Aspects of

Modern Convex Geometry

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1 Brunn-Minkowski revisited

1.1 Theorem [Brunn-Minkowski Inequality]. *Let $K, L \subset \mathcal{K}^n$, and $\lambda \in [0,1]$. Then*

$$\operatorname{vol}\big(\lambda K + (1-\lambda) L\big)^{1/n} \geq \lambda \operatorname{vol}(K)^{1/n} + (1-\lambda)\operatorname{vol}(L)^{1/n}, \tag{1.1.1}$$

and for $0 < \lambda < 1$ equality holds if and only if either K and L are homothetic or K and L lie in parallel hyperplanes.

Proof. For $\lambda \in [0,1]$ let $K_\lambda = \lambda K + (1-\lambda)L$. First we check that the theorems holds for convex bodies $K, L \in \mathcal{K}^n$ of which one is lower dimensional. With out loss of generality let $\dim L < n$.

If also $\dim K < n$ then $\operatorname{vol}(K) = \operatorname{vol}(L) = 0$ and (1.1.1) holds trivally. If equality holds for a $\lambda \in (0,1)$ then $\operatorname{vol}(K_\lambda) = 0$ and so K_λ lies in a hyperplane H, say. But this implies that K and L lie in hyperplanes parallel to H. On the other hand , if K and L lie in parallel hyperplanes then also K_λ is contained in a hyperplane and $\operatorname{vol}(K_\lambda) = 0$ for $0 \leq \lambda \leq 1$.

Now suppose $\dim K = n$. Since $K_\lambda \supseteq \lambda K + (1-\lambda)\boldsymbol{x}$ for any $\boldsymbol{x} \in L$, we have $\operatorname{vol}(K_\lambda) \geq \operatorname{vol}\big(\lambda K + (1-\lambda)\boldsymbol{x}\big) = \lambda^n \operatorname{vol}(K)$ and hence (1.1.1). Now equality holds if and only if $L = \{\boldsymbol{x}\}$ and in this case K and L are homothetic.

Thus, in the following we may assume $\dim K = \dim L = n$, and next we observe that it is sufficient to prove (1.1.1) in the particular situation when $\operatorname{vol}(K) = \operatorname{vol}(L) = 1$. The general case can be reduced to this setting via the normalisation $\overline{K} = \operatorname{vol}(K)^{-1/n} K$, $\overline{L} = \operatorname{vol}(L)^{-1/n} L$, and

$$\overline{\lambda} = \frac{\lambda \operatorname{vol}(K)^{1/n}}{\lambda \operatorname{vol}(K)^{1/n} + (1-\lambda)\operatorname{vol}(L)^{1/n}},$$

since

$$\begin{aligned}
&\operatorname{vol}\big(\overline{\lambda K} + (1-\overline{\lambda})\overline{L}\big) \\
&= \operatorname{vol}\left(\frac{\lambda}{\lambda \operatorname{vol}(K)^{1/n} + (1-\lambda)\operatorname{vol}(L)^{1/n}} K + \frac{1-\lambda}{\lambda \operatorname{vol}(K)^{1/n} + (1-\lambda)\operatorname{vol}(L)^{1/n}} L\right) \\
&= \frac{1}{\big(\lambda \operatorname{vol}(K)^{1/n} + (1-\lambda)\operatorname{vol}(L)^{1/n}\big)^n} \operatorname{vol}\big(\lambda K + (1-\lambda)L\big).
\end{aligned}$$

So let $\operatorname{vol}(K) = \operatorname{vol}(L) = 1$. We will prove the theorem by induction on the dimension. If $n = 1$ the result is certainly true, since K and L are intervals which are homothetic. Let $n \geq 2$. For a fixed $\boldsymbol{u} \in S^{n-1}$ and $\mu \in \mathbb{R}$ we set

$$\begin{aligned}
v_K(\mu) &= \operatorname{vol}_{n-1}\big(K \cap H(\boldsymbol{u}, \mu)\big), & v_L(\mu) &= \operatorname{vol}_{n-1}\big(L \cap H(\boldsymbol{u}, \mu)\big), \\
w_K(\mu) &= \operatorname{vol}\big(K \cap H^-(\boldsymbol{u}, \mu)\big), & w_L(\mu) &= \operatorname{vol}\big(L \cap H^-(\boldsymbol{u}, \mu)\big).
\end{aligned}$$

Then

$$w_K(\mu) = \int_{-h(K,-\boldsymbol{u})}^{\mu} v_K(\boldsymbol{s})\, \mathrm{d}\boldsymbol{s} \quad \text{and} \quad w_L(\mu) = \int_{-h(L,-\boldsymbol{u})}^{\mu} v_L(\boldsymbol{s})\, \mathrm{d}\boldsymbol{s}.$$

The functions v_K, v_L are continuous on the intervals $\big(-h(K,-\boldsymbol{u}), h(K,\boldsymbol{u})\big)$ and $\big(-h(L,-\boldsymbol{u}), h(L,\boldsymbol{u})\big)$, respectively, which ensures that w_K, w_L are differentiable with $w'_K(\mu) = v_K(\mu)$ and $w'_L(\mu) = v_L(\mu)$. Moreover, if we denote by z_K, z_L the inverse function of w_K, w_L respectively, then

$$z'_K(\eta) = \frac{1}{v_K\big(z_K(\eta)\big)}, \qquad z'_L(\eta) = \frac{1}{v_L\big(z_L(\eta)\big)}, \qquad \text{for } \ 0 < \eta < 1.$$

Finally, we write

$$\begin{aligned} &\tilde{K}_\eta = K \cap H\big(\boldsymbol{u}, z_K(\eta)\big), \qquad \tilde{L}_\eta = L \cap H\big(\boldsymbol{u}, z_L(\eta)\big), \\ &z_\lambda(\eta) = \lambda\, z_K(\eta) + (1-\lambda) z_L(\eta). \end{aligned}$$

Observe that $z_\lambda(0) = \lambda\,(-h(K,-\boldsymbol{u})) + (1-\lambda)\,(-h(L,-\boldsymbol{u})) = -h(K_\lambda,-\boldsymbol{u})$ and in the same way we find $z_\lambda(1) = h(K_\lambda,\boldsymbol{u})$. $\tilde{K}_\eta, \tilde{L}_\eta$ are $(n-1)$-dimensional with $\mathrm{vol}_{\,n-1}(\tilde{K}_\eta) = v_K\big(z_K(\eta)\big)$ and $\mathrm{vol}_{\,n-1}(\tilde{L}_\eta) = v_L\big(z_L(\eta)\big)$. Since $\lambda\,\tilde{K}_\eta + (1-\lambda)\tilde{L}_\eta \subseteq K_\lambda \cap H\big(\boldsymbol{u}, z_\lambda(\eta)\big)$ we get in view of our induction hypothesis

$$\begin{aligned} &\mathrm{vol}\,(K_\lambda) = \int_{-h(K_\lambda,-\boldsymbol{u})}^{h(K_\lambda,\boldsymbol{u})} \mathrm{vol}_{\,n-1}\big(K_\lambda \cap H(\boldsymbol{u},\mu)\big)\ \mathrm{d}\mu \\ &= \int_0^1 \mathrm{vol}_{\,n-1}\big(K_\lambda \cap H(\boldsymbol{u}, z_\lambda(\eta))\big) z'_\lambda(\eta)\ \mathrm{d}\eta \\ &\geq \int_0^1 \mathrm{vol}_{\,n-1}\big(\lambda\tilde{K}_\eta + (1-\lambda)\tilde{L}_\eta\big) \left[\frac{\lambda}{v_K\big(z_K(\eta)\big)} + \frac{1-\lambda}{v_L\big(z_L(\eta)\big)}\right]\ \mathrm{d}\eta \\ &\geq \int_0^1 \Big(\lambda\, v_K\big(z_K(\eta)\big)^{\frac{1}{n-1}} + (1-\lambda) v_L\big(z_L(\eta)\big)^{\frac{1}{n-1}}\Big)^{n-1} \left[\frac{\lambda}{v_K\big(z_K(\eta)\big)} + \frac{1-\lambda}{v_L\big(z_L(\eta)\big)}\right]\ \mathrm{d}\eta \\ &\geq \int_0^1 1\ \mathrm{d}\eta = 1, \end{aligned} \tag{1.1.2}$$

where the second to last inequality follows by taking the logarithm of the integrand and using the concavity of the logarithm.

We certainly have equality if K and L are homothetic. Thus, suppose we have equality in (1.1.1). Again we may assume $\mathrm{vol}\,(K) = \mathrm{vol}\,(L) = 1$ and so $\mathrm{vol}\,(K_\lambda) = 1$ for some $\lambda \in (0,1)$. Then we have equality in (1.1.2) and by the strict concavity of the logarithm we conclude $v_K\big(z_K(\eta)\big) = v_L\big(z_L(\eta)\big)$ for $\eta \in [0,1]$ (remember v_K, v_L, z_K, z_L are continuous functions). Hence $z'_K(\eta) = z'_L(\eta)$ for $0 \leq \eta \leq 1$, i.e., $z_K(\eta) - z_L(\eta)$ is constant.

Let g_K, g_L be the centroids of K, L, respectively, and let $g_K = g_L = \mathbf{0}$. Then $\mathbf{0} = \int_K \boldsymbol{x}\ \mathrm{d}\boldsymbol{x}$ and so

$$\begin{aligned} 0 &= \int_K \langle \boldsymbol{x}, \boldsymbol{e}_n \rangle\ \mathrm{d}\boldsymbol{x} \\ &= \int_{-h(K,-\boldsymbol{e}_n)}^{h(K,\boldsymbol{e}_n)} x_n \mathrm{vol}_{\,n-1}\big(K \cap H(\boldsymbol{e}_n, x_n)\big)\ \mathrm{d}x_n \\ &= \int_0^1 \mathrm{vol}_{\,n-1}\Big(K \cap H\big(\boldsymbol{e}_n, z_K(\eta)\big)\Big) z_K(\eta) \frac{1}{v_K\big(z_K(\eta)\big)}\ \mathrm{d}\eta = \int_0^1 z_K(\eta)\ \mathrm{d}\eta. \end{aligned}$$

Analogously we obtain for the body L the relation $0 = \int_0^1 z_L(\eta)\,\mathrm{d}\eta$. Since $z_K(\eta) - z_L(\eta)$ is constant we get $z_K(\eta) = z_L(\eta)$ for $0 \le \eta \le 1$. Hence, $h(K, \boldsymbol{u}) = h(L, \boldsymbol{u})$. By the arbitrarity of $\boldsymbol{u} \in S^{n-1}$ we conclude $h(K, \boldsymbol{u}) = h(L, \boldsymbol{u})$ for all $\boldsymbol{u} \in S^{n-1}$ and thus $K = L$. □

1.2 Lemma. *Let $K, L \in \mathcal{K}^n$ and suppose there exists a hyperplane $H(\boldsymbol{a}, 0)$ such that $K|H(\boldsymbol{a}, 0) = L|H(\boldsymbol{a}, 0)$. Then for $\lambda \in [0, 1]$*

$$\mathrm{vol}\,\big(\lambda K + (1-\lambda)L\big) \ge \lambda\,\mathrm{vol}\,(K) + (1-\lambda)\,\mathrm{vol}\,(L).$$

Proof. Let $|\boldsymbol{a}| = 1$. For abbreviation we write $H = H(\boldsymbol{a}, 0)$, $U = K|H = L|H$, and we set $K_\lambda = \lambda K + (1-\lambda)L$. Then $K_\lambda|H = \lambda(K|H) + (1-\lambda)(L|H) = U$ for all $\lambda \in [0, 1]$, and we may describe the convex body K_λ as

$$K_\lambda = \big\{\boldsymbol{y} + t\boldsymbol{a} : \boldsymbol{y} \in U, f_\lambda(\boldsymbol{y}) \le t \le g_\lambda(\boldsymbol{y})\big\},$$

where the functions f_λ and g_λ satisfy $f_\lambda \le g_\lambda$, f_λ is convex and g_λ is concave. For $\boldsymbol{y} \in U$ and $t_1, t_2 \in \mathbb{R}$ with $\boldsymbol{y} + t_1\boldsymbol{a} \in K$ and $\boldsymbol{y} + t_2\boldsymbol{a} \in L$ we have

$$\boldsymbol{y} + \big(\lambda t_1 + (1-\lambda)t_2\big)\boldsymbol{a} = \lambda(\boldsymbol{y} + t_1\boldsymbol{a}) + (1-\lambda)(\boldsymbol{y} + t_2\boldsymbol{a}) \in K_\lambda.$$

Hence, $f_\lambda(\boldsymbol{y}) \le \lambda t_1 + (1-\lambda)t_2 \le g_\lambda(\boldsymbol{y})$. For $t_1 = f_1(\boldsymbol{y})$ and $t_2 = f_0(\boldsymbol{y})$ we obtain $f_\lambda(\boldsymbol{y}) \le \lambda f_1(\boldsymbol{y}) + (1-\lambda)f_0(\boldsymbol{y})$, and for $t_1 = g_1(\boldsymbol{y})$ and $t_2 = g_0(\boldsymbol{y})$ we get $g_\lambda(\boldsymbol{y}) \ge \lambda g_1(\boldsymbol{y}) + (1-\lambda)g_0(\boldsymbol{y})$. Therefore

$$\begin{aligned}
\mathrm{vol}\,(K_\lambda) &= \int_U \big[g_\lambda(\boldsymbol{y}) - f_\lambda(\boldsymbol{y})\big]\,\mathrm{d}\boldsymbol{y}\\
&\ge \int_U \big[\lambda g_1(\boldsymbol{y}) + (1-\lambda)g_0(\boldsymbol{y}) - \lambda f_1(\boldsymbol{y}) - (1-\lambda)f_0(\boldsymbol{y})\big]\,\mathrm{d}\boldsymbol{y}\\
&= \lambda \int_U \big[g_1(\boldsymbol{y}) - f_1(\boldsymbol{y})\big]\,\mathrm{d}\boldsymbol{y} + (1-\lambda)\int_U \big[g_0(\boldsymbol{y}) - f_0(\boldsymbol{y})\big]\,\mathrm{d}\boldsymbol{y}\\
&= \lambda\,\mathrm{vol}\,(K) + (1-\lambda)\,\mathrm{vol}\,(L),
\end{aligned}$$

which concludes the proof. □

1.3 Theorem. *Let $K, L \in \mathcal{K}^n$ and suppose there exists a hyperplane $H(\boldsymbol{a}, 0)$ such that $\mathrm{vol}_{n-1}(K|H(\boldsymbol{a}, 0)) = \mathrm{vol}_{n-1}(L|H(\boldsymbol{a}, 0))$. Then for $\lambda \in [0, 1]$*

$$\mathrm{vol}\,\big(\lambda K + (1-\lambda)L\big) \ge \lambda\,\mathrm{vol}\,(K) + (1-\lambda)\,\mathrm{vol}\,(L).$$

Proof. For abbreviation we set $H = H(\boldsymbol{a}, 0)$, and let $K' = \mathrm{st}_H(K)$ and $L' = \mathrm{st}_H(L)$ be the Steiner-symmetrals of K and L with respect to the hyperplane H, respectively. By [Proposition 10.3, SkriptWS12] we have $\lambda K' + (1-\lambda)L' \subseteq \mathrm{st}_H\big(\lambda K + (1-\lambda)L\big)$ and since the Steiner symmetrisation preserves the volume, it suffices to prove

$$\mathrm{vol}\,\big(\lambda K' + (1-\lambda)L'\big) \ge \lambda\,\mathrm{vol}\,(K') + (1-\lambda)\,\mathrm{vol}\,(L').$$

Observe that $K|H = K' \cap H$ and $L|H = L' \cap H$. According to [Corollary 10.6, SkriptWS12] we can find hyperplanes $H_i = H(\boldsymbol{a}_i, 0)$, $\boldsymbol{a}_i \in H$, $i \in \mathbb{N}$, such that

$$\begin{aligned}\lim_{i\to\infty} \mathrm{st}_{H_i\cdots H_1}(K'\cap H) &= \left(\frac{\mathrm{vol}_{n-1}(K'\cap H)}{\kappa_{n-1}}\right)^{\frac{1}{n-1}} B_{n-1}\\ &= \left(\frac{\mathrm{vol}_{n-1}(L'\cap H)}{\kappa_{n-1}}\right)^{\frac{1}{n-1}} B_{n-1} = \lim_{i\to\infty} \mathrm{st}_{H_i\cdots H_1}(L'\cap H),\end{aligned} \tag{1.3.1}$$

where B_{n-1} is the ball of radius 1 centered at the origin in H. Now let

$$K'' = \lim_{i\to\infty} \mathrm{st}_{H_i}\cdots \mathrm{st}_{H_1}(K') \text{ and } L'' = \lim_{i\to\infty} \mathrm{st}_{H_i}\cdots \mathrm{st}_{H_1}(L')$$

Since K' is symmetric to H and $\boldsymbol{a}_i \in H$ we have that $\mathrm{st}_{H_i}\cdots \mathrm{st}_{H_1}(K')$ is symmetric to H and so $[\mathrm{st}_{H_i}\cdots \mathrm{st}_{H_1}(K')]|H = \mathrm{st}_{H_i}\cdots \mathrm{st}_{H_1}(K'|H)$. Hence

$$K''|H = \lim_{i\to\infty} \mathrm{st}_{H_i}\cdots \mathrm{st}_{H_1}(K'\cap H) \text{ and } L''|H = \lim_{i\to\infty} \mathrm{st}_{H_i}\cdots \mathrm{st}_{H_1}(L'\cap H).$$

we get with Lemma 1.2 applied to K'', L''

$$\begin{aligned}\mathrm{vol}\left(\lambda K' + (1-\lambda)L'\right) &= \mathrm{vol}\left(\lim_{i\to\infty} \mathrm{st}_{H_i}\cdots \mathrm{st}_{H_1}\left(\lambda K' + (1-\lambda)L'\right)\right)\\ &\geq \mathrm{vol}\left(\lambda K'' + (1-\lambda)L''\right) \geq \lambda\,\mathrm{vol}\,(K'') + (1-\lambda)\,\mathrm{vol}\,(L'')\\ &= \lambda\,\mathrm{vol}\,(K') + (1-\lambda)\,\mathrm{vol}\,(L').\end{aligned}$$

□

1.4 Theorem [Grünbaum]. [1] *Let $K \in \mathcal{K}^n$ with $\dim K = n$ and center of gravity $\boldsymbol{c}(K)$. Let H^+ be a subspace containing $\boldsymbol{c}(K)$. Then*

$$\frac{\mathrm{vol}\,(K\cap H^+)}{\mathrm{vol}\,(K)} \geq \left(\frac{n}{n+1}\right)^n.$$

(Observe that $(n/(n+1))^n \to 1/\mathrm{e}$).

Proof. Without loss of generality let $\boldsymbol{c}(K) = \boldsymbol{0}$ and $H^+ = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle = x_1 \geq 0\}$. Let $H^- = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle \leq 0\}$, and for $t \in \mathbb{R}$ let $H_t = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle = t\}$ and $f(t) = \mathrm{vol}_{n-1}(K \cap H_t)$.

Let $b = h(K, \boldsymbol{e}_1)$ and $a = h(K, -\boldsymbol{e}_1)$, where $h(K,\cdot)$ denotes the support function of K. Since $\boldsymbol{0} = \int_K \boldsymbol{x}\,\mathrm{d}\boldsymbol{x}$ we have

$$0 = \int_K \langle \boldsymbol{x}, \boldsymbol{e}_1\rangle\,\mathrm{d}\boldsymbol{x} = \int_{-a}^{b} t\, f(t)\,\mathrm{d}t. \tag{1.4.1}$$

First we apply Schwarz-Symmetrization (see [, SkripWS12]) to K with respect to the hyperplane H, i.e., we replace $K \cap H_t$ by an $(n-1)$-dimensional ball of volume $f(t)$, i.e., of radius $(f(t)/\kappa_{n-1})^{1/(n-1)}$ and centred at the point

[1] Branko Grünbaum, 1929

$(t, \mathbf{0})^\intercal \in \mathbb{R}^n$. Let $\overline{K}$ be the resulting Schwarz-Symmetral. By construction, $\mathrm{vol}\,(K \cap H_t) = \mathrm{vol}\,(\overline{K} \cap H_t)$ and, in particular, $\mathrm{vol}\,(K) = \mathrm{vol}\,(\overline{K})$, $\mathrm{vol}\,(K \cap H^+) = \mathrm{vol}\,(\overline{K} \cap H^+)$, and we also have $\boldsymbol{c}(\overline{K}) = \boldsymbol{c}(K) = \mathbf{0}$. For the last statement we observe that $\overline{K}$ is rotational symmetric with respect to the axis $\{\lambda\, \boldsymbol{e}_1 : \lambda \in \mathbb{R}\}$. Hence $\boldsymbol{c}(\overline{K})$ has to be on this axis, i.e., $\boldsymbol{c}(\overline{K}) = \gamma\, \boldsymbol{e}_1$. Thus, on account of (1.4.1), we get

$$\begin{aligned} \mathrm{vol}\,(\overline{K})\, \langle \gamma\, \boldsymbol{e}_1, \boldsymbol{e}_1 \rangle &= \int_{\overline{K}} \langle \boldsymbol{x}, \boldsymbol{e}_1 \rangle \, \mathrm{d}\boldsymbol{x} \\ &= \int_{-a}^{b} t\, \mathrm{vol}_{\,n-1}(\overline{K} \cap H_t)\, \mathrm{d}t = \int_{-a}^{b} t\, f(t)\, \mathrm{d}t = 0. \end{aligned}$$

Hence $\gamma = 0$, i.e., $\boldsymbol{c}(\overline{K}) = \mathbf{0}$ and in the following we may assume $K = \overline{K}$.

