambda \left( \frac{\text{vol} (L)}{(1 - \lambda)^n} \right)^{(1-\lambda)} = \left( \frac{\text{vol} (K)}{\lambda^n} \right)^\lambda \left( \frac{\text{vol} (K)}{\lambda^n} \right)^{(1-\lambda)} = \left( \frac{\text{vol} (K)}{\lambda^n} \right) = (\text{vol} (K)^{1/n} + \text{vol} (L)^{1/n})^{n}.
\]

\[\square\]

5.19 Proposition. Let $K_i \subset \mathbb{R}^n$, $\lambda_i \in \mathbb{R}_{\geq 0}$, $1 \leq i \leq r$, and let $K = \sum_{i=1}^r \lambda_i K_i$. Then $h(K, \cdot) = \sum_{i=1}^r \lambda_i h(K_i, \cdot)$ and for $u \in \mathbb{R}^n$ we have
\[
K \cap H(u, h(K, u)) = \sum_{i=1}^r \lambda_i \left[ K_i \cap H(u, h(K_i, u)) \right].
\]

Proof. Let $u \in \mathbb{R}^n \setminus \{0\}$. Since $K_i$ are compact $1 \leq i \leq r$, there exist $x_i \in K_i$ such that $h(K_i, u) = \langle x_i, u \rangle$. Hence $\sum_{i=1}^r \lambda_i h(K_i, u) = \langle \sum_{i=1}^r \lambda_i x_i, u \rangle \leq h(K, u)$. On the other hand, any $x \in K$ can be written as $x = \sum_{i=1}^r \lambda_i x_i$ with $x_i \in K_i$ and so $\langle x, u \rangle = \sum_{i=1}^r \lambda_i \langle x_i, u \rangle \leq \sum_{i=1}^r \lambda_i h(K_i, u)$. Hence $h(K, u) \leq \sum_{i=1}^r \lambda_i h(K_i, u)$.

For the second property we notice that $x = \sum_{i=1}^r \lambda_i x_i \in (K \cap H(u, h(K, u)))$ with $x_i \in K_i$, if and only if $\sum_{i=1}^r \lambda_i \langle x_i, u \rangle = h(K, u) = \sum_{i=1}^r \lambda_i h(x_i, u)$, which is equivalent to $\langle x_i, u \rangle = h(K_i, u)$ for all $i = 1, \ldots, r$, or in other words $x_i \in K_i \cap H(u, h(K_i, u))$ for all $i = 1, \ldots, r$. \[\square\]

5.20 Theorem [Mixed volumes]. Let $r \in \mathbb{N}$, $\lambda_i \in \mathbb{R}_{\geq 0}$ and $K_i \subset \mathbb{R}^n$, $1 \leq i \leq r$. There are non-negative coefficients $\text{V}(K_{i_1}, \ldots, K_{i_n})$, called mixed volumes, which are symmetric in the indices, such that
\[
\text{vol} \left( \sum_{i=1}^r \lambda_i K_i \right) = \sum_{i_1, \ldots, i_n=1}^r \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_n} \text{V}(K_{i_1}, \ldots, K_{i_n}),
\]
i.e., \( \text{vol} \left( \sum_{i=1}^{r} \lambda_i K_i \right) \) is a homogeneous polynomial of degree \( n \) in the variables \( \lambda_1, \ldots, \lambda_r \). Moreover, \( V(K_{i_1}, \ldots, K_{i_n}) \) depends only on the bodies \( K_{i_1}, \ldots, K_{i_n} \).

**Proof.** First we will prove the theorem for polytopes \( P_i \in \mathcal{P}^n, 1 \leq i \leq r \), by induction on the dimension.

If \( n = 1 \) then \( P_i = [\alpha_i, \beta_i] \) with \( \alpha_i, \beta_i \in \mathbb{R}, 1 \leq i \leq r \). Hence \( \sum_{i=1}^{r} \lambda_i P_i = [\sum_{i=1}^{r} \lambda_1 \alpha_i, \sum_{i=1}^{r} \lambda_1 \beta_i] \), and we get \( \text{vol} \left( \sum_{i=1}^{r} \lambda_i P_i \right) = \sum_{i=1}^{r} \lambda_i \text{vol} (P_i) \).

So let \( n \geq 2 \), and we assume for a moment that \( 0 \not\in P_i, 1 \leq i \leq r \), and \( \dim P_r = n, \lambda_r > 0 \). Then \( P = \sum_{i=1}^{r} \lambda_i P_i \) is an \( n \)-dimensional polytope and we define
\[
U(P) = \left\{ u \in S^{n-1} : \dim \left( P \cap H(u, h(P, u)) \right) = n - 1 \right\},
\]
i.e., the set of outer normal vectors to the facets of \( P \). For \( u \in U(P) \) we write \( F_u(P) = P \cap H(u, h(P, u)) \) and \( F_u(P_i) = P_i \cap H(u, h(P_i, u)) \). By Proposition 5.19 we get
\[
h(P, u) = \sum_{i=1}^{r} \lambda_i h(P_i, u) \quad \text{and} \quad F_u(P) = \sum_{i=1}^{r} \lambda_i F_u(P_i).
\]

All \( F_u(P_i), 1 \leq i \leq r \), lie in parallel hyperplanes and since the volume is invariant under translations we may apply our induction hypothesis to \( \sum_{i=1}^{r} \lambda_i F_u(P_i) \) and obtain
\[
\text{vol}_{n-1} (F_u(P)) = \text{vol}_{n-1} \left( \sum_{i=1}^{r} \lambda_i F_u(P_i) \right)
= \sum_{i_1, \ldots, i_{n-1}=1}^{r} \lambda_{i_1} \cdot \cdots \cdot \lambda_{i_{n-1}} V(F_u(P_{i_1}), \ldots, F_u(P_{i_{n-1}})).
\]
Together with Corollary 5.9 we get
\[
\text{vol} (P) = \sum_{u \in U(P)} \frac{h(P, u)}{n} \text{vol}_{n-1} (F_u(P))
= \sum_{u \in U(P)} \left( \sum_{i_1=1}^{r} \lambda_{i_1} \frac{h(P_{i_1}, u)}{n} \right) \left( \sum_{i_1, \ldots, i_{n-1}=1}^{r} \lambda_{i_1} \cdot \cdots \cdot \lambda_{i_{n-1}} V(F_u(P_{i_1}), \ldots, F_u(P_{i_{n-1}})) \right)
= \sum_{i_1, \ldots, i_{n}=1}^{r} \lambda_{i_1} \cdot \cdots \cdot \lambda_{i_{n}} \nabla(P_{i_1}, \ldots, P_{i_{n}}),
\]
where
\[
\nabla(P_{i_1}, \ldots, P_{i_{n}}) = \sum_{u \in U(P)} \frac{h(P_{i_{n}}, u)}{n} V(F_u(P_{i_1}), \ldots, F_u(P_{i_{n-1}})).
\]
Since \( 0 \not\in P_i \) we have \( h(P_i, u) \geq 0 \) and by our induction hypothesis we conclude \( \nabla(P_{i_1}, \ldots, P_{i_{n}}) \geq 0 \). This polynomial can also be rewritten as one with symmetric non-negative coefficients \( V(P_{i_1}, \ldots, P_{i_{n}}) \) and so we know
\[
\text{vol} \left( \sum_{i=1}^{r} \lambda_i P_i \right) = \sum_{i_1, \ldots, i_{n}=1}^{r} \lambda_{i_1} \cdot \cdots \cdot \lambda_{i_{n}} V(P_{i_1}, \ldots, P_{i_{n}}), \tag{5.20.1}
\]
Since the left hand side – and thus the right hand side – is invariant with respect to translations of the $P_i$, (5.20.1) holds also if $0 \notin P_i$. Since the left and right hand side are continuous in $\lambda$, the equation also holds for $\lambda = 0$.

If $\dim P_i \leq n - 1$, $1 \leq i \leq r$, then we choose an arbitrary $n$-polytope $P_{r+1}$, apply (5.20.1) to $\sum_{i=1}^{r+1} \lambda_i P_{r+1}$ and finally set $\lambda_{r+1} = 0$ on both sides of (5.20.1). Hence we have proven the theorem for arbitrary polytopes $P_1, \ldots, P_r$.

Now let $K_i \in \mathcal{K}^n$, $1 \leq i \leq r$, and let $P_i^j \in \mathcal{P}^n$ with $\lim_{j \to \infty} P_i^j = K_i$. For any arbitrary (but fixed) $\lambda_i \in \mathbb{R}_{\geq 0}$ we have $\lambda_i P_i^j \to \lambda_i K_i$ and hence $\sum_{i=1}^r \lambda_i P_i^j \to \sum_{i=1}^r \lambda_i K_i$. Therefore,

$$\text{vol} \left( \sum_{i=1}^r \lambda_i K_i \right) = \lim_{j \to \infty} \text{vol} \left( \sum_{i=1}^r \lambda_i P_i^j \right) = \lim_{j \to \infty} \sum_{i_1, \ldots, i_n=1}^r \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_n} V(P_{i_1}^j, \ldots, P_{i_n}^j).$$

Observe that the coefficients $V(P_{i_1}^j, \ldots, P_{i_n}^j)$ are the same for any choice of (fixed) scalars $\lambda \geq 0$. $V(P_{i_1}^j, \ldots, P_{i_n}^j)$ is non-negative and by setting $\lambda_i = 1$, $1 \leq i \leq r$, we see that they are all bounded by $\text{vol} \left( \sum_{i=1}^r K_i \right)$. Hence each sequence $(V(P_{i_1}^j, \ldots, P_{i_n}^j))_{j \in \mathbb{N}}$ contains a convergent subsequence and so we may assume that $\lim_{j \to \infty} V(P_{i_1}^j, \ldots, P_{i_n}^j) = V(K_{i_1}, \ldots, K_{i_n})$, say. Thus

$$\text{vol} \left( \sum_{i=1}^r \lambda_i K_i \right) = \sum_{i_1, \ldots, i_n=1}^r \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_n} V(K_{i_1}, \ldots, K_{i_n}) \quad (5.20.2)$$

for any arbitrary (but fixed) $\lambda_i \geq 0$, $1 \leq i \leq r$. Finally we observe that $V(K_{i_1}, \ldots, K_{i_n})$ are non-negative and by choosing the proper subsequences we may also assume that these numbers $V(K_{i_1}, \ldots, K_{i_n})$ are symmetric in the indices.

Now let $j_i \in \{1, \ldots, r\}$ for $1 \leq i \leq n$ and let $J = \{j_1, \ldots, j_n\}$. Setting $\lambda_i = 0$ for $i \notin J$ in (5.20.2) gives

$$\text{vol} \left( \sum_{i \in J} \lambda_i K_i \right) = \sum_{i_1, \ldots, i_n=1, i_j \in J}^r \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_n} V(K_{i_1}, \ldots, K_{i_n})$$

which shows that $V(K_{j_1}, \ldots, K_{j_n})$ depends only on $K_{j_1}, \ldots, K_{j_n}$. \hfill $\Box$

5.21 Notation. Let $K_1, \ldots, K_r \in \mathcal{K}^n$ and let $k_i \in \{0, \ldots, n\}$ with $n = \sum_{i=1}^r k_i$.

i) Instead of $V(K_{i_1}, \ldots, K_{i_n})$ we also write $V(K_1, k_1; \ldots; K_r, k_r)$, where the $k_i$ are the multiplicities of the body $K_i$ in the sequence $(K_1, \ldots, K_n)$.

ii) Let $(k_1, \ldots, k_r)^n$ be the multinomial coefficient, i.e., $(k_1, \ldots, k_r)^n = \frac{n!}{k_1! \cdots k_r!}$.

5.22 Remark. With the notation above we may write

$$\text{vol} \left( \sum_{i=1}^r \lambda_i K_i \right) = \sum_{k_1 + \cdots + k_r = n} (k_1, \ldots, k_r)^n \lambda_1^{k_1} \cdot \ldots \cdot \lambda_r^{k_r} V(K_1, k_1; \ldots; K_r, k_r).$$
In particular, for \( r = 2 \) we get

\[
\text{vol} (\lambda_1 K_1 + \lambda_2 K_2) = \sum_{i=0}^{n} \binom{n}{i} \lambda_1^i \lambda_2^{n-i} \text{V}(K_1, i; K_2, n-i).
\]

**5.23 Theorem.** Let \( K_1, \ldots, K_n \in \mathcal{K}^n \).

i) \( \text{V}(K_1, \ldots, K_n) = \text{vol}(K_n) \).

ii) Let \( A \in \mathbb{R}^{n \times n} \), \( \det A \neq 0 \), and let \( t_i \in \mathbb{R}^n \), \( 1 \leq i \leq r \). Then \( \text{V}(t_1 + AK_1, t_2 + AK_2, \ldots, t_n + AK_n) = |\det A| \text{V}(K_1, \ldots, K_n) \).

iii) Mixed volumes are continuous functions on \((\mathcal{K}^n)^n\).

iv) Mixed volumes are linear in each argument, i.e.,

\[
\text{V}(\lambda K_1 + \mu L, K_2, \ldots, K_n) = \lambda \text{V}(K_1, \ldots, K_n) + \mu \text{V}(L, K_2, \ldots, K_n)
\]

for \( \mu, \lambda \in \mathbb{R} \geq 0 \), \( K, L \in \mathcal{K}^n \).

v) Mixed volumes are valuations in each argument, i.e., let \( K, L, K \cup L \in \mathcal{K}^n \) it holds

\[
\text{V}(K \cup L, K_2, \ldots, K_n) = \text{V}(K, K_2, \ldots, K_n) + \text{V}(L, K_2, \ldots, K_n) - \text{V}(K \cap L, K_2, \ldots, K_n).
\]

*Proof.* For i) we just set \( \lambda_m = 1 \) and \( \lambda_i = 0 \) for \( i \neq m \) in the mixed volume formula in Theorem 5.20. For ii) we apply Theorem 5.20 to the bodies \( t_i + AK_i \) and use Remark 5.2 iii)

\[
\sum_{i_1, \ldots, i_n=1}^{n} \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_n} \text{V}(t_{i_1} + AK_{i_1}, \ldots, t_{i_n} + AK_{i_n})
\]

\[
= \text{vol} \left( \sum_{i=1}^{n} \lambda_i (t_i + AK_i) \right) = \text{vol} \left( \sum_{i=1}^{n} \lambda_i t_i + A \left( \sum_{i=1}^{n} \lambda_i K_i \right) \right)
\]

\[
= |\det A| \text{vol} \left( \sum_{i=1}^{r} \lambda_i K_i \right) = \sum_{i_1, \ldots, i_n=1}^{n} \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_n} |\det A| \text{V}(K_{i_1}, \ldots, K_{i_n}).
\]

Comparing the coefficients of the two multivariate polynomials on the left and right hand side gives the results. The other properties are left as an exercise.

**□**

**5.24 Lemma.** Let \( K_1, K_2 \in \mathcal{K}^n \) and \( k_1 > \dim K_1 \). Then

\[
\text{V}(K_1, k_1; K_2, n-k_1) = 0.
\]

*Proof.* Let \( m = \dim K_1 \leq n \), and we assume that

\[
K_1 \subset L_m = \{ x \in \mathbb{R}^n : x_{m+1} = \ldots = x_n = 0 \}.
\]
We write $L_m^\perp = \{ x \in \mathbb{R}^n : x_1 = \cdots = x_m = 0 \}$. Let $\tilde{K}_2$ and $\overline{K}_2$ be, respectively, the orthogonal projections of $K_2$ onto $L_m$ and $L_m^\perp$. Then $K_2 \subseteq K_2 + \tilde{K}_2$ and so

$$
\sum_{i=0}^n \binom{n}{i} \lambda^i \nu(K_1, i; K_2, n - i) = \text{vol}(\lambda K_1 + K_2) \leq \text{vol}(\lambda K_1 + \tilde{K}_2 + \overline{K}_2) \\
= \text{vol}_m(\lambda K_1 + \tilde{K}_2) \cdot \text{vol}_{n-m}(\overline{K}_2) \\
= \sum_{i=0}^m \binom{m}{i} \lambda^i \nu(K_1, i; \tilde{K}_2, m - i) \text{vol}_{n-m}(\overline{K}_2)
$$

by Remark 5.6 iv), where $\nu(K_1; i; \tilde{K}_2, m - i)$ represents the $m$-dimensional mixed volume in $L_m$. Hence the coefficients on the left hand side of the terms $\lambda^i$ for $i > m$ have to be zero.

**5.25 Lemma.** Let $P \in \mathcal{P}^n$, $\dim P \geq n - 1$, and let $u_1, \ldots, u_m \in S^{n-1}$ be the unit outer normal vectors of the $(n-1)$-faces $F_i$ of $P$, $1 \leq i \leq m$. Let $G_1, \ldots, G_p$ be the $(n-2)$-faces of $P$ and let $K \in 
abla^m$. For $\lambda \geq 0$ it holds

$$
\text{vol}(P) + \lambda \sum_{i=1}^m h(K, u_i) \text{vol}_{n-1}(F_i) + \sum_{i=1}^p \text{vol}(G_i + 2\lambda D(K)B_n) \\
\geq \text{vol}(P + \lambda K) \geq \text{vol}(P) + \lambda \sum_{i=1}^m h(K, u_i) \text{vol}_{n-1}(F_i),
$$

where $D(K) = \max\{|x - y| : x, y \in K\}$ denotes the diameter of $K$.

**Proof.** We may assume $0 \in \text{relint } K$ (cf. Corollary 5.8) and $\lambda > 0$ is fixed but arbitrary. Let $v_1, \ldots, v_m \in K$ with $h(K, u_i) = \langle u_i, v_i \rangle \geq 0$ and let $S_i(\lambda) = \text{conv} \{ F_i, \lambda v_i + F_i \}$. $S_i(\lambda)$ is a prism with basis $F_i$ and height $h(K, u_i) = \lambda \langle u_i, v_i \rangle$, and so

$$
\text{vol}(S_i(\lambda)) = \lambda \text{vol}_{n-1}(F_i) h(K, u_i).
$$

Next we observe that two of these prisms have no interior points in common. Otherwise, there exist $y_i \in \text{relint } F_i, y_j \in \text{relint } F_j$ and $\mu_i, \mu_j \in [0, 1]$ with $y_i + \mu_i \lambda v_i = y_j + \mu_j \lambda v_j$. Hence $\langle u_i, y_i \rangle + \mu_i \lambda \langle u_i, v_i \rangle = \langle u_i, y_j \rangle + \mu_j \lambda \langle u_i, v_j \rangle$. Since $\langle u_i, y_i \rangle > \langle u_i, y_j \rangle$ and $\langle u_i, v_i \rangle \geq \langle u_i, v_j \rangle$ we get $\mu_i < \mu_j$. Analogously, taking the inner product with $u_j$ we obtain $\mu_j < \mu_i$, a contradiction. Since $P \cup (\bigcup_{i=1}^m S_i(\lambda)) \subseteq P + \lambda K$ and no interior points of $P$ lie in one the prisms $S_i(\lambda)$, we get the lower bound

$$
\text{vol}(P + \lambda K) \geq \text{vol}(P) + \lambda \sum_{i=1}^m \text{vol}_{n-1}(F_i) h(K, u_i).
$$

In order to prove the upper bound we estimate the volume of $(P + \lambda K) \setminus [P \cup (\bigcup_{i=1}^m S_i(\lambda))]$. To this end let $x \in P + \lambda K$ but $x \notin P \cup (\bigcup_{i=1}^m S_i(\lambda))$. Then we may write $x = x_i + \lambda v$ with $v \in K$, $x_i$ is contained in an $(n-1)$-face $F_i$, say, and $\langle u_i, v \rangle > 0$. 
In the case \( x_i \in \text{relbd} F_i \) there exists an \((n - 2)\)-face \( G_i \) of \( P \) containg \( x_i \) and so
\[
x \in G_i + \lambda K \subseteq G_i + \lambda \text{D}(K) B_n.
\] (5.25.1)

Next let \( x_i \in \text{relin} F_i \). The ray \( \{ x_i + \mu v : \mu \geq 0 \} \) intersects \( \text{bd} S_i(\lambda) \) in a point \( y = x_i + \mu v \) with \( \mu > 0 \). Since \( x \not\in S_i(\lambda) \) we have \( \mu < \lambda \) and since \( y \in \text{bd} S_i(\lambda) \) there exist \( y_i \in F_i \) and \( \mu_i \in [0, 1] \) with \( y = y_i + \mu_i \lambda v_i \). Thus
\[
x = x_i + \lambda v = y + (\lambda - \mu) v = y_i + \mu_i \lambda v_i + (\lambda - \mu) v.
\]

If \( \mu_i = 1 \) then
\[
\langle u_i, x \rangle = \langle u_i, y_i \rangle + \lambda \langle u_i, v_i \rangle + (\lambda - \mu) \langle u_i, v \rangle > h(P, u_i) + h(\lambda K, u_i),
\]
which contradicts \( x \in P + \lambda K \). Hence, \( y \) is not contained in the “upper” boundary of \( S_i(\lambda) \). If \( \mu_i = 0 \) then \( y_i = x_i + \mu v \) and we get the contradiction
\[
0 = \langle u_i, y_i - x_i \rangle = \mu \langle u_i, v \rangle.
\]
Hence, \( y \) is also not contained in the “lower” boundary of \( S_i(\lambda) \) which implies that \( y_i \in \text{relbd} F_i \). Thus there exists an \((n - 2)\)-face \( G_j \) of \( P \) with \( y_i \in G_j \) and so
\[
x \in G_j + 2\lambda \text{D}(K) B_n.
\]
Together with (5.25.1) we have shown \((P + \lambda K) \setminus [P \cup (\bigcup_{i=1}^m S_i(\lambda))] \subset \bigcup_{i=1}^p (G_i + 2\lambda \text{D}(K) B_n)\) which yields the required upper bound.

