The Discrete Planar $L_0$-Minkowski Problem

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In the discrete setting, the $L_0$-Minkowski problem extends the question posed and answered by the classical Minkowski's existence theorem for polytopes. In particular, the planar extension, which we address in this paper, concerns the existence of a convex polygonal body which contains the origin, whose boundary sides have preassigned orientations and each triangle formed by the origin with two consecutive vertices is of prescribed area.

1. INTRODUCTION

Minkowski's existence theorem gives necessary and sufficient conditions for a measure on the unit sphere $S^{n-1}$ to be the surface area measure of a convex body in Euclidean $n$-space. The sufficiency is inherently related, due to Aleksandrov's approach, to the Brunn–Minkowski theory—the study of convex bodies built on the notions of support functions, Minkowski sums of convex bodies, and mixed volumes. Minkowski first proved his result for polytopes in which case the problem can be stated as follows: If $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_N$, $N \geq n + 1$, are pairwise distinct unitary vectors in $S^{n-1}$, not all in a hemisphere, and $a_1, a_2, \ldots, a_N$ are strictly positive numbers such that $\sum_{i=1}^{n} a_i \vec{u}_i = 0$, then there exists a unique, up to translation, convex polytope in the Euclidean $n$-space whose faces have $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_N$ as outer normal vectors and, for each $i = 1, \ldots, N$, the face corresponding to $\vec{u}_i$, has $(n-1)$-dimensional volume $a_i$. For a detailed discussion of Minkowski's existence problem see the book of Schneider [10].

Considering Firey's $p$-sums of convex bodies (for $p = 1$ one has the usual Minkowski sum) Lutwak extended the Brunn–Minkowski theory for each $p \geq 1$ [6]. In the process, he gives a solution to a generalization of the Minkowski problem. A geometric special case of the latter concludes that for any continuous even function $\gamma: S^{n-1} \to (0, \infty)$, there exists a unique
A convex body whose curvature function is $\gamma$. The extension proved by Lutwak states that for any $1 - p < 0$, $(p \neq n)$, and any continuous even function $\gamma: S^{n-1} \to (0, \infty)$, there exists a unique convex body such that $h^{1-p}f = \gamma$, where $h$ is the support function of the convex body and $f$ its curvature function. We should note that, for a sufficiently regular boundary, the curvature function is the reciprocal of the Gauss curvature viewed as a function of the outer normals. Regularity of solutions to the generalized Minkowski problem was studied by Lutwak and Oliker [8]. They conclude that if $\gamma \in C^m(S^{n-1})$, $m \geq 3$, then $h \in C^{m+1}(S^{n-1})$ for any $\alpha \in (0, 1)$, and, if $\gamma$ is analytic, then $h$ is analytic as well.

Following Lutwak, we call the above generalization the $L_p$-Minkowski problem. As recent work [1, 9, 15] shows, besides the intrinsic interest of the problem, its study is related to other questions in convex, affine, and differential geometry. In particular, we note that Umanskyi’s work on Hill’s equation gives a necessary condition for the existence of the solution to the planar $L_2$-problem when the assumption of an even $\gamma$ is dropped.

Even restricted to the Euclidean plane, it is natural to ask if other $L_p$-problems have solutions. Working in the class of convex bodies whose boundaries are, at least, $C^2$, Gage and Li prove the existence of a solution for $p = 0$ and an arbitrary strictly positive $C^2$ function $\gamma$ [5]. Its uniqueness, for an even $\gamma$, follows from an earlier paper by Gage [4]. In higher dimensions, the work of Andrews touches upon this question as well [1].

The focus of this paper is on the existence of solutions to the planar $L_0$-Minkowski problem formulated for polygons. Intuitively, the question we approach is whether there exists a convex polygonal body which contains the origin, whose boundary sides have pre-assigned orientations and each triangle formed by the origin with two consecutive vertices is of prescribed area.

**Definition 1.1.** Let $\mathcal{U} = \{\vec{u}_1, \ldots, \vec{u}_N\}$ be an ordered family of pairwise distinct unitary directions in $S^1$, not all in a half-disk, and let $\Gamma = \{\gamma_1, \ldots, \gamma_N\}$ be an ordered set of strictly positive values, $N \geq n + 1$. We call solution to the discrete planar $L_0$-Minkowski problem associated to $(\mathcal{U}, \Gamma)$ a polygonal convex body whose support number, $h_i$, from the origin to the side of length $l_i$ and outer normal $\vec{u}_i$ satisfies

$$h_i l_i = \gamma_i$$

for each $i = 1, \ldots, N$.

We will show that any discrete $L_0$-Minkowski problem, for which $\mathcal{U}$ contains no pair of opposite directions, has solutions. Otherwise, a condition on $\Gamma$ is imposed to conclude the existence of $L_0$-polygons.

Our main result is the following:
Theorem 1.1. Let $\mathcal{U} = \{\bar{u}_1, ..., \bar{u}_N\}$ be an ordered family of pairwise distinct unitary directions in $S^1$ and let $\Gamma = \{\gamma_1, ..., \gamma_N\}$ be an ordered set of strictly positive values.