Let $K_0 = K \cap H_0$, $K^+ = K \cap H^+$, $K^- = K \cap H^-$, and let $P^+ = \mathrm{conv}\,\{K_0, \beta \boldsymbol{e}_1\}$ be the pyramid with basis K_0 such that $\mathrm{vol}\,(P^+) = \mathrm{vol}\,(K^+)$, i.e., $\beta = n\mathrm{vol}\,(K^+)/\mathrm{vol}_{\,n-1}(K_0)$.

We now extend this pyramid in the other direction such that for $\alpha > 0$ and $P^- = \mathrm{conv}\,\{K_0, -\alpha \boldsymbol{e}_1 + \frac{\alpha+\beta}{\beta} K_0)\}$ holds $\mathrm{vol}\,(P^-) = \mathrm{vol}\,(K^-)$. Let $P = P^+ \cup P^-$, i.e.,

$$P = \mathrm{conv}\left\{ -\alpha \boldsymbol{e}_1 + \frac{\alpha + \beta}{\beta} K_0, \beta \boldsymbol{e}_1 \right\}$$

is a circular pyramid of height $\alpha + \beta$ and $\mathrm{vol}\,(P) = \mathrm{vol}\,(K)$, and we have

$$\frac{\mathrm{vol}\,(K^+)}{\mathrm{vol}\,(K)} = \frac{\mathrm{vol}\,(P^+)}{\mathrm{vol}\,(P)} = \left(\frac{\beta}{\alpha + \beta} \right)^n$$

Hence, it remains to show $\beta/(\alpha + \beta) \geq n/(n+1)$. To this end let

$$l(t) = \mathrm{vol}_{\,n-1}(P \cap H_t) = \left(\frac{\beta - t}{\beta} \right)^{n-1} \mathrm{vol}_{\,n-1}(K_0).$$

The crucial observation here is that due to the concavity of $f(t)^{1/(n-1)}$ the centroid $c(P)$ is on the non-negative x-axis, i.e., $c(P) = (\gamma, \mathbf{0})^\intercal \in \mathbb{R}^n$ with $\gamma \geq 0$. Then we may write

$$\begin{aligned} 0 \leq \gamma = \mathrm{vol}\,(P)\, \langle \boldsymbol{c}(P), \boldsymbol{e}_1 \rangle &= \int_{-\alpha}^{\beta} t\, l(t)\, \mathrm{d}t \\ &= \frac{\mathrm{vol}_{\,n-1}(K_0)}{\beta^{n-1}} \int_{-\alpha}^{\beta} t\, (\beta - t)^{n-1}\, \mathrm{d}t \\ &= \frac{\mathrm{vol}_{\,n-1}(K_0)}{\beta^{n-1}} \left(-\frac{t}{n} (\beta - t)^n - \frac{(\beta - t)^{n+1}}{n(n+1)} \Big|_{-\alpha}^{\beta} \right) \\ &= \frac{\mathrm{vol}_{\,n-1}(K_0)}{\beta^{n-1}} \frac{(\beta + \alpha)^n}{n} \left(-\alpha + \frac{\alpha + \beta}{n+1} \right). \end{aligned}$$

Hence $\alpha/(\alpha + \beta) \leq 1/(n+1)$ which is equivalent to $\beta/(\alpha + \beta) \geq n/(n+1)$.

□

We recall from [Corollary 5.18, SkriptWS12] that the Brunn-Minkowski inequality is equivalent to its multiplicative version

$$\operatorname{vol}\big(\lambda K + (1-\lambda)L\big) \geq \operatorname{vol}(K)^{\lambda} \operatorname{vol}(L)^{1-\lambda}, \quad \text{for all } \lambda \in [0,1]. \tag{1.4.2}$$

If we denote the characteristic functions of K and L by χ_K and χ_L, respectively, and if for a given $\lambda \in (0,1)$ the characteristic function of $\lambda K + (1-\lambda) L$ is denoted by χ_λ, then we have $\chi_\lambda(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \chi_K(\boldsymbol{x})\,\chi_L(\boldsymbol{y}) = \chi_K(\boldsymbol{x})^{\lambda}\,\chi_L(\boldsymbol{y})^{1-\lambda}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and (1.4.2) becomes

$$\int_{\mathbb{R}^n} \chi_\lambda(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \geq \left(\int_{\mathbb{R}^n} \chi_K(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{\lambda} \left(\int_{\mathbb{R}^n} \chi_L(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{1-\lambda}.$$

1.5 Theorem [Prékopa-Leindler Inequality]. [2] [3] *Let $\lambda \in (0,1)$ and let $f, g, h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (Lebesgue-)measurable functions with*

$$h\big(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}\big) \geq f(\boldsymbol{x})^{\lambda} g(\boldsymbol{y})^{1-\lambda}, \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \tag{1.5.1}$$

and let f, g be (Lebesgue-)integrable. Then

$$\int_{\mathbb{R}^n} h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \geq \left(\int_{\mathbb{R}^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{\lambda} \left(\int_{\mathbb{R}^n} g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{1-\lambda}. \tag{1.5.2}$$

Proof. Without loss of generality we may assume that $f, g \neq 0$ and we amy also assume that they are bounded; otherwise, we consider for an arbitrary integer m the function $\min\{f, m\}$ or $\min\{g, m\}$ instead of f or g and apply Beppo Levi's [4] monotone convergence theorem. Since

$$\frac{h\big(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}\big)}{(\sup f)^{\lambda}(\sup g)^{1-\lambda}} \geq \left(\frac{f(\boldsymbol{x})}{\sup f}\right)^{\lambda} \left(\frac{g(\boldsymbol{y})}{\sup g}\right)^{1-\lambda}$$

we may also assume that $\sup f = \sup g = 1$. We show the result by induction on the dimension n.

Let $n = 1$. For a measurable function $\psi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ and a $t \geq 0$ we consider the super level set $[\psi \geq t] = \big\{x \in \mathbb{R} : \psi(x) \geq t\big\}$, and we write $|\psi \geq t|$ to denote the volume of the set $[\psi \geq t]$. Observe that the super level sets are measurable since ψ is measurable and $\int_{\mathbb{R}} \psi(x)\,\mathrm{d}x = \int_0^{\infty} |\psi \geq t|\,\mathrm{d}t$.

If $f(x) \geq t$ and $g(y) \geq t$, then by (1.5.1) we also have $h\big(\lambda x + (1-\lambda)y\big) \geq t$ and so

$$\lambda[f \geq t] + (1-\lambda)[g \geq t] \subseteq [h \geq t].$$

For $t \in [0,1)$ the sets on the left-hand side are non-empty, and thus, the (general) Brunn-Minkowski theorem for compact sets [Theorem 5.16, SkriptWS12]

[2] András Prékopa, 1929
[3] László Leindler, 1935
[4] Beppo Levi, 1875–1961

in the case $n = 1$ gives $|h \geq t| \geq \lambda |f \geq t| + (1-\lambda)|g \geq t|$ and so we obtain

$$\begin{aligned}\int_{\mathbb{R}} h(x)\ \mathrm{d}x \geq \int_0^1 |h \geq t|\ \mathrm{d}t &\geq \lambda \int_0^1 |f \geq t|\ \mathrm{d}t + (1-\lambda)\int_0^1 |g \geq t|\ \mathrm{d}t \\ &= \lambda \int_0^\infty f(x)\ \mathrm{d}x + (1-\lambda)\int_0^\infty g(x)\ \mathrm{d}x \\ &\geq \left(\int_{\mathbb{R}} f(x)\ \mathrm{d}x\right)^{\lambda}\left(\int_{\mathbb{R}} g(x)\ \mathrm{d}x\right)^{1-\lambda},\end{aligned}$$

where the last inequality follows from the arithmetic-geometric mean inequality. This proves the case $n = 1$.

Now let $n > 1$. As usual we identify $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \mathbb{R}$ and for $\boldsymbol{z} \in \mathbb{R}^{n-1}$, $s \in \mathbb{R}$ we define $h_s(\boldsymbol{z}) = h(\boldsymbol{z}, s)$, $f_s(\boldsymbol{z}) = f(\boldsymbol{z}, s)$ and $g_s(\boldsymbol{z}) = g(\boldsymbol{z}, s)$ on $\mathbb{R}^{n-1}$. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n-1}$, $\alpha, \beta \in \mathbb{R}$ and $\gamma = \lambda\alpha + (1-\lambda)\beta$. Then

$$\begin{aligned}h_\gamma\big(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}\big) &= h\big(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}, \lambda\alpha + (1-\lambda)\beta\big) \\ &= h\big(\lambda(\boldsymbol{x}, \alpha) + (1-\lambda)(\boldsymbol{y}, \beta)\big) \\ &\geq f(\boldsymbol{x}, \alpha)^{\lambda} g(\boldsymbol{y}, \beta)^{1-\lambda} = f_\alpha(\boldsymbol{x})^{\lambda} g_\beta(\boldsymbol{y})^{1-\lambda}.\end{aligned}$$

Thus, by our inductive argument we get

$$\int_{\mathbb{R}^{n-1}} h_\gamma(\boldsymbol{z})\ \mathrm{d}\boldsymbol{z} \geq \left(\int_{\mathbb{R}^{n-1}} f_\alpha(\boldsymbol{z})\ \mathrm{d}\boldsymbol{z}\right)^{\lambda}\left(\int_{\mathbb{R}^{n-1}} g_\beta(\boldsymbol{z})\ \mathrm{d}\boldsymbol{z}\right)^{1-\lambda}.$$

With

$$H(\gamma) = \int_{\mathbb{R}^{n-1}} h_\gamma(\boldsymbol{z})\ \mathrm{d}\boldsymbol{z}, \quad F(\alpha) = \int_{\mathbb{R}^{n-1}} f_\alpha(\boldsymbol{z})\ \mathrm{d}\boldsymbol{z}, \quad G(\beta) = \int_{\mathbb{R}^{n-1}} g_\beta(\boldsymbol{z})\ \mathrm{d}\boldsymbol{z},$$

this becomes $H\big(\lambda\alpha + (1-\lambda)\beta\big) \geq F(\alpha)^{\lambda} G(\beta)^{1-\lambda}$. Hence we may apply the case $n = 1$ to these functions and get the desired result:

$$\begin{aligned}\int_{\mathbb{R}^n} h(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}^{n-1}} h_\gamma(\boldsymbol{z})\ \mathrm{d}\boldsymbol{z}\right)\mathrm{d}\gamma \\ &= \int_{\mathbb{R}} H(\gamma)\ \mathrm{d}\gamma \geq \left(\int_{\mathbb{R}} F(\alpha)\ \mathrm{d}\alpha\right)^{\lambda}\left(\int_{\mathbb{R}} G(\beta)\ \mathrm{d}\beta\right)^{1-\lambda} \\ &= \left(\int_{\mathbb{R}^n} f(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x}\right)^{\lambda}\left(\int_{\mathbb{R}^n} g(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x}\right)^{1-\lambda}.\end{aligned}$$

□

1.6 Remark. *The Prékopa-Leindler inequality can be extended to m functions: Let $h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $f_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be measurable functions, $1 \leq i \leq m$, and let $\lambda_i > 0$ with $\sum_{i=1}^m \lambda_i = 1$ such that*

$$h\left(\sum_{i=1}^m \lambda_i\,\boldsymbol{x}_i\right) \geq \prod_{i=1}^m f_i(\boldsymbol{x}_i)^{\lambda_i},$$

and let f_i be integrable, $1 \le i \le m$. Then

$$\int_{\mathbb{R}^n} h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \ge \prod_{i=1}^{m} \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right)^{\lambda_i}.$$

1.7 Remark. *Hölder's-inequality*

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})\,g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \le \left(\int_{\mathbb{R}^n} f(\boldsymbol{x})^p\,\mathrm{d}\boldsymbol{x} \right)^{1/p} \left(\int_{\mathbb{R}^n} g(\boldsymbol{x})^q\,\mathrm{d}\boldsymbol{x} \right)^{1/q}.$$

for integrable functions $f, g : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ and real numbers $p, q > 1$ with $1/p + 1/q = 1$, may be regarded as a counterpart to the Prékopa-Leindler inequality in the special setting $p = 1/\lambda$, $q = 1/(1-\lambda)$, $0 < \lambda < 1$, and with respect to the functions $f^{1/p}, g^{1/q}$:

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})^{\lambda}\,g(\boldsymbol{x})^{1-\lambda}\,\mathrm{d}\boldsymbol{x} \le \left(\int_{\mathbb{R}^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right)^{\lambda} \left(\int_{\mathbb{R}^n} g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right)^{1-\lambda}.$$

2 Applications of the inequalities of Brascamp-Lieb and Barthe

2.1 Remark. *Let $f_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be integrable functions, $1 \leq i \leq m$, and let $\lambda_i \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$. If we set*

$$h(\boldsymbol{y}) = \sup\left\{\prod_{i=1}^m f_i(\boldsymbol{x}_i)^{\lambda_i} : \boldsymbol{y} = \sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i,\, \boldsymbol{x}_i \in \mathbb{R}^n\right\}$$

then $h\left(\sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i\right) \geq \prod_{i=1}^m f_i(\boldsymbol{x}_i)^{\lambda_i}$, for all $\boldsymbol{x}_i \in \mathbb{R}^n$, $1 \leq i \leq m$. Hence, Prékopa-Leindler inequality (1.5.2) (cf. Remark 1.6) can also be equivalently written in the form

$$\int_{\mathbb{R}^n}^{*} \sup\left\{\prod_{i=1}^m f_i(\boldsymbol{x}_i)^{\lambda_i} : \boldsymbol{y} = \sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i,\, \boldsymbol{x}_i \in \mathbb{R}^n\right\} \mathrm{d}\boldsymbol{y} \geq \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{\lambda_i}.$$

Here $\int_{\mathbb{R}^n}^* g(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$ denotes the outer integral, which for non-negative functions $g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is defined as

$$\int_{\mathbb{R}^n}^{*} g(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \inf\left\{\int_{\mathbb{R}^n} u(\boldsymbol{y})\mathrm{d}\boldsymbol{y} : u \geq g, u \text{ measureable}\right\}.$$

2.2 Theorem* [Brascamp-Lieb, Barthe]. *Let $c_i > 0$ and $n_i \in \mathbb{N}$, $1 \leq i \leq m$, satisfying $\sum_{i=1}^m c_i\, n_i = n$. Let $M_i \in \mathbb{R}^{n_i \times n}$, $1 \leq i \leq m$, with $\mathrm{rg}(M_i) = n_i$, and let*

$$\alpha = \inf\left\{\frac{\det\left(\sum_{i=1}^m c_i\, M_i^\intercal A_i\, M_i\right)}{\prod_{i=1}^m (\det A_i)^{c_i}} : A_i \in \mathbb{R}^{n_i \times n_i} \text{ positive definite}\right\}.$$

Let $f_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ be integrable functions. The BRASCAMP-LIEB *inequality states*

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(M_i\, \boldsymbol{x})^{c_i}\right) \mathrm{d}\boldsymbol{x} \leq \frac{1}{\sqrt{\alpha}} \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(\boldsymbol{x}_i)\,\mathrm{d}\boldsymbol{x}_i\right)^{c_i}, \tag{BL-I}$$

and the BARTHE *inequality states*

$$\int_{\mathbb{R}^n}^{*} \sup\left\{\prod_{i=1}^m f_i(\boldsymbol{x}_i)^{c_i} : \boldsymbol{y} = \sum_{i=1}^m c_i\, M_i^\intercal \boldsymbol{x}_i,\, \boldsymbol{x}_i \in \mathbb{R}^{n_i}\right\} \mathrm{d}\boldsymbol{y} \geq \sqrt{\alpha} \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(\boldsymbol{x}_i)\,\mathrm{d}\boldsymbol{x}_i\right)^{c_i}. \tag{B-I}$$

2.3 Corollary [Hölder and Prékopa-Leindler inequalities]. *Let $f_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be integrable functions and let $\lambda_i > 0$, $1 \leq i \leq m$, with $\sum_{i=1}^m \lambda_i = 1$. Then*

$$\begin{aligned}\int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(\boldsymbol{x})^{\lambda_i}\right) \mathrm{d}\boldsymbol{x} &\leq \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{\lambda_i} \\ &\leq \int_{\mathbb{R}^n}^{*} \sup\left\{\prod_{i=1}^m f_i(\boldsymbol{x}_i)^{\lambda_i} : \boldsymbol{y} = \sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i,\, \boldsymbol{x}_i \in \mathbb{R}^n\right\} \mathrm{d}\boldsymbol{y}.\end{aligned}$$

Proof. We start with Hölder's inequality for which we set $n_i = n$, $c_i = \lambda_i$ and $M_i = \mathrm{I}_n$, $1 \le i \le m$. The (BL-I) gives

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(\boldsymbol{x})^{\lambda_i} \right) \mathrm{d}\boldsymbol{x} \le \frac{1}{\sqrt{\alpha}} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right)^{\lambda_i},$$

and it remains to prove

$$1 \le \alpha = \inf \left\{ \frac{\det\left(\sum_{i=1}^m \lambda_i A_i\right)}{\prod_{i=1}^m (\det A_i)^{\lambda_i}} : A_i \in \mathbb{R}^{n\times n} \text{ positive definite} \right\}.$$

Applying Minkowski's Theorem [9.4, SkriptWS12] we get

$$\det \left(\sum_{i=1}^m \lambda_i A_i \right)^{1/n} \ge \sum_{i=1}^m \big(\det(\lambda_i A_i)\big)^{1/n} = \sum_{i=1}^m \lambda_i (\det A_i)^{1/n} \ge \prod_{i=1}^m (\det A_i)^{\lambda_i/n},$$

where the last inequality follows from the arithmetic/geometric-mean inequality. Therefore $\alpha \ge 1$.

In fact, taking $A_i = \mathrm{I}_n$ for all $i = 1, \dots, m$ shows $\alpha = 1$, and Prékopa-Leindler's inequality follows immeaditely from (B-I). □

2.4 Corollary [Young inequality]. *Let $p_i > 0$, $1 \le i \le 3$, with $1/p_1 + 1/p_2 + 1/p_3 = 2$, and let $f_i : \mathbb{R}^n \to \mathbb{R}$, $1 \le i \le 3$, be integrable functions. Then*

$$\int_{\mathbb{R}^n} f_1(\boldsymbol{x}) \left(\int_{\mathbb{R}^n} f_2(\boldsymbol{x}-\boldsymbol{y}) \, f_3(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \le c^n \prod_{i=1}^3 \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x})^{p_i} \, \mathrm{d}\boldsymbol{x} \right)^{1/p_i},$$

where $c = c_{p_1} c_{p_2} c_{p_3}$ and $c_p = p^{1/p}/q^{1/q}$, with q such that $1/q + 1/p = 1$.