5.26 Theorem. Let \( P \in \mathcal{P}^n \), \( \dim P \geq n - 1 \), and let \( u_1, \ldots, u_m \in S^{n-1} \) be the unit outer normal vectors of the \((n - 1)\)-faces \( F_i \) of \( P \), \( 1 \leq i \leq m \). For \( K \in \mathcal{K}^n \) we have
\[
V(P, n - 1; K, 1) = \frac{1}{n} \sum_{i=1}^m h(K, u_i) \text{vol}_{n-1}(F_i).
\]

Proof. For \( \lambda > 0 \) we have
\[
\text{vol}(P + \lambda K) = \text{vol}(P) + n\lambda V(P, n - 1; K, 1) + \sum_{i=0}^{n-2} \binom{n}{i} \lambda^{n-i} V(P, i; K, n - i),
\]
and hence
\[
n V(P, n - 1; K, 1) = \lim_{\lambda \to 0} \frac{\text{vol}(P + \lambda K) - \text{vol}(P)}{\lambda}.
\]
Plugging in the bounds of Lemma 5.25 for the numerator gives
\[
\sum_{i=1}^m h(K, u_i) \text{vol}_{n-1}(F_i) \leq n V(P, n - 1; K, 1)
\]
\[
\leq \sum_{i=1}^m h(K, u_i) \text{vol}_{n-1}(F_i) + \lim_{\lambda \to 0} \sum_{i=1}^p \frac{\lambda}{\text{vol}(G_i + 2\lambda \text{D}(K) B_n)}.
\]
Since $\dim G_i = n - 2$ we get in view of Lemma 5.24

\[
\sum_{i=1}^{p} \text{vol}(G_i + 2\lambda D(K)B_n) = \sum_{i=1}^{p} \left( \sum_{j=0}^{n} \binom{n}{j} (2\lambda D(K))^{n-j} V(G_i, j; B_n, n-j) \right)
\]

\[
= \sum_{i=1}^{p} \left( \sum_{j=0}^{n-2} \binom{n}{j} (2\lambda D(K))^{n-j} V(G_i, j; B_n, n-j) \right)
\]

and so $\lim_{\lambda \to 0} \frac{\sum_{i=1}^{p} \text{vol}(G_i + 2\lambda D(K)B_n)}{\lambda} = 0$. □

5.27 Corollary. Let $K \in \mathcal{K}^n$ and $u \in S^{n-1}$, then

\[V(K, n-1; \text{conv} \{-u, u\}, 1) = \frac{2}{n} \text{vol}_{n-1}(K|u^\perp),\]

where $K|u^\perp$ denotes the orthogonal projection of $K$ onto the hyperplane $u^\perp = H(u, 0)$.

Proof. Let $\dim K \geq n - 1$ and first we show it for a polytope $P \in \mathcal{P}^n$ with outer unit normal vectors $v_1, \ldots, v_m \in S^{n-1}$ of the $(n-1)$-faces $F_i$, $1 \leq i \leq m$. By Theorem 5.26 we get for $u \in S^{n-1}$

\[
V(P, n-1; \text{conv} \{-u, u\}, 1) = \frac{1}{n} \sum_{i=1}^{m} h(\text{conv} \{-u, u\}, v_i) \text{vol}_{n-1}(F_i)
\]

\[
= \frac{1}{n} \sum_{i=1}^{m} |\langle u, v_i \rangle| \text{vol}_{n-1}(F_i) = \frac{1}{n} \sum_{i=1}^{m} \text{vol}_{n-1}(F_i|u^\perp)
\]

\[
= \frac{2}{n} \text{vol}_{n-1}(P|u^\perp);
\]

thus, $V(K, n-1; \text{conv} \{-u, u\}, 1) = (2/n)\text{vol}_{n-1}(K|u^\perp)$ for any convex body $K \in \mathcal{K}^n$ by the usual approximation argument. □

5.28 Theorem*. Let $K_i \in \mathcal{K}^n$, $i = 1, \ldots, r$, $K_1 \subseteq \overline{K}_1 \in \mathcal{K}^n$, and $i_j \in \{1, \ldots, r\}$, $j = 2, \ldots, n$. Then

\[V(K_1, K_{i_2}, \ldots, K_{i_n}) \leq V(\overline{K}_1, K_{i_2}, \ldots, K_{i_n}).\]

5.29 Definition [Surface area]. Let $K \in \mathcal{K}^n$.

\[F(K) = n V(K, n-1; B_n, 1)\]

is called the surface area of $K$.

5.30 Corollary. Let $P \in \mathcal{P}^n$, $\dim P \geq n - 1$, and let $u_1, \ldots, u_m \in S^{n-1}$ be the unit outer normal vectors of the $(n-1)$-faces $F_i$ of $P$, $1 \leq i \leq m$. Then

\[F(P) = \sum_{i=1}^{m} \text{vol}_{n-1}(F_i).\]
Proof. By Theorem 5.26 we have

\[ F(P) = n \, V(P, n - 1; B_n, 1) = \sum_{i=1}^{m} h(B_i, u_i) \text{vol}_{n-1}(F_i) = \sum_{i=1}^{m} \text{vol}_{n-1}(F_i). \]

\[ \square \]

5.31 Proposition.

i) \( F : K^n \to \mathbb{R}_{\geq 0} \) is a monotonous, continuous, rigid motions invariant valuation, which is homogeneous of degree \( n - 1 \).

ii) \( F(K) = 0 \) if (and only if) \( \dim K \leq n - 2 \).

iii) \( F(K) = \lim_{\lambda \to 0} \frac{\text{vol}(K+\lambda B_n) - \text{vol}(K)}{\lambda} \).

iv) \( F(B_n) = n \text{vol}(B_n) \).

Proof. i) is a consequence of the general properties of mixed volumes (cf. Theorem 5.28, Theorem 5.23 ii), iii), v)). The \( (n-1) \)-homogeneity follows from property iv) of Theorem 5.23.

For (the one direction of) ii) see Lemma 5.24, and iii) and iv) are immediate consequences of the definition of the surface area. \( \square \)

5.32 Theorem [Minkowski’s inequalities]. Let \( K, L \in K^n \) with \( \dim K = n \). Then

i) \( V(K, n - 1; L, 1)^n \geq \text{vol}(K)^{n-1} \text{vol}(L) \) and equality holds if and only if \( K \) and \( L \) are homothetic.

ii) \( V(K, n - 1; L, 1)^2 \geq \text{vol}(K) \text{V}(K, n - 2; L, 2) \).

Proof. For \( \lambda \in [0, 1] \) let \( K_\lambda = (1 - \lambda)K + \lambda L \). Then

\[ \text{vol}(K_\lambda) = \sum_{i=0}^{n} \binom{n}{i} (1 - \lambda)^i \lambda^{n-i} \text{V}(K; i; L, n - i). \quad (5.32.1) \]

We consider the function

\[ f(\lambda) = \text{vol}(K_\lambda)^{1/n} - (1 - \lambda)\text{vol}(K)^{1/n} - \lambda\text{vol}(L)^{1/n}. \]

By Brunn-Minkowski’s theorem 5.16 \( f(\lambda) \) is the sum of a concave function and two linear functions; hence it is concave and by Theorem 5.16 we also know that it is non-negative. Moreover, since \( \dim K = n \) we have \( \text{vol}(K_\lambda) > 0 \) for \( \lambda \in [0, 1] \) which shows by (5.32.1) that \( f \) is differentiable on \([0, 1]\). The concavity of \( f \) together with \( f(0) = f(1) = 0 \) gives \( f'(0) \geq 0 \) with \( f'(0) = 0 \) if and only if \( f(\lambda) = 0 \) for all \( \lambda \in [0, 1] \). In view of (5.32.1) we get

\[ 0 \leq f'(0) \]

\[ = \frac{1}{n} \text{vol}(K)^{(1/n)-1} [\text{vol}(K_\lambda)]'_{0} + \text{vol}(K)^{1/n} - \text{vol}(L)^{1/n} \]

\[ = \frac{1}{n} \text{vol}(K)^{(1/n)-1} [n\text{V}(K, n - 1; L, 1) - n\text{vol}(K)] + \text{vol}(K)^{1/n} - \text{vol}(L)^{1/n} \]

\[ = \text{vol}(K)^{-(n-1)/n} \text{V}(K, n - 1; L, 1) - \text{vol}(L)^{1/n}. \]
Hence, $V(K, n - 1; L, 1) \geq \text{vol}(K)^{(n-1)/n} \text{vol}(L)^{1/n}$, with equality if and only if $f(\lambda) = 0$ for all $\lambda \in [0, 1]$, i.e., if and only if $K$ and $L$ are homothetic (cf. Theorem 5.16). This shows i).

Since $f$ is concave we also have $f''(\lambda) \leq 0$, and so by using again (5.32.1)

$$0 \geq f''(0) = -(n - 1)\text{vol}(K)^{-2(n-1)/n} [V(K, n - 1; L, 1)^2 - \text{vol}(K)V(K, n - 2; L, 2)].$$

On account of $\text{vol}(K) > 0$ we get ii).

5.33 Corollary [Isoperimetric inequality]. Let $K \in K^n$ with $\dim K = n$. Then

$$\frac{F(K)^n}{\text{vol}(K)^{n-1}} \geq \frac{F(B_n)^n}{\text{vol}(B_n)^{n-1}} = n^n \text{vol}(B_n),$$

and equality holds if and only if $K$ is an $n$-dimensional ball.

Proof. Minkowski’s first inequality (Theorem 5.32 i)) gives

$$F(K)^n = n^n V(K, n - 1; B_n, 1)^n \geq n^n \text{vol}(K)^{n-1} \text{vol}(B_n),$$

with equality if and only if $K$ and $B_n$ are homothetic.
6 A glimpse on geometry of numbers

6.1 Definition [Lattice]. Let \( b_1, \ldots, b_n \in \mathbb{R}^n \) be linearly independent. The set

\[
\Lambda = \{ z_1 b_1 + z_2 b_2 + \cdots + z_n b_n : z_i \in \mathbb{Z}, 1 \leq i \leq n \}
\]

is called lattice. The set of generating vectors \( \{ b_1, \ldots, b_n \} \) or the matrix \( B = (b_1, \ldots, b_n) \) with columns \( b_i \) is called basis(matrix) of \( \Lambda \). An element \( b \in \Lambda \) is called lattice point of \( \Lambda \). The set of all lattices in \( \mathbb{R}^n \) is denoted by \( \mathcal{L}^n \).

6.2 Remark.

i) The unit vectors \( e_1, \ldots, e_n \in \mathbb{R}^n \) form a basis of the integral lattice (standard lattice) \( \mathbb{Z}^n = \{ z \in \mathbb{R}^n : z_i \in \mathbb{Z} \} \).

ii) Let \( B = (b_1, \ldots, b_n) \) be a basis of \( \Lambda \). Then \( \Lambda = B \mathbb{Z}^n \) and, in particular, \( \Lambda \) is a subgroup of \( \mathbb{R}^n \), i.e., \( b - \bar{b} \in \Lambda \) for all \( b, \bar{b} \in \Lambda \).

iii) \( \Lambda = \begin{pmatrix} 25 & 64 \\ 16 & 41 \end{pmatrix} \mathbb{Z}^2 = \mathbb{Z}^2 \).

6.3 Definition [Unimodular matrix]. An integral matrix \( U \in \mathbb{Z}^{n \times n} \) is called unimodular iff \( |\det U| = 1 \). The group of all unimodular matrices is denoted by \( GL(n, \mathbb{Z}) \).

6.4 Proposition. \( GL(n, \mathbb{Z}) = \{ U \in \mathbb{R}^{n \times n} : U \mathbb{Z}^n = \mathbb{Z}^n \} \).

Proof. \( U \in GL(n, \mathbb{Z}) \) if and only if \( U, U^{-1} \in \mathbb{Z}^{n \times n} \) which is equivalent to \( U \mathbb{Z}^n \subseteq \mathbb{Z}^n \) and \( U^{-1} \mathbb{Z}^n \subseteq \mathbb{Z}^n \). Since the last inclusion is equivalent to \( \mathbb{Z}^n \subseteq U \mathbb{Z}^n \) we are done. \( \square \)

6.5 Lemma. Let \( \Lambda = B \mathbb{Z}^n \in \mathcal{L}^n \). \( A = (a_1, \ldots, a_n) \) is a basis of \( \Lambda \) if and only if there exists a \( U \in GL(n, \mathbb{Z}) \) such that \( A = BU \).

Proof. \( A \) is a basis of \( \Lambda \) if and only if \( A \mathbb{Z}^n = \Lambda = B \mathbb{Z}^n \) which is equivalent to \( B^{-1} A \in GL(n, \mathbb{Z}) \) by Proposition 6.4. \( \square \)

6.6 Definition [Determinant, fundamental cell]. Let \( \Lambda \in \mathcal{L}^n \) with basis \( B = (b_1, \ldots, b_n) \).

i) \( \det \Lambda = |\det B| \) is called determinant of \( \Lambda \).

ii) \( P_B = \{ \rho_1 b_1 + \cdots + \rho_n b_n : 0 \leq \rho_i < 1, 1 \leq i \leq n \} = B [0,1)^n \) is called fundamental cell or fundamental parallelepiped of \( \Lambda \) (w.r.t. the basis \( B \)).

6.7 Remark.

i) \( \det \Lambda \) is independent of the choice of the basis (cf. Lemma 6.5).

ii) \( \det \Lambda = \text{vol} (P_B) \) and \( \det(\mu \Lambda) = |\mu|^n \det \Lambda, \mu \in \mathbb{R} \).
iii) \( \det \Lambda \leq |b_1| |b_2| \cdots |b_n| \), with equality if and only if the vectors \( b_i \) are pairwise orthogonal (Hadamard inequality).

iv) \( P_B \cap \Lambda = \{0\} \). Since \((P_B - P_B) = B(-1, 1)^n \) we even have \((P_B - P_B) \cap \Lambda = \{0\} \).

6.8 Notation. Let \( a_1, \ldots, a_n \in \mathbb{R}^n \) be linearly independent and let \( A = (a_1, \ldots, a_n) \). Let \( x \in \mathbb{R}^n \) with \( x = \sum_{i=1}^n \rho_i a_i \), \( \rho_i \in \mathbb{R} \). Then we write \( [x]_A = \sum_i [\rho_i] a_i \). In particular, \( [x]_A \in AZ^n \) and \( x - [x]_A \in P_A \).

6.9 Proposition. Let \( \Lambda = BZ^n \in \mathcal{L}^n \). Then
\[
\mathbb{R}^n = \bigcup_{b \in \Lambda} (b + P_B),
\]
i.e., \( \mathbb{R}^n \) is the pairwise disjunct union of the lattice translates \( b + P_B \).

Proof. Each \( x \in \mathbb{R}^n \) can be decomposed as \( x = (x - [x]_B) + [x]_B \). The first summand is in \( P_B \) and second is a lattice point of \( \Lambda \). To show that the union is disjunct we observe that the intersection of two lattice translates \( b + P_B, \tilde{b} + P_B \) is non-empty, if and only if \( b - \tilde{b} \in (P_B - P_B) \cap \Lambda \). By Remark 6.7 iv) this is equivalent to \( b = \tilde{b} \). \( \square \)

6.10 Definition [Discrete set]. A set \( S \subset \mathbb{R}^n \) is called discrete iff there exists an \( \epsilon > 0 \) such that \( |s_1 - s_2| \geq \epsilon \) for all \( s_1, s_2 \in S \), \( s_1 \neq s_2 \).

6.11 Theorem. A subset \( S \subset \mathbb{R}^n \) is a lattice if and only if \( S \) a discrete subgroup of \( \mathbb{R}^n \) containing \( n \) linearly independent points.

Proof. Obviously, a lattice is a subgroup of \( \mathbb{R}^n \) containing \( n \) linearly independent points. Let \( B \) be a basis of the lattice and let \( \epsilon \) be the minimum of the continuous function \( |Bx| \) on \( S^{n-1} \). Then for \( z \in \mathbb{Z}^n \setminus \{0\} \) we have \( |Bz| \geq \epsilon |z| \), which shows the discreteness of the lattice.

For the other direction let \( s_1, \ldots, s_n \in S \) be \( n \) linearly independent points. By an inductive construction we show that there exist \( b_1, \ldots, b_n \in S \) such that for \( 1 \leq k \leq n \)
\[
\text{lin} \{s_1, \ldots, s_k\} \cap S = (b_1, \ldots, b_k) \mathbb{Z}^k. \tag{6.11.1}
\]
The case \( k = n \) of (6.11.1) implies the assertion. For \( k = 1 \) let \( b_1 \neq 0 \) be the shortest vector in \( \text{conv} \{0, s_1\} \cap S \). Since \( S \) is discrete such a choice is possible, and since \( S \) is a subgroup we certainly have \( b_1 \mathbb{Z}^1 \subseteq \text{lin} \{s_1\} \cap S \). Now let \( s \in \text{lin} \{s_1\} \cap S \) and let \( \lambda \in \mathbb{R} \) such that \( s = \lambda b_1 \). Then \( s - [\lambda] b_1 = (\lambda - [\lambda]) b_1 \) and by the minimality of \( b_1 \) we must have \( \lambda = [\lambda] \in \mathbb{Z} \). Hence (6.11.1) holds for \( k = 1 \).

Let us assume that we have already found \( b_1, \ldots, b_k \) satisfying (6.11.1). Next we consider the \((k + 1)\)-dimensional parallelepiped \( P_k = \{ \sum_{i=1}^k \alpha_i b_i + \alpha_{k+1} s_{k+1} : 0 \leq \alpha_i \leq 1 \} \). Let \( b_{k+1} \in P_k \cap S \) having minimum positive distance to \( \text{lin} \{b_1, \ldots, b_k\} \), i.e.,
\[
b_{k+1} = \sum_{i=1}^k \alpha_i b_i + \alpha_{k+1} s_{k+1},
\]
and $\alpha_{k+1} > 0$ is minimal among all points in $P_k \cap S$. Obviously, in view of our inductive construction we have $\text{lin} \{b_1, \ldots, b_{k+1}\} = \text{lin} \{s_1, \ldots, s_{k+1}\}$ and by the group properties of $S$ we also know $(b_1, \ldots, b_{k+1})\mathbb{Z}^{k+1} \subseteq \text{lin} \{s_1, \ldots, s_{k+1}\} \cap S$. So let $s \in \text{lin} \{s_1, \ldots, s_{k+1}\} \cap S$ given by $s = \sum_{i=1}^{k+1} \beta_i b_i$. Then we have

$$S \ni s - \sum_{i=1}^{k+1} |\beta_i| b_i = \sum_{i=1}^{k+1} (\beta_i - |\beta_i|) b_i = \sum_{i=1}^{k} (\beta_i - |\beta_i|) b_i + (\beta_{k+1} - |\beta_{k+1}|) b_{k+1}$$

$$= \sum_{i=1}^{k} [(\beta_i - |\beta_i|) + \alpha_i (\beta_{k+1} - |\beta_{k+1}|)] b_i + \alpha_{k+1} (\beta_{k+1} - |\beta_{k+1}|) s_{k+1}.$$ 

For abbreviation we denote by $\mu_i$ all the scalars in front of the vectors. Then we have $0 \leq \mu_{k+1} < \alpha_{k+1}$ and

$$s - \sum_{i=1}^{k+1} |\beta_i| b_i - \sum_{i=1}^{k} |\mu_i| b_i = \sum_{i=1}^{k} (\mu_i - |\mu_i|) b_i + \mu_{k+1} s_{k+1} \in S \cap P_k.$$ 

By the choice of $\alpha_{k+1}$ we must have $\mu_{k+1} = 0$ and so $\beta_{k+1} = |\beta_{k+1}| \in \mathbb{Z}$. Thus

$$s - \beta_{k+1} b_{k+1} = \sum_{i=1}^{k} \beta_i b_i \in \text{lin} \{s_1, \ldots, s_k\} \cap S,$$

and (6.11.1) implies the integrality of $\beta_i$ for $1 \leq i \leq k$. $\square$

6.12 Corollary. Let $a_1, \ldots, a_n \in \Lambda \in \mathcal{L}^n$ linearly independent. Then there exists a basis $b_1, \ldots, b_n$ of $\Lambda$ such that

$$a_k \in (b_1, \ldots, b_k)\mathbb{Z}^k, 1 \leq k \leq n.$$ 

Proof. Follows from (6.11.1) in the proof of Theorem 6.11 by setting $s_i = a_i$, $1 \leq i \leq n$, and $S = \Lambda$. $\square$

6.13 Definition [Index of a sublattice]. Let $\Lambda \in \mathcal{L}^n$ and let $a_1, \ldots, a_n \in \Lambda$ be linearly independent. $\Lambda_0 = (a_1, \ldots, a_n)\mathbb{Z}^n$ is called a sublattice with basis $\{a_1, \ldots, a_n\}$. The number of cosets of the subgroup $\Lambda_0$ with respect to $\Lambda$, i.e., the index of $\Lambda_0$ in $\Lambda$ is denoted by $|\Lambda : \Lambda_0|$.

6.14 Lemma. Let $\Lambda_0 \subseteq \Lambda \in \mathcal{L}^n$ be a sublattice of $\Lambda$. Then

i) $|\Lambda : \Lambda_0| = \#(P_A \cap \Lambda)$ for any basis $A$ of $\Lambda_0$.

ii) $|\Lambda : \Lambda_0| = \det \Lambda_0 / \det \Lambda$.

Proof. For i) it is to show that $\Lambda = \bigcup_{c \in P_A \cap \Lambda} (c + \Lambda_0)$. For if, let $b \in \Lambda$. Then we have $[b]_A \in \Lambda_0 \subseteq \Lambda$ and so $(b - [b]_A) \in P_A \cap \Lambda$. Thus

$$b = (b - [b]_A) + [b]_A \in (P_A \cap \Lambda) + \Lambda_0.$$
If there exist $b_1, b_2 \in P_A \cap \Lambda$ such that $(b_1 + \Lambda_0) \cap (b_2 + \Lambda_0) \neq \emptyset$ then $b_1 - b_2 \in (P_A - P_A) \cap \Lambda_0 = \{0\}$ (cf. Remark 6.7). Hence $b_1 = b_2$ and i) is shown.