Assume that one of the following holds:

(i) $N \geq 4$ and $\mathcal{U}$ consists of pairwise linearly independent vectors, not all in a half-disk.

(ii) $N > 4$ and $\mathcal{U}$ contains, at least, two linearly dependent vectors. For any $j, k$ with $\bar{u}_j = -\bar{u}_k$, we have

$$\gamma_j + \gamma_k < \sum_{i=1, i \neq j, k}^N \gamma_i.$$  \hfill (2)

(iii) $N = 4$, $\mathcal{U}$ contains a unique pair of opposite vectors, $\bar{u}_1 = -\bar{u}_3$, and $\gamma_1 + \gamma_3 < \gamma_2 + \gamma_4$.

Then there exists a solution to the discrete planar $L_0$-Minkowski problem.

The equality in (2) is not always sufficient unless $N = 4$ and $\mathcal{U}$ of the form $\{\bar{u}_1, \bar{u}_2, -\bar{u}_1, -\bar{u}_2\}$. In this case, expressing the area in two different ways, one notes that $\gamma_1 + \gamma_3 = \gamma_2 + \gamma_4$ is also a necessary condition.

On the other hand, for even data with $N > 4$, we infer differently:

Theorem 1.2. The discrete planar $L_0$-Minkowski problem for $\mathcal{U} = \{\bar{u}_1, ..., \bar{u}_M, -\bar{u}_1, ..., -\bar{u}_M\}$ and $\Gamma = \{\gamma_1, ..., \gamma_M, \gamma_1, ..., \gamma_M\}$, $M > 2$, has a solution if and only if

$$\gamma_i < \sum_{j \neq i} \gamma_j,$$  \hfill for any $i = 1, ..., M$.

Moreover, the solution is unique and it is symmetric with respect to the origin.

In other words, the existence of a centrally symmetric solution makes the strict inequality (2) necessary. It is an immediate consequence of the fact that each parallelogram formed by the vertices of parallel equal sides is included in the $L_0$-polygonal body. Theorem 1.2 answers also the polygonal case of Lutwak’s conjecture [7], which states that if an $L_0$-solution to even data exists, then it must be centrally symmetric. This was known for the $L_0$-polygons with 4 or 6 sides. The latter has been positively answered by it Soranzo [11], whose method is likely to hold also for $N = 8$, but it does not in the general case.

However, except for the even $L_0$-problems, the existence of solutions, when $\mathcal{U}$ contains a pair of opposite directions for which (2) does not hold, is still open.
Regarding the uniqueness of solutions for the even problems with \( N > 4 \) under the assumption (2), note that this is also in contrast with the even four-sided case where every parallelogram of outer normals \( \mathcal{U} \) and area \( \gamma_1 + \gamma_3 = \gamma_2 + \gamma_4 \), properly translated, is an \( L_0 \)-solution. The question regarding the number of \( L_0 \)-polygons for arbitrary data is more subtle and it will be treated in a separate paper.

Our approach to the existence of solutions uses a suitable deformation of polygons whose ordered set of outer normals is \( \mathcal{U} \). While a polygon will shrink to a point by such a deformation, a sequence of its properly normalized shapes will converge in the Hausdorff metric to the solution of the polygonal \( L_0 \)-Minkowski problem. This asymptotic behavior has been studied in a different context [13], where our former argument works in the case (i) of Theorem 1.1. The method presented here is different and simpler.

The paper is organized as follows: In the second section we discuss the definition of the continuous deformation and motivate its use for the \( L_0 \)-problem. We then derive the estimates needed to prove the main theorem whose proof is concluded in the last section, as well as that of Theorem 1.2.

2. THE CRYSTALLINE DEFORMATION

Let \( \mathcal{U} \) and \( \Gamma \) be two ordered sets as in Definition 1.1. A convex polygonal body is called admissible if its ordered family of unitary outer normals to the sides of its boundary is \( \mathcal{U} \).

**Definition 2.1.** Let \( K \) be an admissible convex body. We call its crystalline deformation the family of admissible convex bodies parametrized by \( t \), \( \{K(t) | 0 \leq t \leq T\} \), such that the support function of \( K(t) \) in the \( i \)th direction of \( \mathcal{U} \) is a solution of the equation

\[
\frac{dh_i(t)}{dt} = -\frac{\gamma_i}{l_i(t)}, \quad h_i(0) = h_K(u_i), \quad \forall i: 1 \leq i \leq N, \tag{3}
\]

where \( l_i(t) \) is the length of the \( i \)th side at time \( t \) and \( h_K \) is the support function of \( K \).

