Exercise 3.1 *Prove the Gram-Brianchon Theorem, i.e., for a polytope* $P \subset \mathbb{R}^n$ *we have*

$$[P] = \sum_F (-1)^{\dim F} [\mathrm{tcone}(P, F)],$$

where the sum is over all non-empty faces of P*, including* P *itself.*
Hint: Maybe, the "dual" statement is more intuitiv.

Exercise 3.2 *Let* $P \subset \mathbb{R}^n$ *be a polytope. Show that*

$$[\mathrm{int}\, P] \equiv \sum_{\mathbf{v} \in \mathrm{vert}(P)} [\mathrm{int}\, \mathrm{tcone}(P, \mathbf{v})] \bmod \textit{lines}.$$

Exercise 3.3 *Let* $P \subset \mathbb{R}^n$ *be an unbounded* n*-dimensional polyhedron without lines. Let* $f_i^b(P)$*,* $f_i^u(P)$ *be the number of bounded, unbounded* i*-faces of* P*, respectively. Prove that*

$$\sum_{i=0}^{n-1} (-1) f_i^b(P) = 1 \textit{ and } \sum_{i=1}^{n} (-1)^{i+1} f_i^u(P) = 1.$$

Exercise 3.4 *Let* $P \subset \mathbb{R}^n$ *be an* n*-dimensional polytope. Prove that*

$$(-1)^n [\mathrm{int}\, P] = \sum_F (-1)^{\dim F} [F],$$

where the sum is over all non-empty faces of P*, including* P*.*

Exercise 3.5 *Let* $P \subset \mathbb{R}^n$ *be an unbounded* n*-dimensional polyhedron without lines. Prove that* $\chi([\mathrm{int}\, P]) = 0$*.*

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Exercise 2.1 *A (positive) measure* μ *on* $\mathbb{R}^n$ *is called log-concave, if it has a density which is a measurable log-concave function. Show that for such a measure* μ, $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, *and* $A, B \subset \mathbb{R}^n$ *with* $A, B, \alpha A + \beta B$ *measurable holds*

$$\mu(\alpha\, A + \beta\, B) \geq \mu(A)^\alpha\, \mu(B)^\beta.$$

Hint: Prékopa-Leindler could help.

Exercise 2.2 *Let* $K \subset \mathbb{R}^n$ *be a strictly convex body containing* $\mathbf{0}$ *in the interior. Then for every* $\epsilon > 0$ *there exists a* $\delta(\epsilon) > 0$ *such that for all* $\mathbf{x}, \mathbf{y} \in \operatorname{bd} K$ *with* $|\mathbf{x} - \mathbf{y}| \geq \epsilon$ *we have* $(\mathbf{x} + \mathbf{y})/2 \in (1 - \delta(\epsilon))K$. *The function* $\epsilon \to \delta(\epsilon)$ *is called the* modulus of convexity of K.

i) *Show that for the unit ball we my choose* $\delta(\epsilon) = 1 - \sqrt{1 - \epsilon^2/4}$.

ii) *On* $\operatorname{bd} K$ *we define a measure by setting* $\mu(A) = \operatorname{vol}(\operatorname{conv}\{\mathbf{0}, A\})/\operatorname{vol}(K)$, $A \subset \operatorname{bd} K$. *Now let* $A \subset \operatorname{bd} K$ *with* $\mu(A) \geq 1/2$. *Show that for every* $\epsilon > 0$ *with* $\delta(\epsilon) < 1/2$

$$\mu\left(\mathbf{x} \in \operatorname{bd} K : \operatorname{dist}(\mathbf{x}, A) \geq \epsilon\right) \leq 2(1 - \delta(\epsilon))^{2n}.$$

Hint: Let $B = \{\mathbf{x} \in \operatorname{bd} K : \operatorname{dist}(\mathbf{x}, A) \geq \epsilon\}$. First show $(1/2)\operatorname{conv}\{\mathbf{0}, A\} + (1/2)\operatorname{conv}\{\mathbf{0}, B\} \subseteq (1 - \delta(\epsilon))K$.

Exercise 2.3 *Let* $A \subseteq I_n$, $A \neq \emptyset$, *and let* $f : I_n \to \mathbb{R}$ *be given by* $f(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, A)$. *Let* m_f *be the median of* f *and* a_f *its average, i.e.,*

$$a_f = \int_{I_n} f(\mathbf{x})\mathrm{d}\rho_n.$$

Show that $|m_f - a_f| \leq \sqrt{(n \ln n)/2} + 4\sqrt{n}$.

Exercise 2.4 *Let* $A \subseteq I_n$, $A \neq \emptyset$. *Show that*

$$\int_{I_n} \operatorname{dist}(\mathbf{x}, A)\mathrm{d}\rho_n \leq \frac{n}{2}.$$

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Exercise 1.1 *Show that for* $1 \leq p < \infty$ *the space* l_p *is separable, i.e., it contains a countable dense subset. What about* l_∞*?*

Exercise 1.2 *Let* $1 \leq p \leq 2$ *and* $x, y \in \mathbb{R}$*. Show that*

$$|x-y|^p + |x+y|^p \leq 2|x|^p + 2|y|^p.$$

Exercise 1.3 *Let* $1 \leq p \leq 2$*. Infinitely many balls* $B_p(\mathbf{z}, r)$ *can be packed into the unit ball* $B_p(\mathbf{0}, 1)$ *if and only if*

$$r \leq \left[1 + 2^{1-1/p}\right]^{-1}.$$

For $1 \geq r > \left[1 + 2^{1-1/p}\right]^{-1}$ *let* $m_p(r)$ *be the maximum number of* l_p*-balls* $B(\mathbf{z}, r)$ *which can be packed into the unit* l_p*-ball* $B(\mathbf{0}, 1)$*. Then*

$$m_p(r) \leq 1 + \left(\frac{1-r}{r}\right)^p \left(\frac{1 - 2^{1-p}}{1 - 2^{1-p}\left(\frac{1-r}{r}\right)^p}\right).$$

Exercise 1.4 *Let* $x \geq y \in \mathbb{R}_{\geq 0}$*. Show that*

$$x^p - y^p \begin{cases} \leq (x-y)^p, & 0 \leq p \leq 1, \\ \geq (x-y)^p, & 1 \leq p < \infty. \end{cases}$$

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