For ii) we observe that

$$mP_A = \bigcup_{0 \leq m_i < m} (m_1a_1 + \cdots + m_na_n + P_A),$$

where $m_i, m \in \mathbb{N}$. Moreover, for every $a \in \Lambda$ we have $\#((a + P_A) \cap \Lambda) = \#(P_A \cap \Lambda)$ and in view of i) we find

$$\#(mP_A \cap \Lambda) = m^n \#(P_A \cap \Lambda) = m^n |\Lambda : \Lambda_0|.$$ 

Finally, since $P_A$ is Riemann-integrable we may write (cf. Remark 5.2)

$$\det \Lambda_0 = \text{vol}(P_A) = \lim_{m \to \infty} \# \left( (P_A \cap \frac{1}{m}\Lambda) \right) \frac{\det \Lambda}{m^n} = \det \Lambda \lim_{m \to \infty} \frac{\#(mP_A \cap \Lambda)}{m^n} = \det \Lambda |\Lambda : \Lambda_0|.$$

\[ \square \]

**6.15 Corollary.** Let $z_1, \ldots, z_n \in \mathbb{Z}^n$ be linearly independent. Then

$$\#((z_1, \ldots, z_n)[0,1]^n) = |\det(z_1, \ldots, z_n)|.$$

**Proof.** We just apply Lemma 6.14 ii) with $\Lambda = \mathbb{Z}_n$ and $\Lambda_0$ being the lattice generated by $z_1, \ldots, z_n$. \[ \square \]

**6.16 Remark.** Let $\Lambda_0 = AZ^n \subseteq \Lambda \in \mathcal{L}^n$ be a sublattice of $\Lambda$. Then

$A$ is basis of $\Lambda \iff |\Lambda : \Lambda_0| = 1 \iff \Lambda \cap P_A = \{0\}$

$$\iff \Lambda \cap A[0,1]^n = A\{0,1\}^n.$$ 

**6.17 Lemma.** Let $\Lambda \in \mathcal{L}^2$ and let $a_1, a_2 \in \Lambda$ be linearly independent. Then

$$a_1, a_2 \text{ basis of } \Lambda \iff \text{conv} \{0, a_1, a_2\} \cap \Lambda = \{0, a_1, a_2\}.$$

**Proof.** If $a_1, a_2$ are a basis then every point of $\Lambda$ has an unique representation as an integral linear combination of $a_1$ and $a_2$ and so $\text{conv} \{0, a_1, a_2\} \cap \Lambda = \{0, a_1, a_2\}$. For the reverse implication let $b \in P_A = (a_1, a_2)[0,1]^2 \cap \Lambda$. In view of Remark 6.16 we have to show $b = 0$, which, by assumption, is certainly true if $b \in \text{conv} \{0, a_1, a_2\}$. So let $b \notin \text{conv} \{0, a_1, a_2\}$. Then $b = \rho_1a_1 + \rho_2a_2$ with $0 < \rho_1, \rho_2 < 1$ and $\rho_1 + \rho_2 > 1$. So we have $(1 - \rho_1) + (1 - \rho_2) < 1$ and thus

$$(a_1 + a_2) - b = (1 - \rho_1)a_1 + (1 - \rho_2)a_2 \in \text{conv} \{0, a_1, a_2\} \cap \Lambda.$$ 

Hence $b = a_1 + a_2$ contradicting the choice of $b \in P_A$. \[ \square \]
6.18 Remark. An analogous statement to Lemma 6.17 does not exist in dimension $\geq 3$. For $n \geq 3$ and $m \in \mathbb{N}$ let $b(m) = (1, \ldots, 1)^T \in \mathbb{R}^n$ and $T^n(m) = \text{conv} \{0, e_1, \ldots, e_{n-1}, b(m)\}$. Then

$$T^n(m) \cap \mathbb{Z}^n = \{0, e_1, \ldots, e_{n-1}, b(m)\},$$

but the determinant of the lattice with basis $\{e_1, \ldots, e_{n-1}, b(m)\}$ is $m$. $T^n(m)$ are called Reeve simplices.

6.19 Lemma. Let $u_1, \ldots, u_m \in \mathbb{Z}^n$ and let $k_i \in \mathbb{N}$, $k_i \geq 1$, $1 \leq i \leq m$. The set

$$\Lambda = \{z \in \mathbb{Z}^n : \langle u_i, z \rangle \equiv 0 \mod k_i, 1 \leq i \leq m\}$$

is a lattice with $\det \Lambda \leq k_1 k_2 \cdots k_m$.

Proof. By definition $\Lambda$ is a discrete subgroup of $\mathbb{R}^n$, actually of $\mathbb{Z}^n$. Since the $n$ linearly independent vectors $(k_1 \cdots k_m) e_i$, $1 \leq i \leq n$, belong to $\Lambda$, Theorem 6.11 shows that $\Lambda$ is a lattice. By Lemma (6.14) we have $\det \Lambda = |\mathbb{Z}^n : \Lambda|$ and so it suffices to bound the number of cosets of $\mathbb{Z}^n$ w.r.t. $\Lambda$. To this end let $\Phi : \mathbb{Z}^n \to \mathbb{Z}^m$ be given by $\Phi(z)_i = \langle u_i, z \rangle \mod k_i \in \{0, \ldots, k_i - 1\}$, $1 \leq i \leq m$. Then $\#\Phi(\mathbb{Z}^n) = k_1 \cdots k_m$ and it remains to observe that $z, \bar{z}$ belongs to different cosets of $\mathbb{Z}^n$ w.r.t. $\Lambda$ if and only if $(z - \bar{z}) \notin \Lambda$, i.e., if and only if $\Phi(z) \neq \Phi(\bar{z})$. \hfill $\square$

6.20 Lemma. Let $X \subset \mathbb{R}^n$ be a bounded measurable set.

i) If $(z_1 + X) \cap (z_2 + X) = \emptyset$, for all $z_1, z_2 \in \mathbb{Z}^n$, $z_1 \neq z_2$, then $\text{vol} (X) \leq 1$.

ii) If $\mathbb{Z}^n + X = \mathbb{R}^n$ then $\text{vol} (X) \geq 1$.

Proof. Let $P = [0, 1]^n$ be the fundamental cell of $\mathbb{Z}^n$, and let $M = \{z \in \mathbb{Z}^n : (z + P) \cap X \neq \emptyset\}$. By Proposition 6.9) we have $\mathbb{R}^n = \bigcup_{z \in \mathbb{Z}^n} (z + P) = \mathbb{Z}^n + P$ and so

$$\text{vol} (X) = \text{vol} ((\mathbb{Z}^n + P) \cap X) = \sum_{z \in M} \text{vol} ((z + P) \cap X).$$

Now $(z + P) \cap X = (P \cap (X - z)) + z$ and so

$$\text{vol} (X) = \sum_{z \in M} \text{vol} (P \cap (X - z)).$$

In the first case we have $[P \cap (X - z_1)] \cap [P \cap (X - z_2)] = \emptyset$ for $z_1 \neq z_2 \in \mathbb{Z}^n$, and thus

$$\sum_{z \in M} \text{vol} (P \cap (X - z)) \leq \text{vol} (P) = 1.$$

In the second case we observe that $M = \{z \in \mathbb{Z}^n : P \cap (X - z) \neq \emptyset\}$ and on account of $\mathbb{Z}^n + X = \mathbb{R}^n$ we must have $\cup_{z \in M} (P \cap (X - z)) = P$. Hence

$$\sum_{z \in M} \text{vol} (P \cap (X - z)) \geq \text{vol} (P) = 1.$$

$\square$
6.21 Corollary [Blichfeldt]. 30 Let $X \subset \mathbb{R}^n$ with $\text{vol}(X) > 1$. Then there exist $x_1, x_2 \in X$ such that $x_1 - x_2 \in \mathbb{Z}^n \setminus \{0\}$, i.e., $X - X \cap \mathbb{Z}^n \setminus \{0\} \neq \emptyset$.

Proof. With out loss of generality we may assume that $X$ is bounded. By Lemma 6.20 i) there exist $z_1, z_2 \in \mathbb{Z}^n$, $z_1 \neq z_2$, such that $(z_1 + X) \cap (z_2 + X) \neq \emptyset$. Hence $\mathbb{Z}^n \setminus \{0\} \ni z_1 - z_2 \in X - X$.

6.22 Notation. Let $K_0^n = \{K \in K^n : K = -K\}$ be the set of all 0-symmetric convex bodies.

6.23 Theorem [Minkowski, 1896]. Let $K \in K_0^n$ with $\text{vol}(K) \geq 2^n$. Then

$$K \cap \mathbb{Z}^n \setminus \{0\} \neq \emptyset,$$

i.e., a 0-symmetric convex body of volume at least $2^n$ contains a non-trivial lattice point.

Proof. First we assume $\text{vol}(K) > 2^n$. Then we have $\text{vol}(\frac{1}{2}K) > 1$ and by Corollary 6.21 we get $(\frac{1}{2}K - \frac{1}{2}K) \cap \mathbb{Z}^n \setminus \{0\} \neq \emptyset$. By the symmetry we have $\frac{1}{2}K = \frac{1}{2}K$ and thus $K = \frac{1}{2}K - \frac{1}{2}K$.

Now let $\text{vol}(K) = 2^n$ and suppose $K \cap \mathbb{Z}^n = \{0\}$. Since $K$ is compact there exists a $\lambda > 1$ with $\lambda K \cap \mathbb{Z}^n = \{0\}$. However $\text{vol}(\lambda K) > 2^n$ and thus we get a contradiction to the previous case.

6.24 Corollary. Let $\Lambda \in \mathcal{L}^n$ and $K \in K_0^n$ with $\text{vol}(K) \geq 2^n \det \Lambda$. Then

$$K \cap \Lambda \setminus \{0\} \neq \emptyset.$$

Proof. Let $B$ be a basis of $\Lambda$. Then $K \cap \Lambda \setminus \{0\} \neq \emptyset$ if and only if $B^{-1}(K \cap \Lambda \setminus \{0\}) \neq \emptyset$ which is equivalent to $B^{-1}K \cap \mathbb{Z}^n \setminus \{0\} \neq \emptyset$. By assumption we have $\text{vol}(B^{-1}K) = \text{vol}(K)/\det \Lambda \geq 2^n$, and so the corollary is an immediate consequence of Theorem 6.23.

6.25 Theorem [Theorem on linear forms]. Let $a_i \in \mathbb{R}^n$, $1 \leq i \leq n$, be linearly independent, and let $\tau_i \in \mathbb{R}_{>0}$, $1 \leq i \leq n$, such that $\tau_1 \tau_2 \cdots \tau_n \geq |\det(a_1, \ldots, a_n)|$. Then there exists a $z \in \mathbb{Z}^n \setminus \{0\}$ with

$$|\langle a_i, z \rangle| \leq \tau_i, \quad 1 \leq i \leq n.$$

Proof. Let $P = \{x \in \mathbb{R}^n : |\langle a_i, x \rangle| \leq \tau_i, \quad 1 \leq i \leq n\}$. In order to calculate the volume of that 0-symmetric parallelepiped we observe that $P = A^{-1}\{x \in \mathbb{R}^n : |x_i| \leq \tau_i, \quad 1 \leq i \leq n\}$ where $A$ is the matrix with columns $a_i$. Thus

$$\text{vol}(P) = 2^n \frac{\tau_1 \cdots \tau_n}{|\det A|} \geq 2^n.$$ 

The assertion follows from Theorem 6.23.

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30Hans Frederick Blichfeldt, 1873–1945
6.26 Theorem [Dirichlet]. 31 Let \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) and let \( 0 < \epsilon < 1 \). Then there exist \( p_1, \ldots, p_n, q \in \mathbb{Z} \) with \( 1 \leq q \leq \epsilon^{-n} \) such that

\[
\left| \frac{\alpha_i - p_i}{q} \right| < \frac{\epsilon}{q}, \quad 1 \leq i \leq n.
\]

**Proof.** Let \( a_i = -e_i + \alpha_i e_{n+1}, \ 1 \leq i \leq n \), and let \( a_{n+1} = e_{n+1} \). Then \( |\det(a_1, \ldots, a_{n+1})| = 1 \) and by Theorem 6.25 there exists for every \( \tau > 0 \) an integral point \( z = (p_1, \ldots, p_n, q)^T \in \mathbb{Z}^{n+1} \setminus \{0\} \) (depending on \( \tau \)) satisfying the system of linear forms

\[
|\langle a_i, z \rangle| = |\alpha_i q - p_i| \leq \tau^{-1/n}, \quad 1 \leq i \leq n, \quad \text{and} \quad |\langle a_{n+1}, z \rangle| = |q| \leq \tau.
\]

Now we choose a \( \tau > \epsilon^{-n} \) such that \( \lfloor \tau \rfloor \leq \epsilon^{-n} \) and get

\[
|\langle a_i, z \rangle| = |\alpha_i q - p_i| < \epsilon, \quad 1 \leq i \leq n, \quad \text{and} \quad |\langle a_{n+1}, z \rangle| = |q| \leq \tau \leq \epsilon^{-n}.
\]

Finally, we observe that \( q \neq 0 \), because otherwise \( p_i = 0 \) for \( i = 1, \ldots, n \) since \( \epsilon < 1 \) and so \( z = 0 \). Thus we may assume \( q \geq 1 \). \( \square \)

6.27 Proposition. Let \( p \) be prime. Then there exist \( a, b \in \mathbb{N} \) with

\[
a^2 + b^2 + 1 \equiv 0 \mod p.
\]

**Proof.** For \( p = 2 \) the statement is certainly true. So let \( p \) be odd. For \( 0 \leq a \leq \frac{1}{2}(p-1) \) the numbers \( a^2 \) belong to pairwise distinct residue classes \( \mod p \), because

\[
a^2 \equiv \bar{a}^2 \mod p \iff (a - \bar{a})(a + \bar{a}) \equiv 0 \mod p.
\]

The same is true if we look at the residue classes of \( -b^2 - 1 \) for \( 0 \leq b \leq \frac{1}{2}(p-1) \). Since there are precisely \( p \) different residue classes \( \mod p \) we can find integers \( 0 \leq a, b \leq \frac{1}{2}(p-1) \) such that \( a^2 \) and \( -(b^2 + 1) \) belong to the same residue class \( \mod p \) which proves the proposition. \( \square \)

6.28 Theorem [Lagrange]. 32 Every positive number \( m \in \mathbb{N} \) can be written as the sum of four integer squares, i.e., there exist \( m_1, m_2, m_3, m_4 \in \mathbb{N} \) such that

\[
m = (m_1)^2 + (m_2)^2 + (m_3)^2 + (m_4)^2.
\]

**Proof.** First we observe that it suffices to prove the theorem for integers \( m \) which are not divisible by a square other than \( 1 \). Hence let \( m = p_1 \cdots p_k \) with distinct primes \( p_i, \ 1 \leq i \leq k \). According to Proposition 6.27 we choose \( a_i, b_i, \ 1 \leq i \leq k, \) such that

\[
(a_i)^2 + (b_i)^2 + 1 \equiv 0 \mod p_i, \quad 1 \leq i \leq k. \quad (6.28.1)
\]

\[\text{Gustav Lejeune Dirichlet, 1805–1859}
\[\text{Joseph-Louis Lagrange, 1736–1813}\]
With this notation we set
\[ \Lambda = \{ z \in \mathbb{Z}^4 : z_1 \equiv (a_i z_3 + b_i z_4) \mod p_i, z_2 \equiv (b_i z_3 - a_i z_4) \mod p_i, 1 \leq i \leq k \}. \]

By Lemma 6.19 we know that \( \Lambda \subset \mathbb{R}^4 \) is a lattice with \( \det \Lambda \leq (p_1)^2 \cdots (p_k)^2 = m^2 \). Thus we get
\[ \text{vol}(\sqrt{2m} B_4) = (2m)^2 \text{vol}(B_4) = (2m)^2 \frac{\pi^2}{2} > 2^4 m^2 \geq 2^4 \det \Lambda. \]

Hence by Corollary 6.24 there exists a \( z \in \Lambda \setminus \{0\} \cap \text{int } B_4 \), i.e.,
\[ 0 < (z_1)^2 + (z_2)^2 + (z_3)^2 + (z_4)^2 < 2m. \]

In order to prove the theorem it suffices to show that \( m \) is a divisor of the sum \( (z_1)^2 + (z_2)^2 + (z_3)^2 + (z_4)^2 \).

By the choice of \( \Lambda \) we get for \( 1 \leq i \leq k \)
\[ \begin{aligned}
(z_1)^2 + (z_2)^2 + (z_3)^2 + (z_4)^2 &\equiv ((a_i z_3 + b_i z_4)^2 + (b_i z_3 - a_i z_4)^2 + (z_3)^2 + (z_4)^2) \mod p_i \\
&\equiv ((z_3)^2((a_i)^2 + (b_i)^2 + 1) + (z_4)^2((a_i)^2 + (b_i)^2 + 1)) \mod p_i \\
&\equiv 0 \mod p_i,
\end{aligned} \]
where the last relation is a consequence of (6.28.1). Thus all the distinct \( p_i \) are divisors of \( (z_1)^2 + (z_2)^2 + (z_3)^2 + (z_4)^2 \) and so \( m \) is a divisor of this sum. \( \square \)

**6.29 Theorem.** Let \( k \in \mathbb{N} \) and let \( X \subset \mathbb{R}^n \) be a Jordan measurable set with \( \text{vol}(X) > k \). Then there exist \( x_1, \ldots, x_{k+1} \in X \) such that \( x_i - x_j \in \mathbb{Z}^n \setminus \{0\}, i \neq j \).

**Proof.** Since we have a Jordan-measurable set we know
\[ k < \text{vol}(X) = \lim_{m \to \infty} \# \left( X \cap \frac{1}{m} \mathbb{Z}^n \right) \frac{1}{m^n}. \]
Thus there exists an \( m \in \mathbb{N} \) such that \( \# (X \cap \frac{1}{m} \mathbb{Z}^n) > k m^n \). Since there are \( m^n \) cosets of the sublattice \( \mathbb{Z}^n \) with respect to the lattice \( \frac{1}{m} \mathbb{Z}^n \) there exist (at least) \( (k + 1) \) different \( x_1, \ldots, x_{k+1} \in X \cap \frac{1}{m} \mathbb{Z}^n \) belonging to the same coset and thus \( x_i - x_j \in \mathbb{Z}^n \setminus \{0\}, i \neq j \). \( \square \)

**6.30 Corollary.** Let \( \Lambda \in \mathcal{L}^n \) and let \( K \in \mathcal{K}_0^n \) with \( \text{vol}(K) \geq k 2^n \det \Lambda \). Then
\[ \# (K \cap \Lambda) \geq 2k + 1. \]

**Proof.** Without loss of generality let \( \Lambda = \mathbb{Z}^n \) and \( \text{vol}(K) > k 2^n \) (cf. the proofs of Corollary 6.24 and Theorem 6.23). By Theorem 6.29 there exist \( (k + 1) \) different points \( x_1, \ldots, x_{k+1} \in \frac{1}{k} K \) with \( x_i - x_j \in \mathbb{Z}^n \). Let us assume that \( x_1 \) is one with maximal Euclidean length among these \( (k + 1) \) points and let \( z_i = x_{i+1} - x_1, 1 \leq i \leq k \). Then we have \( z_i \neq z_j, i \neq j \), and \( z_i \in K \cap \mathbb{Z}^n \setminus \{0\} \). Furthermore, by the choice of \( x_1 \) all these points satisfy \( \langle x_1, z_i \rangle < 0 \) which implies that the \( 2k \) points \( \pm z_i, 1 \leq i \leq k \), are pairwise distinct. Together with the point \( 0 \) this gives the desired lower bound. \( \square \)
6.31 Definition [Successive minima]. Let $K \in K^n_0$, $\Lambda \in \mathcal{L}^n$. For $1 \leq i \leq n$,

$$\lambda_i(K, \Lambda) = \min \{ \lambda > 0 : \dim (\lambda K \cap \Lambda) \geq i \}$$

is called the $i$-th successive minimum of $K$ w.r.t. $\Lambda$.

6.32 Remark.

i) $\lambda_i(K, \Lambda) \geq \lambda_{i-1}(K, \Lambda)$, $2 \leq i \leq n$.

ii) $\lambda_i(\mu K, \Lambda) = \frac{1}{\mu} \lambda_i(K, \Lambda)$, $\mu \in \mathbb{R}$, $\mu \neq 0$.

iii) $\lambda_i(\mu K, \Lambda) = \lambda_i(K, 1/\mu \Lambda)$, $\mu \in \mathbb{R}$, $\mu \neq 0$.

iv) $\lambda_i(B_n, \Lambda) = \min \{ |b| : b \in \Lambda \setminus \{0\} \}$.

6.33 Proposition. Let $K \in K^n_0$, $\Lambda \in \mathcal{L}^n$. Let $a_1, \ldots, a_n \in \Lambda$ be linearly independent such that $a_i \in \lambda_i(K, \Lambda) K$, $1 \leq i \leq n$. Then

$$\text{int} \left( \lambda_i(K, \Lambda) K \right) \cap \Lambda \subseteq \text{lin} \{a_1, \ldots, a_{i-1}\}, 1 \leq i \leq n,$$

where we set $\text{lin} \emptyset = \{0\}$.