Equation (3) is a version of the crystalline flow proposed by Angenent and Gurtin [2] for studying the motion of a piecewise linear curve separating a two-phase system. It was also, independently, defined and studied by Taylor [14], for a particular set \( \Gamma \). The flow is well defined as all the sides of the evolving polygon become of zero length simultaneously,
at finite time $T$ [14]. It should also be noted that the shape of evolving polygons is independent on the choice of the origin [13], thus we may assume, without any loss of generality, that the origin is the shrinking point $K(T)$. An immediate consequence is the strict positivity of the support numbers to the sides of the boundary $K(t)$ for $t < T$. For any other choice of the origin, the support numbers, $h_i(t)$, will become signed distances according to the definition $h_i(t) := \sup_{x \in K(t)} \langle x, \tilde{u}_i \rangle$.

Remark 2.1. The extinction time $T$ can be determined a priori by the area of the polygonal body at $t = 0$. The length of each boundary segment can be described by its neighboring support functions, $l_i(t) = \frac{h_{i+1}(t)}{\sqrt{1 - (\tilde{u}_i \cdot \tilde{u}_{i+1})^2}} + \frac{h_{i-1}(t)}{\sqrt{1 - (\tilde{u}_{i-1} \cdot \tilde{u}_i)^2}}$. For any other choice of the origin, the support numbers, $h_i(t)$, will become signed distances according to the definition $h_i(t) := \sup_{x \in K(t)} \langle x, \tilde{u}_i \rangle$.

We then have

\[ \frac{d \text{Area}(K(t))}{dt} = \frac{1}{2} \sum_{i=1}^{N} \frac{dh_i(t)}{dt} l_i(t) + \frac{1}{2} \sum_{i=1}^{N} h_i(t) \frac{dl_i(t)}{dt} \]  

and formula (4) leads, by reindexing, to

\[ \frac{d \text{Area}(K(t))}{dt} = \sum_{i=1}^{N} \frac{dh_i(t)}{dt} l_i(t) = -\sum_{i=1}^{N} \gamma_i. \]  

Since $\text{Area}(K(T)) = 0$, we obtain

\[ \text{Area}(K(t)) = (T - t) \sum_{i=1}^{N} \gamma_i \]  

and $T = \text{Area}(K(0))/(\sum_{i=1}^{N} \gamma_i)$.

For simplicity, we will assume that all $K$’s are enclosing at time $t = 0$ an area equal to $\alpha := \frac{1}{2} \sum_{i=1}^{N} \gamma_i$. In this way, one can notice that the extinction time will always be $T = 1/2$.

The motivation for using the crystalline flow to solve the discrete $L_0$-Minkowski problem arises naturally. The unique existence of a convex body $\bar{K}$ with $\bar{h}_i = \gamma_i$, for all $i$’s, is equivalent to a unique admissible polygonal shape which is deformed homothetically by the crystalline flow. Indeed, if $K(0) = \bar{K}$ is as in Definition 1.1, the evolving polygons $K(t)$
determined by (3) are defined by the support functions: \( h_i(t) = h_i \cdot \sqrt{2T - 2t} \).

In other words, if we normalize the flow, magnifying \( R^2 \) by \( \mu = (2T - 2t)^{-1/2} \), so that all evolving curves enclose a fixed area \( \alpha \), and pushing \( T \) to infinity via \( \tau = -\frac{1}{2} \log(2T - 2t) \), then the rescaled evolution equations are

\[
\frac{dh_i'}{d\tau} = -\frac{\gamma_i}{l_i'} + h_i', \quad 1 \leq i \leq N, \tag{7}
\]

where \( h_i' := \mu h_i \) and \( l_i' := \mu l_i \).

Finding the convex body which solves the \( L_0 \)-Minkowski problem becomes the search of a fixed point of the normalized crystalline flow in the class of admissible convex curves. As the fixed points are often attractors of other solutions of the flow, the idea is to study the asymptotic behavior of the \( \Gamma \)-deformation, in other words the final shape of an evolving polygon.

It is precisely along this line that we obtain the existence of the \( L_0 \)-solution.

As an admissible polygon evolves by the \( \Gamma \)-flow, we will show that a discrete sequence from the family \( \{\mu K(t) | 0 \leq t \leq T\} \) converges in the Hausdorff metric to a solution of the \( L_0 \)-Minkowski problem.

We will start with an analytic estimate which is essential to our reasoning. Let

\[
\mathcal{E}(t) := \sum_{i=1}^{N} \gamma_i \log \frac{\gamma_i}{l_i}
\]

be the entropy of \( K(t) \), as in [13]. Then:

**Lemma 2.1.** The rate of change of the entropy satisfies, for all time, the inequality

\[
\frac{d\mathcal{E}(t)}{dt} \leq \frac{\alpha}{T - t}. \tag{8}
\]

**Proof.** Consider

\[
y_i := \left( \log \left( \frac{\gamma_i}{l_i} \right) \right) = -\left( \frac{\gamma_i}{l_i} \right) = \frac{1}{l_i} \left( \frac{\gamma_i}{l_i} \cdot A_i + \frac{\gamma_{i+1}}{l_{i+1}} \cdot B_i + \frac{\gamma_{i-1}}{l_{i-1}} \cdot B_{i-1} \right),
\]

where \( l_i := l_i(t), \ y_i := y_i(t) \) and we have used Eq. (4) for the last equality.