Proof. We apply (BL-I) to the functions $f_i^{p_i}$, $c_i = 1/p_i$, $n_i = n$ and to the matrices $M_i \in \mathbb{R}^{n\times 2n}$, $1 \le i \le 3 = m$, given by $M_1(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x}$, $M_2(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x}-\boldsymbol{y}$ and $M_3(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{y}$. Then $\sum_{i=1}^m c_i n_i = \sum_{i=1}^3 n/p_i = 2n$ and

$$\int_{\mathbb{R}^{2n}} f_1(M_1(\boldsymbol{x},\boldsymbol{y})) \, f_2(M_2\,(\boldsymbol{x},\boldsymbol{y}) \, f_3(M_3(\boldsymbol{x},\boldsymbol{y})) \, \mathrm{d}(\boldsymbol{x},\boldsymbol{y}) \le \frac{1}{\sqrt{\alpha}} \prod_{i=1}^3 \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x})^{p_i} \, \mathrm{d}\boldsymbol{x} \right)^{1/p_i}.$$

The computation of the constant α is quite involved and we omit it here. □

2.5 Lemma. *Let $c_i > 0$ and let $\boldsymbol{v}_i \in S^{n-1}$, $1 \le i \le m$, $m \ge n$, such that $\sum_{i=1}^m c_i \, (\boldsymbol{v}_i \boldsymbol{v}_i^\intercal) = \mathrm{I}_n$. Then*

$$\inf \left\{ \frac{\det\left(\sum_{i=1}^m c_i \, \alpha_i \, (\boldsymbol{v}_i \boldsymbol{v}_i^\intercal)\right)}{\prod_{i=1}^m \alpha_i^{c_i}} : \alpha_i \in \mathbb{R}_{>0} \right\} = 1.$$

Proof. Let the infimum on the left hand side be denoted by α. Setting $\alpha_i = 1$, $1 \le i \le m$, shows $\alpha \le 1$, and in the following we prove $\alpha \ge 1$. Let

$\boldsymbol{w}_i = \sqrt{c_i}\boldsymbol{v}_i$, $1 \le i \le m$. Then $\mathrm{I}_n = \sum_{i=1}^m c_i\,(\boldsymbol{v}_i\boldsymbol{v}_i^\intercal) = \sum_{i=1}^m(\boldsymbol{w}_i\boldsymbol{w}_i^\intercal)$, and thus $1 = \det\big(\sum_{i=1}^m(\boldsymbol{w}_i\boldsymbol{w}_i^\intercal)\big)$. With $W = (\boldsymbol{w}_1, \cdots, \boldsymbol{w}_m) \in \mathbb{R}^{n\times m}$ we get

$$\begin{aligned}\boldsymbol{e}_k^\intercal\left(\sum_{i=1}^m(\boldsymbol{w}_i\boldsymbol{w}_i^\intercal)\right)\boldsymbol{e}_l &= \sum_{i=1}^m(\boldsymbol{e}_k)^\intercal(\boldsymbol{w}_i\boldsymbol{w}_i^\intercal)\boldsymbol{e}_l = \sum_{i=1}^m(\boldsymbol{e}_k^\intercal\boldsymbol{w}_i)(\boldsymbol{w}_i^\intercal\boldsymbol{e}_l)\\ &= \sum_{i=1}^m w_{ik}w_{il} = \boldsymbol{e}_k^\intercal(W\,W^\intercal)\boldsymbol{e}_l\end{aligned}$$

for any $k, l \in \{1,\dots,n\}$, and therefore $\sum_{i=1}^m(\boldsymbol{w}_i\boldsymbol{w}_i^\intercal) = W\,W^\intercal$. For each $I \subset \{1,\dots,m\}$ with $\#I = n$, we write $\beta_I = \big(\det(\boldsymbol{w}_i : i \in I)\big)^2$. The Cauchy-Binet formula[5] gives

$$1 = \det(WW^\intercal) = \sum_{\#I=n}\beta_I. \tag{2.5.1}$$

In the same way we see that

$$\begin{aligned}\det\left(\sum_{i=1}^m\alpha_i(\boldsymbol{w}_i\boldsymbol{w}_i^\intercal)\right) &= \sum_{\#I=n}\big(\det\big(\sqrt{\alpha_i}\,\boldsymbol{w}_i : i\in I\big)\big)^2\\ &= \sum_{\#I=n}\left(\prod_{i\in I}\alpha_i\right)\beta_I = \sum_{\#I=n}\alpha_I\beta_I,\end{aligned}$$

with $\alpha_I = \prod_{i\in I}\alpha_i$. In view of (2.5.1) we may apply the arithmetic/geometric-mean inequality and get

$$\det\left(\sum_{i=1}^m\alpha_i(\boldsymbol{w}_i\boldsymbol{w}_i^\intercal)\right) \ge \prod_{\#I=n}\alpha_I^{\beta_I}.$$

The exponent of a fixed α_i in the right-hand side is given by

$$\begin{aligned}\sum_{i\in I,\#I=n}\beta_I &= \sum_{\#I=n}\beta_I - \sum_{i\notin I,\#I=n}\beta_I\\ &= 1 - \det\left(\sum_{j=1,j\ne i}^m\boldsymbol{w}_j\,\boldsymbol{w}_j^\intercal\right) = 1 - \det\big(\mathrm{I}_n - \boldsymbol{w}_i\boldsymbol{w}_i^\intercal\big).\end{aligned}$$

Since the eigenvalues of the matrix $\mathrm{I}_n - (\boldsymbol{w}_i\boldsymbol{w}_i^\intercal)$ are $1 - |w_i|^2$ (with eigenvector $\boldsymbol{w}_i$) and $(n-1)$-times 1 (with eigenvectors orthogonal to $\boldsymbol{w}$) the expoent of a fixed α_i is $|\boldsymbol{w}_i|^2 = c_i$. Hence $\prod_{\#I=n}\alpha_I^{\beta_I} = \prod_{i=1}^m\alpha_i^{c_i}$ and so

$$\frac{\det\big(\sum_{i=1}^m c_i\,\alpha_i\,(\boldsymbol{v}_i\boldsymbol{v}_i^\intercal)\big)}{\prod_{i=1}^m\alpha_i^{c_i}} = \frac{\det\big(\sum_{i=1}^m\alpha_i\,(\boldsymbol{w}_i\boldsymbol{w}_i^\intercal)\big)}{\prod_{i=1}^m\alpha_i^{c_i}} \ge \frac{\prod_{\#I=n}\alpha_I^{\beta_I}}{\prod_{i=1}^m\alpha_i^{c_i}} = 1,$$

which shows that $\alpha \ge 1$, as required. □

[5]In its general form it says, that for $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$ we have $\det AB = \sum_{I\in\binom{m}{n}}\det A^I\det B_I$, where A^I, B_I are the $\times n$ submatrices of A and B with row and column indices in I, respectively. For $m < n$ it just gives/means $\det AB = 0$.

2.6 Theorem. *Let $f_i : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be measurable functions, and let $c_i > 0$ and $\boldsymbol{v}_i \in S^{n-1}$, $1 \leq i \leq m$, such that $\sum_{i=1}^m c_i \, (\boldsymbol{v}_i \boldsymbol{v}_i^\intercal) = \mathrm{I}_n$. Then*

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(\langle \boldsymbol{v}_i, \boldsymbol{x} \rangle)^{c_i} \right) \mathrm{d}x \leq \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t) \, \mathrm{d}t \right)^{c_i}$$
$$\leq \int_{\mathbb{R}^n}^* \sup \left\{ \prod_{i=1}^m f_i(t_i)^{c_i} : \boldsymbol{y} = \sum_{i=1}^m c_i \, t_i \, \boldsymbol{v}_i, \, t_i \in \mathbb{R} \right\} \mathrm{d}y. \tag{2.6.1}$$

For $m = n$ we have equality in both inequalities and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ build an orthonormal basis.

Proof. Since $\mathrm{I}_n = \sum_{i=1}^m c_i \, (\boldsymbol{v}_i \boldsymbol{v}_i^\intercal)$ we may write $\boldsymbol{x} = \sum_{i=1}^m c_i \, \langle \boldsymbol{v}_i, \boldsymbol{x} \rangle \, \boldsymbol{v}_i$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and thus, in particular, we have $\mathrm{lin} \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\} = \mathbb{R}^n$. First we assume $m = n$. For any $k \in \{1, \ldots, n\}$ we can write $\boldsymbol{v}_k = \sum_{i=1}^n c_i \, \langle \boldsymbol{v}_i, \boldsymbol{v}_k \rangle \, \boldsymbol{v}_i$ and thus we get

$$0 = \sum_{i \neq k} c_i \, \langle \boldsymbol{v}_i, \boldsymbol{v}_k \rangle \, \boldsymbol{v}_i + (c_k - 1) \boldsymbol{v}_k,$$

which implies that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ has to be an orthonormal basis and $c_i = 1$, $1 \leq i \leq n$. Hence we get with $V = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)^\intercal$

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^n f_i(\langle \boldsymbol{v}_i, \boldsymbol{x} \rangle) \right) \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} \left(\prod_{i=1}^n f_i(\boldsymbol{e}_i^\intercal V \boldsymbol{x}) \right) \mathrm{d}\boldsymbol{x}$$
$$= \int_{\mathbb{R}^n} \left(\prod_{i=1}^n f_i(z_i) \right) |\det V| \, \mathrm{d}\boldsymbol{z} = \prod_{i=1}^n \left(\int_{\mathbb{R}} f_i(t) \, \mathrm{d}t \right).$$

For the upper bound we observe that for $\boldsymbol{y} = \sum_{i=1}^n t_i \boldsymbol{v}_i$ we have $t_i = \langle \boldsymbol{v}_i, \boldsymbol{y} \rangle$ and hence we get as before

$$\int_{\mathbb{R}^n}^* \sup \left\{ \prod_{i=1}^n f_i(t_i) : \boldsymbol{y} = \sum_{i=1}^n t_i \, \boldsymbol{v}_i, \, t_i \in \mathbb{R} \right\} \mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^n} \left(\prod_{i=1}^n f_i(\langle \boldsymbol{v}_i, \boldsymbol{y} \rangle) \right) \mathrm{d}\boldsymbol{y}$$
$$= \prod_{i=1}^n \left(\int_{\mathbb{R}} f_i(t) \, \mathrm{d}t \right).$$

So the case $m = n$ gives the desired equality. The general case $m \geq n$ is reduced to the (BL-I) and (B-I). Since

$$n = \mathrm{tr} \mathrm{I}_n = \sum_{i=1}^m c_i \, \mathrm{tr}(\boldsymbol{v}_i \boldsymbol{v}_i^\intercal) = \sum_{i=1}^m c_i \, |\boldsymbol{v}_i|^2 = \sum_{i=1}^m c_i$$

and $\mathrm{lin} \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\} = \mathbb{R}^n$, we may apply (BL-I) and (B-I) with $n_i = 1$ and $M_i = \boldsymbol{v}_i^\intercal$, for $1 \leq i \leq m$, and get

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(\langle \boldsymbol{v}_i, \boldsymbol{x} \rangle)^{c_i} \right) \mathrm{d}\boldsymbol{x} \leq \frac{1}{\sqrt{\alpha}} \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t) \, \mathrm{d}t \right)^{c_i} \qquad \text{and}$$
$$\int_{\mathbb{R}^n}^* \sup \left\{ \prod_{i=1}^m f_i(t_i)^{c_i} : \boldsymbol{y} = \sum_{i=1}^m c_i \, t_i \, v_i, \, t_i \in \mathbb{R} \right\} \mathrm{d}\boldsymbol{y} \geq \sqrt{\alpha} \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t) \, \mathrm{d}t \right)^{c_i},$$

with

$$\alpha = \inf\left\{ \frac{\det\left(\sum_{i=1}^m c_i\, \alpha_i (\boldsymbol{v}_i \boldsymbol{v}_i^\intercal)\right)}{\prod_{i=1}^m \alpha_i^{c_i}} : \alpha_i > 0 \right\}.$$

Lemma 2.5 shows that $\alpha = 1$, which concludes the proof. □

2.7 Theorem*. *Let $n \geq 2$, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in S^{n-1}$ be pairwise different and let $c_i > 0$ such that $\sum_{i=1}^m c_i\, \boldsymbol{v}_i \boldsymbol{v}_i^\intercal = I_n$. Let $f_i : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be measureable function which are neither identical 0 or Gaussian, i.e., of the type $c\,\mathrm{e}^{-\gamma x}$ for positive numbers c, γ. If we have equality in one of the inequalities (2.6.1), then $m = n$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ build an orthonormal basis.*

2.8 Proposition.

i) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function. Via polar coordinates we get*

$$\begin{aligned} \int_{\mathbb{R}^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= \int_0^\infty \int_{S^{n-1}} r^{n-1} f(r\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}r \\ &= n\kappa_n \int_0^\infty \int_{S^{n-1}} r^{n-1} f(r\boldsymbol{u})\,\mathrm{d}\sigma(\boldsymbol{u})\,\mathrm{d}r. \end{aligned}$$

Here $\mathrm{d}\boldsymbol{u}$ denotes the rotational invariant area measure on the sphere of total mass $\mathrm{F}(S^{n-1}) = n\mathrm{vol}\,(B_n)$, and $\mathrm{d}\sigma(\boldsymbol{u})$ its normalization to a probability measure, i.e., $\int_{S^{n-1}} 1\,\mathrm{d}\sigma(\boldsymbol{u}) = 1$.

ii) *Let $K \in \mathcal{K}^n$ with $\boldsymbol{0} \in K$. For $\boldsymbol{u} \in S^{n-1}$ let $r_K(\boldsymbol{u}) = \max\{\rho \geq 0 : \rho\,\boldsymbol{u} \in K\}$ be its* radial function. *Then*

$$\mathrm{vol}\,(K) = \kappa_n \int_{S^{n-1}} r_K(\boldsymbol{u})^n\,\mathrm{d}\sigma(\boldsymbol{u}).$$

iii) *Let $K \in \mathcal{K}_0^n$. Then*

$$\mathrm{vol}\,(K) = \kappa_n \int_{S^{n-1}} |\boldsymbol{u}|_K^{-n}\,\mathrm{d}\sigma(\boldsymbol{u}).$$

iv) [6] *Let $K \in \mathcal{K}_0^n$ and $1 \leq p < \infty$. Then*

$$\mathrm{vol}\,(K) = \frac{1}{\Gamma\left(\frac{n}{p}+1\right)} \int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}|_K^p}\,\mathrm{d}\boldsymbol{x}.$$

v)

$$\kappa_n = \mathrm{vol}\,(B_n) = \frac{\pi^{n/2}}{\Gamma\big((n/2)+1\big)} \approx \left(\frac{2\pi e}{n}\right)^{n/2} \frac{1}{\sqrt{\pi n}}.$$

[6] It is $\Gamma(x) = \int_0^\infty \mathrm{e}^{-t} t^{x-1} \mathrm{d}t$. In particular, $\Gamma(t+1) = t\Gamma(t)$, $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$.

Proof. i) is just a coordinate transformation. For ii) we apply i) to the characteristic function χ_K of K

$$\begin{aligned}\mathrm{vol}\,(K) &= \int_{\mathbb{R}^n} \chi_K(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = n\kappa_n \int_0^\infty \int_{S^{n-1}} \chi_K(r\boldsymbol{u}) r^{n-1}\,\mathrm{d}\sigma(\boldsymbol{u})\,\mathrm{d}r \\ &= n\kappa_n \int_{S^{n-1}} \left(\int_0^{r_K(\boldsymbol{u})} r^{n-1}\,\mathrm{d}r \right) \mathrm{d}\sigma(\boldsymbol{u}) = \kappa_n \int_{S^{n-1}} r_K(\boldsymbol{u})^n\,\mathrm{d}\sigma(\boldsymbol{u}).\end{aligned}$$

In iii) K is 0-symmetric, and so we have $r_K(\boldsymbol{u}) = 1/\left|\boldsymbol{u}\right|_K$ (it was an exercise). Hence, in this case iii) is just a reformulation of ii).

For iv) we observe that

$$\begin{aligned}\int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}|_K^p}\,\mathrm{d}\boldsymbol{x} &= n\,\kappa_n \int_0^\infty \int_{S^{n-1}} r^{n-1}\mathrm{e}^{-|r\boldsymbol{u}|_K^p}\,\mathrm{d}\sigma(\boldsymbol{u})\,\mathrm{d}r \\ &= n\,\kappa_n \int_{S^{n-1}} \int_0^\infty r^{n-1}\mathrm{e}^{-|r\boldsymbol{u}|_K^p}\,\mathrm{d}r\,\mathrm{d}\sigma(\boldsymbol{u}) \\ &\overset{s=r|\boldsymbol{u}|_K}{=} n\,\kappa_n \left(\int_{S^{n-1}} |u|_K^{-n}\,\mathrm{d}\sigma(u) \right) \left(\int_0^\infty \mathrm{e}^{-s^p} s^{n-1}\,\mathrm{d}s \right) \\ &\overset{\text{ii)}}{=} n\,\mathrm{vol}\,(K) \int_0^\infty \mathrm{e}^{-s^p} s^{n-1}\,\mathrm{d}s \\ &\overset{t=s^p}{=} \frac{n}{p}\,\mathrm{vol}\,(K) \int_0^\infty \mathrm{e}^{-t}\, t^{n/p-1}\,\mathrm{d}t \\ &= \frac{n}{p}\,\mathrm{vol}\,(K)\,\Gamma\left(\frac{n}{p}\right) = \mathrm{vol}\,(K)\,\Gamma\left(\frac{n}{p}+1\right).\end{aligned}$$

For v) let $f : \mathbb{R}^n \to \mathbb{R}$ be the function $f(\boldsymbol{x}) = \mathrm{e}^{-|\boldsymbol{x}|^2}$. Then by iv) we have

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \kappa_n\,\Gamma\left(\frac{n}{2}+1\right).$$

On the other hand we may write

$$\begin{aligned}\int_{\mathbb{R}^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^n} \mathrm{e}^{\sum_{i=1}^n x_i^2}\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} \left(\prod_{i=1}^n \mathrm{e}^{-x_i^2} \right) \mathrm{d}\boldsymbol{x} = \prod_{i=1}^n \int_{-\infty}^\infty \mathrm{e}^{-x_i^2}\,\mathrm{d}x_i \\ &= 2^n \prod_{i=1}^n \int_0^\infty \mathrm{e}^{-x_i^2}\,\mathrm{d}x_i = \prod_{i=1}^n \int_0^\infty \mathrm{e}^{-t} t^{-1/2}\,\mathrm{d}t \\ &= \Gamma\left(\frac{1}{2}\right)^n = \left(\sqrt{\pi}\right)^n.\end{aligned}$$

Hence together with *Stirling's formula* $n! \approx \sqrt{2\pi n}(n/e)^n$ we get

$$\mathrm{vol}\,(B_n) = \kappa_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \approx \frac{\pi^{n/2}}{\sqrt{2\pi}\sqrt{\frac{n}{2}}\left(\frac{n}{2e}\right)^{n/2}} = \left(\frac{2\pi e}{n}\right)^{n/2} \frac{1}{\sqrt{\pi n}}.$$

□

2.9 Remark.

i) *The radius of an n-dimensional ball of radius 1 is about $\sqrt{\frac{n}{2\pi \mathrm{e}}}$.*

ii) *Let $C_n = [-1,1]^n$ the cube of volume 2^n. From*

$$2^n = \mathrm{vol}\,(C_n) = \kappa_n \int_{S^{n-1}} r_{C_n}(\boldsymbol{u})^n \,\mathrm{d}\sigma(\boldsymbol{u})$$

we get that the average value of the radial function of C_n is $2/\kappa_n^{1/n} \approx \sqrt{2\,n/(\pi e)}$. Observe, that $1 \leq r_{C_n}(\boldsymbol{u}) \leq \sqrt{n}$.

iii) *For the crosspolytope ${C_n}^\star = \mathrm{conv}\,\{\pm \boldsymbol{e}_i : 1 \leq i \leq n\}$ we find*

$$\frac{2^n}{n!} = \mathrm{vol}\,(C_n^\star) = \kappa_n \int_{S^{n-1}} r_{C_n^\star}(\boldsymbol{u})^n \,\mathrm{d}\sigma(\boldsymbol{u})$$

and hence the average value of the radial function of the crosspolytope ${C_n}^\star$ is

$$\frac{2}{\left(\kappa_n\, n!\right)^{1/n}} \approx 2\frac{\sqrt{n}\,e}{\sqrt{2\pi e}\left(\sqrt{2\pi n}\right)^{1/n} n} \approx \sqrt{\frac{2e/\pi}{n}}.$$

Observe, that $\frac{1}{\sqrt{n}} \leq r_{{C_n}^\star}(\boldsymbol{u}) \leq 1$.

2.10 Lemma. *Let $c_i > 0$ and $\boldsymbol{v}_i \in S^{n-1}$, $1 \leq i \leq m$, such that $\sum_{i=1}^m c_i\, \boldsymbol{v}_i \boldsymbol{v}_i^\intercal = \mathrm{I}_n$. Moreover, let $\alpha_i > 0$, $1 \leq i \leq m$ and $1 \leq p < \infty$.*

i) *Let $K \in \mathcal{K}_0^n$ be the 0-symmetric convex body with gauge function*

$$|\boldsymbol{x}|_K = \left(\sum_{i=1}^m \alpha_i \big|\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\big|^p\right)^{1/p}.$$

Then

$$\mathrm{vol}\,(K) \leq 2^n \frac{\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} \prod_{i=1}^m \left(\frac{c_i}{\alpha_i}\right)^{c_i/p}.$$

Observe, if $V \in \mathbb{R}^{m\times n}$ denotes the matrix with rows $\boldsymbol{v}_i$ then $K = B_n^P \cap V\mathbb{R}^n$.

ii) *Let Z be the zonotope $Z = \sum_{i=1}^m \mathrm{conv}\,\{-\alpha_i \boldsymbol{v}_i, \alpha_i \boldsymbol{v}_i\}$.[7] Then*

Exercise: volume and support function

$$\mathrm{vol}\,(Z) \geq 2^n \prod_{i=1}^m \left(\frac{\alpha_i}{c_i}\right)^{c_i}.$$

Equality holds if and only if $m = n$, $c_i = 1$, and $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ form an orthonormal basis of $\mathbb{R}^n$, i.e., Z is an orthogonal box.

[7] The volume of such a zonotope Z is $\mathrm{vol}\,(Z) = 2^n \sum_{I \in \binom{m}{n}} |\det(v_{j_i} : i \in I)|$.