Proof. We set $\lambda_0(K, \Lambda) = 0$, $a_0 = 0$, and let $0 \leq j < i$ be the maximal index with $\lambda_j(K, \Lambda) < \lambda_i(K, \Lambda)$. Then each lattice point in $\lambda K$ with $\lambda_j(K, \Lambda) \leq \lambda < \lambda_i(K, \Lambda)$ must be linearly dependent on $\{a_0, \ldots, a_j\}$. Hence

$$\text{int} \left( \lambda_i(K, \Lambda) K \right) \cap \Lambda \subset \text{lin} \{a_0, \ldots, a_j\}.$$

6.34 Theorem [Minkowski’s first theorem on successive minima]. Let $K \in K^n_0$ and $\Lambda \in \mathcal{L}^n$. Then

$$\lambda_1(K, \Lambda)^n \text{vol}(K) \leq 2^n \det \Lambda.$$

Proof. By the definition of $\lambda_1(K, \Lambda)$ we have $\text{int} (\lambda_1(K, \Lambda) K) \cap \Lambda \setminus \{0\} = \emptyset$. Hence by Corollary 6.24 we get $\text{vol} (\lambda_1(K, \Lambda) K) \leq 2^n \det \Lambda$.

6.35 Theorem* [Minkowski’s second theorem on successive minima]. Let $K \in K^n_0$ and $\Lambda \in \mathcal{L}^n$. Then

$$\frac{2^n}{n!} \det \Lambda \leq \lambda_1(K, \Lambda) \lambda_2(K, \Lambda) \cdot \ldots \cdot \lambda_n(K, \Lambda) \text{vol}(K) \leq 2^n \det \Lambda.$$
Proof. Without loss of generality let $\Lambda = \mathbb{Z}^n$. For convenience we write $\lambda_i = \lambda_i(K, \mathbb{Z}^n)$, and let $z_1, \ldots, z_n$ be $n$ linearly independent lattice points with

$$z_i \in \lambda_i K \cap \mathbb{Z}^n, \quad 1 \leq i \leq n. \quad (6.35.1)$$

Here we only prove the easy lower bound, for which we just observe that (6.35.1) implies \( \text{conv} \{ \pm \frac{1}{\lambda_i} z_i : 1 \leq i \leq n \} \subseteq K \).

Thus \( \text{vol} (K) \geq \text{vol} \left( \text{conv} \{ \pm \frac{1}{\lambda_i} z_i : 1 \leq i \leq n \} \right) = \frac{2^n}{n!} \prod_{i=1}^{n} \lambda_i \cdot \ldots \cdot \lambda_n \).

\[ \square \]

6.36 Theorem. Let $K \in \mathcal{K}_0^n$ and $\Lambda \in \mathcal{L}^n$. Then

$$\# (K \cap \Lambda) \leq \left( \left\lfloor \frac{2}{\lambda_1(K, \Lambda)} + 1 \right\rfloor \right)^n.$$  

In particular: Let $K \in \mathcal{K}_0^n$ with $\text{int} K \cap \Lambda = \{ 0 \}$. Then $\# (K \cap \Lambda) \leq 3^n$.

Proof. Without loss of generality let $\Lambda = \mathbb{Z}^n$ and we write $\lambda_1$ instead of $\lambda_1(K, \mathbb{Z}^n)$. Let $k = \lfloor 2/\lambda_1 \rfloor + 1$. Since there are $k^n$ different cosets of $\mathbb{Z}^n$ with respect to the sublattice $k\mathbb{Z}^n$ it suffices to show that two different lattice points $z, \bar{z} \in K \cap \mathbb{Z}^n$ belong to two different cosets. Suppose there exist $z, \bar{z} \in K \cap \mathbb{Z}^n$ such that $(z - \bar{z}) \in k\mathbb{Z}^n$. Then we find \( \mathbb{Z}^n \ni \frac{1}{k} (z - \bar{z}) = \frac{2}{k} \left( \frac{1}{2} z - \frac{1}{2} \bar{z} \right) \in \frac{2}{k} K \subset \text{int} (\lambda_1 K) \), since $2/k < \lambda_1$. Thus, by definition of $\lambda_1$ we must have $z = \bar{z}$. \[ \square \]

6.37 Conjecture. Let $K \in \mathcal{K}_0^n$ and $\Lambda \in \mathcal{L}^n$. Then

$$\# (K \cap \Lambda) \leq \prod_{i=1}^{n} \left( \left\lfloor \frac{2}{\lambda_i(K, \Lambda)} + 1 \right\rfloor \right).$$

6.38 Remark. Conjecture 6.37 would imply the upper bound in Minkowski’s second theorem 6.35, because

$$\text{vol} (K) = \lim_{m \to \infty} \left( \frac{1}{m} \right)^n \# (K \cap \frac{1}{m} \mathbb{Z}^n) = \lim_{m \to \infty} \left( \frac{1}{m} \right)^n \# (m K \cap \mathbb{Z}^n) \leq \lim_{m \to \infty} \left( \frac{1}{m} \right)^n \prod_{i=1}^{n} \left( \frac{2}{\lambda_i(m K)} + 1 \right) = \left( \frac{1}{m} \right)^n \lim_{m \to \infty} \prod_{i=1}^{n} \left( \frac{2 m}{\lambda_i(K)} + 1 \right) = \prod_{i=1}^{n} \frac{2}{\lambda_i(K)}.$$
7 Packing

7.1 Definition [Packing sets]. A subset $D \subset \mathbb{R}^n$ is called a packing set of $K \in \mathcal{K}^n$ if for all $x, y \in D$, $x \neq y$,

$$(x + \text{int } K) \cap (y + \text{int } K) = \emptyset.$$  

The family of all packing sets of $K$ is denoted by $\mathcal{P}(K)$.

7.2 Definition [Density of a Packing]. Let $K \in \mathcal{K}^n$ and $D \in \mathcal{P}(K)$.

$$\delta(K, D) = \limsup_{\lambda \to \infty} \frac{\text{vol } (K) \# \{ x \in D : x + \lambda K \subseteq \lambda C_n \}}{\text{vol } (\lambda C_n)}$$  

is called the density of the packing $D + K$ (with respect to the gauge body $C_n = [-1, 1]^n$). Here a gauge body is an arbitrary convex body $G$ with $0 \in \text{int } G$.

7.3 Remark. This definition of the density depends on the chosen gauge body. For instance, let $D = \{ z \in \mathbb{Z}^n : z \geq 0 \} \in \mathcal{P}([0, 1]^n)$. According to the Definition 7.2 we get $\delta([0, 1]^n, D) = \frac{1}{2^n}$. Now let $H_n = C_n \cap \{ x \in \mathbb{R}^n : |x_1 - x_2 + x_3 + \cdots + x_n| \leq n - 1 \}$. Then $[0, 1]^n \subset H_n$, $\text{vol } (H_n) = 2^n - 2/n!$ and changing in Definition 7.2 the gauge body $C_n$ to $H_n$ gives

$$\limsup_{\lambda \to \infty} \frac{\text{vol } ([0, 1]^n) \# \{ x \in D : x + [0, 1]^n \subset \lambda H_n \}}{\text{vol } (\lambda H_n)} = \frac{1}{2^n - \frac{2}{n!}}.$$  

7.4 Theorem. Let $K \in \mathcal{K}^n$. The supremum $\sup \{ \delta(K, D) : D \in \mathcal{P}(K) \}$ is independent of the chosen gauge body, and there exists a packing set $D_K \in \mathcal{P}(K)$ such that $\sup \{ \delta(K, D) : D \in \mathcal{P}(K) \} = \delta(K, D_K)$.

The proof of Theorem 7.4 is split into several lemmas, which are of interest in their own. To this end we need the following notation.

7.5 Notation. Let $K \in \mathcal{K}^n$. For $G \in \mathcal{K}^n$ with $0 \in \text{int } G$ and for $\lambda > 0$, we write

$$\Phi(\lambda, G) = \max \{ \# D : D \in \mathcal{P}(K) \text{ and } D + K \subseteq \lambda G \} \quad \text{and} \quad \Psi(\lambda, G) = \frac{\text{vol } (K) \Phi(\lambda, G)}{\text{vol } (\lambda G)}.$$  

7.6 Lemma. Let $K \in \mathcal{K}^n$ and $G \in \mathcal{K}^n$ with $0 \in \text{int } G$. Then $\lim_{\lambda \to \infty} \Psi(\lambda, G)$ exists and is independent of $G$.

Proof. First we show that in the case $G = C_n$ the limit exists. For short we write $\Phi(\lambda) = \Phi(\lambda, C_n)$, $\Psi(\lambda) = \Psi(\lambda, C_n)$, and for a given $\lambda > 0$ we denote by $D(\lambda) \in \mathcal{P}(K)$ a maximal packing set, i.e., $\# D(\lambda) = \Phi(\lambda)$ and $D(\lambda) + K \subset \lambda C_n$. Furthermore, let $\gamma \geq 1$ be such that $K - K \subset \gamma C_n$.

Now let $\lambda > \gamma$ and $q \in \mathbb{N}$. The cube $q \lambda C_n$ can be filled by $q^n$ non-overlapping copies of $\lambda C_n$, and let $t_i \in \mathbb{R}^n$, $1 \leq i \leq q^n$, be the centers of these copies, i.e.,

$$q \lambda C_n = \{ t_1, \ldots, t_{q^n} \} + \lambda C_n.$$
In particular, we see that
\[ \Phi(q\lambda) \geq q^n \Phi(\lambda). \]  (7.6.1)

Next we want to find an upper bound on \( \Phi(q\lambda) \) in terms of \( \Phi(\lambda) \). To this end we set \( D_1(q\lambda) = \{ x \in D(q\lambda) : \exists t_i s. t. x + K \subset t_i + \lambda C_n \} \) and let \( D_2(q\lambda) = D(q\lambda) \setminus D_1(q\lambda) \). Obviously,
\[ \Phi(q\lambda) = \#D_1(q\lambda) + \#D_2(q\lambda) \leq q^n \Phi(\lambda) + \#D_2(q\lambda). \]  (7.6.2)

On the other hand, since \( D(q\lambda) + K \subset q\lambda C_n \) and since \( K - K \subset \gamma C_n \) we conclude for the translates by points in \( D_2(q\lambda) \) (cf. Exercise ??)
\[ D_2(q\lambda) + K \subset \bigcup_{i=1}^q \{ t_i + \lambda C_n \} \setminus (t_i + (\lambda - \gamma) C_n). \]

Thus we have
\[ \text{vol } (K) \#D_2(q\lambda) \leq q^n [\lambda^n - (\lambda - \gamma)^n] \text{ vol } (C_n) \leq q^n \lambda^{n-1} \text{ vol } (C_n) c, \]
where \( c \) is a constant depending only on \( n \) and \( \gamma \). Together with (7.6.2) and (7.6.1) we get
\[ 0 \leq \Phi(q\lambda) - q^n \Phi(\lambda) \leq \frac{q^n \lambda^{n-1} c \text{ vol } (C_n)}{\text{ vol } (K)}. \]

Multiplying by \( \text{ vol } (K)/\text{ vol } (q\lambda C_n) \) yields
\[ 0 \leq \Psi(q\lambda) - \Psi(\lambda) \leq \frac{c}{\lambda}. \]  (7.6.3)

Now let \( p_1, p_2 \in \mathbb{Q} \) with \( p_1 \geq p_2 > \gamma \) and let \( m_1, m_2 \in \mathbb{N} \) with \( m_1 p_1 = m_2 p_2 \). Then we get from (7.6.3)
\[ |\Psi(p_1) - \Psi(p_2)| \leq |\Psi(p_1) - \Psi(m_1 p_1)| + |\Psi(p_2) - \Psi(m_2 p_2)| \leq \frac{2c}{p_2}. \]  (7.6.4)

This shows that \( \Psi(\lambda) \) forms a Cauchy-sequence on the rational numbers. Since \( \Phi(\Lambda) \) is a right-continuous function, for every \( \lambda \in \mathbb{R} \) there exists a rational number \( p_\lambda \in \mathbb{Q} \) with
\[ 0 \leq p_\lambda - \lambda \leq \frac{1}{2} \quad \text{and} \quad \Phi(\lambda) = \Phi(p_\lambda). \]

So we have
\[ 0 \leq \Psi(\lambda) - \Psi(p_\lambda) = \frac{\text{ vol } (K) \Phi(\lambda)}{\text{ vol } (p_\lambda C_n)} \left( \left( \frac{p_\lambda}{\lambda} \right)^n - 1 \right) \leq \frac{\left( \frac{p_\lambda}{\lambda} \right)^n}{\lambda} - 1 \leq \frac{\bar{c}}{\lambda}, \]
where \( \bar{c} \) is a constant only depending on \( n \). Hence for \( \mu, \lambda \in \mathbb{R}, \mu \geq \lambda \), we find together with (7.6.4)
\[ |\Psi(\mu) - \Psi(\lambda)| \leq |\Psi(p_\mu) - \Psi(p_\lambda)| + |\Psi(\mu) - \Psi(p_\mu)| + |\Psi(\lambda) - \Psi(p_\lambda)| \leq \frac{2(c + \bar{c})}{\lambda}, \]

which finally shows that \( \Psi(\lambda) \) is a Cauchy-sequence. Hence \( \lim_{\lambda \to \infty} \Psi(\lambda) \) exists and it remains to show its independency from the gauge body. This is left as an exercise for the reader. \( \square \)
7.7 Lemma. Let $K \in \mathcal{K}^n$ and $G \in \mathcal{K}^n$ with $0 \in \text{int } G$. Then there exists $D_K \in \mathcal{P}(K)$ such that

$$
\limsup_{\lambda \to \infty} \frac{\text{vol}(K) \# \{x \in D_K : x + K \subseteq \lambda G\}}{\text{vol}(\lambda G)} = \lim_{\lambda \to \infty} \Psi(\lambda, G).
$$

Proof. By definition of $\Psi(\lambda, G)$ it suffices to show that the left hand side is not less than the limit on the right. Let $K - K \subseteq \gamma G$ and for $\lambda > 0$ let $D(\lambda) \in \mathcal{P}(K)$ be a packing set of maximal cardinality such that $D(\lambda) + K \subseteq \lambda G$. In particular, we have $\#D(\lambda) = \Phi(\lambda, G)$, and for $m \in \mathbb{N}$ we define

$$
D_m = \left\{ x \in D(2^{m^2}) : (x + K) \cap 2^{(m-1)^2} G = \emptyset \right\}
$$

and set $D_K = \bigcup_{m \in \mathbb{N}} D_m$. Since $D_m \subset D(2^{m^2})$ and $(D_m + K) \cap (D_{m-1} + K) = \emptyset$, the set $D_K$ is a packing set for $K$. Moreover, for $x \in D(2^{m^2}) \setminus D_m$ we have $(x + K) \cap 2^{(m-1)^2} G \neq \emptyset$ and so

$$
0 \leq \Phi(2^{m^2}, G) - \#D_m \leq \frac{\text{vol} \left( \left( 2^{(m-1)^2} + \gamma \right) G \right)}{\text{vol}(K)}.
$$

Hence we obtain

$$
\frac{\# \{ x \in D_K : x + K \subseteq 2^{m^2} G \} \text{vol}(K)}{\text{vol}(2^{m^2} G)} \geq \frac{\#D_m \text{vol}(K)}{\text{vol}(2^{m^2} G)} \geq \frac{\Phi(2^{m^2}, G) \text{vol}(K)}{\text{vol}(2^{m^2} G)} - \frac{\text{vol} \left( \left( 2^{(m-1)^2} + \gamma \right) G \right)}{\text{vol}(2^{m^2} G)} \geq \Psi(2^{m^2}, G) - \left( \frac{1 + \gamma}{2^m} \right)^n.
$$

By Lemma 7.6 we know $\lim_{\lambda \to \infty} \Psi(\lambda, G) = \lim_{m \to \infty} \Psi(2^{m^2}, G)$, and so (7.7.1) implies the assertion. □

Proof. [Proof of Theorem 7.4] Let $K \in \mathcal{K}^n$. For $G \in \mathcal{K}^n$ with $0 \in \text{int } G$ and $D \in \mathcal{P}(K)$ we set

$$
\delta_G(K, D) = \limsup_{\lambda \to \infty} \frac{\text{vol}(K) \# \{ x \in D : x + K \subseteq \lambda G \}}{\text{vol}(\lambda G)}.
$$

Obviously we have $\delta_G(K, D) \leq \lim_{\lambda \to \infty} \Psi(\lambda, G)$ and on account of Lemma 7.7 and Lemma 7.6 we get

$$
\sup \{ \delta_G(K, D) : D \in \mathcal{P}(K) \} = \max \{ \delta_G(K, D) : D \in \mathcal{P}(K) \} = \lim_{\lambda \to \infty} \Psi(\lambda, G) = \lim_{\lambda \to \infty} \Psi(\lambda, C_n).
$$

□
7.8 Definition [Density of a Densest Packing]. Let $K \in \mathbb{K}^n$.

$$\delta(K) = \sup \{ \delta(K, D) : D \in \mathcal{P}(K) \}$$

is called the density of a densest packing of $K$ and a set $D_K \in \mathcal{P}(K)$ with $\delta(K) = \delta(K, D_K)$ is called a densest packing set of $K$.

7.9 Proposition. Let $K \in \mathbb{K}^n$. Then

i) $0 < \delta(K) \leq 1$.

ii) $\delta(t + AK) = \delta(K)$ for all $A \in \text{GL}(n, \mathbb{R})$ and $t \in \mathbb{R}^n$.

iii) Let $K \in \mathbb{K}^n$ and $D \in \mathcal{P}(K)$. Then

$$\delta(K, D) = \limsup_{\lambda \to \infty} \frac{\text{vol}(K) \#(D \cap \lambda C_n)}{\text{vol}(\lambda C_n)}.$$

iv) $\mathcal{P}(K) = \mathcal{P}\left(\frac{1}{2}(K - K)\right)$ and for $D \in \mathcal{P}(K)$ we have

$$\delta(K, D) = \delta\left(\frac{1}{2}(K - K), D\right) \frac{\text{vol}(K)}{\text{vol}\left(\frac{1}{2}(K - K)\right)},$$

and consequently

$$\delta(K) = \delta\left(\frac{1}{2}(K - K)\right) \frac{\text{vol}(K)}{\text{vol}\left(\frac{1}{2}(K - K)\right)}.$$

v) Let $K \in \mathbb{K}^n_0$. Then $D \in \mathcal{P}(K)$ if and only if $|x - y|_K \geq 2$ for all $x, y \in D, x \neq y$. Here $|x|_K = \min\{\lambda \geq 0 : x \in \lambda K\}$ is the norm induced by $K$.

Proof. Items i), ii) follow immediately from the definition. For iii) let $\gamma > 0$ with $K, K - K \subset \gamma C_n$, and for a given $\lambda > \gamma$ let

$$m_1(\lambda) = \#\{x \in D : x + K \subset \lambda C_n\} \quad \text{and} \quad m_2(\lambda) = \#(D \cap \lambda C_n).$$

If $x \in D$ with $x + K \subset \lambda C_n$, but $x \notin \lambda C_n$ then $x + K \subset \lambda C_n \setminus (\lambda - \gamma)C_n$ and so

$$\text{vol}(K) m_1(\lambda) \leq \text{vol}(K) m_2(\lambda) + [\lambda^n - (\lambda - \gamma)^n]\text{vol}(C_n) \leq \text{vol}(K) m_2(\lambda) + \lambda^{n-1}\text{vol}(C_n) \ c,$$

where $c$ is a constant only depending on $\gamma$ and $n$. On the other hand, if $x \in D \cap \lambda C_n$ but $x + K \notin \lambda C_n$ then $x + K \subset (\lambda + \gamma)C_n \setminus (\lambda - \gamma)C_n$. Hence

$$\text{vol}(K) m_2(\lambda) \leq \text{vol}(K) m_1(\lambda) + [(\lambda + \gamma)^n - (\lambda - \gamma)^n]\text{vol}(C_n) \leq \text{vol}(K) m_1(\lambda) + \lambda^{n-1}\text{vol}(C_n) \ c,$$

for another constant $\bar{c}$. So we obtain

$$-\frac{\bar{c}}{\lambda} \leq \frac{\text{vol}(K) \#\{x \in D : x + K \subset \lambda C_n\}}{\text{vol}(\lambda C_n)} - \frac{\text{vol}(K) \#(D \cap \lambda C_n)}{\text{vol}(\lambda C_n)} \leq \frac{c}{\lambda},$$
which shows iii). For iv) we note that
\[(x + \text{int } K) \cap (y + \text{int } K) \neq \emptyset\]
\[\iff x - y \in \text{int } K - \text{int } K = \frac{1}{2} \text{int } (K - K) - \frac{1}{2} \text{int } (K - K)\]
\[\iff \left(x + \frac{1}{2} \text{int } (K - K)\right) \cap \left(y + \frac{1}{2} \text{int } (K - K)\right) \neq \emptyset\]
and thus \(P(K) = P\left(\frac{1}{2}(K - K)\right)\). Here we have used the fact that \(\text{int } (K + L) = \text{int } K + \text{int } L\) for convex bodies \(K, L\) (cf. Exercise []). v) is just a reformulation of the packing property of an \(o\)-symmetric convex body. □

7.10 Lemma. Let \(S \subset \mathbb{R}^n\) be a bounded and measurable set with \(\text{vol } (S) > 0\) and let \(D \in P(K)\). Then there exist \(v, w \in \mathbb{R}^n\) such that
\[
\frac{\text{vol } (K) \#((w + S) \cap D)}{\text{vol } (S)} \leq \delta(K, D) \leq \frac{\text{vol } (K) \#((w + S) \cap D)}{\text{vol } (S)}.
\]

Proof. We just prove the upper bound, the lower bound can be done analogously. Let \(\gamma > 0\) such that \(S \subset \gamma C_n\) and let \(\varepsilon(\lambda) \in \mathbb{R}\) with \(\varepsilon(\lambda) \rightarrow 0\) as \(\lambda\) tends infinity and (cf. Proposition 7.9 iv))
\[
\varepsilon(\lambda) + \frac{\delta(K, D)}{\text{vol } (K)} = \frac{\#(D \cap \lambda C_n)}{\text{vol } (\lambda C_n)}.
\] (7.10.1)

Let \(x \in D \cap \lambda C_n\). Since \(\{v \in \mathbb{R}^n : x \in v + S\} = x - S \subset (\lambda + \gamma) C_n\) we get
\[
\int_{(\lambda + \gamma) C_n} \#((v + S) \cap D) \, dv \geq \text{vol } (S) \#(D \cap \lambda C_n).
\]

Hence there exist \(v_\lambda \in \mathbb{R}^n\) such that
\[
\#((v_\lambda + S) \cap D) \geq \text{vol } (S) \frac{\#(D \cap \lambda C_n)}{\text{vol } ((\lambda + \gamma) C_n)}
\]
\[
= \text{vol } (S) \left(\frac{\#(D \cap \lambda C_n)}{\text{vol } (\lambda C_n)}\right) \frac{(\lambda + \gamma)^n}{\lambda^n}.
\]
and with (7.10.1) we obtain
\[
\#((v_\lambda + S) \cap D) \geq \left(\frac{\delta(K, D)\text{vol } (S)}{\text{vol } (K)} + \varepsilon(\lambda) \text{vol } (S)\right) \left(1 - \frac{\gamma}{\lambda + \gamma}\right)^n
\]
\[
\geq \frac{\delta(K, D)\text{vol } (S)}{\text{vol } (K)} + \rho(\lambda),
\]
for suitable numbers \(\rho(\lambda)\) satisfying \(\lim_{\lambda \to \infty} \rho(\lambda) = 0\). Since the left hand side is an integer we can find \(\lambda \in \mathbb{R}_{>0}\) such that \(\#((v_\lambda + S) \cap (D \cap \lambda C_n)) \geq \delta(K, D)\text{vol } (S)/\text{vol } (K)\), which gives the upper bound. □
7.11 Remark. Let $K \in \mathcal{K}_n$.

$$R(K) = \min \{ R > 0 : \exists x \in \mathbb{R}^n \text{ with } K \subseteq x + R B_n \}$$

is called the circumradius of $K$. The uniquely determined point $t_c \in \mathbb{R}^n$ with $K \subseteq t_c + R(K) B_n$ is called circumcenter of $K$. There exist $k + 1$ affinely independent points $x_0, \ldots, x_k \in \text{bd} K \cap \text{bd} (t_c + R(K) B_n)$ and $\lambda_i > 0$, $0 \leq i \leq k$, with $\sum_{i=0}^k \lambda_i = 1$ such that $t_c = \sum_{i=0}^k \lambda_i x_i$ (cf. Exercise [ ]), and these points are extreme points of $K$.