To simplify the presentation we will sometime drop the \( t \) argument.

Then

\[
y_i = \left( \frac{2\gamma_i}{l_i^2} \cdot A_i + \frac{\gamma_{i+1}}{l_{i+1}^2} \cdot B_i + \frac{\gamma_{i-1}}{l_{i-1}^2} \cdot B_{i-1} \right) (l_i),
\]

\[
= \frac{\gamma_{i+1}(l_{i+1})}{l_{i+1}^2 l_i} B_i - \frac{\gamma_{i-1}(l_{i-1})}{l_{i-1}^2 l_i} B_{i-1}
\]
and, since \((l_j)_j = -y_j l_j\) for all \(j\)'s, it follows that
\[
(y_i)_i = 2y_i^2 + \frac{y_{i+1}}{l_{i+1}l_i} B_i (y_{i+1} - y_i) + \frac{y_{i-1}}{l_{i-1}l_i} B_{i-1} (y_{i-1} - y_i)
\]
for all \(i = 1, \ldots, N\).

Thus
\[
\mathcal{E}_t = \left(\sum_{i=1}^N y_i \gamma_i, \gamma_i \right) = \sum_{i=1}^N 2\gamma_i y_i^2 \geq 2 \left(\frac{\sum_{i=1}^N \gamma_i y_i}{\sum_{i=1}^N \gamma_i}\right)^2 = \frac{\gamma_i}{\alpha}.
\]

(9)

Denoting \(y(t) := \min_{i=1, \ldots, N} y_i(t)\) and, given that \(B_i > 0\), we have

\[
y_i \geq 2y^2,
\]

(10)

where, since the function \(t \mapsto y(t)\) may not be differentiable, but it is a Lipschitz function, \(y_i\) is interpreted as the lim inf of the forward differences: \(y_i(t) := \lim \inf_{e \to 0} (\frac{y(t+e) - y(t)}{e})\). Thus each \(y_i\) is bounded below by \(y(0)\).

On the other hand, \(K(t+\epsilon) \subset K(t)\) for any \(\epsilon > 0\) and any \(t \in [0, T)\), their convexity implying that \(L(t) := \text{Length}(K(t))\) is a decreasing function of time. Hence, at any given moment at least one side must have a decreasing length and we conclude that \(Y(t) := \max_{i=1, \ldots, N} y_i(t)\) is strictly positive.

Furthermore, since

\[
Y(t) \cdot \sum_{i=1}^N (h_i l_i) \geq \sum_{i=1}^N (h_i l_i) y_i = \frac{d \text{Area}(K(t))}{dt} = 2\alpha,
\]

(11)

we have \(Y(t) \geq 1/(2T - 2t)\).

Therefore the derivative of the entropy blows up at time \(T\). The inequality (8) follows now from a comparison principle: Once \(\mathcal{E}_t\) is bigger than a solution of \(z_t = z^2/\alpha\), it must stay bigger. So if \(\mathcal{E}_t\) is bigger than \(\alpha/(T - \delta - t)\) for any \(t\) and any positive \(\delta\), then we obtain a contradiction, since in this case the blow-up of \(\mathcal{E}_t\) must occur at an earlier time than \(T\).

The estimate on the increase of the entropy has an important geometric consequence:

**Proposition 2.1.** If the ordered sets \(U\) and \(\Gamma\) are as in the cases (i), (ii), or (iii) of Theorem 1.1, then the homotheties \(K(t)/\sqrt{2T - 2t}\) of the evolving polygonal bodies \(K(t)\) remain bounded.

**Proof.** Note that \(\text{Area}(K(t)/\sqrt{2T - 2t}) = \alpha\) and no sides can disappear before the extinction time \(T\). Therefore, if \(U = \{\tilde{u}_1, \ldots, \tilde{u}_N\}\) consists of
pairwise linearly independent unitary directions in $S^1$, then the diameter of $K(t)/\sqrt{2T-2t}$ is bounded uniformly from above. It is an immediate consequence of the fact that a polygon with no parallel sides cannot be elongated indefinitely while keeping its outer normals fixed and area constant.

Thus, it remains to show that the condition (2) suffices to ensure an upper bound on the diameter of $K(t)/\sqrt{2T-2t}$ when $\mathcal{U}$ contains two opposite unitary directions.

Here we use the bound on the rate of change of the entropy, (8), which implies that

$$\frac{d}{dt} \left( \sum_{i=1}^{N} \gamma_i \log \left( \frac{\gamma_i}{l_i(t)} \cdot \sqrt{2T-2t} \right) \right) \leq 0. \tag{12}$$

Thus the entropy of the normalized curves is bounded by the initial data and we have

$$\prod_{i=1}^{N} \left( \frac{l_i(t)}{\sqrt{2T-2t}} \right)^{\gamma_i} \geq C, \tag{13}$$

where $C > 0$ is a constant independent of time.