Proof. By Proposition 2.8 iv) we have

$$\int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}|_K^p}\,\mathrm{d}\boldsymbol{x} = \mathrm{vol}\,(K)\,\Gamma\left(\frac{n}{p}+1\right). \tag{2.10.1}$$

On the other hand, due to the definition of the gauge function we may evaluate the integral as

$$\int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}|_K^p}\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n}\left(\prod_{i=1}^m \mathrm{e}^{-\alpha_i|\langle \boldsymbol{v}_i,\boldsymbol{x}\rangle|^p}\right)\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n}\left(\prod_{i=1}^m f_i(\langle \boldsymbol{v}_i,\boldsymbol{x}\rangle)^{c_i}\right)\mathrm{d}\boldsymbol{x},$$

with $f_i(t) = \mathrm{e}^{-(\alpha_i/c_i)|t|^p}$. Now we apply the upper bound of Theorem 2.6 and get[8]

$$\begin{aligned}\int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}|_K^p}\,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^n}\left(\prod_{i=1}^m f_i(\langle \boldsymbol{v}_i,\boldsymbol{x}\rangle)^{c_i}\right)\mathrm{d}\boldsymbol{x} \le \prod_{i=1}^m\left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{c_i}\\ &= \prod_{i=1}^m\left(2\int_0^\infty \mathrm{e}^{-\frac{\alpha_i}{c_i}t^p}\,\mathrm{d}t\right)^{c_i} = \prod_{i=1}^m\left(2\left(\frac{c_i}{\alpha_i}\right)^{1/p}\Gamma\left(\frac{1}{p}+1\right)\right)^{c_i}\\ &= 2^n\Gamma\left(\frac{1}{p}+1\right)^n\prod_{i=1}^m\left(\frac{c_i}{\alpha_i}\right)^{c_i/p},\end{aligned}$$

since, by assumption, $\sum_{i=1}^m c_i = \sum_{i=1}^m c_i\,|v_i|^2 = \sum_{i=1}^m c_i\,\mathrm{tr}(v_i v_i^\intercal) = \mathrm{tr} \mathrm{I}_n = n$. Together with (2.10.1) we get the result.

ii) For $1\le i\le m$ let $f_i(t) = \chi_{[-\alpha_i/c_i,\alpha_i/c_i]}(t)$ be the characteristic functions of the intervals $[-\alpha_i/c_i,\alpha_i/c_i]$, i.e., $f_i(t)=1$ if and only if $c_i t\in[-\alpha_i,\alpha_i]$. Hence, if $\boldsymbol{z}\in\mathbb{R}^n$ satisfies

$$\sup\left\{\prod_{i=1}^m f_i(t_i)^{c_i} : \boldsymbol{z} = \sum_{i=1}^m c_i\,t_i\,\boldsymbol{v}_i,\ t_i\in\mathbb{R}\right\} = 1,$$

then $\boldsymbol{z}$ admits a representation as $\boldsymbol{z} = \sum_{i=1}^m c_i\,t_i\,\boldsymbol{v}_i$ with $|c_i\,t_i|\le\alpha_i$, $1\le i\le m$. Thus $\boldsymbol{z}\in Z = \sum_{i=1}^m \mathrm{conv}\,\{-\alpha_i\boldsymbol{v}_i,\alpha_i\boldsymbol{v}_i\}$. With the upper bound in Theorem 2.6 we get

$$\begin{aligned}\mathrm{vol}\,(Z) &= \int_{\mathbb{R}^n}\chi_Z(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \ge \int_{\mathbb{R}^n}^{*}\sup\left\{\prod_{i=1}^m f_i(t_i)^{c_i} : \boldsymbol{z} = \sum_{i=1}^m c_i\,t_i\,\boldsymbol{v}_i,\ t_i\in\mathbb{R}\right\}\mathrm{d}\boldsymbol{z}\\ &\ge \prod_{i=1}^m\left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{c_i} = \prod_{i=1}^m\left(2\frac{\alpha_i}{c_i}\right)^{c_i} = 2^n\prod_{i=1}^m\left(\frac{\alpha_i}{c_i}\right)^{c_i},\end{aligned}$$

since, again, $\sum_{i=1}^m c_i = n$. The characterization of the equality case follows from Theorem 2.7. □

[8] Observe that $\int_0^\infty \mathrm{e}^{-\gamma t^p}\,\mathrm{d}t = \gamma^{-1/p}(1/p)\int_0^\infty \mathrm{e}^{-s}s^{1/p-1}\,\mathrm{d}s = \gamma^{-1/p}\Gamma(1/p+1)$ for $\gamma>0$.

2.11 Theorem. *Let $K \in \mathcal{K}_0^n$ and let B_n be the maximum volume ellipsoid contained in K.*[9] *Then*

$$\mathrm{vol}\,(K) \le \mathrm{vol}\,(C_n) = 2^n.$$

Moreover equality holds if and only if K is a cube of edge length 2, i.e., $K = \big\{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle| \le 1,\, 1 \le i \le n\big\}$, where $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ form an orthonormal basis.

Proof. Since B_n is the maximum volume ellipsoid contained in K, we know by Theorem [9.11, SkriptWS12] that there exist $\boldsymbol{v}_i \in S^{n-1} \cap \mathrm{bd}\, K$ and $\lambda_i > 0$, $1 \le i \le m$, with $\sum_{i=1}^m \lambda_i = n$ and $\mathrm{I}_n = \sum_{i=1}^m \lambda_i\,(\boldsymbol{v}_i \boldsymbol{v}_i^\intercal)$. Let $U = \big\{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle| \le 1,\, 1 \le i \le m\big\}$. Clearly $K \subseteq U$ and so $\mathrm{vol}\,(K) \le \mathrm{vol}\,(U)$. With $f_i = \chi_{[-1,1]}$, $1 \le i \le m$, we can write

$$U = \left\{\boldsymbol{x} \in \mathbb{R}^n : \prod_{i=1}^m f_i\big(\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\big)^{\lambda_i} = 1\right\}$$

and with lower bound in Theorem 2.6 we get

$$\mathrm{vol}\,(U) = \int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i\big(\langle v_i, x\rangle\big)^{\lambda_i}\right) \mathrm{d}x \le \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{\lambda_i} = 2^{\sum_{i=1}^m \lambda_i} = 2^n.$$

The characterization of the equality case follows from Theorem 2.7. □

2.12 Theorem. *Let $K \in \mathcal{K}^n$ and let B_n be the maximum volume ellipsoid contained in K. Then*

$$\mathrm{vol}\,(K) \le \mathrm{vol}\,(T_n) = \frac{(\sqrt{n})^n(\sqrt{n+1})^{n+1}}{n!},$$

where T_n is a regular simplex with inradius 1. Moreover, equality holds if and only if K is a regular simplex with inradius 1, i.e., $K = \big\{x \in \mathbb{R}^n : \langle \boldsymbol{v}_i, \boldsymbol{x}\rangle \le 1,\, 1 \le i \le n+1\big\}$ with $\boldsymbol{v}_i \in S^{n-1}$ and $\langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle = -\frac{1}{n}$ for $i \ne j$.

Proof. Since B_n is the maximum volume ellipsoid contained in K, we know by Theorem [9.14, SkriptWS12] that there exist $\boldsymbol{v}_i \in S^{n-1} \cap \mathrm{bd}\, K$ and $\lambda_i > 0$, $1 \le i \le m$ and $m \ge n+1$, with $\sum_{i=1}^m \lambda_i = n$, $\mathrm{I}_n = \sum_{i=1}^m \lambda_i\,(\boldsymbol{v}_i \boldsymbol{v}_i^\intercal)$ and $\sum_{i=1}^m \lambda_i\,\boldsymbol{v}_i = 0$. For $U = \big\{x \in \mathbb{R}^n : \langle \boldsymbol{v}_i, \boldsymbol{x}\rangle \le 1,\, 1 \le i \le m\big\}$ it holds $K \subseteq U$ and so $\mathrm{vol}\,(K) \le \mathrm{vol}\,(U)$. Now let

$$\boldsymbol{w}_i = \sqrt{\frac{n}{n+1}} \begin{pmatrix} -\boldsymbol{v}_i \\ \frac{1}{\sqrt{n}} \end{pmatrix} \in \mathbb{R}^{n+1} \quad \text{and} \quad c_i = \frac{n+1}{n}\lambda_i, \quad 1 \le i \le m.$$

Then $\boldsymbol{w}_i \in S^n$ for all $1 \le i \le m$, $\sum_{i=1}^m c_i = n+1$ and

$$\sum_{i=1}^m c_i(\boldsymbol{w}_i \boldsymbol{w}_i^\intercal) = \sum_{i=1}^m \lambda_i \begin{pmatrix} (\boldsymbol{v}_i \boldsymbol{v}_i^\intercal) & -\frac{1}{\sqrt{n}}\boldsymbol{v}_i \\ -\frac{1}{\sqrt{n}}\boldsymbol{v}_i^\intercal & \frac{1}{n} \end{pmatrix} = \mathrm{I}_{n+1}.$$

[9] According to Theorem [9.11, SkriptWS12] B_n is the uniquely determined ellipsoid of maximum volume in K if and only if there exist $\boldsymbol{u}_i \in S^{n-1} \cap \mathrm{bd}\, K$ and $\lambda_i > 0$, $1 \le i \le m$, with $n \le m \le n(n+1)/2$ such that $\mathrm{I}_n = \sum_{i=1}^m \lambda_i\,(\boldsymbol{u}_i \boldsymbol{u}_i^\intercal)$.

Now let

$$f_i : \mathbb{R} \to \mathbb{R}_{\geq 0} \quad \text{given by} \quad f_i(t) = \begin{cases} e^{-t} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases}$$

for all $1 \leq i \leq m$. Applying Theorem 2.6 we obtain

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} \left(\prod_{i=1}^{m} f_i\big(\langle \boldsymbol{w}_i, \boldsymbol{x} \rangle\big)^{c_i} \right) \mathrm{d}\boldsymbol{x} &\leq \prod_{i=1}^{m} \left(\int_{\mathbb{R}} f_i(t)\, \mathrm{d}t \right)^{c_i} \\ &= \prod_{i=1}^{m} \left(\int_0^{\infty} e^{-t}\, \mathrm{d}t \right)^{c_i} = 1. \end{aligned} \tag{2.12.1}$$

For the sake of brevity we write $F(\boldsymbol{x}) = \prod_{i=1}^{m} f_i\big(\langle \boldsymbol{w}_i, \boldsymbol{x} \rangle\big)^{c_i}$ and we decompose $\boldsymbol{x} = \binom{\boldsymbol{y}}{r} \in \mathbb{R}^{n+1}$ with $\boldsymbol{y} \in \mathbb{R}^n$, $r \in \mathbb{R}$. Then

$$\langle \boldsymbol{w}_i, \boldsymbol{x} \rangle = \frac{r}{\sqrt{n+1}} - \sqrt{\frac{n}{n+1}}\, \langle \boldsymbol{v}_i, \boldsymbol{y} \rangle .$$

Due to definition of $F(\boldsymbol{x})$ we have $F(\boldsymbol{x}) \neq 0$, if and only if

$$\langle \boldsymbol{v}_i, \boldsymbol{y} \rangle \leq \frac{r}{\sqrt{n}}, \quad 1 \leq i \leq m. \tag{2.12.2}$$

Since $\sum_{i=1}^{m} \lambda_i \boldsymbol{v}_i = 0$, $\lambda_i > 0$, we know that for each $\boldsymbol{y} \in \mathbb{R}^n \backslash \{0\}$ there exists $j \in \{1, \ldots, m\}$ such that $\langle \boldsymbol{v}_j, \boldsymbol{y} \rangle \geq 0$, and hence, if $r < 0$ (2.12.2) is never fulfilled. In the case $r \geq 0$, (2.12.2) is equivalent to $\boldsymbol{y} \in \frac{r}{\sqrt{n}} U$, and for such a $\boldsymbol{y} \in \frac{r}{\sqrt{n}} U$, $r \geq 0$, and $\boldsymbol{x} = \binom{\boldsymbol{y}}{r}$, we have

$$F(\boldsymbol{x}) = e^{-\sum_{i=1}^{m} c_i \langle \boldsymbol{w}_i, \boldsymbol{x} \rangle} = e^{-\sum_{i=1}^{m} c_i \frac{r}{\sqrt{n+1}} + \sum_{i=1}^{m} c_i \sqrt{\frac{n}{n+1}} \langle v_i, y \rangle} = e^{-r\sqrt{n+1}}.$$

Together with Fubini's theorem we get

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} F(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} &= \int_0^{\infty} \int_{\frac{r}{\sqrt{n}} U} F(\boldsymbol{y}, r)\, \mathrm{d}\boldsymbol{y}\, \mathrm{d}r = \int_0^{\infty} \int_{\frac{r}{\sqrt{n}} U} e^{-r\sqrt{n+1}}\, \mathrm{d}\boldsymbol{y}\, \mathrm{d}r \\ &= \int_0^{\infty} e^{-r\sqrt{n+1}} \left(\frac{r}{\sqrt{n}} \right)^n \mathrm{vol}\,(U)\, \mathrm{d}r \\ &= \mathrm{vol}\,(U) \left(\frac{1}{\sqrt{n}} \right)^n \int_0^{\infty} e^{-t} \left(\frac{t}{\sqrt{n+1}} \right)^n \frac{1}{\sqrt{n+1}}\, \mathrm{d}t \\ &= \frac{n!\, \mathrm{vol}\,(U)}{(\sqrt{n})^n (\sqrt{n+1})^{n+1}}. \end{aligned}$$

Together with (2.12.1), the result follows. The characterization of the equality case follows from Theorem 2.7; observe, if we have equality we have $m = n+1$ and the vectors $\boldsymbol{w}_i$ build an orthonormal basis in $\mathbb{R}^{n+1}$. Hence $\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = -\frac{1}{n}$ for $i \neq j$, which shows that U and thus K is a regular simplex. □

exercise?

2.13 Theorem [Reverse Isoperimetric Inequality]. *Let $K \in \mathcal{K}^n$, $\dim K = n$. There exists a regular affine transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\frac{\mathrm{F}(T\,K)^n}{\mathrm{vol}\,(T\,K)^{n-1}} \leq \frac{\mathrm{F}(T_n)^n}{\mathrm{vol}\,(T_n)^{n-1}} = n^n \frac{(\sqrt{n})^n (\sqrt{n+1})^{n+1}}{n!},$$

where T_n is a regular simplex. If K is o-symmetric then

$$\frac{\mathrm{F}(TK)^n}{\mathrm{vol}\,(TK)^{n-1}} \leq \frac{\mathrm{F}(C_n)^n}{\mathrm{vol}\,(C_n)^{n-1}} = (2\,n)^n,$$

where C_n is a cube. Both inequalities are best possible.

Proof. For the general case let the affine transformation T be chosen such that B_n is the maximum volume ellipsoid contained in TK. By the formula for the surface area given in Remark [5.31 iii), SkriptWS12], we obtain

$$\begin{aligned}\mathrm{F}(TK) &= \lim_{\lambda\to 0}\frac{\mathrm{vol}\,(TK+\lambda B_n)-\mathrm{vol}\,(TK)}{\lambda} \leq \lim_{\lambda\to 0}\frac{\mathrm{vol}\,(TK+\lambda TK)-\mathrm{vol}\,(TK)}{\lambda}\\ &= \lim_{\lambda\to 0}\frac{(1+\lambda)^n-1}{\lambda}\,\mathrm{vol}\,(TK) = n\,\mathrm{vol}\,(TK)\\ &= n\,\mathrm{vol}\,(TK)^{1/n}\mathrm{vol}\,(TK)^{(n-1)/n} \leq n\,\mathrm{vol}\,(T_n)^{1/n}\mathrm{vol}\,(TK)^{(n-1)/n},\end{aligned}$$

where the last inequality follows from Theorem 2.12 and T_n is here a regular simplex with inradius 1. Hence $\mathrm{F}(T_n) = n\mathrm{vol}\,(T_n)$ and we get

$$\frac{\mathrm{F}(TK)^n}{\mathrm{vol}\,(TK)^{n-1}} \leq n^n\mathrm{vol}\,(T_n) = \frac{\mathrm{F}(T_n)^n}{\mathrm{vol}\,(T_n)^{n-1}}.$$

Now suppose that S is an n-simplex. Then we have $r(S)\,\mathrm{F}(S) = n\mathrm{vol}\,(S)$ and hence we have

$$\frac{\mathrm{F}(TS)^n}{\mathrm{vol}\,(TS)^{n-1}} = n^n\frac{\mathrm{vol}\,(TS)}{r(TS)^n}$$

for any regular affine transformation T. Now for a given inradius the regular simplex has smallest volume and hence we conclude Exercise

$$\frac{\mathrm{F}(TS)^n}{\mathrm{vol}\,(TS)^{n-1}} = n^n\frac{\mathrm{vol}\,(TS)}{r(TS)^n} \geq n^n\mathrm{vol}\,(T_n) = \frac{\mathrm{F}(T_n)^n}{\mathrm{vol}\,(T_n)^{n-1}}.$$

Hence for a simplex the inequality can be improved.

The proof for symmetric convex bodies is analogous, but now we apply Theorem 2.11. □

2.14 Theorem. *Let $K \in \mathcal{K}^n$, and let $c_i > 0$ and $\boldsymbol{v}_i \in S^{n-1}$, for $1 \leq i \leq m$, be such that $\sum_{i=1}^m c_i\,(\boldsymbol{v}_i\boldsymbol{v}_i^\intercal) = \mathrm{I}_n$. Then*

$$\mathrm{vol}\,(K)^{n-1} \leq \prod_{i=1}^m \mathrm{vol}_{\,n-1}(K|\boldsymbol{v}_i^\perp)^{c_i}.$$

Here $K|\boldsymbol{v}_i^\perp$ denotes the orthogonal projection of K onto the orthogonal complement of $\boldsymbol{v}_i$, i.e., the hyperplane $H(\boldsymbol{v}_i, 0)$.

Moreover, equality holds if and only if $m = n$, $\boldsymbol{v}_1,\dots,\boldsymbol{v}_n$ are an orthonormal basis and K is an orthogonal box with respect to $\boldsymbol{v}_i$, i.e., $K = \{\boldsymbol{x} \in \mathbb{R}^n : \alpha_i \leq \langle \boldsymbol{v}_i, \boldsymbol{x}\rangle \leq \beta_i, 1 \leq i \leq n\}$ and $\alpha_i < \beta_i \in \mathbb{R}$.

Proof. Let $\alpha_i = c_i/\mathrm{vol}_{n-1}(K|\boldsymbol{v}_i^{\perp})$, $1 \le i \le m$, and let Z be the zonotope $Z = \sum_{i=1}^m \mathrm{conv}\{-\alpha_i \boldsymbol{v}_i, \alpha_i \boldsymbol{v}_i\}$. Applying first Minkowski's inequality [Theorem 5.32, SkriptWS12], then [Corollary 5.27, SkriptWS12] and finally using the linearity of the mixed volumes [Lemma 5.23 iv), SkriptWS12], we get

$$\begin{aligned}\mathrm{vol}(Z)^{1/n}\mathrm{vol}(K)^{(n-1)/n} &\le \mathrm{V}(K, n-1; Z, 1)\\ &= \sum_{i=1}^m \mathrm{V}\big(K, n-1; \mathrm{conv}\{-\alpha_i\,\boldsymbol{v}_i, \alpha_i\,\boldsymbol{v}_i\}, 1\big)\\ &= \frac{2}{n}\sum_{i=1}^m \alpha_i \mathrm{vol}_{n-1}(K|\boldsymbol{v}_i^{\perp}) = \frac{2}{n}\sum_{i=1}^m c_i = 2.\end{aligned}$$

Therefore, $\mathrm{vol}(K)^{n-1} \le 2^n/\mathrm{vol}(Z)$. Finally, from Lemma 2.10 we get

$$\mathrm{vol}(Z) \ge 2^n \prod_{i=1}^m \left(\frac{\alpha_i}{c_i}\right)^{c_i} = 2^n \prod_{i=1}^m \left(\frac{1}{\mathrm{vol}_{n-1}(K|\boldsymbol{v}_i^{\perp})}\right)^{c_i},$$

and the result follows.

Now, if we have equality then by the equality case of the Minkowski inequality, K and Z are homothetic and from Lemma 2.10 we conclude $m = n$, $c_i = 1$, and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are an orthonormal basis. Hence K is an orthogonal box with edge directions $\boldsymbol{v}_i$. □

2.15 Corollary [Loomis-Whitney inequality]. *Let $K \in \mathcal{K}^n$. Then*

$$\mathrm{vol}(K)^{n-1} \le \prod_{i=1}^n \mathrm{vol}_{n-1}(K|\boldsymbol{e}_i^{\perp}),$$

and equality hols if and only if K is an orthogonal box with facets parallel to the coordinate axes.

Proof. Apply Theorem 2.14 with $m = n$, $\boldsymbol{v}_i = \boldsymbol{e}_i$ and $c_i = 1$, $1 \le i \le n$. □

2.16 Theorem [Meyer-Inequality]. *Let $K \in \mathcal{K}^n$ and let $H_i = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{e}_i, \boldsymbol{x}\rangle = 0\}$, $1 \le i \le n$. Then*

$$\mathrm{vol}(K)^{n-1} \ge \frac{(n-1)!}{n^{n-1}} \prod_{i=1}^n \mathrm{vol}_{n-1}(K \cap H_i),$$

and equality holds if and only if K is a generalized cross-polytope, i.e., $K = \mathrm{conv}\{-\alpha_i \boldsymbol{e}_i, \beta_i \boldsymbol{e}_i : \alpha_i, \beta_i > 0, 1 \le i \le n\}$.

