Exercise 2.1 *A (positive) measure μ on $\mathbb{R}^n$ is called log-concave, if it has a density which is a measurable log-concave function. Show that for such a measure μ, $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, and $A, B \subset \mathbb{R}^n$ with $A, B, \alpha A + \beta B$ measurable holds*

$$\mu(\alpha\, A + \beta\, B) \geq \mu(A)^\alpha\, \mu(B)^\beta.$$

Hint: Prékopa-Leindler could help.

Exercise 2.2 *Let $K \subset \mathbb{R}^n$ be a strictly convex body containing $\mathbf{0}$ in the interior. Then for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that for all $\mathbf{x}, \mathbf{y} \in \operatorname{bd} K$ with $|\mathbf{x} - \mathbf{y}| \geq \epsilon$ we have $(\mathbf{x} + \mathbf{y})/2 \in (1 - \delta(\epsilon))K$. The function $\epsilon \to \delta(\epsilon)$ is called the* modulus of convexity of K.

i) *Show that for the unit ball we my choose $\delta(\epsilon) = 1 - \sqrt{1 - \epsilon^2/4}$.*

ii) *On $\operatorname{bd} K$ we define a measure by setting $\mu(A) = \operatorname{vol}(\operatorname{conv}\{\mathbf{0}, A\})/\operatorname{vol}(K)$, $A \subset \operatorname{bd} K$. Now let $A \subset \operatorname{bd} K$ with $\mu(A) \geq 1/2$. Show that for every $\epsilon > 0$ with $\delta(\epsilon) < 1/2$*

$$\mu\left(\mathbf{x} \in \operatorname{bd} K : \operatorname{dist}(\mathbf{x}, A) \geq \epsilon\right) \leq 2(1 - \delta(\epsilon))^{2n}.$$

Hint: Let $B = \{\mathbf{x} \in \operatorname{bd} K : \operatorname{dist}(\mathbf{x}, A) \geq \epsilon\}$. First show $(1/2)\operatorname{conv}\{\mathbf{0}, A\} + (1/2)\operatorname{conv}\{\mathbf{0}, B\} \subseteq (1 - \delta(\epsilon))K$.

Exercise 2.3 *Let $A \subseteq I_n$, $A \neq \emptyset$, and let $f : I_n \to \mathbb{R}$ be given by $f(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, A)$. Let m_f be the median of f and a_f its average, i.e.,*

$$a_f = \int_{I_n} f(\mathbf{x}) \mathrm{d}\rho_n.$$

Show that $|m_f - a_f| \leq \sqrt{(n \ln n)/2} + 4\sqrt{n}$.

Exercise 2.4 *Let $A \subseteq I_n$, $A \neq \emptyset$. Show that*

$$\int_{I_n} \operatorname{dist}(\mathbf{x}, A) \mathrm{d}\rho_n \leq \frac{n}{2}.$$

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Exercise 1.1 *Show that for* $1 \leq p < \infty$ *the space* l_p *is separable, i.e., it contains a countable dense subset. What about* l_∞*?*

Exercise 1.2 *Let* $1 \leq p \leq 2$ *and* $x, y \in \mathbb{R}$*. Show that*

$$|x-y|^p + |x+y|^p \leq 2|x|^p + 2|y|^p.$$

Exercise 1.3 *Let* $1 \leq p \leq 2$*. Infinitely many balls* $B_p(\mathbf{z}, r)$ *can be packed into the unit ball* $B_p(\mathbf{0}, 1)$ *if and only if*

$$r \leq \left[1 + 2^{1-1/p}\right]^{-1}.$$

For $1 \geq r > \left[1 + 2^{1-1/p}\right]^{-1}$ *let* $m_p(r)$ *be the maximum number of* l_p*-balls* $B(\mathbf{z}, r)$ *which can be packed into the unit* l_p*-ball* $B(\mathbf{0}, 1)$*. Then*

$$m_p(r) \leq 1 + \left(\frac{1-r}{r}\right)^p \left(\frac{1 - 2^{1-p}}{1 - 2^{1-p}\left(\frac{1-r}{r}\right)^p}\right).$$

Exercise 1.4 *Let* $x \geq y \in \mathbb{R}_{\geq 0}$*. Show that*

$$x^p - y^p \begin{cases} \leq (x-y)^p, & 0 \leq p \leq 1, \\ \geq (x-y)^p, & 1 \leq p < \infty. \end{cases}$$

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