Consider the width of $K(t)/\sqrt{2T-2t}$, in an arbitrary direction $\vec{u}_0 = (\cos \phi_0, \sin \phi_0)$,

$$w(\phi_0) = \frac{1}{2} \sum_{i=1}^{N} \frac{l_i(t)}{\sqrt{2T-2t}} \cdot |\sin(\phi_i - \phi_0)|, \tag{14}$$

where $\phi_i$ corresponds to the $i$th direction of $\mathcal{U}$, $\vec{u}_i = (\cos \phi_i, \sin \phi_i)$.

Assume that the diameter of $K(t)/\sqrt{2T-2t}$ is not bounded. Since its area is constant, $K(t)/\sqrt{2T-2t}$ will be enclosed in a rectangle that will become infinitely long and thin. Moreover, since the directions of the outer normals are fixed, the blow-up of the diameter occurs in the direction of two parallel sides. Thus $\exists j, \vec{u}_j \in \mathcal{U}$, with $w(\phi_j) \sim 0$ and $w(\phi_j + \pi/2) \sim \infty$.

Then

$$w(\phi_j) = \frac{1}{2} \sum_{i=1}^{N} \frac{l_i(t)}{\sqrt{2T-2t}} \cdot |\sin(\phi_i - \phi_j)| \geq \sum_{i=1, \phi_i \neq \phi_j, \phi_j + \pi}^{N} c \cdot \frac{l_i(t)}{\sqrt{2T-2t}},$$

where $c := \min_{i=1, \ldots, N; \phi_i \neq \phi_j, \phi_j + \pi} |\sin(\phi_i - \phi_j)| > 0$, and,

$$w(\phi_j + \pi/2) = \frac{1}{2} \sum_{i=1}^{N} \frac{l_i(t)}{\sqrt{2T-2t}} \cdot |\cos(\phi_i - \phi_j)| \leq \sum_{i=1, \phi_i \neq \phi_j, \phi_j + \pi}^{N} \left( \frac{l_i(t)}{\sqrt{2T-2t}} + \frac{l_k(t)}{\sqrt{2T-2t}} \right), \tag{15}$$

where $\phi_k := \phi_j + \pi$. 

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We infer that the assumption on the blow-up of the diameter, combined with the area constraint, forces $N-2$ sides of $\partial K(t)/\sqrt{2T-2t}$ to approach zero length, and its remaining two parallel sides to become infinite. However, the product $w(\phi_j) \cdot w(\phi_j + \pi/2)$ remains bounded from both above and below

$$\alpha \leq w(\phi_j) \cdot w(\phi_j + \pi/2) \leq \sum_{i=1, i \neq j}^{N} \left( \frac{l_i(t)}{\sqrt{2T-2t}} \right) \cdot w(\phi_j) + \frac{l_j(t)}{\sqrt{2T-2t}} \cdot w(\phi_j) \leq C',$$

as the first term goes to zero as the time approaches $T$, while each of the last two terms is twice the area of a triangle included in $K(t)/\sqrt{2T-2t}$, thus less than $2\alpha$.

On the other hand,

$$l_j/\sqrt{2T-2t}, l_k/\sqrt{2T-2t} \leq w(\phi_j + \pi/2)$$

and

$$l_i/\sqrt{2T-2t} \leq c_i \cdot w(\phi_j), \quad \forall i \neq j, k,$$

where $c_i := 1/\sqrt{1-(\bar{u}_i \cdot \bar{u}_j)^2}$ is a strictly positive constant.

Use (16) and (17), together with $w(\phi_j) \leq C'/w(\phi_j + \pi/2)$, in the inequality (13). Since $w(\phi_j + \pi/2) \nearrow \infty$, we reach a contradiction with the hypothesis (2) as

$$w(\phi_j + \pi/2)^{2\gamma_1 N-\sum_{i=1, i \neq j,k}^{N} \gamma_i} \geq C'' > 0 \Rightarrow \gamma_j + \gamma_k - \sum_{i=1, i \neq j,k}^{N} \gamma_i \geq 0.$$

Therefore the diameter of the rescaled evolving bodies is bounded for all time by a constant depending only on the initial conditions. Recall now that our choice of the origin is $K(T)$, thus all support numbers $h_i(t)$ remain strictly positive for $t < T$. In conclusion, there exists $M > 0$ such that

$$\frac{h_i}{\sqrt{2T-2t}} \leq \text{Diameter}(K(t)/\sqrt{2T-2t}) \leq M, \quad \forall i = 1, ..., N,$$

for all time, so the normalized convex polygons remain within a fixed ball centered at the origin.

Using the upper bound on the diameter in conjunction with inequality (13) leads further to:
Corollary 2.1. If the ordered sets \( \mathcal{U} \) and \( \Gamma \) are as in the cases (i)–(iii) of Theorem 1.1, the lengths of the segments belonging to \( \partial K(t)/\sqrt{2T-2t} \) are uniformly bounded below by a strictly positive constant, independent of time.