7.12 Theorem. Let $K \in \mathcal{K}_0^n$. Then

i) $\delta(K) \geq 2^{-n}$,

ii) $\delta(B_n) \leq (n + 1) \sqrt{2}^{-n}$.

Proof. i) Let $D_s \in \mathcal{P}(K)$ be a saturated packing, i.e., for every $x \in \mathbb{R}^n$ we have $(x + K) \cap (D_s + K) \neq \emptyset$. Thus $(x + 2K) \cap D_s \neq \emptyset$ and now we use the lower bound of Lemma 7.10 with respect to the set $S = 2K$ and the packing set $D_s$.

ii) For $r \in \mathbb{R}_{>0}$ let $f(r, n) = \max \{ \# (D \cap \text{int} (r B_n)) : D \in \mathcal{P}(B_n) \}$. In view of the upper bound given in Lemma 7.10, applied to $S = \sqrt{2} \text{int} B_n$, it suffices to show

$$f(\sqrt{2}, n) \leq n + 1. \quad (7.12.1)$$

Let $l = f(\sqrt{2}, n)$ and $D \in \mathcal{P}(B_n)$ be a packing set attaining this bound, i.e., there exists $x_i \in \text{int} (\sqrt{2} B_n) \cap D$, $1 \leq i \leq l$, and let $\overline{D} = \{ x_1, \ldots, x_l \}$. Let $R < \sqrt{2}$ be the circumradius of $\text{conv} \overline{D}$ and without loss of generality we may assume $\overline{D} \subseteq R B_n$, i.e., the circumcenter of $\text{conv} \overline{D}$ is $0$. Due to Remark 7.11, among the $l$ points there exists $k + 1$ affinely independent points $x_0, \ldots, x_k$ with $k \in \{ 1, \ldots, n \}$, say, and $\lambda_i > 0$, $0 \leq i \leq k$, such that

$$\sum_{i=0}^k \lambda_i x_i = 0. \quad (7.12.2)$$

We further have for $1 \leq i \neq j \leq l$,

$$4 \leq |x_i - x_j|^2 \leq 2 R^2 - 2 \langle x_i, x_j \rangle < 4 - 2 \langle x_i, x_j \rangle .$$

In particular, for any $x_j \in \overline{D} \setminus \{ x_1, \ldots, x_{k+1} \}$ we would have $\langle x_i, x_j \rangle < 0$, $0 \leq i \leq k$, which is impossible on account of (7.12.2). Hence $\overline{D} = \{ x_1, \ldots, x_{k+1} \}$ and so $f(\sqrt{2}, n) \leq n + 1$. On the other hand, for instance, the circumradius of a regular simplex of edge length 2 is equal to $\sqrt{2} \sqrt{n/(n + 1)}$, and so we also know $f(\sqrt{2}, n) \geq n + 1$. \qed

7.13 Remark. Chronologically, the first upper bound is due to Blichfeldt \[33\] (1928), who proved

$$\delta(B_n) \leq \frac{n + 2}{2} \sqrt{2}^{-n}.$$

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\[33\] Hans Frederick Blichfeldt, 1873–1945
This was slightly improved by Rogers\textsuperscript{34} (1958) by a factor of 2/e, roughly speaking. In 1973/74 Sidelnikov\textsuperscript{35} showed that
\[ \delta(B_n) \leq 2^{-(0.509+o(1))n}, \quad n \text{ large.} \]
Subsequently, this bound was improved by Levenshtein\textsuperscript{36} (1975), and Kabatiansky\textsuperscript{37} and Levenshtein (1978) to
\[ \delta(B_n) \leq 2^{-(0.599+o(1))n}, \quad n \text{ large,} \]
which is still the best known bound.

7.14 Theorem. Let $K \in \mathcal{K}^n$ and $\Lambda \in \mathcal{L}^n \cap \mathcal{P}(K)$. Then
\[ \delta(K, \Lambda) = \frac{\text{vol}(K)}{\det \Lambda}. \]

Proof. Let $P_B$ be a fundamental cell of $\Lambda$. By Proposition 6.9, $\mathbb{R}^n$ is the pairwise disjoint union of the translates $b + P_B, b \in \Lambda$. Hence $\#((x + P_B) \cap \Lambda) \leq 1$ for all $x \in \mathbb{R}^n$, whereas the property that $\mathbb{R}^n$ is covered by all translates is equivalent to the lower bound $\#((x + P_B) \cap \Lambda) \geq 1$ for all $x \in \mathbb{R}^n$. Hence $\#((x + P_B) \cap \Lambda) = 1$ for all $x \in \mathbb{R}^n$. Together with $\text{vol}(P_B) = \det \Lambda$ the identity follows from Lemma 7.10. \qed

7.15 Definition [Density of a densest Lattice Packing]. For $K \in \mathcal{K}^n$ the set $\mathcal{P}_L(K) = \mathcal{L}^n \cap \mathcal{P}(K)$ is called the family of all packing lattices of $K$. For $\Lambda \in \mathcal{P}_L(K)$ the arrangement $\Lambda + K$ is called a lattice packing of $K$ and
\[ \delta_L(K) = \sup\{\delta(K, \Lambda) : \Lambda \in \mathcal{P}_L(K)\} \]
is called the density of a densest lattice packing of $K$.

7.16 Definition [Critical determinant and admissible lattices]. Let $K \in \mathcal{K}_0^n$. A lattice $\Lambda$ is called admissible for $K$ (or $K$-admissible) if $\text{int} K \cap \Lambda = \{0\}$.
\[ \Delta(K) = \inf \{\det \Lambda : \Lambda \text{ admissible for } K\} \]
is called the critical determinant of $K$.

7.17 Remark. Let $K \in \mathcal{K}^n_0$ and $\Lambda \in \mathcal{L}^n$. Then
i) $\Lambda$ is a packing lattice iff $\lambda_1(K, \Lambda) \geq 2$, and $\Lambda$ is admissible iff $\lambda_1(K, \Lambda) \geq 1$.
ii) $\Lambda$ is a admissible iff $2 \Lambda$ is a packing lattice.

\textsuperscript{34}Claude Ambrose Rogers, 1920–2005
\textsuperscript{35}Vladimir Michilovich Sidelnikov, 1940–2008
\textsuperscript{36}Vladimir Iosifovich Levenshtein, 1935
\textsuperscript{37}Grigory A. Kabatiansky, 1949
iii) $\frac{1}{\lambda_1(K, \Lambda)} \Lambda$ is admissible for $K$, and $\frac{2}{\lambda_1(K, \Lambda)} \Lambda$ is a packing lattice.

7.18 Proposition* [Critical lattice]. For $K \in \mathcal{K}_0^n$ there exists a $K$-admissible lattice $\Lambda_K$ with $\det \Lambda_K = \Delta(K)$. Such a lattice will be called a critical lattice of $K$.

7.19 Proposition. Let $K \in \mathcal{K}^n$. Then
\[
\delta_L(K) = \frac{\text{vol}(K)}{\Delta(K - K)}.
\]
Proof. By Proposition 7.9 v) and the definition of admissible lattices (cf. Definition 7.16) the family of packing sets $\mathcal{P}_L(K) = \mathcal{P}_L\left(\frac{1}{2}(K - K)\right)$ coincides with the set of all admissible lattices for $K - K$. From Corollary 7.14 and Proposition 7.18 we get the desired identity. □

7.20 Proposition. Let $K \in \mathcal{K}_0^n$ and $\Lambda \in \mathcal{L}^n$. Then
\[
\frac{\text{vol}(K)}{2^n} \leq \Delta(K) \leq \frac{\det \Lambda}{\prod_{i=1}^n \lambda_i(K, \Lambda)}.
\]
Proof. By Minkowski’s theorem 6.34 we get $\lambda_1(K, \Lambda_K)^n \text{vol}(K) \leq 2^n \Delta(K)$. Since $\lambda_1(K, \Lambda_K) \geq 1$ we get the lower bound. For the upper bound we just notice that by Remark 7.17 iii) we have $\Delta(K) \leq \det\left(\frac{1}{\lambda_1(K, \Lambda)} \Lambda\right)$. □

7.21 Conjecture [Davenport]. 38 Let $K \in \mathcal{K}_0^n$ and $\Lambda \in \mathcal{L}^n$. Then
\[
\Delta(K) \leq \det \Lambda \prod_{i=1}^n \frac{1}{\lambda_i(K, \Lambda)}.
\]

7.22 Remark.

i) $0 < \delta_L(K) \leq \delta(K) \leq 1$.

ii) $\delta_L(AK + t) = \delta_L(K)$ for all $A \in \text{GL}(n, \mathbb{R})$ and $t \in \mathbb{R}^n$.

iii) For $K \in \mathcal{K}_0^n$ we have $\delta_L(K) = 2^{-n} \text{vol}(K)/\Delta(K)$.

7.23 Theorem [Minkowski-Hlawka, 1943]. 39 Let $S \subset \mathbb{R}^n$ be a bounded Jordan-measurable set with $\text{vol}(S) < 1$. Then there exists a lattice $\Lambda \in \mathcal{L}^n$ with
\[
\det \Lambda = 1 \quad \text{and} \quad S \cap \Lambda \setminus \{0\} = \emptyset.
\]

---

38 Harold Davenport, 1907 – 1969
Proof. Since $S$ is bounded and Jordan-measurable of $\text{vol}(S) < 1$, there exists a prime $p$ such that

\begin{align}
\text{i)} & \quad \frac{1}{p^{n-1}} \# \left( S \cap \frac{1}{p^{(n-1)/n}} \mathbb{Z}^n \right) < 1, \\
\text{ii)} & \quad S \subset \left\{ x \in \mathbb{R}^n : |x_i| < p^{1/n}, 1 \leq i \leq n \right\}. \tag{7.23.1}
\end{align}

If we suppose that we can find $p^{n-1}$ sublattices of $\mathbb{Z}^n$ of determinant $p^{n-1}$, such that $\{0\}$ is the only common point in each two of them, then (7.23.1) i) would immediately imply the assertion. However, although we can not find those sublattices, we can find sublattices whose common points different from $0$ are "far away", which is good by (7.23.1) ii).

Let $U_p = \{ u \in \mathbb{Z}^n : u_1 = 1, 0 \leq u_i < p, i = 2, \ldots, n \}$. For $u \in U_p$ let $\Lambda(u)$ be the lattice with basis $u, pe_2, \ldots, pe_n$. Obviously, we have $\det(\Lambda(u)) = p^{n-1}$ and there are $p^{n-1}$ sublattices of that type. Since $u_1 = 1$ we observe that

\begin{equation}
\Lambda(u) \cap \Lambda(\bar{u}) \subset \{0\} \cup \{ x \in \mathbb{R}^n : \exists x_i \text{ with } |x_i| \geq p \}. \tag{7.23.3}
\end{equation}

To see this let $z \in \Lambda(u) \cap \Lambda(\bar{u})$. By (7.23.2) we have $z_1(u_i - \bar{u}_i) \equiv 0 \mod p$ and since $-(p-1) \leq u_i - \bar{u}_i \leq p - 1$, we conclude $z_1 \equiv 0 \mod p$. If $z_1 \neq 0$ we are done, and if $z_1 = 0$ we get from (7.23.2) that $z_i \equiv 0 \mod p, i = 2, \ldots, n$. Thus, either $z = 0$ or at least one coordinate is not less than $p$ in absolute value.

In view of (7.23.1) ii) the inclusion (7.23.3) shows that the $p^{n-1}$ sets

\begin{equation}
S \cap \frac{1}{p^{(n-1)/n}} \Lambda(u) \setminus \{0\}, \quad u \in U_p,
\end{equation}

are pairwise disjoint, and so (cf. (7.23.1) i))

\begin{equation}
\sum_{u \in U_p} \# \left( S \cap \frac{1}{p^{(n-1)/n}} \Lambda(u) \setminus \{0\} \right) \leq \# \left( S \cap \frac{1}{p^{(n-1)/n}} \mathbb{Z}^n \right) < p^{n-1}.
\end{equation}

Thus, at least one of the lattices $\frac{1}{p^{(n-1)/n}} \Lambda(u)$ has the desired properties. \hfill $\square$

7.24 Remark. Theorem 7.23 remains true for Jordan-measurable, unbounded, closed sets.

7.25 Corollary. Let $S \subset \mathbb{R}^n$ be a bounded Jordan-measurable set with $S = -S$ and with $\text{vol}(S) < 2$. Then there exists a lattice $\Lambda \in \mathcal{L}^n$ with

\begin{equation}
\det \Lambda = 1 \quad \text{and} \quad S \cap \Lambda \setminus \{0\} = \emptyset.
\end{equation}

Proof. Apply Theorem 7.23 to the set $\overline{S} = \{ x \in S : x_1 > 0 \}$. \hfill $\square$
7.26 Corollary. Let $K \in \mathcal{K}^{n}_0$. Then

$$\delta_L(K) \geq 2^{-(n-1)} \left[ \iff \Delta(K) \leq \frac{\text{vol}(K)}{2} \right].$$

Proof. Without loss of generality let $\text{vol}(K) = 2 - \varepsilon$ for some $\varepsilon > 0$. Corollary 7.25 shows that there exist a lattice $\Lambda_{\varepsilon}$, with $\text{det} \Lambda_{\varepsilon} = 1$, which is admissible for $K$ and thus $2\Lambda_{\varepsilon} \in \mathcal{P}_L(K)$. Hence we get

$$\delta_L(K) \geq \frac{\text{vol}(K)}{\text{det}(2\Lambda_{\varepsilon})} = 2^{-(n-1)} - \frac{\varepsilon}{2^n}.$$ 

Since this is true for every $\varepsilon > 0$ the statement is shown. □

7.27 Theorem*. Let $S \subset \mathbb{R}^n$, $n \geq 2$, be a bounded ray set (i.e., if $x \in S$ then $\lambda x \in S$ for all $\lambda \in [0, 1]$) with $\text{vol}(S) < \zeta(n)$. Then there exists a lattice $\Lambda \in \mathcal{L}^n$ with $\text{det} \Lambda = 1$ and $S \cap \Lambda \setminus \{0\} = \emptyset$. Here $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ denotes the (Riemann) zeta function ($\zeta$-function).

7.28 Theorem* [K. Ball, 1992]. \[40 \delta_L(B_n) \geq \frac{(n-1)2^{-(n-1)} \zeta(n)}{2^{n/(n-1)}}.\]

7.29 Lemma [Lagrange, 1773]. \[41 \text{Let } \Lambda \in \mathcal{L}^2 \text{ be a planar lattice. There exists a basis } b_1, b_2 \text{ of } \Lambda \text{ such that}

$$\text{det } \Lambda \geq \frac{\sqrt{3}}{2} |b_1| |b_2|. $$

Proof. Let $b_1, b_2$ be a basis of $\Lambda$, and let $\tilde{b}_2$ be the orthogonal projection of $b_2$ onto the line perpendicular to $\text{lin}(b_1)$, i.e.,

$$\tilde{b}_2 = b_2 - \frac{\langle b_1, b_2 \rangle}{|b_1|^2} b_1. $$

Then $\text{det } \Lambda = |b_1| |\tilde{b}_2|$. Replacing $b_2$ by $b_2 + z b_1$ with $z \in \mathbb{Z}$ does not change $\tilde{b}_2$ and for any $z \in \mathbb{Z}$ the vectors $b_1, b_2 + z b_1$ form a basis of $\Lambda$ as well. So we may choose a $\tilde{z} \in \mathbb{Z}$ such that for $\tilde{b}_2 = b_2 + \tilde{z} b_1$

$$\frac{|\langle b_1, \tilde{b}_2 \rangle|}{|b_1|^2} = \frac{|\langle b_1, b_2 \rangle|}{|b_1|^2} + \tilde{z} \leq \frac{1}{2}. $$

Hence we get

$$|\tilde{b}_2|^2 = |b_2|^2 - \frac{|\langle b_1, \tilde{b}_2 \rangle|^2}{|b_1|^2} \geq |b_2|^2 - \frac{1}{4} |b_1|^2 \geq \frac{3}{4} |\tilde{b}_2|^2, $$

since $b_1$ was a shortest vector. Combing this bound with $\text{det } \Lambda = |b_1| |\tilde{b}_2|$ gives the desired inequality. □

\[40 \text{Keith M. Ball, 1960}
41 \text{Joseph-Louis Lagrange, 1736–1813} \]
7.30 Corollary [Lagrange, 1773].

\[ \delta_{\mathcal{L}}(B_2) = \frac{\pi}{2\sqrt{3}} \quad \iff \Delta(B_2) = \frac{\sqrt{3}}{2} \]

Proof. Let \( \Lambda \) be a packing lattice of the unit circle \( B_2 \) with basis \( b_1, b_2 \). According to Theorem 7.14 and Lemma 7.29 we have

\[ \delta(B_2, \Lambda) = \frac{\text{vol}(B_2)}{\det \Lambda} \leq \frac{2}{\sqrt{3}|b_1||b_2|} \leq \frac{\pi}{2\sqrt{3}}. \]

On the other hand, the hexagonal lattice \( \Lambda_{\text{hex}} \) with basis \((2, 0)^\top\) and \((1, \sqrt{3})^\top\) is a packing lattice of \( B_2 \) with \( \delta(B_2, \Lambda_{\text{hex}}) = \pi/(2\sqrt{3}) \) \( \square \)

7.31 Theorem* [Thue, 1890, 1910]. \(^{42}\) \( \delta(B_2) = \delta_{\mathcal{L}}(B_2) \).

7.32 Lemma [Gauß, 1840]. \(^{43}\) Let \( \Lambda \in \mathcal{L}^3 \). Then there exists a basis \( b_1, b_2, b_3 \) of \( \Lambda \) such that

\[ \det \Lambda \geq \frac{1}{\sqrt{2}} |b_1| |b_2| |b_3|. \]

Proof. Let \( b_1, b_2, b_3 \) be a basis of \( \Lambda \) such that \( b_1 \) is a shortest lattice vector, \( b_2 \) is a shortest vector among all vectors \( c \in \Lambda \) which can be together with \( b_1 \) extended to a basis of \( \Lambda \). Finally, let \( b_3 \) be a shortest lattice vector such that \( b_1, b_2, b_3 \) is a basis of \( \Lambda \). Then we have

\[ |b_i + b_j|^2 \geq |b_i|^2, \quad 1 \leq i \neq j \leq 3, \]

and thus

\[ 2|b_i| \leq b_{i,j}, \quad 1 \leq i \neq j \leq 3, \quad (7.32.1) \]

with \( b_{i,j} = \langle b_i, b_j \rangle \), \( 1 \leq i \leq j \leq 3 \). Then

\[ (\det \Lambda)^2 = \beta_{1,1}\beta_{2,2}\beta_{3,3} - \beta_{1,1}\beta_{2,2}^2 - \beta_{2,2}\beta_{3,3} - \beta_{3,3}\beta_{1,1}^2 + 2\beta_{1,2}\beta_{1,3}\beta_{2,3}. \]

We can assume that either \( \beta_{i,j} \geq 0 \) or \( \beta_{i,j} \leq 0 \) for \( 1 \leq i < j \leq 3 \).