Proof. Applying Steiner-Symmetrization to K with respect to the hyperplane H_1 yields a convex body K_1, say, with $\mathrm{vol}(K_1) = \mathrm{vol}(K)$, $\mathrm{vol}_{n-1}(K \cap H_1) \le \mathrm{vol}_{n-1}(K_1 \cap H_1) = \mathrm{vol}_{n-1}(K|\boldsymbol{e}_1^{\perp})$ and $\mathrm{vol}_{n-1}(K \cap H_i) = \mathrm{vol}_{n-1}(K_1 \cap H_i)$ for all $i > 1$. Hence it suffices to prove the inequality for K_1 and repeating this argument to all coordinate hyperplanes H_i shows that it suffices to prove the inequality for a body L which is symmetric to all coordinate hyperplanes, i.e., for $\boldsymbol{x} \in L$ we also have $(\pm x_1, \ldots, \pm x_n)^\intercal \in L$. Such a convex body is also called an unconditional convex body. In particular, we have $L \cap H_i = L|\boldsymbol{e}_i^\perp$ for such a body.

Let $L^s = L \cap \mathbb{R}^n_{\geq 0}$ and $L^s_i = L^s \cap H_i$. By the symmetries we have $\mathrm{vol}\,(L) = 2^n \mathrm{vol}\,(L^s)$ and $\mathrm{vol}\,(L \cap H_i) = 2^{n-1}\mathrm{vol}_{\,n-1}(L^s_i)$. Hence it suffices to verify the inequality for L^s and its sections L^s_i, for which we rewrite the inequality as

$$\mathrm{vol}\,(L^s) \leq \frac{n^{n-1}}{(n-1)!} \prod_{i=1}^n \frac{\mathrm{vol}\,(L^s)}{\mathrm{vol}\,(L^s_i)}. \tag{2.16.1}$$

For $\boldsymbol{y} \in L^s$ the intersection of the pyramid $\mathrm{conv}\,\{L^s_i, \boldsymbol{y}\}$ with another pyramid $\mathrm{conv}\,\{L^s_j, \boldsymbol{y}\}$, $j \neq i$, is contained in an $(n-1)$-dimensional subspace. Thus we have

$$\mathrm{vol}\,(L^s) \geq \sum_{i=1}^n \mathrm{vol}\,(\mathrm{conv}\,\{L^s_i, \boldsymbol{y}\}) = \frac{1}{n} \sum_{i=1}^n y_i \,\mathrm{vol}_{\,n-1}(L^s_i)$$

for every $\boldsymbol{y} \in L^s$, and so

$$L^s \subseteq \left\{ \boldsymbol{y} \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^n y_i \,\mathrm{vol}_{\,n-1}(L^s_i) \leq n \mathrm{vol}\,(L^s) \right\}.$$

Now the set on the left hand side is an n-simplex with vertices $\mathbf{0}$, $\frac{n\,\mathrm{vol}\,(L^s)}{\mathrm{vol}_{\,n-1}(L^s_i)} \boldsymbol{e}_i$, $1 \leq i \leq n$, and of volume

$$\frac{1}{n!} \prod_{i=1}^n \frac{n\,\mathrm{vol}\,(L^s)}{\mathrm{vol}_{\,n-1}(L^s_i)}.$$

and (2.16.1) is proven.

Now if we equality for K then the proof above shows that we also have equality for all the bodies K_i created by successively Steiner-symmetrizations at the planes H_i. In particular, we have equality for the unconditional body $K_n = L$, i.e., we have equality in (2.16.1). Hence L^s is a simplex with vertices $\mathbf{0}$ and at the coordinate axis, and thus $L = \mathrm{conv}\,\{\pm\gamma_i\, e_i : 1 \leq i \leq n\}$, with $\gamma_i \in \mathbb{R}_{>0}$, and $\mathrm{vol}\,(L) = \mathrm{vol}\,(K)$.

By the equality assumption we have $K_{n-1} \cap H_n = K_{n-1}|H_n = L \cap H_n$ and so we conclude $\mathrm{conv}\,\{L \cap H_n, -\alpha_n \boldsymbol{e}_n, \beta_n \boldsymbol{e}_n\} \subseteq K_{n-1}$ with $\alpha_n + \beta_n = 2\gamma_n$. Comparing the volumes of the two sets shows that we have indeed $K_{n-1} = \mathrm{conv}\,\{L \cap H_n, -\alpha_n \boldsymbol{e}_n, \beta_n \boldsymbol{e}_n\}$ and $\alpha_n, \beta_n > 0$. Hence $-\alpha_n \boldsymbol{e}_n, \beta_n \boldsymbol{e}_n \in K$. Repeating backwards this argument shows that K has to be a generalized crosspolytope. □

2.17 Theorem. *Let L be a k-dimensional linear subspace of $\mathbb{R}^n$. Then*

$$\mathrm{vol}_{\,k}(C_n \cap L) \leq 2^k \left(\frac{n}{k}\right)^{k/2}.$$

If k is a divisor of n then the inequality is best possible.

Proof. Here we want to apply Theorem 2.6 in the k-dimensional space L. To this end let P be the orthogonal projection of $\mathbb{R}^n$ onto L. Let $\boldsymbol{u}_i = P\boldsymbol{e}_i$ and $c_i = |\boldsymbol{u}_i|^2$, $1 \le i \le n$. If $c_j = 0$ then $L \subseteq \boldsymbol{e}_j^\perp$ and $C_n \cap L = (C_n \cap \boldsymbol{e}_j^\perp) \cap L$. Hence the problem is reduced to a cube of dimension one less and the result follows inductively.

So we assume $c_i > 0$ for $1 \le i \le n$. For $\boldsymbol{x} \in \mathbb{R}^n$ we have

$$P\boldsymbol{x} = P\left(\sum_{i=1}^n \langle \boldsymbol{x}, \boldsymbol{e}_i\rangle \boldsymbol{e}_i\right) = \sum_{i=1}^n \langle \boldsymbol{x}, \boldsymbol{e}_i\rangle P\boldsymbol{e}_i = \sum_{i=1}^n \langle \boldsymbol{x}, \boldsymbol{e}_i\rangle \boldsymbol{u}_i,$$

and since for $\boldsymbol{x} \in L$ we also have $\langle \boldsymbol{x}, \boldsymbol{e}_i\rangle = \langle \boldsymbol{x}, \boldsymbol{u}_i\rangle$ we get

$$\boldsymbol{x} = P\boldsymbol{x} = \sum_{i=1}^n \langle \boldsymbol{x}, \boldsymbol{u}_i\rangle \boldsymbol{u}_i = \sum_{i=1}^n (\boldsymbol{u}_i \boldsymbol{u}_i^\intercal)\boldsymbol{x} = \sum_{i=1}^n c_i\,(\boldsymbol{v}_i \boldsymbol{v}_i^\intercal)\boldsymbol{x},$$

with $\boldsymbol{v}_i = \boldsymbol{u}_i/\sqrt{c_i} \in L \cap S^{n-1}$. Hence, in L the unit vectors $\boldsymbol{v}_i$ give a decomposition of the identity and since $\mathrm{tr} P = k$ we get $\sum_{i=1}^n c_i = k$. Moreover we have

$$\begin{aligned} C_n \cap L &= \left\{x \in L : \left|\langle \boldsymbol{x}, \boldsymbol{e}_i\rangle\right| \le 1,\ 1 \le i \le n\right\} \\ &= \left\{x \in L : \left|\langle \boldsymbol{x}, \boldsymbol{v}_i\rangle\right| \le \frac{1}{\sqrt{c_i}},\ 1 \le i \le n\right\}, \end{aligned}$$

and with $f_i = \chi_{[-1/\sqrt{c_i}, 1/\sqrt{c_i}]}$ we get by Theorem 2.6 (applied in L)

$$\begin{aligned} \mathrm{vol}_k(C_n \cap L) &= \int_L \left(\prod_{i=1}^n f_i\big(\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\big)^{c_i}\right) \mathrm{d}\boldsymbol{x} \le \prod_{i=1}^n \left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{c_i} \\ &= \prod_{i=1}^n \left(\frac{2}{\sqrt{c_i}}\right)^{c_i} = 2^k \left(\prod_{i=1}^n c_i^{c_i}\right)^{-1/2}. \end{aligned}$$

The continuous function $\prod_{i=1}^n x_i^{x_i}$ attains a minimum on the compact set $\big\{\boldsymbol{x} \in \mathbb{R}^n_{\ge 0} : \sum_{i=1}^n x_i = k\big\}$. Inductively it can be shown that such a minimum satisfies $x_1 = \cdots = x_n$. Therefore, exercise

$$\prod_{i=1}^n c_i^{c_i} \ge \prod_{i=1}^n \left(\frac{\sum_{i=1}^n c_i}{n}\right)^{\sum_{i=1}^n c_i/n} = \left(\frac{k}{n}\right)^k.$$

Thus $\mathrm{vol}_k(C_n \cap L) \le 2^k (n/k)^{k/2}$.

Now suppose that k is a divisor of n, and let $m = n/k$. For each $1 \le i \le k$, let $\boldsymbol{u}_i = \sum_{j=1}^m \boldsymbol{e}_{(i-1)m+j}$. These vectors are pairwise orthogonal and with $L = \mathrm{lin}\,\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\}$ we get that

$$Q = \left\{\sum_{i=1}^n \lambda_i u_i : \lambda_i \in [-1, 1]\right\} \subset C_n \cap L.$$

Since Q is a k-dimensional cube with edge-length $\sqrt{m} = \sqrt{n/k}$ we have equality in this case. ∎

2.18 Theorem*. *Let L be a k-dimensional linear subspace of $\mathbb{R}^n$. Then*

$$\mathrm{vol}_k(C_n \cap L) \le 2^k\, 2^{(n-k)/2}.$$

If $k \ge n/2$ the inequality is best possible.

2.19 Theorem [Vaaler]. *Let L be an $(n-1)$-dimensional linear subspace of $\mathbb{R}^n$. Then*

$$\mathrm{vol}_{n-1}(C_n \cap L) \ge 2^{n-1},$$

and equality holds if and only if L is a coordinate hyperplane.

Proof. For an arbitrary but fixed $\boldsymbol{u} \in S^{n-1}$ let $H(t) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{x}, \boldsymbol{u}\rangle = t\}$, $t \in \mathbb{R}$, and let $f(t) = \mathrm{vol}_{n-1}\big(C_n \cap H(t)\big)$. Since C_n is a 0-symmetric convex body, we get by the Brunn-Minkowski theorem that $f(t) \le f(0)$ for all $t \in \mathbb{R}$.

$$F(t) = \int_0^t f(s)\,\mathrm{d}s,$$

we know $F(t) \le t\, f(0)$ for $t \ge 0$ where for $t > 0$ equality holds if and only if $f(t) = f(0)$ for all $t > 0$ with $f(t) \ne 0$; also observe that $F(t)' = f(t)$. Therefore we find

$$\begin{aligned}
\int_{C_n} \langle \boldsymbol{x}, \boldsymbol{u}\rangle^2\,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}} t^2\, f(t)\,\mathrm{d}t = \frac{2}{f(0)^2}\int_0^\infty \big(t\, f(0)\big)^2 f(t)\,\mathrm{d}t \\
&\ge \frac{2}{f(0)^2}\int_0^\infty F(t)^2 f(t)\,\mathrm{d}t = \frac{2}{3\, f(0)^2}\int_0^\infty \big[F(t)^3\big]'\,\mathrm{d}t \\
&= \frac{2}{3\, f(0)^2}\big[F(\infty)^3 - F(0)^3\big] = \frac{2}{3 f(0)^2}\left(\frac{\mathrm{vol}\,(C_n)}{2}\right)^3 \\
&= \frac{2}{3\, f(0)^2}\left(2^{n-1}\right)^3,
\end{aligned}$$

with equality if and only if $F(t) = t\, f(0)$ for all $t \in \mathbb{R}$ with $f(t) > 0$, i.e., if and only if $f(t) = f(0)$ for all $t \in \mathbb{R}$ with $f(t) > 0$. By the continuity of $f(t)$ on its support we conclude that this is equivalent to $\boldsymbol{u} \in \{\pm \boldsymbol{e}_i,\ 1 \le i \le n\}$.

On the other hand, we may evaluate the left hand side integral by

$$\begin{aligned}
\int_{C_n} \langle \boldsymbol{x}, \boldsymbol{u}\rangle^2\,\mathrm{d}\boldsymbol{x} &= \int_{C_n}\left(\sum_{i=1}^n u_i\, x_i\right)^2 \mathrm{d}\boldsymbol{x} \\
&= \int_{C_n}\left(\sum_{i=1}^n u_i^2 x_i^2 + 2\sum_{1\le i<j\le n} u_i u_j x_i x_j\right)\mathrm{d}\boldsymbol{x} \\
&= \sum_{i=1}^n \int_{C_n} u_i^2 x_i^2\,\mathrm{d}\boldsymbol{x} + 2\int_{C_n}\left(\sum_{1\le i<j\le n} u_i x_i\, u_j x_j\right)\mathrm{d}\boldsymbol{x} \qquad (2.19.1) \\
&= \sum_{i=1}^n \left(\int_{-1}^1 \cdots \int_{-1}^1 u_i^2 x_i^2\right)\mathrm{d}x_1 \ldots \mathrm{d}x_n \\
&= \frac{2}{3}\, 2^{n-1}\sum_{i=1}^n u_i^2 = \frac{2}{3}\, 2^{n-1}.
\end{aligned}$$

Comparing the two integrals gives $f(0) \geq 2^{n-1}$ with equality if and only if $\boldsymbol{u} \in \{\pm \boldsymbol{e}_i, 1 \leq i \leq n\}$. □

2.20 Remark [Busemann-Petty problem]. *The Busemann-Petty problem was the question whether for two* 0*-symmetric convex bodies* $K, L \in \mathcal{K}_0^n$*, the inequalities*

$$\mathrm{vol}_{n-1}\big(K \cap H(\boldsymbol{u},0)\big) \geq \mathrm{vol}_{n-1}\big(L \cap H(\boldsymbol{u},0)\big), \quad \text{for all } \boldsymbol{u} \in S^{n-1},$$

imply $\mathrm{vol}(K) \geq \mathrm{vol}(L)$*? Taking* $K = C_n$ *and* L *a ball of volume* 2^n*, Theorem 2.18 gives*

$$\mathrm{vol}(K \cap H(\boldsymbol{u},0)) \leq 2^{n-1}\sqrt{2} < \mathrm{vol}(L \cap H(\boldsymbol{u},0)) = 2^{n-1}\frac{\Gamma(\frac{n}{2}+1)^{\frac{n-1}{n}}}{\Gamma(\frac{n-1}{2}+1)} \to 2^{n-1}\sqrt{\mathrm{e}}.$$

And so the answer is No *for* $n \geq 10$*. In the meantime the problem has been completely solved: the answer is affirmative for* $n \leq 4$ *and negative for* $n \geq 5$.

2.21 Definition. *Let* $K \in \mathcal{K}^n$*. The set*

$$\Pi(K) = \big\{x \in \mathbb{R}^n : |\langle \boldsymbol{u}, \boldsymbol{x}\rangle| \leq \mathrm{vol}_{n-1}(K|\boldsymbol{u}^\perp), \text{ for all } \boldsymbol{u} \in S^{n-1}\big\}$$

is called the projection body *of* K.

2.22 Proposition. *Let* $K \in \mathcal{K}^n$*. Then*

i) $h\big(\Pi(K), \boldsymbol{u}\big) = \mathrm{vol}_{n-1}(K|\boldsymbol{u}^\perp)$.

ii) $\Pi(K)$ *is o-symmetric.*

iii) $\Pi(A\,K) = |\det A|\, A^{-\intercal}\Pi(K)$ *for* $A \in \mathrm{GL}(n, \mathbb{R})$*, and* $\Pi(\boldsymbol{t} + K) = \Pi(K)$ *for* $\boldsymbol{t} \in \mathbb{R}^n$.

2.23 Theorem. *Let* $K \in \mathcal{K}^n$*. There exists a regular affine transformation* $T : \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$\mathrm{vol}_{n-1}\big((TK)|\boldsymbol{u}^\perp\big)^n \geq \mathrm{vol}(TK)^{n-1}, \quad \text{for all } \boldsymbol{u} \in S^{n-1}.$$

Proof. According to Proposition 2.22 iii), we may find a linear transformation $A \in \mathrm{GL}(n, \mathbb{R})$ such that B_n is the maximum volume ellipsoid contained in $A^{-\intercal}\Pi(K) = |\det A|^{-1}\Pi(A\,K)$. In particular, $|\det A| B_n \subseteq \Pi(A\,K)$, and hence, with Proposition 2.22 ii) we get

$$\mathrm{vol}_{n-1}((A\,K)|\boldsymbol{u}^\perp) = h\big(\Pi(A\,K), \boldsymbol{u}\big) \geq |\det A|,$$

for all $\boldsymbol{u} \in S^{n-1}$.

By [Theorem 9.11, SkriptWS12] there exist $\boldsymbol{u}_i \in S^{n-1} \cap \mathrm{bd}\big(|\det A|^{-1}\Pi(AK)\big)$ and $\lambda_i > 0$, $1 \leq i \leq m$, such that $\mathrm{I}_n = \sum_{i=1}^n \lambda_i\, (\boldsymbol{u}_i \boldsymbol{u}_i^\intercal)$ and so $\sum_{i=1}^m \lambda_i = n$. In particular,

$$\mathrm{vol}_{n-1}((AK)|\boldsymbol{u}_i^\perp) = h\big(\Pi(AK), \boldsymbol{u}_i\big) = |\det A|.$$

Together with the Loomis-Whitney inequality (Theorem 2.14) we get

$$\begin{aligned}\operatorname{vol}(AK)^{n-1} &\le \prod_{i=1}^{m} \operatorname{vol}_{n-1}(AK|\boldsymbol{u}_i^{\perp})^{\lambda_i} = \prod_{i=1}^{m} |\det A|^{\lambda_i} \\ &= |\det A|^{\sum_{i=1}^{m}\lambda_i} = |\det A|^n \le \operatorname{vol}_{n-1}\big(AK|\boldsymbol{u}^{\perp}\big)^n,\end{aligned}$$

for all $\boldsymbol{u} \in S^{n-1}$. $\square$

Remarks on sections and projections of the unit cube

2.24 Theorem* [Vaaler, 1979]. *Let L be a k-dimensional linear subspace. Then* $\mathrm{vol}_k(C_n \cap L) \geq 2^k$.

2.25 Theorem*. *Let* $H = \{\boldsymbol{x} \in \mathbb{R}^n : \langle a, \boldsymbol{x} \rangle = b\}$, $\boldsymbol{a} \in S^{n-1}$, $b \in \mathbb{R}$, *and assume* $a_i \neq 0$, $1 \leq i \leq n$. *Then*

$$\mathrm{vol}\,(C_n \cap H) = \frac{1}{2\,(n-1)!} \left(\prod_{i=1}^{n} a_i \right)^{-1} \sum_{\boldsymbol{v} \in \mathrm{vert}\, C_n} (\langle \boldsymbol{a}, \boldsymbol{v} \rangle + b)^{n-1} \mathrm{sgn}\,(\langle \boldsymbol{a}, \boldsymbol{v} \rangle + b) \prod_{i=1}^{n} \boldsymbol{v}_i.$$

2.26 Theorem* [Chakerian& Filliman, 1986]. *Let L be a k-dimensional linear subspace. Then*

$$\mathrm{vol}_k(C_n | L) \leq 2^k \min \left\{ \frac{\kappa_{k-1}^{k}}{\kappa_k^{k-1}} \left(\frac{n}{k} \right)^{\frac{k}{2}}, \sqrt{\binom{n}{k}} \right\}.$$

2.27 Theorem* [McMullen, 1984]. *Let L be a k-dimensional linear subspace with orthogonal complement $L^{\perp}$. Then* $\mathrm{vol}_k(C_n | L) = \mathrm{vol}_{n-k}(C_n | L^{\perp})$.

3 A few remarks on isotropic positions

3.1 Notation. *Let $\mathcal{K}^n_c \subset \mathcal{K}^n$ be the set of all convex bodies with* $\mathrm{vol}\,(K) = 1$ *and centroid at the origin, i.e.,* $\int_K \boldsymbol{x}\,\mathrm{d}\boldsymbol{x} = \boldsymbol{0}$.