3. THE EXISTENCE OF SOLUTIONS TO THE DISCRETE PLANAR \( L_0 \)-MINKOWSKI PROBLEM

We will proceed now with the proof of the main result.

Proof of Theorem 1.1. For a crystalline deformation of an arbitrary admissible body, consider the function:

\[
\mathcal{H} : [0, T] \rightarrow \mathbb{R}
\]

\[
\mathcal{H}(t) := \sum_{i=1}^{N} \gamma_i \log \frac{h_i}{\sqrt{2T-2t}}.
\]  

(19)

Then

\[
\frac{d \mathcal{H}}{dt} = -\sum_{i=1}^{N} \gamma_i^2 \frac{h_i}{h_i^2} + \sum_{i=1}^{N} \gamma_i \leq \frac{(\sum_{i=1}^{N} \gamma_i)^2}{\sum_{i=1}^{N} h_i^2} + \frac{\sum_{i=1}^{N} \gamma_i}{2T-2t} = 0.
\]  

(20)

We will show first that \( \mathcal{H} \) has a lower bound. Note that Corollary 2.1 provides, via (14), a uniform lower bound for the minimum width of \( K(t)/\sqrt{2T-2t} \). Consequently, the radius of the largest circle inscribed in \( K(t)/\sqrt{2T-2t} \), called the inner radius and denoted \( r_{in}(t) \), is uniformly bounded below via the classical inequality for convex bodies \( r_{in}(K) \geq \text{min width}(K)/3 \), [3].

Let \( \{t_n\} \) be a sequence of times converging to \( T \). For each \( t_n \), take \((\tilde{a}_n, \tilde{b}_n)\) to be the center of the largest circle inscribed in \( K(t_n)/\sqrt{2T-2t_n} \). Then \((\tilde{a}_n, \tilde{b}_n)\) belongs to the ball of radius \( M \) (the upper bound on the diameter) centered at the origin and

\[
\frac{h_i(t_n)}{\sqrt{2T-2t_n}} - \tilde{a}_n \cos \phi_i - \tilde{b}_n \sin \phi_i \geq r_{in}(t_n), \quad \forall i = 1, \ldots, N.
\]

Considering \( a_n = \sqrt{2T-2t_n} \) and \( b_n = \sqrt{2T-2t_n} \), the previous inequality corresponds to

\[
\frac{h_i^{(a_n, b_n)}(t_n)}{\sqrt{2T-2t_n}} \geq r_{in}(t_n), \quad \forall i = 1, \ldots, N,
\]
where \( h^{(a_n, b_n)}_{i}(t) = h_i(t) - a_n \cos \phi_i - b_n \sin \phi_i \), \( i = 1, \ldots, N \), are the support numbers of the evolving body \( K(t) \) with respect to the point \((a_n, b_n)\). Since \( h^{(a_n, b_n)}_{i}(t)/\sqrt{2T - 2t} > 0 \), then \( h^{(a_n, b_n)}_{i}(t)/\sqrt{2T - 2t} > 0 \) for all \( t < t_n \). Otherwise \( h^{(a_n, b_n)}_{i}(t)/\sqrt{2T - 2t} \leq 0 \) at some earlier time and it will remain so as the following holds, for any choice of the origin,

\[
\frac{d}{dt} \left( \frac{h_i(t)}{\sqrt{2T - 2t}} \right) = \frac{1}{2T - 2t} \left( -\frac{\gamma_i}{l_i/\sqrt{2T - 2t}} + \frac{h_i}{\sqrt{2T - 2t}} \right).
\]

In particular, we may conclude that \( h^{(a_n, b_n)}_{i}(0) > 0 \) for all \( i \)'s, thus \((a_n, b_n) \in K(0)\).

Moreover (20) holds independently of the support function chosen to describe the evolution, as long as the support numbers remain positive so \( \mathcal{H} \) is well defined. Therefore, for every \( n \),

\[
\sum_{i=1}^{N} \gamma_i \log \frac{h^{(a_n, b_n)}_{i}(t)}{\sqrt{2T - 2t}} \geq \log r_m(t) \cdot \sum_{i=1}^{N} \gamma_i, \quad \forall t \leq t_n.
\] (21)

As the sequence \( \{(\tilde{a}_n, \tilde{b}_n)\}_n \) lies then in a compact set, and \( t_n \to T \) as \( n \to \infty \), then \( \{(a_n, b_n)\}_n := \{ (\tilde{a}_n \sqrt{2T - 2t_n}, \tilde{b}_n \sqrt{2T - 2t_n}) \}_n \) converges to the origin.