1. \( \beta_{i,j} \geq 0, 1 \leq i < j \leq 3. \)

2. \( \beta_{i,j} \leq 0, 1 \leq i < j \leq 3. \)

\[ 2(\det \Lambda)^2 = \beta_{1,1}\beta_{2,2}\beta_{3,3} + \beta_{1,1}\beta_{2,2}(\beta_{2,2} + 2\beta_{3,3}) + \beta_{2,2}\beta_{1,3}(\beta_{3,3} - 2\beta_{1,3}) + \beta_{3,3}\beta_{1,2}(\beta_{1,1} - 2\beta_{1,2}) + \beta_{1,2}(\beta_{3,3} - 2\beta_{2,3}) + \beta_{1,2}(\beta_{3,3} - 2\beta_{2,3}) + \beta_{1,2}(\beta_{3,3} - 2\beta_{2,3}) + \beta_{1,2}(\beta_{3,3} - 2\beta_{2,3}) \]

\[ \geq \beta_{1,1}\beta_{2,2}\beta_{3,3}, \]

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42 Axel Thue, 1863–1922
43 Carl Friedrich Gauss, 1777–1855
where the last inequality follows from \( \beta_{i,j} \geq 0, 1 \leq i \neq j \leq 3 \), and (7.32.1).

II. \( \beta_{i,j} \leq 0, 1 \leq i < j \leq 3 \). Here we need one more inequality, namely
\[
|b_1 + b_2 + b_3|^2 \geq |b_i|^2, 1 \leq i \leq 3,
\]
which may be rewritten as
\[
\alpha_{i,j} = \beta_{i,j} + 2 \beta_{1,2} + 2 \beta_{2,3} + 2 \beta_{1,3} \geq 0, \quad 1 \leq i < j \leq 3.
\] (7.32.2)

Furthermore, let \( \gamma_{i,j} = 2 \beta_{i,j} + \beta_{j,j} \). By (7.32.1) we have
\[
\gamma_{i,j} \geq 0, \quad 1 \leq i, j \leq 3.
\] (7.32.3)

where the last inequality is an immediate consequence of \( \beta_{i,j} \leq 0, 1 \leq i \neq j \leq 3 \), cf. (7.32.2) and (7.32.3).

\[ \square \]

7.33 Corollary [Gauss].

\[
\delta_L(B_3) = \frac{\pi}{\sqrt{18}} \quad \Leftrightarrow \quad \Delta(B_3) = \frac{1}{\sqrt{2}}.
\]

Proof. Analogously to the proof of Corollary 7.30 we argue via Lemma 7.32 that for any packing lattice \( \Lambda \) of \( B_3 \) we have
\[
\delta(B_3, \Lambda) \geq \frac{\pi}{\sqrt{18}}.
\]

On the other hand let \( \Lambda_{fcc} = \{ z \in \mathbb{Z}^3 : z_1 + z_2 + z_3 \equiv 0 \mod 2 \} \). Then \( \det \Lambda_{fcc} = 2 \) and \( \lambda_1(B_3, \Lambda_{fcc}) = \sqrt{2} \). Hence \( \sqrt{2} \Lambda_{fcc} \) is a packing lattice of \( B_3 \) with determinant \( 2 \sqrt{2} \), given the density \( \delta(B_3, \Lambda_{fcc}) = \frac{\pi}{\sqrt{18}} \). \( \Lambda_{fcc} \) is called the face-centered-cubic lattice.

\[ \square \]

7.34 Theorem* [Hales, 1998/2005, Proof of the “Kepler-conjecture”].

\[
\delta(B_3) = \delta_L(B_3) = \frac{\pi}{\sqrt{18}}.
\]

\[ ^{44} \text{Thomas Callister Hales, 1958} \]
7.35 Theorem* [Korkin-Zolotarev, 1872/73; Blichfeldt, 1934].

\[ \delta_L(B_4) = \frac{\pi^2}{16} \quad \left[ \Leftrightarrow \Delta(B_4) = \frac{1}{2} \right] \]
\[ \delta_L(B_5) = \frac{\pi^2}{15\sqrt{2}} \quad \left[ \Leftrightarrow \Delta(B_5) = \frac{1}{2\sqrt{2}} \right] \]
\[ \delta_L(B_6) = \frac{\pi^3}{48\sqrt{3}} \quad \left[ \Leftrightarrow \Delta(B_6) = \frac{\sqrt{3}}{8} \right] \]
\[ \delta_L(B_7) = \frac{\pi^3}{105} \quad \left[ \Leftrightarrow \Delta(B_7) = \frac{1}{8} \right] \]
\[ \delta_L(B_8) = \frac{\pi^4}{384} \quad \left[ \Leftrightarrow \Delta(B_8) = \frac{1}{16} \right] \]

7.36 Theorem* [Cohn & Kumar, 2004]. The so-called Leech lattice \( \Lambda_{\text{Leech}} \) is the optimal packing lattice in dimension 24, and it is

\[ \delta_L(B_{24}) = \frac{\pi^{12}}{12!} \quad \left[ \Leftrightarrow \Delta(B_{24}) = \frac{1}{2^n} \right] \]

7.37 Theorem [Swinnerton-Dyer, 1953]. Let \( K \in K^0_n \) and let \( \Lambda_K \in \mathcal{L}^n \) be a critical lattice of \( K \). Then

\[ \#(K \cap \Lambda_K \setminus \{0\}) \geq n(n+1). \]

**Proof.** Without loss of generality let \( \Lambda_K = \mathbb{Z}^n \), and let \( \{\pm a_1, \ldots, \pm a_k\} = K \cap \mathbb{Z}^n \setminus \{0\} \). For \( T \in \mathbb{R}^{n \times n} \) with entries \( t_{l,m} \) and for \( \rho \in \mathbb{R} \) let \( I_{\rho,T} = I_n + \rho T \).

In the following we show that we can choose parameters \( t_{l,m} \) and \( \rho \) such that the corresponding lattice \( \Lambda_{\rho,T} = I_{\rho,T} \mathbb{Z}^n \) is still admissible for \( K \), but \( \det \Lambda_{\rho,T} < \det \Lambda_K = 1 \).

Let \( H_i = \{ x \in \mathbb{R}^n : \langle u_i, x \rangle = 1 \} \) be a supporting hyperplane of \( K \) at the point \( a_i \) and let \( a_{i,\rho,T} = I_{\rho,T} a_i = a_i + \rho Ta_i \). Then we have

\[ a_{i,\rho,T} \in H_i \Leftrightarrow \langle u_i, I_{\rho,T} a_i \rangle = 1 \Leftrightarrow \langle u_i, Ta_i \rangle = 0 \Leftrightarrow \sum_{l,m=1}^n c_{l,m,i} t_{l,m} = 0, \]

where \( c_{l,m,i} \) are certain numbers, depending on \( a_i, u_i \). Since \( k < n(n+1)/2 \) we can find non-trivial scalars \( t_{l,m} \in \mathbb{R} \), say, such that

i) \( t_{l,m} = \bar{t}_{m,l} \),

ii) \( a_{i,\rho,T} \in H_i \), \( 1 \leq i \leq k \), and \( \rho \in \mathbb{R} \). \hfill (7.37.1)

Next we argue that we can find a \( \overline{\rho} \in \mathbb{R} \) such that \( \Lambda_{\rho,T} \) is a \( K \)-admissible lattice for all \( |\rho| \leq \overline{\rho} \). First we notice that for sufficiently small \( \rho \), \( \Lambda_{\rho,T} \in \mathcal{L}^n \), i.e.,

\[ 45 \text{Yegor Ivanovich Zolotarev, 1847–1878} \]
\[ 46 \text{Aleksandr Korkin, 1837–1908} \]
\[ 47 \text{Henry Peter Francis Swinnerton-Dyer, 1927} \]
\[ \det \Lambda_{\rho,T} > 0. \] Next, suppose that there exists a sequence \( \rho_i \to 0 \) such that there exists \( u_{\rho_i,T} \in \Lambda_{\rho_i,T} \setminus \{0\} \cap \text{int} K \) for \( i \in \mathbb{N} \). Since \( \Lambda_{\rho_i,T} \to \Lambda_K \) we may assume that \( u_{\rho_i,T} \to a_1 \), say. We also know that \( \mathbf{a}_{1,\rho_i,T} \to \mathbf{a}_1 \) and so we have \( \mathbf{a}_{1,\rho_i,T} - u_{\rho_i,T} \to 0 \), which implies \( \mathbf{a}_{1,\rho_i,T} = u_{\rho_i,T} \) for sufficiently large \( i \). Since \( \mathbf{a}_{1,\rho_i,T} \in H_1 \) we have \( u_{\rho_i,T} \notin \text{int} K \), a contradiction.

Now since \( \Lambda_{\rho,T} \) is \( K \)-admissible for all \( |\rho| \leq \bar{p} \) we must have \( \det \Lambda_{\rho,T} \geq \det \Lambda_K \) and so \( \det (I_n + \rho T) \geq 1 \) for all \( |\rho| \leq \bar{p} \). Expressing \( \det (I_n + \rho T) \) as the sum over all permutations leads to

\[
\det (I_n + \rho T) = \prod_{i=1}^{n} (1 + \rho \bar{t}_{i,i}) - \rho^2 \sum_{1 \leq i < j \leq n} \bar{t}_{i,j} \bar{t}_{j,i} \prod_{k \notin \{i,j\}} (1 + \rho \bar{t}_{k,k}) + \rho^3 \gamma_3 + \cdots + \rho^n \gamma_n,
\]

where the first two terms reflect the identity permutation and the transpositions, and \( \gamma_i \) are certain constants depending on \( T \). Hence we get

\[
\det (I_n + \rho T) = 1 + \tau_1 \rho + \tau_2 \rho^2 + \rho^3 \gamma_3 + \cdots + \rho_n \gamma_n,
\]

with

\[
\tau_1 = \sum_{j=1}^{n} \bar{t}_{j,j}, \quad \tau_2 = \sum_{1 \leq i < j \leq n} (\bar{t}_{i,i} \bar{t}_{j,j} - \bar{t}_{i,j} \bar{t}_{j,i}),
\]

and some other constants \( \gamma_i \). Since \( \det (I_n + \rho T) \geq 1 \) for all \( |\rho| \leq \bar{p} \) we first get \( \tau_1 = 0 \) and then \( \tau_2 \geq 0 \), which gives by (7.37.1) i)

\[
0 \leq 2\tau_2 - (\tau_1)^2 = 2 \sum_{i<j} \bar{t}_{i,i} \bar{t}_{j,j} - 2 \sum_{i<j} (\bar{t}_{i,j})^2 - \sum_{i=1}^{n} (\bar{t}_{i,i})^2 - 2 \sum_{i<j} \bar{t}_{i,i} \bar{t}_{j,j} = -\sum_{i=1}^{n} (\bar{t}_{i,i})^2 - 2 \sum_{i<j} (\bar{t}_{i,j})^2.
\]

Thus we have \( T = 0 \), which contradicts the choice of \( T \). \( \square \)

### 7.38 Theorem

Let \( K \in \mathcal{K}_{0}^2 \).

i) Let \( b_1, b_2 \in \text{bd} K \) such that \( b_2 - b_1 \in \text{bd} K \). Then the lattice \( \Lambda = (b_1, b_2) \mathbb{Z}^2 \) is admissible for \( K \).

ii) Let \( \Lambda \in \mathcal{L}^2 \) be a critical lattice of \( K \). Then there exists a basis \( b_1, b_2 \) of \( \Lambda \) such that \( b_1, b_2, b_2 - b_1 \in \text{bd} K \).

**Proof.** For i) we first notice that, since \( b_1, b_2, b_2 - b_1 \in \text{bd} K \) then

\[
\text{int} K \cap \{ z_1 b_1 + z_2 b_2 : z_2 \in \{0, \pm 1\}, z_1 \in \mathbb{Z} \} = \{0\}.
\]

Now assume that there exists \( \mathbf{b} = \overline{z_1} \mathbf{b}_1 + \overline{z_2} \mathbf{b}_2 \in \text{int} K \) and without loss of generality let \( \overline{z}_2 \geq 2 \). Further let \( \varepsilon > 0 \) such that \( \mathbf{b} \pm \varepsilon \mathbf{b}_1 \in \text{int} K \) and let
\( P = \text{conv}\{\pm b_1, b \pm \varepsilon b_1\}. \) Then \( \text{vol}_1(P \cap (\overline{P} b_2 + \text{lin}\{b_1\})) \geq 2\varepsilon |b_1|, \) and since \( \text{vol}_1(P \cap \text{lin}\{b_1\}) = 2|b_1| \) we conclude
\[
\text{vol}_1(P \cap (b_2 + \text{lin}\{b_1\})) > |b_1|
\]
This shows, however, that either \( b_2 \) or \( b_2 - b_1 \) belong to the interior of \( K \).

For ii) we observe that on account of \( \det \Lambda = \Delta(K) \) we can find two linearly independent points \( b_1, b_2 \in \Lambda \cap \text{bd} K \). On account of Lemma 6.17 we may assume that \( b_1, b_2 \) build a basis of \( \Lambda \). Now let \( \alpha_1, \alpha_2 \in \mathbb{R}_{\geq 0} \) be maximal such that \( b_2 - \alpha_1 b_1 \in \text{bd} K \) and \( b_2 + \alpha_2 b_1 \in \text{bd} K \).

If \( \alpha_1 \geq 1 \) then we must have \( b_2 - b_1 \in \text{bd} K \) and we are done. Similarly, if \( \alpha_2 \geq 1 \) we have \( b_2 + b_1 \in \text{bd} K \) and the basis \( b_1, b_1 + b_2 \) of \( \Lambda \) has the required property. So we may assume \( \alpha_1 < 1 \). If \( \alpha_1, \alpha_2 > 0 \) then \( \pm b_2 + \text{lin}\{b_1\} \) are supporting lines of \( K \), and from this we conclude that \( K \cap \Lambda \setminus \{0\} = \{\pm b_1, \pm b_2\} \), which contradicts Theorem 7.37. Hence we may assume that \( \alpha_2 = 0 \) and since \( \alpha_1 < 1 \) we can find \( \lambda \in (0, 1) \) such that \( \text{vol}_1(K \cap (\lambda b_2 + \text{lin}\{b_1\})) = |b_1| \). Let \( u \) and \( v \) be the corresponding points in the boundary, such that \( v = u - b_1 \). By i) we know that the lattice with basis \( b_1, u \) is admissible for \( K \), but \( \left| \det(b_1, u) \right| = \left| \det(b_1, \lambda b_2) \right| < \Delta(K) \), which contradicts the definition of \( \Delta(K) \).

\begin{align*}
7.39 \text{ Corollary.} & \quad \text{Let } K \in \mathcal{K}^3_0 \text{ and } H_K \text{ be an affinely regular hexagon of minimal volume with vertices on } \text{bd} K. \text{ Then} \\
& \quad \delta_L(K) = \frac{3}{4} \frac{\text{vol}(K)}{\text{vol}(H_K)} \quad \left[ \Leftrightarrow \Delta(K) = \frac{1}{3} \text{vol}(H_K) \right].
\end{align*}

\textbf{Proof.} Let \( \pm v_1, \pm v_2, \pm v_3 \) be the vertices of \( H_K \) on the boundary of \( K \). Since \( H_K \) is affinely regular we may assume that \( v_3 = v_2 - v_1 \). Hence, by Theorem 7.38 i), the lattice \( \Lambda(H_K) \) with basis \( v_1, v_2 \) is admissible for \( K \) and it is \( \det \Lambda(H_K) = \text{vol}(H_K)/3 \). Thus \( \Delta(K) \leq \text{vol}(H_K)/3 \).

On the other hand, by Theorem 7.38 ii), each critical lattice \( \Lambda_K \) gives rise to such an affinely regular hexagon \( H(\Lambda_K) \) with vertices on the boundary and \( \text{vol}(H(\Lambda_K)) = 3\Delta(K) \). Thus we also have \( \Delta(K) \geq \text{vol}(H(\Lambda_K))/3 \).

\begin{align*}
7.40 \text{ Theorem}^* \quad \text{[Fejes Tóth, 1950; Rogers, 1951].} & \quad \text{Let } K \in \mathcal{K}^2. \text{ Then} \\
& \quad \delta(K) = \delta_L(K).
\end{align*}
8 Count and generate

8.1 Definition [Lattice polytope]. A polytope \( P = \text{conv} \{ v_1, \ldots, v_m \} \subset \mathbb{R}^n \) is called a lattice polytope if \( v_i \in \mathbb{Z}^n \), \( 1 \leq i \leq m \). The set of all lattice polytopes is denoted by \( P^m_n \mathbb{Z} \).

8.2 Notation.

i) For \( S \subset \mathbb{R}^n \) we denote by \( G(S) \) the lattice point enumerator, i.e., \( G(S) = \#(S \cap \mathbb{Z}^n) \).

ii) For integers \( m, n \) we denote by 
\[
\left( \frac{x + m}{n} \right) = \frac{1}{n!} \prod_{i=0}^{n-1} (x + m - i)
\]
the polynomial of degree \( n \) with roots \( i - m, i = 0, \ldots, n - 1 \), and leading coefficient \( 1/n! \). In particular, the polynomials \( \left( \frac{x + n - i}{n} \right) \), \( i = 0, \ldots, n \), form a basis of the space of all polynomials of degree at most \( n \).

8.3 Lemma. Let \( T = \text{conv} \{ 0, v_1, \ldots, v_n \} \in P^m_n \mathbb{Z} \) be a lattice simplex, i.e., \( v_1, \ldots, v_n \in \mathbb{Z}^n \) are linearly independent, and for \( 0 \leq i \leq n \) let
\[
U_i = \left\{ \sum_{j=1}^{n} \lambda_j v_j \in \mathbb{Z}^n : 0 \leq \lambda_j < 1, i - 1 < \sum_{j=1}^{n} \lambda_j \leq i \right\}
\]
Then for all \( k \in \mathbb{N}, k \geq 1 \), we have
\[
G(kT) = \sum_{i=0}^{n} \#U_i \cdot \left( \frac{k + n - i}{n} \right).
\]
In particular, \( G(kT) \) is a polynomial of degree \( n \) in \( k \) whose coefficients are integral multiples of \( n! \).

Proof. For \( m \in \mathbb{Z} \) let
\[
Q_m = \left\{ \sum_{i=1}^{n} q_i v_i : q_i \in \mathbb{N} \text{ and } \sum_{i=1}^{n} q_i \leq m \right\},
\]
where we set \( Q_m = \emptyset \) if \( m < 0 \). Furthermore, let \( U = \{ \sum_{i=1}^{n} \lambda_i v_i \in \mathbb{Z}^n : 0 \leq \lambda_i < 1 \} \) be the half open fundamental cell of the lattice generated by \( v_1, \ldots, v_n \). Then \( U \) is the disjoint union of the \( U_i \)'s, and each integral point \( z \in kT \cap \mathbb{Z}^n \) has a unique representation as
\[
z = u_z + w_z \quad (8.3.1)
\]
with \( u_z \in U_i \) and \( w_z \in Q_{k-i} \) for a suitable \( i \in \{0, \ldots, n\} \). For if, let \( z = \sum_{i=1}^{n} \mu_i v_i \in kT \cap \mathbb{Z}^n \). Then \( 0 \leq \mu_i \leq k, \sum_{i=1}^{n} \mu_i \leq k \) and we may write
\[
z = \sum_{i=1}^{n} (\mu_i - \lfloor \mu_i \rfloor) v_i + \sum_{i=1}^{n} \lfloor \mu_i \rfloor v_i.
\]
The first sum on the right hand side is in $U$, and thus in some $U_j$, say, and on account of $0 \leq \mu_i \leq k$, we also get that the second sum is in $Q_{k-j}$. Regarding the uniqueness we observe that $u + w = \overline{u} + \overline{w}$ for $u \in U$, $\overline{u} \in \overline{U}$, $w \in Q_m$, $\overline{w} \in \overline{Q}_m$ implies that
\[
\mathbf{u} - \mathbf{u} \in (U - U) \cap (Q_m - Q_m).
\]
By Remark 6.7 iv), the intersection on the right hand side contains only $0$. Hence the representation (8) is unique. Since we obviously have $U_i + Q_{k-i} \subset kT \cap \mathbb{Z}^n$ we have shown
\[
kT \cap \mathbb{Z}^n = \bigcup_{i=0}^n (U_i + Q_{k-i}),
\]
and $\#(U_i + Q_{k-i}) = \#U_i \#Q_{k-i}$. It finally remains to verify that
\[
\#Q_m = \binom{n + m}{n},
\]
which follows easily, e.g., by induction on $m, n$. \hfill \Box

8.4 Remark. The statement of the lemma above is also true for lower dimensional simplices and arbitrary lattices.

8.5 Lemma [Inclusion-Exclusion Formula]. Let $A_i \subseteq \mathbb{R}^n$, $1 \leq i \leq m$, with characteristic functions $\chi(A_i)$. Then
\[
\chi(A_1 \cup A_2 \cup \cdots \cup A_m) = \sum_{I \subseteq \{1, \ldots, m\}} (-1)^{#I-1} \chi \left( \bigcap_{j \in I} A_j \right).
\]

Proof. First we observe that $\chi(A) \cdot \chi(B) = \chi(A \cap B)$ for any two subsets $A, B \subseteq \mathbb{R}^n$. Hence the right hand side can be rewritten as
\[
\mathbf{1} - \prod_{i=1}^n (\mathbf{1} - \chi(A_i)),
\]
where $\mathbf{1}$ is the 1-constant function. Now the function in (8.5.1) takes the value 1 exactly for those $\mathbf{x} \in \mathbb{R}^n$ for which one of the functions $\mathbf{1} - \chi(A_i)$ takes the value 0, i.e., if and only if $\mathbf{x} \in A_1 \cup A_2 \cup \cdots \cup A_m$. \hfill \Box

8.6 Definition [Triangulation]. A triangulation of a convex $n$-polytope $P$ is a finite collection $T$ of $n$-simplices such that

i) $P$ is the union of all simplices in $T$.

ii) For any two simplices $\tau_1, \tau_2 \in T$ their intersection $\tau_1 \cap \tau_2$ is a face common to both $\tau_1$ and $\tau_2$. 
8.7 Theorem. Every \( n \)-polytope \( P \) can be triangulated such that the vertices of any simplex in the triangulation are vertices of \( P \).