3.2 Definition [Isotropic Position]. *Let $K \in \mathcal{K}^n_c$. K is said to be* in isotropic position *if there exists a positive constant* L_K*, called the* isotropic constant, *such that for all $\boldsymbol{u} \in S^{n-1}$*

$$\int_K \langle \boldsymbol{u}, \boldsymbol{x}\rangle^2 \,\mathrm{d}\boldsymbol{x} = \mathrm{L}_K^2.$$

3.3 Definition [Moment matrix]. *For $K \in \mathcal{K}^n$ the $n \times n$-matrix*

$$A_K = \int_K \boldsymbol{x}\boldsymbol{x}^\intercal \,\mathrm{d}\boldsymbol{x}$$

is called the moment matrix *or* matrix of inertia *of K.*

3.4 Remark [Binet ellipsoid]. *Observe that for $\boldsymbol{u} \in \mathbb{R}^n$*

$$\boldsymbol{u}^\intercal A_K \boldsymbol{u} = \int_K \boldsymbol{u}^\intercal \boldsymbol{x}\boldsymbol{x}^\intercal \boldsymbol{u}\,\mathrm{d}\boldsymbol{x} = \int_K \langle \boldsymbol{u}, \boldsymbol{x}\rangle^2 \,\mathrm{d}\boldsymbol{x}.$$

Hence A_K is a positive definite symmetric matrix and it can be decomposed as $A_K = (D\,U)^\intercal\,(D\,U)$ where $U \in \mathrm{O}(n,\mathbb{R})$ is an orthogonal matrix and D is a diagonalmatrix. The ellipsoid $\mathcal{E}_K = (DU)^{-1}\,B_n$ has norm $|\boldsymbol{u}|_{\mathcal{E}_K} = \sqrt{\boldsymbol{u}^\intercal A_K \boldsymbol{u}}$ and it is called the Binet (fundamental) Ellipsoid *of K.*

3.5 Proposition.

i) *$K \in \mathcal{K}^n_c$ is in isotropic position if and only if $\mathcal{E}_K = (1/\mathrm{L}_K)\,B_n$ which is equivalent to $A_K = \mathrm{L}_K^2\,\mathrm{I}_n$.*

ii) *If $K \in \mathcal{K}^n_c$ is in isotropic position, then*

$$\int_K |\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x} = n\,\mathrm{L}_K^2.$$

iii) *Let $K \in \mathcal{K}^n$ and let $T \in \mathrm{GL}(n,\mathbb{R})$. Then*

$$A_{T\,K} = |\det T|\,T\,A_K\,T^\intercal.$$

iv) *If $K \in \mathcal{K}^n_c$ is in isotropic position then $T\,K$ is in isotropic position for all $T \in \mathrm{O}(n,\mathbb{R})$, and it is $L_K = L_{T\,K}$.*

Proof. K is isotropic if and only if $\boldsymbol{u}^\intercal(A_k - \mathrm{L}_K^2\,\mathrm{I}_n)\boldsymbol{u} = 0$ for all $\boldsymbol{u} \in S^{n-1}$. Since $A_k - \mathrm{L}_K^2\,\mathrm{I}_n$ is symmetric we conclude $A_K = \mathrm{L}_K^2\,\mathrm{I}_n$ which is equivalent to the fact that $\mathcal{E}_K$ is a ball of radius $1/\mathrm{L}_K$. This shwows i).

ii) is obvious since $\int_K x_i^2\,\mathrm{d}\boldsymbol{x} = \mathrm{L}_K^2$ for each $1 \le i \le n$, if K is in isotroic position. For iii) we note

$$\begin{aligned}\boldsymbol{u}^\intercal T\,A_K\,T^\intercal\boldsymbol{u} &= \int_K \langle T^\intercal\boldsymbol{u},\boldsymbol{x}\rangle^2\ \mathrm{d}\,\boldsymbol{x} = \int_K \langle \boldsymbol{u}, T\,\boldsymbol{x}\rangle^2\ \mathrm{d}\,\boldsymbol{x}\\ &= \frac{1}{|\det T|}\int_{TK} \langle \boldsymbol{u},\boldsymbol{x}\rangle^2\ \mathrm{d}\,\boldsymbol{x} = \frac{1}{|\det T|}\boldsymbol{u}^\intercal A_{TK}\boldsymbol{u}\end{aligned}$$

for all $\boldsymbol{u} \in S^{n-1}$. Hence $|\det T|\,T\,A_K\,T^\intercal = A_{TK}$. For $T \in \mathrm{O}(n,\mathbb{R})$ and K isotropic, iii) and i) gives $A_{TK} = T A_k T^\intercal = T\,\mathrm{L}_K^2\mathrm{I}_n\,T^\intercal = \mathrm{L}_K^2\,\mathrm{I}_n$ and so by i) TK is isotropic as well.

□

3.6 Remark. *Let* $\overline{C}_n = \frac{1}{2}C_n$, $\overline{B}_n = (1/\kappa_n)^{1/n}B_n$ *be the cube and ball of volume 1, respectively. Then*

$$\mathrm{L}_{\overline{C}_n} = \sqrt{\frac{1}{12}} \quad \textit{and} \quad \mathrm{L}_{\overline{B}_n} = \sqrt{\frac{1}{n+2}\,\frac{\Gamma(\frac{n}{2}+1)^{\frac{2}{n}}}{\pi}} \sim \sqrt{\frac{1}{2\pi\mathrm{e}}}.$$

Proof. First we note that for $K \in \mathcal{K}^n$ and $\lambda \in \mathbb{R}_{>0}$

$$\int_{\lambda K} \langle \boldsymbol{u},\boldsymbol{x}\rangle^2\ \mathrm{d}\boldsymbol{x} = \lambda^n \int_K \langle \boldsymbol{u},\lambda\,\boldsymbol{x}\rangle^2\ \mathrm{d}\boldsymbol{x} = \lambda^{n+2}\int_K \langle \boldsymbol{u},\boldsymbol{x}\rangle^2\ \mathrm{d}\boldsymbol{x}. \tag{3.6.1}$$

Form (2.19.1) we know already $\int_{C_n} \langle \boldsymbol{u},\boldsymbol{x}\rangle^2\ \mathrm{d}\boldsymbol{x} = \frac{2}{3}2^{n-1}$, and hence $\overline{C}_n$ is in isotropic position. With (3.6.1) we find $\mathrm{L}^2_{\overline{C}_n} = 1/12$.

Obviously, the ball $\overline{B}_n$ is in isotropic position, since the value of $\int_{\overline{B}_n} \langle \boldsymbol{u},\boldsymbol{x}\rangle^2\ \mathrm{d}\boldsymbol{x}$ is independent of the chosen direction $\boldsymbol{u}$. For the diagonal entries of the moment matrix we compute

$$\begin{aligned}\frac{1}{\kappa_{n-1}}\int_{B_n} x_i^2\,\mathrm{d}\boldsymbol{x} &= \int_{-1}^{1} t^2\sqrt{1-t^2}^{\,n-1}\ \mathrm{d}t \overset{t=\sin\phi}{=} \int_{-\pi/2}^{\pi/2} \sin^2\cos^n\ \mathrm{d}\phi\\ &= -\frac{\cos^2}{2}\sin\cos^{n-1}\Big|_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2}\frac{\cos^2}{2}\left(\cos^n - (n-1)\sin^2\cos^{n-2}\right)\ \mathrm{d}\phi\\ &= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos^2\cos^n\ \mathrm{d}\phi - \frac{n-1}{2}\int_{-\pi/2}^{\pi/2}\sin^2\cos^n\ \mathrm{d}\phi\\ &= \frac{1}{2}\int_{-\pi/2}^{\pi/2}(1-\sin^2)\cos^n\ \mathrm{d}\phi - \frac{n-1}{2}\int_{-\pi/2}^{\pi/2}\sin^2\cos^n\ \mathrm{d}\phi\end{aligned}$$

$$\begin{aligned}&= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos^n\ \mathrm{d}\phi - \frac{n}{2}\int_{-\pi/2}^{\pi/2}\sin^2\cos^n\ \mathrm{d}\phi\\ &= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos^n\ \mathrm{d}\phi - \frac{n}{2}\frac{1}{\kappa_{n-1}}\int_{B_n} x_i^2\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Hence we have found

$$\int_{B_n} x_i^2\,\mathrm{d}\boldsymbol{x} = \frac{\kappa_{n-1}}{n+2}\int_{-\pi/2}^{\pi/2}\cos^n\ \mathrm{d}\phi \overset{t=\sin\phi}{=} \frac{\kappa_{n-1}}{n+2}\int_{-1}^{1}\sqrt{1-t^2}^{\,n-1}\,\mathrm{d}t = \frac{\kappa_n}{n+2},$$

and thus

$$L_{B_n}^2 = \left(\frac{1}{\kappa_n}\right)^{\frac{n+2}{n}}\frac{\kappa_n}{n+2} = \frac{1}{n+2}\kappa_n^{-2/n} = \frac{1}{n+2}\frac{\Gamma(\frac{n}{2}+1)^{2/n}}{\pi}.$$

□

3.7 Lemma. *Let $K \in \mathcal{K}_c^n$. Then there exists a $T \in \mathrm{VP}(n,\mathbb{R})$ such that $T\,K$ is in isotropic position. Moreover, T is uniquely determined up to orthogonal transformations.*

Proof. Since A_K is a positive definite symmetric matrix there exists an $U \in \mathrm{O}(n,\mathbb{R})$ such that $U A_k U^\intercal = \mathrm{diag}\,(s_1,\ldots,s_n)$ with $s_i > 0$. Let $\sigma = (s_1\cdot\ldots\cdot s_n)^{1/n}$ and let $V = \mathrm{diag}\,(\sqrt{\sigma/s_1},\ldots,\sqrt{\sigma/s_n})$. Then $VU \in \mathrm{VP}(n,\mathbb{R})$ and

$$VU\,A_K\,(VU)^\intercal = \sigma \mathrm{I}_n$$

and with Proposition 3.5 iii) and i), we see that VUK is in isotropic position. Moreover, VU is uniquely determined up to orthogonal transformations, because $SA_KS^\intercal = \sigma\mathrm{I}_n = TA_KT^\intercal$ implies $S^{-1}S^{-\intercal} = T^{-1}T^{-\intercal}$ or $\mathrm{I}_n = (TS^{-1})(TS^{-1})^\intercal$. Thus $TS^{-1} \in \mathrm{O}(n,\mathbb{R})$ which means $T = U\,S$ for an $U \in \mathrm{O}(n,\mathbb{R})$. □

3.8 Notation. *Due to Lemma 3.7 and Proposition 3.5 iv) we may define for any $K \in \mathcal{K}^n$ with centroid at $\mathbf{0}$ independent of its position and its volume its isotropic constant L_K as $\mathrm{L}_{T\,K}$ where $T \in \mathrm{GL}(n,\mathbb{R})$ is chosen such that $\mathrm{vol}\,(TK) = 1$, i.e., $TK \in \mathcal{K}_c^n$, and $T\,K$ is in isotropic position.*

In particular, the so defined isotropic constant is an affine invariant functional, i.e., $L_{A\,K} = L_K$ for any $A \in \mathrm{GL}(n,\mathbb{R})$.

3.9 Proposition. *Let $K \in \mathcal{K}_c^n$. Then $\det A_K = \mathrm{L}_K^{2n}$ and (thus) $\mathrm{vol}\,(\mathcal{E}_K) = \mathrm{L}_K^{-n}\,\kappa_n$. Moreover, there exist $\hat{\boldsymbol{u}}, \tilde{\boldsymbol{u}} \in S^{n-1}$ such that*

$$\int_K \langle\hat{\boldsymbol{u}},\boldsymbol{x}\rangle^2\,\mathrm{d}\boldsymbol{x} \le \mathrm{L}_K^2 \le \int_K \langle\tilde{\boldsymbol{u}},\boldsymbol{x}\rangle^2\,\mathrm{d}\boldsymbol{x}.$$

Proof. According to Lemma 3.7 we can find a $T \in \mathrm{VP}(n,\mathbb{R})$ such that $T\,K$ is in isotropic position, i.e., $\det(A_{TK}) = L_K^{2n}$. By Proposition 3.5 iii) it is $A_{TK} = |\det T| T A_K T^\intercal$, and hence $\det A_K = \mathrm{L}_K^{2n}$. Due to the definition of $\mathcal{E}_K$ we have $\mathrm{vol}\,(\mathcal{E}_K) = \kappa_n\,\det(A_k)^{-1/2}$. Hence, together with Proposition 2.8 iii) we have

$$\mathrm{L}_K^{-n} = \frac{\mathrm{vol}\,(\mathcal{E}_K)}{\kappa_n} = \int_{S^{n-1}} (|\boldsymbol{u}|_{\mathcal{E}_K})^{-n}\,\mathrm{d}\sigma(\boldsymbol{u}).$$

Thus we can find $\tilde{\boldsymbol{u}} \in S^{n-1}$ such that (cf. Remark 3.4)

$$\mathrm{L}_K^2 \leq |\tilde{\boldsymbol{u}}|_{\mathcal{E}_K}^2 = \tilde{\boldsymbol{u}}^\intercal A_K \tilde{\boldsymbol{u}} = \int_K \langle \tilde{\boldsymbol{u}}, \boldsymbol{x} \rangle^2 \, \mathrm{d}\boldsymbol{x}.$$

In the same way we can find a lower bound on L_K. □

3.10 Theorem. *Let $K \in \mathcal{K}^n$. Then $\mathrm{L}_K \geq \mathrm{L}_{B_n}$, and equality holds if and only if K is an ellipsoid.*

Proof. Let $T \in \mathrm{GL}(n, \mathbb{R})$ such that $L = T\,K$ is in isotropic position and let $\overline{B}_n$ be the ball of volume 1. Since L and $\overline{B}_n$ have the same volume we conclude $\mathrm{vol}\,(L \setminus \overline{B}_n) = \mathrm{vol}\,(\overline{B}_n \setminus L)$. Hence

$$\int_{L \setminus \overline{B}_n} |\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x} \geq \left(\frac{1}{\kappa_n}\right)^{\frac{2}{n}} \mathrm{vol}\,(L \setminus \overline{B}_n) = \left(\frac{1}{\kappa_n}\right)^{\frac{2}{n}} \mathrm{vol}\,(\overline{B}_n \setminus L) \geq \int_{\overline{B}_n \setminus L} |\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x}$$

and so

$$\begin{aligned} n\,\mathrm{L}_L^2 &= \int_K |\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x} = \int_{L \cap \overline{B}_n} |\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x} + \int_{L \setminus \overline{B}_n} |\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x} \\ &\geq \int_{\overline{B}_n \cap L} |\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x} + \int_{\overline{B}_n \setminus L} |\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x} = \int_{\overline{B}_n} |\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x} = n\,\mathrm{L}_{\overline{B}_n}^2, \end{aligned}$$

and here we have equality if and only if $L = \overline{B}_n$, and hence if and only if K is an ellipsoid. □

3.11 Theorem. *Let $K \in \mathcal{K}_c^n$. K is in isotropic position if and only if*

$$\int_K |\boldsymbol{x}|^2 \;\mathrm{d}\,\boldsymbol{x} \leq \int_{TK} |\boldsymbol{x}|^2 \;\mathrm{d}\,\boldsymbol{x}$$

for all $T \in \mathrm{GL}(n, \mathbb{R})$, $|\det T| = 1$.

Proof. First we note that via eigenvalues and the arithmetic/geometric mean inequality we have for any matrix $T \in \mathrm{GL}(n, \mathbb{R})$, $|\det T| = 1$

$$\mathrm{tr}(TT^\intercal) \geq n\, \det(T\,T^\intercal)^{1/n} = n.$$

Let K be in isotropic position, then $A_K = \mathrm{L}_K^2\,\mathrm{I}_n$ and in view (cf. Proposition 3.5) we get for any $T \in \mathrm{GL}(n, \mathbb{R})$, $|\det T| = 1$

$$\begin{aligned} \int_K |\boldsymbol{x}|^2 \;\mathrm{d}\,\boldsymbol{x} &= n\,\mathrm{L}_K^2 \leq \mathrm{tr}(TT^\intercal)\,\mathrm{L}_K^2 = \mathrm{tr}(T\,A_K\,T^\intercal) \\ &= \mathrm{tr}(A_{TK}) = \int_{TK} |\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

For the reverse direction let $B \in \mathbb{R}^{n \times n}$. Then

$$\det(\mathrm{I}_n + \epsilon\, B) = 1 + \epsilon\,\mathrm{tr}(B) + \cdots + \epsilon^n \det B.$$

For all sufficiently small $|\epsilon|$ we have $\det(\mathrm{I}_n+\epsilon\, B)\neq 0$, and due to our assumption we find

$$\begin{aligned}\int_K |\boldsymbol{x}|^2 \;\mathrm{d}\,\boldsymbol{x} &\leq \int_{\left(\frac{1}{\det(\mathrm{I}_n+\epsilon\, B)}\right)^{1/n}(\mathrm{I}_n+\epsilon\, B)\, K} |\boldsymbol{x}|^2 \;\mathrm{d}\,\boldsymbol{x} \\ &= |\det(\mathrm{I}_n+\epsilon\, B)|^{\frac{-2}{n}} \int_K |\boldsymbol{x}+\epsilon B\boldsymbol{x}|^2 \;\mathrm{d}\,\boldsymbol{x}.\end{aligned}$$

Or equivalently,

$$\begin{aligned}|\det(\mathrm{I}_n+\epsilon\, B)|^{\frac{2}{n}} \int_K |\boldsymbol{x}|^2 \;\mathrm{d}\,\boldsymbol{x} &\leq \int_K |\boldsymbol{x}+\epsilon B\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x} \\ &= \int_K |\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x} + 2\epsilon \int_K \langle \boldsymbol{x}, B\boldsymbol{x}\rangle \,\mathrm{d}\boldsymbol{x} + \epsilon^2 \int_K |B\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{3.11.1}$$

Looking at the Taylor series of $|\det(\mathrm{I}_n+\epsilon\, B)|^{\frac{2}{n}}$ at $\epsilon=0$ gives

$$|\det(\mathrm{I}_n+\epsilon\, B)|^{\frac{2}{n}} = 1+\epsilon\,\frac{2}{n}\mathrm{tr}B + O(\epsilon^2)$$

for $\epsilon\to 0$, and hence (3.11.1) yields

$$\frac{\mathrm{tr}B}{n}\int_K |\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x} \leq \int_K \langle \boldsymbol{x}, B\boldsymbol{x}\rangle \,\mathrm{d}\boldsymbol{x}.$$

Replacing B by $-B$ shows that we must equality in the inequality above and, especially, for $B=\boldsymbol{e}_i\boldsymbol{e}_j^{\intercal}$ we get

$$\frac{\delta_{i,j}}{n}\int_K |\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x} = \int_K x_i\, x_j \,\mathrm{d}\boldsymbol{x} = (A_K)_{i,j}.$$

Hence, the moment matrix A_K is a multiple of I_n which means that K is in isotropic position. □

3.12 Corollary. *Let $K\in\mathcal{K}_c^n\cap\mathcal{K}_o^n$. Then $\mathrm{L}_K\leq c\sqrt{n}$ for an universal constant c.*

Proof. Let K be in isotropic position. Let $T\in\mathrm{GL}(n,\mathbb{R})$, $|\det T|=1$, such that the volume maximal ellipsoid in $T\,K$ is a ball of radius α, say. Observe that the volume maximal ellipsoid is centered at the origin, since $K\in\mathcal{K}_o^n$. By Corollary [9.16, SkriptWS12] we have $\alpha\, B_n\subseteq T\,K\subset\sqrt{n}\alpha\, B_n$ and thus with Theorem 3.11 we get

$$n\,\mathrm{L}_K^2\leq\int_{T\,K}|\boldsymbol{x}|^2 \;\mathrm{d}\boldsymbol{x}\leq\alpha^2 n.$$

Hence $\mathrm{L}_K\leq\alpha$. By the inclusion $\alpha\, B_n\subseteq T\,K$ we have $\alpha^n\leq \mathrm{vol}\,(T\,K)/\kappa_n$ and so

$$\alpha\leq\kappa_n^{-1/n}\leq c\sqrt{n},$$

for a suitable constant c. Hence we have found the bound $\mathrm{L}_K\leq c\sqrt{n}$. □

3.13 Conjecture [Isotropic Constant Conjecture]. *There exists an absolute constant $C > 0$ such that $\mathrm{L}_K \le C$ for any $K \in \mathcal{K}_o^n$ with $\mathrm{vol}\,(K) = 1$.*[10]

3.14 Proposition. *Let $K \in \mathcal{K}_o^n$ with $\mathrm{vol}\,(K) = 1$. Then (cf. Notation 3.8)*

$$\mathrm{L}_K^2 \cdot \mathrm{L}_{K^\star}^2 \le c\, n \frac{1}{\mathrm{vol}\,(K)\mathrm{vol}\,(K^\star)} \int_K \int_{K^\star} \langle \boldsymbol{y}, \boldsymbol{x}\rangle^2 \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}.$$

In particular, $\mathrm{L}_K \cdot \mathrm{L}_{K^\star} \le c\sqrt{n}$.[11]

Proof. Let

$$\Phi(K) = \frac{1}{\mathrm{vol}\,(K)\mathrm{vol}\,(K^\star)} \int_K \int_{K^\star} \langle \boldsymbol{y}, \boldsymbol{x}\rangle^2 \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}.$$

Oberve, that $\Phi(K) \le 1$. For every $T \in \mathrm{GL}(n, \mathbb{R})$ we have $(TK)^\star = T^{-\intercal}K^\star$ and thus $\Phi(K) = \Phi(TK)$. Hence for the proof we may assume that $K^\star$ is in isotropic position. Then

$$\begin{aligned}\left(\frac{1}{\mathrm{vol}\,(K)}\right)^{\frac{n+2}{n}} \int_K \int_{K^\star} \langle \boldsymbol{y}, \boldsymbol{x}\rangle^2 \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} &= \left(\frac{1}{\mathrm{vol}\,(K)}\right)^{\frac{n+2}{n}} \mathrm{L}_{K^\star}^2 \int_K |\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x} \\ &= \mathrm{L}_{K^\star}^2 \int_{K/\mathrm{vol}\,(K)^{1/n}} |\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x} \\ &\ge n\mathrm{L}_{K^\star}^2\mathrm{L}_K^2,\end{aligned}$$

by Theorem 3.11. On account of $\mathrm{vol}\,(K^\star) = 1$ we may write

$$n\,\mathrm{L}_{K^\star}^2\mathrm{L}_K^2 \le \left(\frac{1}{\mathrm{vol}\,(K)\mathrm{vol}\,(K^\star)}\right)^{\frac{2}{n}} \Phi(K).$$

By [12] we know $\mathrm{vol}\,(K)\mathrm{vol}\,(K^\star) \ge c^n/n!$ and thus $n\,\mathrm{L}_{K^\star}^2\mathrm{L}_K^2 \le c\,n^2\Phi(K)$. □

3.15 Theorem. *Let $K \in \mathcal{K}_c^n$ be in isotropic position. Then $K \subset (n+1)\,\mathrm{L}_K\,B_n$.* [13]

Proof. For $\boldsymbol{w} \in K$ we consider the local radial function $\rho_{\boldsymbol{w}} : S^{n-1} \to \mathbb{R}_{\ge 0}$ with $\rho_{\boldsymbol{w}}(\boldsymbol{v}) = \max\{\rho \ge 0 : \boldsymbol{w} + \rho\,\boldsymbol{v} \in K\}$. Let $\boldsymbol{u} \in S^{n-1}$. Then

$$\begin{aligned}\mathrm{L}_K^2 &= \int_K \langle \boldsymbol{u}, \boldsymbol{x}\rangle^2 \,\mathrm{d}\boldsymbol{x} \\ &= n\,\kappa_n \int_{S^{n-1}} \int_0^{\rho_{\boldsymbol{w}}(\boldsymbol{v})} t^{n-1} \langle \boldsymbol{u}, \boldsymbol{w} + t\,\boldsymbol{v}\rangle^2 \,\mathrm{d}t\,\mathrm{d}\sigma(\boldsymbol{v}) \\ &= n\,\kappa_n \int_{S^{n-1}} \int_0^{\rho_{\boldsymbol{w}}(\boldsymbol{v})} \Big(t^{n-1} \langle \boldsymbol{u}, \boldsymbol{w}\rangle^2 + 2\,t^n \langle \boldsymbol{u}, \boldsymbol{v}\rangle \langle \boldsymbol{u}, \boldsymbol{w}\rangle \\ &\qquad + t^{n+1} \langle \boldsymbol{u}, \boldsymbol{v}\rangle^2 \Big) \,\mathrm{d}t\,\mathrm{d}\sigma(\boldsymbol{v})\end{aligned}$$

---

[10] Bourgain, 1991 proved an upper bound of order $n^{1/4}\log(n)$ for o-symmetric bodies, which also holds true for arbitrary bodies (Paouris, 2000). The current general best bound is of size $n^{1/4}$ due to Klartag, 2006

[11] Here c is always an absolute constant, which may vary from statement to statement.