Then for each \( t \in [0, T) \)

\[
\frac{h^{(a_n, b_n)}_{i}(t)}{\sqrt{2T - 2t}} \to \frac{h_i(t)}{\sqrt{2T - 2t}} \quad \text{as} \quad (a_n, b_n) \to (0, 0).
\]

Moreover the set of support numbers \( h^{(a_n, b_n)}_{i}(t)/\sqrt{2T - 2t}, i = 1, \ldots, N \) is uniformly bounded above by the maximum of the diameter of the normalized polygon. So,

\[
\mathcal{H}(t) = \sum_{i=1}^{N} \gamma_i \log \frac{h_i(t)}{\sqrt{2T - 2t}} = \lim_{n \to \infty} \sum_{i=1}^{N} \gamma_i \log \frac{h^{(a_n, b_n)}_{i}(t)}{\sqrt{2T - 2t}} \geq \log r_m(t) \cdot \sum_{i=1}^{N} \gamma_i \geq \text{constant}.
\] (22)

The decrease of \( \mathcal{H} \) and its lower bound imply that \( \mathcal{H} \) has a limit as \( t \to T \). Furthermore, use \( \sum_{i=1}^{N} \gamma_i = (2T - 2t) \cdot \sum_{i=1}^{N} l_i \) in (20) to assert that

\[
\frac{d\mathcal{H}}{dt} = -\sum_{i=1}^{N} \frac{1}{h_i} \frac{1}{\sqrt{2T - 2t}} \left( \frac{h_i}{\sqrt{2T - 2t}} - \frac{\gamma_i \sqrt{2T - 2t}}{l_i} \right)^2 \cdot \frac{1}{2T - 2t}.
\]
Therefore, by the Corollary 2.1, and the upper bound on the normalized support numbers (18), there exists a strictly positive constant $C$ such that

$$ \frac{d \mathcal{H}}{dt} \leq - \frac{C}{T-t} \sum_{i=1}^{N} \left( \frac{h_i}{\sqrt{2T-2t} - \gamma_i \sqrt{2T-2t}/l_i} \right)^2. $$

Let now $\{t_j\}$ be an arbitrary sequence of times converging to $T$. The functions $h_i(t)/\sqrt{2T-2t}$ and $\gamma_i \sqrt{2T-2t}/l_i(t)$ are continuous, and bounded from both sides. Therefore one can extract, at least, a subsequence of times going to $T$, denoted for simplicity the same as the initial sequence, so that $\sum_i (h_i(t_j)/\sqrt{2T-2t_j} - \gamma_i \sqrt{2T-2t_j}/l_i(t_j))^2$ converges to zero.

Otherwise, $\mathcal{H}_i \leq -\varepsilon/(T-t)$ for some $\varepsilon > 0$ and

$$ \mathcal{H}(T) - \mathcal{H}(t) \leq \varepsilon \ln(T-t) |^T_t = -\infty, $$

contradicting the bound from below of $\mathcal{H}(t)$.

Thus, there exists a sequence of times $t_j \to T$ such that

$$ \lim_{t_j \to T} \left( \frac{h_i(t_j)}{\sqrt{2T-2t_j} - \gamma_i \sqrt{2T-2t_j}/l_i(t_j)} \right) = 0, \quad \forall i = 1, \ldots, N $$

and, consequently, there exists a sequence of polygons converging, in the Hausdorff metric, to a solution of the discrete planar $L_0$-Minkowski problem.

Under the assumptions (i)–(iii), in addition to the existence of solutions to the discrete $L_0$-Minkowski problem, we conclude that for any admissible body deformed by the crystalline flow there exists a sequence of its normalized evolving shapes which approaches an $L_0$-shape. In fact, note from the proof above that the conclusion regarding the behavior of the flow can be stated in a stronger form:

**Theorem 3.1.** Let the ordered sets $\mathcal{U}$ and $\Gamma$ be as in the cases of Theorem 1.1. Then the crystalline deformation (3) shrinks the body $K$ to a point such that, for any sequence of times $\{t_j\}$ converging to $T$, there exists a convergent subsequence along which the shape of $K(t)$ approaches, in the Hausdorff metric, a solution to the $L_0$-Minkowski problem defined by $(\mathcal{U}, \Gamma)$.

We address now the Proof of Theorem 1.2.

($\Rightarrow$) It suffices to choose the origin to be the center of symmetry and apply the crystalline deformation to a centrally symmetric polygonal shape...
of ordered set of outer normals \( U \). The crystalline deformation associated to an even \( \Gamma \) preserves the central symmetry of the evolving polygons (in particular the shrinking point \( K(T) \) coincides with the origin). Then the \( L_0 \)-solution, obtained as the limit of a convergent sequence of normalized evolving shapes, and whose existence is guaranteed by Theorem 1.1, must be symmetric with respect to the origin.

\( (\Rightarrow) \) If there exists a solution symmetric about the origin, the converse follows from considering all parallelograms formed by two opposite equal sides of the \( L_0 \)-solution and comparing their areas to the area of the entire polygon.

Assume that there exists an \( i \in \{1, \ldots, M\} \) such that \( \gamma_i \geq \sum_{j \neq i} \gamma_j \). Then the discrete Minkowski problem has no solutions symmetric with respect to the origin, but by hypothesis there exists an \( L_0 \)-polygon which is not centrally symmetric. Connect the extremities of the sides with outer normals \( \bar{u}_i \) and, respectively, \( -\bar{u}_i \) to obtain a trapezoid, \( Q \).