Proof. Let \( V = \{ \mathbf{v}_1, \ldots, \mathbf{v}_m \} \) be the set of vertices of \( P \). We want to find non-negative numbers \( \eta_1, \ldots, \eta_m \) such that for any choice of \( n + 1 \) affinely independent points \( \mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_{n+1}} \in V \) the unique hyperplane containing the lifted points \( (\mathbf{v}_{i_j}, \eta_j)^T, 1 \leq j \leq n + 1 \) contains no other lifted points. Then all faces of the convex hull of this lifted configuration, i.e., \( \text{conv} \{ (\mathbf{v}_{i_j}, \eta_j)^T : 1 \leq i \leq m \} \) are simplices and the projected faces of the lower convex hull (i.e., faces, whose outer unit normal vector has a negative coordinate) gives the desired regular triangulation. Now for a given set \( \mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_{n+1}} \in V \) this is equivalent to say that

\[
\det \begin{pmatrix}
1 & 1 & \cdots & 1 & 1 \\
\mathbf{v}_{i_1} & \mathbf{v}_{i_2} & \cdots & \mathbf{v}_{i_{n+1}} & \mathbf{v}_k \\
\eta_{i_1} & \eta_{i_2} & \cdots & \eta_{i_{n+1}} & \eta_k
\end{pmatrix}
\]

is non zero for all \( k \in \{1, \ldots, m\} \setminus \{i_1, \ldots, i_{n+1}\} \). Evaluating this determinant with respect to the last row shows that the determinant is zero if and only if \( (\eta_{i_1}, \eta_{i_2}, \ldots, \eta_{i_{n+1}}, \eta_k)^T \) satisfies a non-trivial linear equation. Thus, except points lying in a hyperplane of the form \( \langle \mathbf{w}, (\eta_1, \ldots, \eta_m)^T \rangle = 0 \) for a certain \( \mathbf{w} \in \mathbb{R}^m \) yielding a non-zero determinant in (8.7.1).

Hence for almost any choice of \( (\eta_1, \ldots, \eta_m)^T \in \mathbb{R}_{\geq 0}^m \) all the determinants of type (8.7.1) for any choice of affinely independent points \( \mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_{n+1}} \in V \) and \( \mathbf{v}_k \in V \setminus \{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_{n+1}}\} \) are non zero. \( \square \)

8.8 Theorem [Ehrhart, 1967]. \( ^{48} \) Let \( P \in \mathcal{P}_n^\mathbb{Z} \). Then there exist numbers \( G_i(P) \in \mathbb{Q}, 0 \leq i \leq n \), depending only on \( P \), such that for all \( k \in \mathbb{N}_{\geq 1} \)

\[
G(kP) = \sum_{i=0}^{n} G_i(P) k^i.
\]

The right hand side is called Ehrhart-polynomial.

Proof. Without loss of generality let \( \text{dim} P = n \). On account of Theorem 8.7 there exists a triangulation \( T = \{ \tau_1, \ldots, \tau_m \} \) of \( P \) where the vertices of each \( \tau_i \) are vertices of \( P \). Then each \( \tau_i \) as well as the intersection of two \( \tau_i \)'s are \( m \)-dimensional lattice simplices or empty. Hence with Lemma 8.5 and Lemma 8.3 we get

\[
G(kP) = G \left( \bigcup_{i=1}^{m} k \tau_i \right) = \sum_{I \subseteq \{1, \ldots, m\}} (-1)^{\#I-1} G \left( k \bigcap_{j \in I} \tau_j \right)
\]

\[
= \sum_{I \subseteq \{1, \ldots, m\}} (-1)^{\#I-1} \sum_{i=0}^{\dim(\bigcap_{j \in I} \tau_j)} G_i \left( \bigcap_{j \in I} \tau_j \right) k^i,
\]

which also implies \( G_i(P) \in \mathbb{Q} \). \( \square \)

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\( ^{48} \) Eugénie Ehrhart, 1906–2000
8.9 Proposition. Let $P \in \mathcal{P}^n_Z$.

i) $G_n(P) = \text{vol}(P)$.

ii) $G_i : \mathcal{P}^n_Z \to \mathbb{R}$ is

a) homogeneous of degree $i$,

b) invariant with respect to unimodular transformations, i.e., for any $U \in \text{GL}(n, \mathbb{Z})$, $t \in \mathbb{Z}^n$, it is $G_i(t + UP) = G_i(P)$,

iii) additive, i.e., for $P, Q \in \mathcal{P}^n_Z$ with $P \cup Q, P \cap Q \in \mathcal{P}^n_Z$ it is $G_i(P \cup Q) = G_i(P) + G_i(Q) - G_i(P \cap Q)$.

iii) $G_i(P)$ are independent of the dimension of the space in which $P$ is embedded, i.e., let $P \in \mathcal{P}^n_Z$ and let $\tilde{P} = \text{conv}\{ (v,0)^\top : v \in P \} \in \mathcal{P}^{n+1}_Z$. Then $G_i(P) = G_i(\tilde{P})$, $i = 0, \ldots, n$.

Proof. By the Riemann integrability of the characteristic function of $P$ we get in view of Theorem 8.8

$$
\text{vol}(P) = \lim_{m \to \infty} \frac{\#(P \cap \frac{1}{m}\mathbb{Z}^n)}{m^n} = \lim_{m \to \infty} \frac{G(mP)}{m^n} = \lim_{m \to \infty} \sum_{i=0}^n G_i(P)m^{i-n} = G_n(P).
$$

For ii) we observe that for $k, m \in \mathbb{N}_{\geq 1}, U \in \text{GL}(n, \mathbb{Z}), t \in \mathbb{Z}^n$ and $P, Q \in \mathcal{P}^n_Z$

$$
\sum_{i=0}^n (G_i(P)m^i) k^i = G((km)P) = G(k(mP)) = \sum_{i=0}^n (G_i(mP)) k^i,
$$

$$
\sum_{i=0}^n G_i(t + UP)k^i = G(k(t + UP)) = G(kP) = \sum_{i=0}^n G_i(P)k^i,
$$

$$
\sum_{i=0}^n (G_i(P) + G_i(Q)) k^i = G(kP) + G(kQ) = G(k(P \cup Q)) + G(k(P \cap Q))
$$

$$
= \sum_{i=0}^n (G_i(P \cup Q) + G_i(P \cap Q)) k^i,
$$

where for the last we also assume $P \cup Q, P \cap Q \in \mathcal{P}^n_Z$. Comparing the coefficients in all three equations shows the required properties of $G_i(P)$.

Obviously, $G(kP) = G(k\tilde{P})$ for $k \in \mathbb{N}_{\geq 1}$, and this shows iii). \qed

8.10 Theorem* [Betke& Kneser, 1985]. 49 50 Every additive and unimodular invariant functional on the space of all lattice polytopes is a linear combination of the $n+1$ functionals $G_i(\cdot)$.

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49 Martin Kneser, 1928 – 2004

50 Ulrich Betke, 1948 – 2008
8.11 Remark. Some of the coefficients $G_i(P)$ might be negative. One family of standard examples in this context are the so-called Reeve-simplices: let $R_m = \text{conv} \{0, e_1, e_2, (1, 1, m)^T\} \in \mathcal{P}_Z^3$ for $m \in \mathbb{N}$. The only lattice points contained in $R_m$ are the four vertices, the volume, however, can be arbitrarily large. Hence some $G_i(R_m)$ must be negative for large $m$. More precisely, it is $G_3(R_m) = m/6$, $G_2(R_m) = 1$, $G_1(R_m) = (12 - m)/6$ and $G_0(R_m) = 1$.

8.12 Corollary. Let $P \in \mathcal{P}_Z^n$. Then there exist numbers $a_i(P) \in \mathbb{Q}$, $0 \leq i \leq n$, such that for all $k \in \mathbb{N}_{\geq 1}$

$$G(kP) = \sum_{i=0}^{n} a_i(P) \binom{k + n - i}{n}.$$

8.13 Example.

i) Let $T_n = \text{conv} \{0, e_1, e_2, \ldots, e_n\}$. Then

$$\#(kT_n \cap \mathbb{Z}^n) = \binom{n + k}{n},$$

and so we have $a_i(T_n) = 0$ for $1 \leq i \leq n$, and $a_0(T_n) = 1$. The $G_i(T_n)$ are up to $\pm 1$ – Stirling numbers of the first kind.

ii) Let $C_n = [-1, 1]^n$. Then $G(kC_n) = (2k + 1)^n$ and so $G_i(C_n, \mathbb{Z}^n) = 2^i \binom{n}{i}$, $0 \leq i \leq n$. Here the $a_i(C_n)$ are some combinatorial numbers, the so-called Eulerian numbers.

iii) Let $C_n^* = \text{conv} \{\pm e_i : 1 \leq i \leq n\}$. Then

$$G(kC_n^*) = \sum_{i=0}^{n} \binom{n}{i} \binom{k + n - i}{n},$$

and so $a_i(C_n^*) = \binom{n}{i}$, $i = 0, \ldots, n$.

8.14 Definition [Generating function]. For $S \subseteq \mathbb{R}^n$ the function (formal power series)

$$\gamma(S; x) = \sum_{m \in S \cap \mathbb{Z}^n} x^m$$

is called the generating function or the integer-point transform of $S$. Here $x^m$ denotes the monomial $x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}$.

8.15 Example. Let $S_1 = [0, \infty)$ and $S_2 = (-\infty, 2]$. Then

$$\gamma(S_1; x) = \sum_{i=0}^{\infty} x^i = \frac{1}{1 - x}, \quad |x| < 1,$$

$$\gamma(S_2; x) = x^2 + x + \sum_{i=0}^{\infty} x^i = x^2 + x + \frac{1}{1 - 1/x}, \quad |x| > 1.$$
For $P = [0, 2] = S_1 \cap S_2$ we observe that
\[
\gamma(P; x) = 1 + x + x^2 = \frac{1}{1-x} + x^2 + x + \frac{1}{1-1/x} = \gamma(S_1; x) + \gamma(S_2; x).
\]

8.16 Definition [Simplicial, rational, pointed cones]. Let $v_1, \ldots, v_k \in \mathbb{R}^n$. Then $C = \text{pos}\{v_1, \ldots, v_k\}$ is called a polyhedral cone generated by $v_1, \ldots, v_k$. If $v_i \in \mathbb{Z}^n$, then $C$ is called a rational cone. $C$ is called a polyhedral pointed cone, if $0$ is a vertex of $C$. If $v_1, \ldots, v_k$ are linearly independent, then $C$ is called a simplicial cone.

8.17 Lemma. Let $C = \text{pos}\{v_1, \ldots, v_k\}$ be a rational simplicial cone, $W = \{x \in \mathbb{C}^n : |x^{v_i}| < 1 : 1 \leq i \leq k\}$, and let $t \in \mathbb{R}^n$.

i) $\gamma(t + C; x)$ converges absolutely and uniformly on all compact subsets of $W$ to the rational function
\[
\gamma(t + C; x) = \frac{\gamma(t + U; x)}{(1 - x^{v_1}) \cdot (1 - x^{v_2}) \cdots (1 - x^{v_k})},
\]
where $U = \{\sum_{i=1}^k \alpha_i v_i : 0 \leq \alpha_i < 1\}$.

ii) $\gamma(t + \text{relint} C; x)$ converges absolutely and uniformly on all compact subsets of $W$ to the rational function
\[
\gamma(t + \text{relint} C; x) = \frac{\gamma(t + U; x)}{(1 - x^{v_1}) \cdot (1 - x^{v_2}) \cdots (1 - x^{v_k})},
\]
where $U = \{\sum_{i=1}^k \alpha_i v_i : 0 < \alpha_i \leq 1\}$.

iii) If $(t + \text{relbd} C) \cap \mathbb{Z}^n = \emptyset$, then $U$ or $\overline{U}$ may be replaced by any set $\tilde{U}$ with $\text{relint} U \subseteq \tilde{U} \subseteq \text{cl}U$.

Proof. First we note that the given set $W$ is non-empty. To this end let $C^* = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 0, \text{ for all } x \in C\}$ be the polar cone. Since $C$ is a pointed cone we have $\dim C^* = n$, and for $y \in \text{int} C^*$, $s \in \mathbb{C}^n$ let $e^{y+is} \in \mathbb{C}^n$ be the vector with entries $e^{y+is}_i$. Then $|\langle e^{y+is} v_k \rangle| = |e^{\langle v_k, y \rangle}| |e^{i\langle v_k, s \rangle}| < |e^{\langle v_k, s \rangle}| \leq 1$, thus
\[
\{e^{y+is} : y \in \text{int} C^*, s \in \mathbb{C}^n\} \subseteq W.
\]

Now let $C_N = \{\sum_{i=1}^k q_i v_i : q_i \in \mathbb{N}\}$. For i) we once again exploit the fact that any vector $z \in (t + C) \cap \mathbb{Z}^n$ can be uniquely written as $t + u + h$ with $u \in U$ and $h \in C_N$. For if, let $z = t + \sum_{i=1}^k \rho_i v_i$, $\rho_i \geq 0$, then
\[
z = t + \sum_{i=1}^k (\rho_i - \lfloor \rho_i \rfloor) v_i + \sum_{i=1}^k \lfloor \rho_i \rfloor v_i = t + u + h.
\]
with \( u \in U \) and \( h \in C_N \), and the uniqueness follows from Remark 6.7 iv). Hence
\[
\gamma(t + C; x) = \sum_{m \in (t + C) \cap \mathbb{Z}_n} x^m = \sum_{g \in t + U, h \in C_N} x^g x^h
\]
\[
= \sum_{g \in t + U} x^g \left( \sum_{q_1, \ldots, q_k = 0}^{\infty} x^{g_i} v_1 + \cdots + q_k v_k \right)
\]
\[
= \sum_{g \in t + U} x^g \left( \prod_{i=1}^k \frac{1}{t_i} \right) = \sum_{g \in t + U} x^g \prod_{i=1}^k \frac{1}{1 - x^{v_i}}
\]
where the last equation means absolutely convergence for \( x \in W \). Hence we also have uniformly convergence on compact subsets of \( W \).

In order to get a unique representation of a point \( z = (t + \text{relint} C) \cap \mathbb{Z}_n \), i.e., \( z = t + \sum_{i=1}^k \rho_i v_i \in \mathbb{Z}_n \) with \( \rho_i > 0 \) (cf. Theorem 0.21), we write
\[
z = t + \left( \sum_{\rho_i \notin \mathbb{N}} (\rho_i - [\rho_i]) v_i + \sum_{\rho_i \in \mathbb{N}} v_i \right) + \left( \sum_{\rho_i \notin \mathbb{N}} [\rho_i] v_i + \sum_{\rho_i \in \mathbb{N}_1} (\rho_i - 1) v_i \right)
\]
with \( t + \bar{U} = C_N \), and as before such a representation is unique. It just remains to note that also \( t + \bar{U} = C_N \subseteq (t + \text{relint} C) \cap \mathbb{Z}_n \) (cf. Theorem 0.21), and we can proceed as before.

For iii) we observe that there are no lattice points \( z = t + \sum_{i=1}^k \alpha_i v_i \), \( 0 \leq \alpha_i \leq 1 \), in the cone with some \( \alpha_i = 0 \) or some \( \alpha_i = 1 \). Of course, \( \alpha_i = 0 \) implies \( z \in t + \text{relbd} C \) and \( \alpha_i = 1 \) would imply \( t + \sum_{i=1}^k (1 - \alpha_i) v_i \in t + \text{relbd} C \). Hence the sets \( U \) and \( \bar{U} \) in i) and ii) may be replaced by any set \( U \) with the required property.

8.18 Lemma. Every polyhedral pointed cone \( C = \text{pos} \{v_1, \ldots, v_k\} \subset \mathbb{R}_n \), can be triangulated into \((\dim C)\)-dimensional simplicial cones generated by \( v_1, \ldots, v_k \), i.e., there exists a collection \( S = \{\sigma_1, \ldots, \sigma_m\} \) of \((\dim C)\)-dimensional simplicial cones \( \sigma_i \) such that \( C = \cup_{i=1}^m \sigma_i \), and for any \( \sigma_1, \sigma_2 \in S \), \( \sigma_1 \cap \sigma_2 \) is the face common to both \( \sigma_1 \) and \( \sigma_2 \).

Proof. Without loss of generality let \( \dim C = n \). Since 0 is a vertex \( C \), there exists an \( a \in \mathbb{R}_n \) with \( \langle a, v_i \rangle > 0 \) for all \( v_i \). Let \( P = \text{conv} \{v_i/\langle a, v_i \rangle : 1 \leq i \leq k\} \). Then \( C = \text{pos} P \) and each triangulation \( T = \{\tau_1, \ldots, \tau_m\} \) of \( P \) into \((\dim C)\)-dimensional simplices \( \tau_i \) yields a triangulation \( S = \{\sigma_1, \ldots, \sigma_m\} \) of \( C \) into simplicial cones \( \sigma_i = \text{pos} \tau_i \). Hence the assertion follows from Theorem 8.7.

8.19 Corollary. Let \( C \subset \mathbb{R}_n \) be a rational pointed cone, and let \( t \in \mathbb{R}_n \). Then there exists an open subset \( W \subset \mathbb{C}_n \) such that \( \gamma(t + C; x) \) converges absolutely and uniformly to a rational function on all compact subsets of \( W \).
Proof. Let $C = \text{pos} \{v_1, \ldots, v_m\}$, $v_i \in \mathbb{Z}^n$, and let $S = \{\sigma_1, \ldots, \sigma_l\}$ be a triangulation of the cone $C$ via its generators $v_1, \ldots, v_m$ (cf. Lemma 8.18). Applying the Inclusion-Exclusion formula (Lemma 8.5) gives as formal power series the identity

$$\gamma(t+C; x) = \sum_{I} (-1)^{|I|-1} \gamma(t + \bigcap_{j \in I} \sigma_j; x).$$

Each $\bigcap_{j \in I} \sigma_j$ is a rational simplicial cone and by Lemma 8.17 we know that $\gamma(t + \bigcap_{j \in I} \sigma_j; x)$ converges absolutely and uniformly to a rational function on all compact subsets of $W = \{x \in \mathbb{C}^n : |x^\nu| < 1 : 1 \leq i \leq m\}$. As in the proof of Lemma 8.17 it can be argued that $W \neq \emptyset$.

8.20 Lemma. Let $C \subset \mathbb{R}^n$ be a rational pointed $n$-dimensional cone. Let $v \in \mathbb{R}^n$, and let $S = \{\sigma_1, \ldots, \sigma_l\}$ be a triangulation of $C$. Then there exists an $s \in \mathbb{R}^n$ such that

i) $(v + \text{int} (C)) \cap \mathbb{Z}^n = (s + C) \cap \mathbb{Z}^n$ and

ii) bd $(\pm s + \sigma_i) \cap \mathbb{Z}^n = \emptyset, 1 \leq i \leq l.$

In particular, for such an $s$ it holds $(v + C) \cap \mathbb{Z}^n = (s + C) \cap \mathbb{Z}^n$.

Proof. Let $C = \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq 0, 1 \leq i \leq m\}$ with $a_i \in \mathbb{Z}^n$, and we may assume gcd$(a_i) = 1$. Then $v + C = \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq \langle a_i, v \rangle, 1 \leq i \leq m\}$, and the points $s$ satisfying i) are given by the points for which $\{z \in \mathbb{Z}^n : \langle a_i, z \rangle < \langle a_i, v \rangle\} = \{z \in \mathbb{Z}^n : \langle a_i, z \rangle \leq \langle a_i, s \rangle\}, 1 \leq i \leq m$. Hence the set of all possible points $s$ fulfilling i) is given by

$S = \left\{x \in \mathbb{R}^n : \floor{\langle a_i, v \rangle - 1} \leq \langle a_i, x \rangle < \floor{\langle a_i, v \rangle}, 1 \leq i \leq m\right\}$.

In particular, for such an $s$ we also have $(v + C) \cap \mathbb{Z}^n = (s + C) \cap \mathbb{Z}^n$. $S$ is a full dimensional set; on the other hand, ii) imposes only finitely many conditions on $s$ and so ii) is satisfied for almost all points in $S$, in particular, for any vector $s \in S$ whose entries together with the number 1 are linearly independent over $\mathbb{Q}$.

8.21 Theorem [Stanley’s Reciprocity Theorem]. Let $C$ be a rational pointed cone and let $v \in \mathbb{R}^n$. Then

$$\gamma(v+C; x) = (-1)^{\dim C} \gamma(-v + \text{int} (C); x^{-1}),$$

where $x^{-1} = (\frac{1}{x_1}, \ldots, \frac{1}{x_m})$.