[12] Bourgain& Milman proved in 1987 that there exists an absolute constant c such that $\mathrm{vol}\,(K)\mathrm{vol}\,(K^\star) \ge c^n/n!$ for every $K \in \mathcal{K}_o^n$. See also Wikipedia Mahler volume.

[13] Kannan, Lovasz, Simonovits proved for an isotropic body $(n+2)/n\,\mathrm{L}_K\,B_n \subset K \subset \sqrt{n(n+2)}\,\mathrm{L}_K\,B_n$, which are best possible.

$$
\begin{aligned}
&= n\,\kappa_n \int_{S^{n-1}} \Big(\frac{\rho_{\boldsymbol{w}}(\boldsymbol{v})^n}{n} \langle \boldsymbol{u}, \boldsymbol{w}\rangle^2 + 2\frac{\rho_{\boldsymbol{w}}(\boldsymbol{v})^{n+1}}{n+1} \langle \boldsymbol{u}, \boldsymbol{v}\rangle \langle \boldsymbol{u}, \boldsymbol{w}\rangle \\
&\qquad\qquad + \frac{\rho_{\boldsymbol{w}}(\boldsymbol{v})^{n+2}}{n+2} \langle \boldsymbol{u}, \boldsymbol{v}\rangle^2 \Big) \,\mathrm{d}\sigma(\boldsymbol{v}) \\
&= n\,\kappa_n \int_{S^{n-1}} \Big(\frac{1}{n(n+1)^2} \rho_{\boldsymbol{w}}(\boldsymbol{v})^n \langle \boldsymbol{u}, \boldsymbol{w}\rangle^2 + \frac{\rho_{\boldsymbol{w}}(\boldsymbol{v})^n}{n} \times \\
&\qquad\qquad \left[\sqrt{\frac{n}{n+2}} \rho_{\boldsymbol{w}}(\boldsymbol{v}) \langle \boldsymbol{u}, \boldsymbol{v}\rangle + \frac{\sqrt{n(n+2)}}{n+1} \langle \boldsymbol{u}, \boldsymbol{w}\rangle \right]^2 \Big) \,\mathrm{d}\sigma(\boldsymbol{v}) \\
&\geq \frac{\langle \boldsymbol{u}, \boldsymbol{w}\rangle^2}{(n+1)^2} \kappa_n \int_{S^{n-1}} \rho_{\boldsymbol{w}}(\boldsymbol{v})^n \,\mathrm{d}\sigma(\boldsymbol{v}) = \frac{\langle \boldsymbol{u}, \boldsymbol{w}\rangle^2}{(n+1)^2},
\end{aligned}
$$

where the last identity comes from $\mathrm{vol}\,(K) = 1$ and Proposition 2.8 ii). Replacing $\boldsymbol{u}$ by $\boldsymbol{w}/|\boldsymbol{w}|$ shows that $|\boldsymbol{w}|^2 \leq (n+1)^2 \mathrm{L}_K^2$ for all $\boldsymbol{w} \in K$. □

3.16 Lemma. *Let $K \in \mathcal{K}^n$ with centroid in $\mathbf{0}$. Let $\boldsymbol{a} \in S^{n-1}$, and for $t \in \mathbb{R}$ let $f(t) = \mathrm{vol}_{n-1}(K \cap H(\boldsymbol{a}, t))$. Then*

$$
\max\{f(t) : t \in \mathbb{R}\} \leq \left(\frac{n+1}{n}\right)^{n-1} f(0).
$$

Observe that $\left(\frac{n+1}{n}\right)^{n-1} \to \mathrm{e}$ as n to infinity.

Proof. Let $H_t \cap K \neq \emptyset$ for $t \in [-a, b]$, say, with $a, b > 0$. Let $t^\star$ such that $f(t^\star) = \max\{f(t) : t \in [-a, b]\}$. With loss of generality we may assume $f(0) < f(t^\star)$ and $t^\star > 0$. For if $f(t^\star) = f(0)$ we are done and if $t^\star < 0$ we replace $\boldsymbol{a}$ by $-\boldsymbol{a}$. Since $\mathbf{0}$ is the centroid we know $0 = \int_{-a}^{b} t\, f(t)\mathrm{d}t$. Hence

$$
\int_{-a}^{0} (-t)\, f(t)\,\mathrm{d}t = \int_0^b t\, f(t)\,\mathrm{d}t \geq \int_0^{t^\star} t\, f(t)\,\mathrm{d}t. \tag{3.16.1}
$$

Now let $h : [-a, b] \to \mathbb{R}_{\geq 0}$ be the concave function $h(t) = f(t)^{\frac{1}{n-1}}$, and let $g : [-a, b] \to \mathbb{R}_{\geq 0}$ be the linear function

$$
g(t) = h(0) + \frac{t}{t^\star}\left(h(t^\star) - h(0)\right).
$$

Then $g(0) = h(0)$ and $g(t^\star) = h(t^\star)$ and due to the concavity of h we conclude

$$
g(t) \geq h(t), \quad t \in [-a, 0] \text{ and } g(t) \leq h(t), \quad t \in [0, t^\star].
$$

Let $g(-c) = 0$ for some $c \geq a$, and let $g(t) = \alpha(t + c)$ with $\alpha > 0$. Then with (3.16.1)

$$
\begin{aligned}
\int_{-c}^{0} (-t)\, g(t)^{n-1}\,\mathrm{d}t \geq \int_{-a}^{0} (-t)\, g(t)^{n-1}\,\mathrm{d}t \geq \int_{-a}^{0} (-t)\, h(t)^{n-1}\,\mathrm{d}t \\
\geq \int_0^{t^\star} t\, h(t)^{n-1}\,\mathrm{d}t \geq \int_0^{t^\star} t\, g(t)^{n-1}\,\mathrm{d}t.
\end{aligned}
$$

Thus

$$\begin{aligned} 0 &\geq \int_{-c}^{t^\star} t\, g(t)^{n-1}\,\mathrm{d}t = \alpha^{n-1}\int_{-c}^{t^\star} t(t+c)^{n-1}\,\mathrm{d}t \\ &= \alpha^{n-1}\int_0^{t^\star+c}(t-c)t^{n-1}\,\mathrm{d}t = \alpha^{n-1}\left(\frac{1}{n+1}t^{n+1}-\frac{c}{n}t^n\right)\Big|_0^{t^\star+c} \\ &= \alpha^{n-1}(t^\star+c)^n\left(\frac{t^\star+c}{n+1}-\frac{c}{n}\right). \end{aligned}$$

Hence $(t^\star+c)/c \leq (n+1)/n$, and so

$$\frac{f(t^\star)}{f(0)} = \frac{h(t^\star)^{n-1}}{h(0)^{n-1}} = \left(\frac{g(t^\star)}{g(0)}\right)^{n-1} = \left(\frac{t^\star+c}{c}\right)^{n-1} \leq \left(\frac{n+1}{n}\right)^{n-1}.$$

□

3.17 Theorem. *There exists absolute constants c_1, c_2 such that for every $K \in \mathcal{K}_c^n$ and for every $\boldsymbol{u} \in S^{n-1}$*

$$c_1\,\frac{1}{\mathrm{vol}_{n-1}(K\cap H(\boldsymbol{u},0))} \leq \left(\int_K (\boldsymbol{u}^\intercal\boldsymbol{x})^2\mathrm{d}\,\boldsymbol{x}\right)^{1/2} \leq c_2\,\frac{1}{\mathrm{vol}_{n-1}(K\cap H(\boldsymbol{u},0))}.$$

Proof. Let $\boldsymbol{u} \in S^{n-1}$ and for $t \in \mathbb{R}$ let $f(t) = \mathrm{vol}_{n-1}(K \cap H(\boldsymbol{u},t))$. Here $H(\boldsymbol{u},t) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u},\boldsymbol{x}\rangle = t\}$. Moreover, let $t^\star \in \mathbb{R}$ such that $f(t^\star) = \max\{f(t) : t \in \mathbb{R}\}$ and let

$$A = \int_{-\infty}^{0} f(t)\,\mathrm{d}t \text{ and } B = \int_0^\infty f(t)\,\mathrm{d}t.$$

We note that $1 = \mathrm{vol}\,(K) = A + B$ and

$$\int_K \langle \boldsymbol{u},\boldsymbol{x}\rangle^2\,\mathrm{d}\boldsymbol{x} = \int_{-\infty}^{\infty} t^2 f(t)\,\mathrm{d}t = \int_{-\infty}^{0} t^2 f(t)\,\mathrm{d}t + \int_0^\infty t^2 f(t)\,\mathrm{d}t. \tag{3.17.1}$$

First we prove the lower bound. To this end let $g(t) = f(t^\star)\chi_{[0,B/f(t^\star)]}$. Then we certainly have $f(t) \leq g(t)$ for all $t \in [0, B/f(t^\star)]$ and so

$$\int_0^s f(t)\,\mathrm{d}t \leq \int_0^s g(t)\,\mathrm{d}t$$

for all $s \in [0, B/f(t^\star)]$. Since we also have

$$\int_0^\infty g(t)\,\mathrm{d}t = B = \int_0^\infty f(t)\,\mathrm{d}t$$

we conclude

$$\int_s^\infty f(t)\,\mathrm{d}t \geq \int_s^\infty g(t)\,\mathrm{d}t$$

for all $s \geq 0$. Hence

$$\begin{aligned}\int_0^\infty t^2 f(t)\,\mathrm{d}t &= \int_0^\infty \int_0^t 2\,s\,f(t)\,\mathrm{d}s\,\mathrm{d}t = \int_0^\infty \left(\int_s^\infty 2\,s\,f(t)\,\mathrm{d}t\right)\mathrm{d}s \\ &\geq \int_0^\infty 2s\left(\int_s^\infty g(t)\,\mathrm{d}t\right)\mathrm{d}s = \int_0^\infty t^2 g(t)\,\mathrm{d}t \\ &= \int_0^{B/f(t^\star)} t^2\,f(t^\star)\,\mathrm{d}t = \frac{B^3}{3}\frac{1}{f(t^\star)^2}.\end{aligned}$$

In the same way it can be shown

$$\int_{-\infty}^0 t^2 f(t)\,\mathrm{d}t \geq \frac{A^3}{3}\frac{1}{f(t^\star)^2}.$$

Thus, in view of (3.17.1) we have found

$$\int_K \langle \boldsymbol{u}, \boldsymbol{x}\rangle^2\,\mathrm{d}\boldsymbol{x} \geq \frac{A^3 + B^3}{3 f(t^\star)^2}.$$

Since $A + B = \mathrm{vol}\,(K) = 1$ we conclude $A^3 + B^3 \geq 1/8 + 1/8 = 1/4$, and due to Lemma 3.16 we also know $f(t^\star) \leq \mathrm{e} f(0)$. Therefore,

$$\int_K \langle \boldsymbol{u}, \boldsymbol{x}\rangle^2\,\mathrm{d}\boldsymbol{x} \geq \frac{1}{12\,\mathrm{e}}\frac{1}{f(0)^2},$$

which shows the lower bound. For the upper bound we distinguish two cases. Firstly we assume that there exists an $\bar{t} > 0$ with the property that

$$f(\bar{t}) = f(0)/2. \tag{3.17.2}$$

By the Brunn-Minkowski inequality the function $f(t)$ is log-concave and so for every $\lambda \in [0, 1]$ we find

$$f(\lambda\,\bar{t}) = f((1-\lambda)\,0 + \lambda\,\bar{t}) \geq f(0)^{1-\lambda} f(\bar{t})^\lambda \geq f(\bar{t}).$$

Hence $f(t) \geq f(\bar{t})$ for all $t \in [0, \bar{t}]$ and so

$$1 \geq B = \int_0^{\bar{t}} f(t)\,\mathrm{d}t \geq \bar{t} f(\bar{t}) = \bar{t}\frac{f(0)}{2}.$$

This shows

$$\bar{t} \leq 2\frac{1}{f(0)}. \tag{3.17.3}$$

Now, for $t > \bar{t}$ we have

$$f(\bar{t}) = f\left((\bar{t}/t)t\right) = f((1 - \bar{t}/t)0 + (\bar{t}/t)t) \geq f(0)^{(1-\bar{t}/t)} f(t)^{\bar{t}/t}$$

Hence

$$f(t) \leq \left(f(\bar{t})\,f(0)^{(\bar{t}/t-1)}\right)^{\frac{t}{\bar{t}}} = \left(\frac{1}{2}\right)^{\frac{t}{\bar{t}}} f(0). \tag{3.17.4}$$

Using these bounds as well as Lemma 3.16 leads to

$$\begin{aligned}\int_0^\infty t^2\, f(t)\,\mathrm{d}t &= \int_0^{\bar t} t^2\, f(t)\,\mathrm{d}t + \int_{\bar t}^\infty t^2\, f(t)\,\mathrm{d}t\\ &\le f(t^\star)\int_0^{\bar t} t^2\,\mathrm{d}t \overset{(3.17.4)}{+} f(0)\int_{\bar t}^\infty t^2\, 2^{-t/\bar t}\,\mathrm{d}t\\ &\overset{\text{Lemma 3.16}}{\le} f(0)\left(\mathrm{e}\int_0^{\bar t} t^2\,\mathrm{d}t + \int_{\bar t}^\infty t^2\,2^{-t/\bar t}\,\mathrm{d}t\right) \qquad (3.17.5)\\ &= f(0)\left(\mathrm{e}\frac{\bar t^3}{3} \overset{u=t/\bar t}{+} \bar t^3\int_1^\infty u^2 2^{-u}\,\mathrm{d}u\right)\\ &= f(0)\,\bar t^3\cdot c \overset{(3.17.3)}{\le} 8\,c\frac{1}{f(0)^2}.\end{aligned}$$

Now assume the assumption (3.17.2) does not hold. Then by the continuity of f in the interior of its support we know $f(t) > f(0)/2$ for all $t > 0$ for which $f(t) \neq 0$. Let $\widehat{t} = \sup\{t > 0 : f(t) \neq 0\}$. Then $1 \ge B = \int_0^{\widehat{t}} f(t)\,\mathrm{d}t \ge \widehat{t} f(0)/2$ and so

$$\widehat{t} \le 2\frac{1}{f(0)}$$

and so

$$\int_0^\infty t^2 f(t)\,\mathrm{d}t = \int_0^{\widehat{t}} t^2 f(t)\,\mathrm{d}t \overset{\text{Lemma 3.16}}{\le} \mathrm{e}\frac{\widehat{t}^3}{3} f(0) \le \frac{8\mathrm{e}}{3}\frac{1}{f(0)^2}.$$

Together with (3.17.5) we have shown that there exists a constant $\bar c$, say, such that

$$\int_0^\infty t^2\, f(t)\,\mathrm{d}t \le \bar c\frac{1}{f(0)^2}.$$

Via the same argumentation we can bound the integral on the negative x-axis, i.e., $\int_{-\infty}^0 t^2\, f(t)\,\mathrm{d}t$ and we are done (cf.(3.17.1)). □

3.18 Corollary. *There exists absolute constants c_1, c_2 such that for every $K \in \mathcal{K}_c^n$ in isotropic position and for every $\boldsymbol{u} \in S^{n-1}$*

$$\frac{c_1}{\mathrm{L}_K} \le \mathrm{vol}_{n-1}(K \cap H(\boldsymbol{u}, 0)) \le \frac{c_2}{\mathrm{L}_K}.$$

Proof. Immediate consequence of Theorem 3.17. □

3.19 Conjecture [Slicing conjecture]. *There exists an absolute constant $c > 0$ with the property that for every $K \in \mathcal{K}_o^n$ with $\mathrm{vol}(K) = 1$ there exits an $\boldsymbol{v} \in S^{n-1}$ such that*

$$\mathrm{vol}_{n-1}(K \cap H(\boldsymbol{v}, 0)) \ge c.$$

3.20 Remark. *The Slicing conjecture 3.19 is equivalent to the Isotropic constant conjecture 3.13*

Proof. If the Slicing conjecture is true we get from the upper bound in Corollary 3.18 that the isotropic constant is bounded from above by an absolute constant. On the other hand, if the Isotropic constant conjecture holds true then we know by Proposition 3.9 that for every $K \in \mathcal{K}_o^n$ there exists a $\hat{\boldsymbol{u}} \in S^{n-1}$ that $\int_K \langle \hat{\boldsymbol{u}}, \boldsymbol{x} \rangle^2 \, \mathrm{d}\boldsymbol{x} \leq L_K^2 \leq c^2$. Hence from the lower bound in Theorem 3.17 we get $\mathrm{vol}_{n-1}(K \cap H(\hat{\boldsymbol{u}}, 0)) \geq c_1/c$. □

3.21 Notation. [14] *For $K \in \mathcal{K}^n$ with $\mathrm{vol}\,(K) = 1$ let*

$$S(K) = \int_K \cdots \int_K \mathrm{vol}\,(\mathrm{conv}\,\{\mathbf{0}, \boldsymbol{x}_1, \dots, \boldsymbol{x}_n\})^2 \, \mathrm{d}\boldsymbol{x}_n \dots \mathrm{d}\boldsymbol{x}_1.$$

3.22 Theorem. *Let $K \in \mathcal{K}^n$ with $\mathrm{vol}\,(K) = 1$. Then*

$$n!\, S(K) = \det(A_K) = \mathrm{L}_K^{2n}.$$

Proof. It is $\mathrm{vol}\,(\mathrm{conv}\,\{\mathbf{0}, \boldsymbol{x}_1, \dots, \boldsymbol{x}_n\}) = \frac{1}{n!} |\det(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)|$, and so

$$\begin{aligned}
n!^2\, S(K) &= \int_K \cdots \int_K |\det(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)|^2 \, \mathrm{d}\boldsymbol{x}_n \cdots \mathrm{d}\boldsymbol{x}_1 \\
&= \int_K \cdots \int_K \left(\sum_{\sigma \in S_n} \mathrm{sgn}\,\sigma \prod_{i=1}^n (\boldsymbol{x}_i)_{\sigma(i)} \right) \left(\sum_{\tau \in S_n} \mathrm{sgn}\,\tau \prod_{i=1}^n (\boldsymbol{x}_i)_{\tau(i)} \right) \mathrm{d}\boldsymbol{x}_n \cdots \mathrm{d}\boldsymbol{x}_1 \\
&= \int_K \cdots \int_K \left(\sum_{\sigma, \tau \in S_n} \mathrm{sgn}\,\sigma \mathrm{sgn}\,\tau \prod_{i=1}^n (\boldsymbol{x}_i)_{\sigma(i)} (\boldsymbol{x}_i)_{\tau(i)} \right) \mathrm{d}\boldsymbol{x}_n \cdots \mathrm{d}\boldsymbol{x}_1 \\
&= \int_K \cdots \int_K \left(\sum_{\sigma, \tau \in S_n} \mathrm{sgn}\,\tau \prod_{i=1}^n (\boldsymbol{x}_i)_{\sigma(i)} (\boldsymbol{x}_i)_{\tau(\sigma(i))} \right) \mathrm{d}\boldsymbol{x}_n \cdots \mathrm{d}\boldsymbol{x}_1 \\
&= \sum_{\sigma, \tau \in S_n} \mathrm{sgn}\,\tau \left(\prod_{i=1}^n \int_K (\boldsymbol{x}_i)_{\sigma(i)} (\boldsymbol{x}_i)_{\tau(\sigma(i))} \, \mathrm{d}\boldsymbol{x}_i \right) \\
&= \sum_{\sigma, \tau \in S_n} \mathrm{sgn}\,\tau \left(\prod_{i=1}^n \int_K y_{\sigma(i)} y_{\tau(\sigma(i))} \, \mathrm{d}\boldsymbol{y} \right) \\
&= \sum_{\sigma, \tau \in S_n} \mathrm{sgn}\,\tau \left(\prod_{i=1}^n \int_K y_i \, y_{\tau(i)} \, \mathrm{d}\boldsymbol{y} \right) \\
&= n! \sum_{\tau \in S_n} \mathrm{sgn}\,\tau \prod_{i=1}^n a_{i,\tau(i)} = n! \det A_K.
\end{aligned}$$

By Proposition 3.9 we also know $\det A_K = \mathrm{L}_K^{2n}$. □

[14] The Sylvester problem asks for the convex bodies $K \in \mathcal{K}^n$, $\mathrm{vol}\,(K) = 1$, for which $\int_K \dots \int_K \mathrm{vol}\,(\mathrm{conv}\,\{\mathbf{0}, \boldsymbol{x}_1, \dots, \boldsymbol{x}_n, \boldsymbol{x}_{n+1}\}) \, \mathrm{d}\boldsymbol{x}_{n+1} \dots \mathrm{d}\boldsymbol{x}_1$ is minimized or maximized. It is known to be minimized for a ball and it is conjectured that it is maximized for a simplex.