Since \( M > 2 \), due to the convexity of the \( L_0 \)-solution we have that

\[
\text{Area}(Q) < \gamma_i + \sum_{j \neq i} \gamma_j \leq 2\gamma_i. \tag{25}
\]

On the other hand,

\[
\text{Area}(Q) = \frac{1}{2} (h_i + h_{i+M})(l_i + l_{i+M}) \\
= \frac{1}{2} (h_i + h_{i+M}) \left( \frac{\gamma_i}{h_i} + \frac{\gamma_{i+M}}{h_{i+M}} \right) = \frac{1}{2} \frac{(h_i + h_{i+M})^2}{h_i h_{i+M}} \cdot \gamma_{i}, \tag{26}
\]

where \( \{h_i\} \) and \( \{l_i\} \) denote the support numbers, respectively, the lengths of the sides of the \( L_0 \)-solution.

Note that (25) and (26) imply \( (h_i - h_{i+M})^2 < 0 \) contradicting the possibility of an even \( L_0 \)-problem with a non-centrally symmetric solution.

As a byproduct of the one-to-one correspondence between the \( L_0 \)-solutions and the shapes which evolve homothetically under the \( \Gamma \)-deformation, the uniqueness of the \( L_0 \)-polygon follows from a result on the number of self-similar solutions to the crystalline flow, [12]. We have proved that for even data, \( U = \{\bar{u}_i, \ldots, \bar{u}_M, \bar{u}_i, \ldots, \bar{u}_M\} \) and \( \Gamma = \{\gamma_1, \ldots, \gamma_M, \gamma_1, \ldots, \gamma_M\}, M > 2 \), if a centrally symmetric homothetic flow solution exists, then it is the only one among all deformations of admissible polygons. \( \blacksquare \)
Remark 3.1. The necessary condition for the existence of solutions to the $L_0$-Minkowski problem if $N=4$ and $\mathcal{U} = \{\bar{u}_1, \bar{u}_2, -\bar{u}_1, -\bar{u}_2\}$ can be seen also via the $C$-flow. Consider the ratio of two consecutive sides of an evolving parallelogram:

$$\frac{d}{dt} \left( \frac{l_1(t)}{l_2(t)} \right) = -\frac{\gamma_2 + \gamma_4}{l_2^2(t) \sqrt{1 - (\bar{u}_1 \cdot \bar{u}_2)^2}} + \frac{\gamma_1 + \gamma_3}{l_2^2(t) \sqrt{1 - (\bar{u}_1 \cdot \bar{u}_2)^2}}.$$ 

The existence of a homothetic solution to the flow requires $l_1(t)/l_2(t)$ to be constant. If $\gamma_1 + \gamma_3 \neq \gamma_2 + \gamma_4$, there is none. The scaled parallelograms become unbounded. On the other hand, if $\gamma_1 + \gamma_3 = \gamma_2 + \gamma_4$, any parallelogram of area $\alpha$ is a homothetic solution to the flow. Each one, rescaled, will remain fixed, thus bounded. However, we can find a sequence of rescaled parallelograms $K_n$, each an $L_0$-solution to the same data, such that Diameter$(K_n) \to \infty$. It is worth noting that they all have the same entropy.

Recall now the case $N > 4$, where $\mathcal{U} = \{\bar{u}_1, ..., \bar{u}_N\}$ is an ordered family of pairwise distinct unitary normal directions in $S^1$ containing, at least, one pair of opposite vectors, and $\Gamma = \{\gamma_1, ..., \gamma_N\}$ is an ordered set of strictly positive values such that for a pair $j, k$ with $\bar{u}_j = -\bar{u}_k$, then $\gamma_j + \gamma_k = \sum_{i=1, i \neq j, k}^{N} \gamma_i$. Thus (2) holds for any other pair of opposite outer normals. The equality implies a lower bound on the entropy of the normalized evolving polygons, but this is not sufficient to conclude the existence of an $L_0$-solution.

By Theorem 1.1, a solution to the corresponding discrete planar $L_0$-Minkowski problem exists if and only if there exists a polygonal evolution by the $C$-flow whose normalization remains bounded.

Suppose that the diameter of any rescaled evolving polygon becomes unbounded. Then there exists a sequence of times $t_n \to T$ such that the polygons $K(t_n)/\sqrt{2T-2t_n}$ come arbitrarily close to a sequence of parallelograms $K_n$ as above. These parallelograms are $L_0$-solutions to a $C'$ set with $\gamma'_j + \gamma'_k : = \sum_{i=1, i \neq j, k}^{N} \gamma_i = \gamma_j + \gamma_k =: \gamma'_j + \gamma'_k$. Therefore the lower bound on the normalized entropy of $K(t_n)$ is the constant normalized entropy of the parallelograms $K_n$.

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Note added in proof. Similar results with the ones stated in Theorem 3.1 have been obtained independently by Ben Andrews in Singularities in crystalline curvature flows, Asian J. Math. 6 (2002), 101–122.
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