Proof. Let $\dim C = n$, and let $\{\sigma_1, \ldots, \sigma_m\}$ be a triangulation of $C$, and let $-s$ be satisfying Lemma 8.20 with respect to the vector $-v$. Then we have

$$\gamma(-v + \text{int} C; x) = \gamma(-s + C; x) = \sum_{i=1}^{m} \gamma(-s + \sigma_i; x) = \sum_{i=1}^{m} \gamma(-s + \text{int} \sigma_i; x)$$

$$\gamma(v + C; x) = \gamma(s + C; x) = \sum_{i=1}^{m} \gamma(s + \sigma_i; x),$$

$\text{[51]}$Here the equality between the generating functions is meant as equality between the rational functions represented by these functions (cf. Lemma 8.17).
and it suffices to show the assertion for a cone $s + C$, say, where $C$ is a rational \textit{simplicial} cone generated by the vectors $v_1, \ldots, v_n \in \mathbb{Z}^n$. Let $U = \{ \sum_{i=1}^n \alpha_i v_i : 0 \leq \alpha_i < 1 \}$ and $\overline{U} = \{ \sum_{i=1}^n \alpha_i v_i : 0 < \alpha_i \leq 1 \}$. With this notion we have $s + U = s + (v_1 + \cdots + v_n) - \overline{U}$ and thus

$$\sum_{z \in (s+U) \cap \mathbb{Z}^n} x^z = \sum_{z \in (-s+\overline{U})+(v_1+\cdots+v_n) \cap \mathbb{Z}^n} x^z = \sum_{w \in (-s+\overline{U}) \cap \mathbb{Z}^n} x^{-w} x^{v_1} \cdots x^{v_n},$$

and so

$$\gamma(s + U; x) = \gamma(-s + \overline{U}; x^{-1}) x^{v_1} \cdots x^{v_n}.$$ 

Finally, with Lemma 8.17 we get

$$\gamma(-s + \text{int } C; x^{-1}) = \frac{\gamma(-s + \overline{U}; x^{-1})}{(1 - x^{-v_1}) \cdots (1 - x^{-v_n})} = \frac{\gamma(-s + \overline{U}; x^{-1}) x^{v_1} \cdots x^{v_n}}{(x^{v_1} - 1) \cdots (x^{v_n} - 1)} = (-1)^n \gamma(s + C; x).$$

\[\square\]

8.22 Definition \textit{[Ehrhart series]}. For bounded $S \subset \mathbb{R}^n$ and $t \in \mathbb{R}$ the formal series

$$L(S; t) = 1 + \sum_{k \geq 1} G(kS) t^k$$

is called the Ehrhart series of $S$.

8.23 Example. Let $Q \subset \mathbb{R}^{n-1} \times \{0\}$ be an $(n-1)$-dimensional lattice polytope.

i) If $P$ is the pyramid $P = \text{conv} \{Q, e_n\}$ then

$$L(P; t) = \frac{1}{1-t} L(Q; t), \quad |t| < 1.$$ 

In particular, we obtain for the standard simplex: $L(T_n; t) = \frac{1}{(1-t)^{n+1}}$.

To see this we note that for such a pyramid $P$ and an integer $k$ the section of $kP$ with a hyperplane $\{ x \in \mathbb{R}^n : x_n = l \}$, $0 \leq l \leq k-1$, is just $le_n + (k-l)Q$. So $G(kP) = 1 + \sum_{l=0}^{k-1} G((k-l)Q) = 1 + G(kQ) + \sum_{l=1}^{k-1} G(lQ)$,
and we obtain for \(|t| < 1\)

\[
L(P; t) = 1 + \sum_{k \geq 1} G(k P) t^k
\]

\[
= 1 + \sum_{k \geq 1} t^k + \sum_{k \geq 1} G(k Q) t^k + \sum_{k \geq 1} \sum_{l=1}^{k-1} G(l Q) t^k
\]

\[
= L(Q; t) + \frac{t}{1 - t} + \sum_{k \geq 1} \sum_{l=1}^{k-1} G(l Q) t^k
\]

\[
= L(Q; t) + \frac{t}{1 - t} + \sum_{l \geq 1} G(l Q) \sum_{k \geq l+1} t^k
\]

\[
= L(Q; t) + \frac{t}{1 - t} + \sum_{l \geq 1} G(l Q) \frac{t^{l+1}}{1 - t}
\]

\[
= L(Q; t) + \frac{t}{1 - t} L(Q; t) = \frac{1}{1 - t} L(Q; t).
\]

In the case \(T_n\) we observe that \(L(T_n; t) = 1 + \sum_{k \geq 1} (k + 1) t^k = \sum_{k \geq 0} (k + 1) t^k = 1/(1 - t)^2\), and together with the previous recursion for pyramids we get \(L(T_n; t) = 1/(1 - t)^{n+1}\).

ii) If \(P\) is the bypyramid \(P = \text{conv} \{Q, \pm e_n\}\) then

\[
L(P; t) = \frac{1 + t}{1 - t} L(Q; t), \quad |t| < 1.
\]

In particular, we obtain for the crosspolytope: \(L(C_n^*; t) = \frac{(1+t)^n}{(1-t)^{n+1}}\).

This can be easily calculated in the same way as before, or we just observe that \(G(k P) = 2 G(k \tilde{P}) - G(k Q)\), where \(\tilde{P}\) is the pyramid \(\text{conv} \{Q, e_n\}\).

Hence by the previous example we find for \(|t| < 1\)

\[
L(P; t) = 1 + \sum_{k \geq 1} G(k P) = 2 (1 + \sum_{k \geq 1} G(k \tilde{P}) t^k) - (1 + \sum_{k \geq 1} G(k Q) t^k)
\]

\[
= 2 L(\tilde{P}; t) - L(Q; t) = 2 \frac{1}{1 - t} L(Q; t) - L(Q; t)
\]

\[
= \frac{1 + t}{1 - t} L(Q; t).
\]

Finally, we note that \(L(C_n^*; t) = (1 + t)/(1 - t)^2\).

8.24 Proposition. Let \(P \in \mathcal{P}^n\) and let \(\hat{P} = \text{conv} \{(\begin{pmatrix} 1 \\ x \end{pmatrix}) : x \in P\}\) be its canonical embedding into \(\mathbb{R}^{n+1}\). Then \(L(P; t) = \gamma(\text{pos} \hat{P}; (1, t))\), where \(1 \in \mathbb{R}^n\) is the all 1-vector.

Proof. Obviously, a point \((y_1, \ldots, y_n, y_{n+1})^T\) belongs to \(\text{pos} \hat{P}\) iff \((y_1, \ldots, y_n)^T \in y_{n+1} P\). Thus

\[
\gamma(\text{pos} \hat{P}; (x, t)) = 1 + \sum_{(z, k) \in \text{pos} \hat{P} \cap \mathbb{Z}^{n+1}} x^z t^k = 1 + \sum_{k \geq 1} t^k \sum_{z \in k P \cap \mathbb{Z}^n} x^z,
\]
and the right hand side evaluates to \( L(P; t) \) for \( x = 1 \). \( \square \)

**8.25 Lemma.** Let \( f : \mathbb{N} \to \mathbb{R} \) and \( c_i \in \mathbb{R} \), \( 0 \leq i \leq n \). Then

\[
\sum_{k \geq 0} f(k) t^k = \frac{\sum_{i=0}^{n} c_i t^i}{(1 - t)^{n+1}}
\]

for all \( |t| < 1 \) if and only if \( f(k) = \sum_{i=0}^{n} c_i \left( \frac{k+n-i}{n} \right) \) for all \( k \in \mathbb{N} \).

**Proof.** By taking the \( n + 1 \)-st power (or the \( n \)-the derivative) of the geometric series \( \sum_{k \geq 0} t^k = 1/(1 - t) \), \( |t| < 1 \), we see that

\[
\frac{1}{(1 - t)^{n+1}} = \sum_{k \geq 0} \binom{n+k}{n} t^k,
\]

(8.25.1)

and so

\[
\sum_{i=0}^{n} c_i t^i \frac{1}{(1 - t)^{n+1}} = \sum_{k \geq 0} \binom{n+k}{n} \frac{1}{(1 - t)^{n+1}} = \sum_{k \geq 0} \sum_{i=0}^{\min\{k,n\}} \binom{n+k-i}{n} c_i t^k = \sum_{k \geq 0} \left( \sum_{i=0}^{n} c_i \binom{n+k-i}{n} \right) t^k.
\]

(8.25.2)

Comparing the coefficients of \( \sum_{k \geq 0} f(k) t^k \) with the powers series on the right hand side finishes the proof. \( \square \)

**8.26 Corollary.** Let \( P \in \mathcal{P}^n_2 \), and let \( a_i(P) \in \mathbb{Q} \), \( 0 \leq i \leq n \), given by Corollary 8.12. Then

\[
L(P; t) = \frac{\sum_{i=0}^{n} a_i(P) t^i}{(1 - t)^{n+1}}, \quad |t| < 1,
\]

and, in particular, \( a_0(P) = 1 = G_0(P) \).

**Proof.** If \( P \) is an \( l \)-dimensional lattice simplex \( T \), say, then we know by Lemma 8.3 that \( G(kT) = \sum_{i=0}^{l} \tilde{a}_i(T) \binom{l+k-i}{l} \), \( k \geq 1 \), and \( \tilde{a}_0(T) = 1 \). Hence, rewriting as a polynomial in the basis \( \binom{n+k-i}{n} \) gives \( G(kT) = \sum_{i=0}^{n} a_i(T) \binom{n+k-i}{n} \), \( k \geq 1 \), with \( a_0(T) = 1 \). Setting \( f(k) = G(kT) \), \( k \geq 1 \), and \( f(0) = 1 \) we may write \( f(k) = \sum_{i=0}^{n} a_i(T) \binom{n+k-i}{n} \), \( k \geq 0 \), and

\[
L(T; t) = 1 + \sum_{k \geq 1} G(kT) t^k = \sum_{k \geq 0} f(k) t^k.
\]

Thus, by Lemma 8.25, the statement is proved for lattice simplices.

Now, let \( \{\tau_1, \ldots, \tau_m\} \) be a triangulation of \( P \), and let \( \tau_1, \overline{P} \) be the corresponding canonical embeddings. By Proposition 8.24 and the Inclusion-
Exclusion principle we may write

\[ L(P; t) = \gamma(\overline{P}; (1, t)) = \sum_{I \subseteq \{1, \ldots, m\} \backslash \{t\} \neq \emptyset} (-1)^{|I|-1}\gamma \left( \bigcap_{j \in I} \overline{\tau}_j; (1, t) \right) \]

\[ = \sum_{I \subseteq \{1, \ldots, m\}} (-1)^{|I|-1}L \left( \bigcap_{j \in I} \tau_j; t \right). \]

Since the lemma is verified for lattice simplices, we conclude that there exist numbers \( c_i \in \mathbb{Z} \) such that

\[ L(P; t) = \sum_{i=0}^{n} c_i \frac{t^i}{(1-t)^{n+1}}, \quad |t| < 1. \]

Hence by Lemma 8.25 we get \( G(kP) = \sum_{i=0}^{n} c_i \binom{n+k-i}{n} \), and in view of Corollary 8.12 it is \( c_i = a_i(P), \) \( 0 \leq i \leq n. \)

\( a_0(P) \) is the constant coefficient in the polynomial representation of \( G(kP) \) via the binomial basis and with respect to the monomial basis \( G_0(P) \) is the constant coefficient. Hence \( a_0(P) = G_0(P) \) and from the identity is the statement we get \( a_0(P) = L(P; 0) = 1. \)

**8.27 Theorem [Stanley’s Non-Negativity Theorem].** Let \( P \in \mathbb{P}_n^2 \), \( \dim P = n. \) Then \( a_i(P) \in \mathbb{N}_{\geq 0}, \) \( 0 \leq i \leq n. \)

**Proof.** Let \( \overline{P} \) be the canonical embedding of \( P \) into \( \mathbb{R}^{n+1} \), and let \( \{\sigma_1, \ldots, \sigma_m\} \) be a triangulation of \( \text{pos} \overline{P} \) into simplicial cones generated by the generators of \( \text{pos} \overline{P} \). According to Lemma 8.20 we choose an \( -s \in \mathbb{R}^{n+1} \) with respect to the cone \( \text{pos} \overline{P} \), i.e., \( \text{pos} \overline{P} \cap \mathbb{Z}^{n+1} = (s + \text{pos} \overline{P}) \cap \mathbb{Z}^{n+1} \), and thus \( -s \in \text{pos} \overline{P} \), and \( \{s + \sigma_1, \ldots, s + \sigma_m\} \) is a triangulation of \( s + \text{pos} \overline{P} \) such that no lattice points are contained in the boundaries of \( s + \sigma_i \). Hence we have

\[ L(P; t) = \gamma(\text{pos} \overline{P}; (1, t)) = \gamma(s + \text{pos} \overline{P}; (1, t)) = \sum_{i=1}^{m} \gamma(s + \sigma_i; (1, t)). \] (8.27.1)

Now let \( \sigma_i = \text{pos} \left( \{v_1^{(i)}, \ldots, v_{n+1}^{(i)}\} \right), \) where \( v_1^{(i)}, \ldots, v_{n+1}^{(i)} \) are vertices of \( P \), and let

\[ U_i = \left\{ \sum_{j=1}^{m} \alpha_j \frac{v_j^{(i)}}{1} : 0 \leq \alpha_j < 1 \right\}. \]

Then by Lemma 8.17 i) we can continue (8.27.1) for \( |t| < 1 \) as

\[ L(P; t) = \sum_{i=1}^{m} \frac{\gamma(s + \sigma_i; (1, t))}{(1-t)^{n+1}} = \frac{1}{(1-t)^{n+1}} \sum_{i=1}^{m} \sum_{z \in (s + U_i) \cap \mathbb{Z}^{n+1}} t^{z_{n+1}}. \] (8.27.2)
So we have written \( L(P; t) \) as a (Laurent) polynomial with non-negative coefficients. Comparing (8.27.2) with the representation in Corollary 8.26 shows
\[
a_i(P) = \# \{ z \in \bigcup_{i=1}^{n} (s + U_i) : z_{n+1} = i \}, \quad 0 \leq i \leq n.
\] (8.27.3)

8.28 Theorem [Stanley’s Monotonicity Theorem]. Let \( P, Q \in \mathcal{P}_n^\mathbb{Z} \), \( \dim P = \dim Q = n \) with \( P \subseteq Q \). Then \( a_i(P, \mathbb{Z}^n) \leq a_i(Q, \mathbb{Z}^n) \), \( 0 \leq i \leq n \).

Proof. We start as in the proof of Theorem 8.27 with a triangulation of the cone \( \text{pos } P \) by the generators of \( \text{pos } P \). Then we extend this triangulation to a triangulation of the cone \( \text{pos } Q \), where \( Q \) is the canonical embedding of \( Q \) into \( \mathbb{R}^{n+1} \). Next, with respect to the cone \( \text{pos } Q \) we choose a translation vector \( s \) as in Lemma 8.20 and by the interpretation of \( a_i(Q) \) as the number of "level \( i \) lattice points of a triangulation" (see (8.27.3)) we have \( a_i(P) \leq a_i(Q) \), \( 0 \leq i \leq n \). \( \square \)

8.29 Proposition. The functionals \( a_i : \mathcal{P}_n^\mathbb{Z} \to \mathbb{R} \), \( 0 \leq i \leq n \), are invariant with respect to unimodular transformations, monotonous and non-negative on \( P \in \mathcal{P}_n^\mathbb{Z} \) with \( \dim P = n \).

8.30 Lemma. Let \( p : \mathbb{Z} \to \mathbb{R} \) be a polynomial, and let \( g_+(t) = \sum_{k \geq 1} p(k) t^k \) and \( g_-(t) = \sum_{k \leq 0} p(k) t^k \). Then both series evaluate to rational functions for certain (and usually different) values of \( t \), and these rational functions coincide, i.e., \( g_+(t) + g_-(t) = 0 \) (wherever they are defined).

Proof. Let \( p \) be a polynomial of degree \( n \), and let \( p(k) = \sum_{i=0}^{n} c_i \binom{k+n-i}{n} \) for some \( c_i \in \mathbb{R} \). By Lemma 8.25 we have for \( |t| < 1 \)
\[
g_+(t) = \sum_{i=0}^{n} c_i t^n (1 - t)^{n+1} - p(0).
\] (8.30.1)

In order to evaluate \( g_-(t) \) we proceed as in the proof of Lemma 8.25: First we note that substituting \( t \) by \( 1/t \) in (8.25.1) leads for \( |t| > 1 \) to
\[
\frac{1}{(1-t)^{n+1}} = \sum_{k \leq -(n+1)} \binom{n+k}{n} t^k,
\]
which then yields analogously to (8.25.2)
\[
-\sum_{i=0}^{n} \frac{c_i t^i}{(1-t)^{n+1}} = \sum_{k \leq -1} p(k) t^k = g_-(t) - p(0).
\]

Together with (8.30.1) this shows the assertion. \( \square \)

8.31 Theorem [Ehrhart-Macdonald Reciprocity]. Let \( P \in \mathcal{P}_n^\mathbb{Z} \), \( \dim P = n \). Then
\[
G(\text{int } k P) = (-1)^n \sum_{i=0}^{n} G_i(P)(-k)^i.
\]
Proof. For \( k \in \mathbb{Z} \) let \( p(k) = \sum_{i=0}^{n} G_i(P)k^i \). We have to show that \( G(\text{int } k \ P) = (-1)^n p(-k) \) for \( k \in \mathbb{N} \). To this end let \( L_{\text{int}}(P; t) = \sum_{k \geq 1} G(\text{int } k \ P) t^k \) be the corresponding Erhart series for the interior lattice points, and, as before, let \( \overline{P} \) be the canonical embedding of \( P \) into \( \mathbb{R}^{n+1} \). As in Proposition 8.24 we have
\[
L_{\text{int}}(P; t) = \gamma(\text{int } P; (1, t)).
\]
Applying the "Reciprocity Theorem" 8.21 to the corresponding rational functions (cf. Lemma 8.17)
\[
\sum_{k \geq 1} G(\text{int } k \ P) t^k = L_{\text{int}}(P; t) = \gamma(\text{int } \overline{P}; (1, t)) = (-1)^{n+1} \gamma(\text{pos } \overline{P}; (1, t^{-1})) = (-1)^{n+1} L(P; t^{-1}) = (-1)^{n+1} \sum_{k \geq 0} G(k \ P) t^{-k} = (-1)^{n+1} \sum_{k \leq 0} p(-k) t^k = (-1)^{n} \sum_{k \geq 1} p(-k) t^k,
\]
where the last identity follows from Lemma 8.30. Comparing the coefficients gives \( G(\text{int } k \ P) = (-1)^n p(-k) \).

8.32 Theorem. Let \( P \in \mathcal{P}_{\mathbb{Z}}^n \) with \( \dim P = n \). Then for \( i \neq n \) mod 2
\[
G_i(P) = \frac{1}{2} \sum_{j=1}^{n-1} (-1)^{i+j} \sum_{F \text{ is } j\text{-face of } P} G_i(F).
\]
In particular,
\[
G_{n-1}(P) = \frac{1}{2} \sum_{F \text{ is facet of } P} \frac{\text{vol}_{n-1}(F)}{\det(\text{aff } F \cap \mathbb{Z}^n)}.
\]

Proof. For \( k \in \mathbb{Z} \) and a face \( F \) of \( P \) let \( p(F; k) = \sum_{i=0}^{\dim F} G_i(F) k^i \), and for \( 0 \leq j \leq n \) let \( H_j(k) = \sum_{F \text{ is } j\text{-face of } P} p(F; k) \). Since \( G(k \ P) = \sum_{F \text{ is face of } P} G(\text{int } k \ F) \) we get from the Reciprocity-Theorem 8.31
\[
G(k \ P) = p(P; k) = \sum_{F \text{ face of } P} (-1)^{\dim F} p(F; -k)
\]
\[
= \sum_{j=0}^{n} (-1)^j \sum_{F \text{ is } j\text{-face of } P} p(F; -k) = \sum_{j=0}^{n} (-1)^j H_j(-k).
\]
Hence
\[
G(k \ P) - G(\text{int } k \ P) = p(P; k) - (-1)^n p(P; -k) = \sum_{j=0}^{n-1} (-1)^j H_j(-k).
\]
On the other hand we have by Theorem 8.31
\[
G(k \ P) - G(\text{int } k \ P) = \sum_{i=0}^{n} G_i(P) (1 - (-1)^{n+i}) k^i = 2 \sum_{i \neq n \text{ mod } 2} G_i(P) k^i,
\]
and so
\[
\sum_{i \not\equiv n \mod 2} G_i(P) k^i = \frac{1}{2} \sum_{j=0}^{n-1} (-1)^j H_j(-k) = \frac{1}{2} \sum_{j=0}^{n-1} (-1)^j \sum_{F \text{ is } j\text{-face of } P} p(F; -k)
\]
\[
= \frac{1}{2} \sum_{j=0}^{n-1} \sum_{F \text{ is } j\text{-face of } P} \sum_{i=0}^{j} (-1)^{i+j} G_i(F) k^i.
\]
Comparing the coefficients on the left and on the right hand side gives
\[
G_i(P) = \frac{1}{2} \sum_{j=i}^{n-1} (-1)^{i+j} \sum_{F \text{ is } j\text{-face of } P} G_i(F).
\]
Finally, we note that for a facet \( F \) of \( P \) by Proposition 8.9 i) gives \( G_{n-1}(F) = \text{vol}_{n-1}(F)/\det(\text{aff } F \cap \mathbb{Z}^n) \), and so we get the formula for \( G_{n-1}(P) \).

**8.33 Proposition.** Let \( P \in \mathcal{P}^n_{\mathbb{Z}} \). Then
\[
G_0(P) = a_0(P) = 1,
\]
\[
G_n(P) = \frac{1}{n!} (a_0(P) + \cdots + a_n(P)) = \text{vol } (P),
\]
\[
a_1(P) = G(P) - (n + 1),
\]
\[
a_n(P) = G(\text{int } P).
\]
**Proof.** For the constant coefficients see Corollary 8.26. Comparing the leading coefficient in the two polynomials \( \sum_{i=0}^{n} G_i(P) x^i \) and \( \sum_{i=0}^{n} a_i(P) \binom{x+n}{n} \) yields the relations for \( G_n(P) \) (cf. Proposition 8.9 i)). Comparing the polynomials at \( x = -1 \) gives
\[
\sum_{i=0}^{n} G_i(P) (-1)^i = \sum_{i=0}^{n} a_i(P) \binom{-1 + n - i}{n} = (-1)^n a_n(P),
\]
which shows iii) by Theorem 8.31. Finally, evaluating at \( x = 1 \) gives \( G(P) = a_0(P) (n + 1) + a_1(P) = (n + 1) + a_1(P) \).
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