3.23 Corollary. *Let $K \in \mathcal{K}^n$ with centroid* $\mathbf{0}$. *Then* $\mathrm{L}_K \leq c\sqrt{n}$, *where* c *is an absoulte constant (cf. Corollary 3.12).*

Proof. By Theorem 3.22 we have $\mathrm{L}_K^{2n} \leq n!$ and thus $\mathrm{L}_K \leq \sqrt[2n]{n!} \leq c\sqrt{n}$. □

3.24 Definition. $K \in \mathcal{K}^n$ *is called* unconditional *if it is symmetric to each coordinate hyperplane, i.e.,* $(\pm x_1, \ldots, \pm x_n)^\intercal \in K$ *for all* $\boldsymbol{x} \in K$.

3.25 Proposition. *Let* $K \in \mathcal{K}^n$ *be unconditional.*

i) $\int_K x_i x_j \,\mathrm{d}\boldsymbol{x} = 0$ *for all* $1 \leq i \neq j \leq n$.

ii) *For each coordinate hyperplane we have* $K \cap H = K|H$.

iii) *If* $\mathrm{vol}\,(K) = 1$ *then there exists a diagonal matrix* $T \in \mathrm{VP}(n, \mathbb{R})$ *such* TK *is in isotropic position.*

3.26 Theorem. *There exists an absolute constant* c *such that* $\mathrm{L}_K \leq c$ *for all unconditional convex bodies* $K \in \mathcal{K}^n$.

Proof. According to Proposition 3.25 there exists a diagonalmatrix T such that $TK \in \mathcal{K}_c^n$ and TK is in isotropic position. Since T is a diagonalmatrix, TK is still an unconditional body.

For $1 \leq i \leq n$ let H_i be the coordinate hyperplanes. By Corollary 3.18 there exists an absolute constant c_2 such that for $1 \leq i \leq n$

$$\mathrm{L}_k \leq c_2 \frac{1}{\mathrm{vol}_{\,n-1}(TK \cap H_i)} = c_2 \frac{1}{\mathrm{vol}_{\,n-1}(TK|H_i)}, \tag{3.26.1}$$

where for the last identity we have used Proposition 3.25 ii). By Corollary 2.15 we know that

$$1 = \mathrm{vol}\,(TK)^{n-1} \leq \prod_{i=1}^{n} \mathrm{vol}_{\,n-1}(TK|H_i),$$

and thus there exists and $i^\star \in \{1, \ldots, n\}$ with $\mathrm{vol}_{\,n-1}(TK|H_{i^\star}) \geq 1$. Hence, for that $i^\star$, (3.26.1) gives the desired bound. □

4 Distances to the something round

We briefly recall (and extend) some basic facts on the Banach-Mazur distance and on volume maximal and minimal ellipsoids from last term.

4.1 Theorem.

i) *Let $K \in \mathcal{K}^n$. Then there exists a uniquely determined ellipsoid of maximum (minimum) volume contained in (containing) K.*

ii) *Let $K \in \mathcal{K}^n$ with $B_n \subseteq K$ ($K \subseteq B_n$). The following statements are equivalent:*

a) *B_n is the uniquely determined ellipsoid of maximum (minimum) volume contained in (containing) K.*

b) *There exists $\boldsymbol{u}_i \in S^{n-1} \cap \operatorname{bd} K$ (contact points) and $\lambda_i > 0$, $1 \leq i \leq m$, with $n+1 \leq m \leq n(n+3)/2$ such that*

$$\sum_{i=1}^{m} \lambda_i = n, \quad \mathrm{I}_n = \sum_{i=1}^{n} \lambda_i \left(\boldsymbol{u}_i \boldsymbol{u}_i^\intercal\right) \quad \text{and} \quad \mathbf{0} = \sum_{i=1}^{m} \lambda_i \, \boldsymbol{u}_i.$$

iii) *If $K \in \mathcal{K}_o^n$, then the condition ii) b) on the number of contact points can be replaced by $n \leq m \leq n(n+1)/2$ and, in this case, the condition $\mathbf{0} = \sum_{i=1}^{m} \lambda_i \, \boldsymbol{u}_i$ must be omitted.*

4.2 Corollary. *Let $K \in \mathcal{K}^n$. B_n is the maximum volume ellipsoid contained in K if and only if B_n is the minimum volume ellipsoid containing $K^\star$.*

Proof. Suppose B_n is the maximum volume ellipsoid contained in K. Then, by Theorem 4.1, there exist $\boldsymbol{u}_i \in S^{n-1} \cap \operatorname{bd} K$ and $\lambda_i > 0$, $1 \leq i \leq m$, with $n+1 \leq m \leq n(n+3)/2$ such that

$$\sum_{i=1}^{m} \lambda_i = n, \quad \mathrm{I}_n = \sum_{i=1}^{n} \lambda_i \left(\boldsymbol{u}_i \boldsymbol{u}_i^\intercal\right) \quad \text{and} \quad \sum_{i=1}^{m} \lambda_i \, \boldsymbol{u}_i = \mathbf{0}.$$

In particular, we have $\langle \boldsymbol{u}_i, \boldsymbol{x} \rangle \leq 1$ for all $\boldsymbol{x} \in K$, which implies $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m \in K^\star$. Moreover, since $\langle \boldsymbol{u}_i, \boldsymbol{u}_i \rangle = 1$ it we even know $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m \in \operatorname{bd} K^\star$, i.e., $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m \in S^{n-1} \cap \operatorname{bd} K^\star$. On the other hand, from $B_n \subseteq K$ we get $K^\star \subseteq B_n$ and thus using Theorem 4.1 we get the result. The proof of the reverse implication is the same. □

4.3 Definition [Banach-Mazur distance]. *Let $K, L \in \mathcal{K}^n$ be n-dimensional convex bodies. Then*

$$\begin{aligned} \mathrm{d_{BM}}(K, L) = \inf\bigl\{\lambda > 0 : \exists\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n,\ T \in \mathrm{GL}(n, \mathbb{R}) \ \textit{with} \\ \boldsymbol{y} + K \subseteq T(\boldsymbol{x} + L) \subseteq \lambda(\boldsymbol{y} + K)\bigr\} \end{aligned}$$

is called the Banach-Mazur *distance of K and L.*

4.4 Remark.

i) *If $K, L \in \mathcal{K}_o^n$ then we can assume $\boldsymbol{x} = \boldsymbol{y} = \mathbf{0}$, and $\mathrm{d_{BM}}(K, L)$ may be interpreted as a distance between norms defined by K and L.*

ii) *Let $K, L, M \in \mathcal{K}^n$. Then $\mathrm{d_{BM}}(K, L) = \mathrm{d_{BM}}(L, K)$, $\mathrm{d_{BM}}(K, L) = \mathrm{d_{BM}}(\boldsymbol{x} + TK, \boldsymbol{y} + \tilde{T}L)$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $T, \tilde{T} \in \mathrm{GL}(n, \mathbb{R})$.*

iii) *$\mathrm{d_{BM}}$ verifies the multiplicative triangular inequality, i.e.,*

$$\mathrm{d_{BM}}(K, M) \leq \mathrm{d_{BM}}(K, L)\,\mathrm{d_{BM}}(L, M).$$

4.5 Proposition.

i) *$\mathrm{d_{BM}}(K, B_n) \leq \sqrt{n}$ for all $K \in \mathcal{K}_o^n$, and $\mathrm{d_{BM}}(C_n, B_n) = \mathrm{d_{BM}}(C_n^\star, B_n) = \sqrt{n}$.*

ii) *$\mathrm{d_{BM}}(K, B_n) \leq n$ for all $K \in \mathcal{K}^n$, and $\mathrm{d_{BM}}(S_n, B_n) = n$ for an n-simplex S_n.*

iii) *$\mathrm{d_{BM}}(K, L) \leq n$ for all $K, L \in \mathcal{K}_o^n$, and $\mathrm{d_{BM}}(K, L) \leq n^2$ for all $K, L \in \mathcal{K}^n$.*

Proof. This is Proposition [9.21, SkriptWS12].[15] □

In the following we want to prove

4.6 Theorem. *Let $K \in \mathcal{K}^n$ with $\mathrm{d_{BM}}(K, B_n) = n$. Then K is a simplex.*

To this end we need the next two lemmas.

4.7 Lemma. *Let $K \in \mathcal{K}^n$ and let B_n be the minimum volume ellipsoid containing K. Then for every $\boldsymbol{u} \in S^{n-1}$*

$$h(K, \boldsymbol{u})\, h(K, -\boldsymbol{u}) \geq \frac{1}{n}.$$

Proof. Let $\alpha = h(K, -\boldsymbol{u})$ and $\beta = h(K, \boldsymbol{u})$. Since B_n is the minimum volume ellipsoid containing K we know $\mathbf{0} \in \operatorname{int} K$ and hence we have $1 \geq \alpha, \beta > 0$. Without loss of generality we may assume that $\boldsymbol{u} = \boldsymbol{e}_1$ and $\beta > \alpha$: indeed, if $\beta = \alpha$ then either $\beta = \alpha = 1$ or we increase slightly β by an arbitrary small number.

Let $M = \{\boldsymbol{x} \in B_n : -\alpha \leq x_1 \leq \beta\}$. Then clearly $K \subset M$. For $\varepsilon \in \mathbb{R}$ let

$$E_\varepsilon = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \gamma_1(\varepsilon)(x_1 - \varepsilon)^2 + \gamma_2(\varepsilon) \sum_{i=2}^n x_i^2 \leq 1 \right\}$$

be an ellipsoid with center $\varepsilon \boldsymbol{e}_1$. The values $\gamma_i(\varepsilon) > 0$ are chosen such that the points $\left(\beta, \sqrt{1-\beta^2}, 0, \ldots, 0\right)^\intercal$ and $\left(-\alpha, \sqrt{1-\alpha^2}, 0, \ldots, 0\right)^\intercal$ lie on the boundary of E_ε. Hence they are determined by

$$1 = \gamma_1(\varepsilon)(\beta - \varepsilon)^2 + \gamma_2(\varepsilon)(1 - \beta^2) \quad \text{and} \quad 1 = \gamma_1(\varepsilon)(\alpha + \varepsilon)^2 + \gamma_2(\varepsilon)(1 - \alpha^2).$$

[15] It was shown by Gluskin (1981) that there exists $K, L \in \mathcal{K}_o^n$ such that $\mathrm{d_{BM}}(K, L) \geq c\,n$, where c is an absolute constant. In the general case, Rudelson (2001) proved an upper bound of order $n^{4/3} \log^9(n)$. See also Banach-Mazur compactum.

Differentiating these identities together with $\gamma_1(0) = \gamma_2(0) = 1$ leads to

$$\gamma_1'(0) = 2\frac{1-\alpha\,\beta}{\beta-\alpha} \geq 0, \quad \gamma_2'(0) = -2\frac{\alpha\,\beta}{\beta-\alpha} < 0. \tag{4.7.1}$$

Next we show that for sufficiently small $\varepsilon > 0$ the inclusion $M \subseteq E_\varepsilon$ holds. Let $\boldsymbol{x} \in M$. Then $|\boldsymbol{x}| \leq 1$ and so

$$\gamma_1(\varepsilon)(x_1-\varepsilon)^2 + \gamma_2(\varepsilon)\sum_{i=2}^{n} x_i^2 \leq \gamma_1(\varepsilon)(x_1-\varepsilon)^2 + \gamma_2(\varepsilon)(1-x_1^2).$$

For $t \in [-\alpha, \beta]$ let $\rho_\varepsilon(t) = \gamma_1(\varepsilon)(t-\varepsilon)^2 + \gamma_2(\varepsilon)(1-t^2)$. Then $\rho_\varepsilon(-\alpha) = \rho_\varepsilon(\beta) = 1$. Since $\gamma_1'(0) \geq 0$, $\gamma_2'(0) < 0$ we have $\gamma_1(\varepsilon) - \gamma_2(\varepsilon) > 0$ for sufficiently small positive ε and hence, for those values of ε, the quadratic polynomial is a convex function and so $p_\varepsilon(t) \leq 1$ for all $t \in [-\alpha, \beta]$. Therefore,

$$\gamma_1(\varepsilon)(x_1-\varepsilon)^2 + \gamma_2(\varepsilon)\sum_{i=2}^{n} x_i^2 \leq \rho_\varepsilon(x_1) \leq 1,$$

which shows $M \subseteq E_\varepsilon$ and thus $K \subseteq E_\varepsilon$. Since B_n is the minimum volume ellipsoid containing K we must have

$$\mathrm{vol}\,(B_n) \leq \mathrm{vol}\,(E_\varepsilon) = \frac{\mathrm{vol}\,(B_n)}{\sqrt{\gamma_1(\varepsilon)\gamma_2(\varepsilon)^{n-1}}},$$

and thus $\gamma_1(\varepsilon)\,\gamma_2(\varepsilon)^{n-1} \leq 1$. Let $f(\varepsilon) = \gamma_1(\varepsilon)\gamma_2(\varepsilon)^{n-1}$. Then $f(0) = 1$ and $f(\varepsilon) \leq 1$ for all sufficiently small positive ε. Hence

$$\begin{aligned} 0 \geq \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f(\varepsilon) &= \gamma_1'(0)\gamma_2(0)^{n-1} + (n-1)\gamma_1(0)\gamma_2(0)^{n-2}\gamma_2'(0) \\ &= \gamma_1'(0) + (n-1)\gamma_2'(0) = \frac{2}{\beta-\alpha}(1-n\alpha\beta). \end{aligned}$$

Thus $\alpha\beta \geq 1/n$. □

4.8 Lemma. *Let $K \in \mathcal{K}^n$ with $\mathrm{d_{BM}}(K, B_n) = n$ and let B_n be the minimum volume ellipsoid containing K. Then*

$$\mathbf{0} \in \mathrm{conv}\left(\frac{1}{n}S^{n-1} \cap \mathrm{bd}\,K\right).$$

Proof. Since B_n is the minimum volume ellipsoid containing K, we get $(1/n)B_n \subset K$ (cf. Corollary 4.2). First we note that

$$(1/n)S^{n-1} \cap \mathrm{bd}\,K \neq \emptyset.$$

Otherwise there exists a $\delta > 0$ such that $(1+\delta)/nB_n \subset K$ which gives $\mathrm{d_{BM}}(K, B_n) \leq n/(1+\delta) < n$, a contradiction. Next we note a property of contact points of $(1/n)\,S^{n-1}$. Let $\boldsymbol{v} \in (1/n)S^{n-1} \cap \mathrm{bd}\,K$. Then for all $\boldsymbol{x} \in K$ we have

$$\langle \boldsymbol{v}, \boldsymbol{x}\rangle \leq |\boldsymbol{v}|^2 = \frac{1}{n^2}. \tag{4.8.1}$$

Thus $\langle n\,\boldsymbol{v}, \boldsymbol{x}\rangle \leq 1/n$ and with Lemma 4.7 we get

$$\frac{1}{n} h(K, -n\boldsymbol{v}) \geq h(K, n\boldsymbol{v})\, h(K, -n\boldsymbol{v}) \geq \frac{1}{n},$$

i.e., $h(K, -n\boldsymbol{v}) \geq 1$. Hence there exists $\boldsymbol{z} \in K$ with $\langle -n\boldsymbol{v}, \boldsymbol{z}\rangle \geq 1$. But since $-n\boldsymbol{v} \in S^{n-1}$ and $\boldsymbol{z} \in K \subseteq B_n$ we must have

$$\boldsymbol{z} = -n\boldsymbol{v} \in K. \tag{4.8.2}$$

In the following we assume

$$\mathbf{0} \notin D = \operatorname{conv}\big((1/n)S^{n-1} \cap \operatorname{bd} K\big)$$

and we are going to derive a contradiction.

Let $\rho = \min\big\{|\boldsymbol{x}| : \boldsymbol{x} \in D\big\}$ and let $\boldsymbol{u} \in D$ with $|\boldsymbol{u}| = \rho$. Without loss of generality we suppose that $\boldsymbol{u} = -\rho \boldsymbol{e}_1$. Then

$$\langle \boldsymbol{u}, \boldsymbol{x}\rangle \geq |\boldsymbol{u}|^2 = \rho^2 \quad \text{for all } \ \boldsymbol{x} \in D. \tag{4.8.3}$$

For $\varepsilon > 0$ let

$$E_\varepsilon = \left\{\boldsymbol{x} \in \mathbb{R}^n : \frac{1}{(1+\varepsilon)^2}(x_1 - \varepsilon)^2 + \frac{1}{1+\varepsilon}\sum_{i=2}^{n} x_i^2 \leq 1\right\}.$$

It is easy to check that $B_n \subseteq E_\varepsilon$ and thus $K \subseteq B_n \subseteq E_\varepsilon$.

Claim I. $\rho \geq 1/n^2$.

Since $\boldsymbol{u} \in D$, there exist $\lambda_i > 0$, $\boldsymbol{v}_i \in (1/n)S^{n-1} \cap \operatorname{bd} K$, $1 \leq i \leq m$, with $\sum_{i=1}^m \lambda_i = 1$ and $\boldsymbol{u} = \sum_{i=1}^m \lambda_i \boldsymbol{v}_i$. In view of (4.8.1) and (4.8.2) we find $\langle \boldsymbol{v}_j, -n\boldsymbol{v}_i\rangle \leq 1/n^2$ and so $\langle \boldsymbol{v}_j, \boldsymbol{v}_i\rangle \geq -1/n^3$ for $1 \leq i, j \leq m$. On the other hand we have

$$\langle \boldsymbol{u}, \boldsymbol{u}\rangle = \sum_{i=1}^m \lambda_i \langle \boldsymbol{u}, \boldsymbol{v}_i\rangle \geq \sum_{i=1}^m \lambda_i \langle \boldsymbol{u}, \boldsymbol{u}\rangle = \langle \boldsymbol{u}, \boldsymbol{u}\rangle,$$

which implies that $\langle \boldsymbol{u}, \boldsymbol{v}_i\rangle = \langle \boldsymbol{u}, \boldsymbol{u}\rangle$ for $i = 1, \ldots, m$. Therefore, all points $\boldsymbol{v}_i$, $1 \leq i \leq m$, are contained in the same affine hyperplane and by Carathéodory's Theorem [2.7, SkriptWS12] we may assume $m \leq n$. Hence there exists a $\lambda_i \geq 1/n$ and without loss of generality let $\lambda_1 \geq 1/n$. Then we may write

$$\begin{aligned}
\rho^2 = \langle \boldsymbol{u}, \boldsymbol{u}\rangle = \langle \boldsymbol{u}, \boldsymbol{v}_1\rangle &= \sum_{i=1}^m \lambda_i \langle \boldsymbol{v}_i, \boldsymbol{v}_1\rangle = \lambda_1 \langle \boldsymbol{v}_1, \boldsymbol{v}_1\rangle + \sum_{i=2}^m \lambda_i \langle \boldsymbol{v}_i, \boldsymbol{v}_1\rangle \\
&\geq \lambda_1 \frac{1}{n^2} - \sum_{i=2}^m \lambda_i \frac{1}{n^3} \\
&= \frac{1}{n^2}\lambda_1 - \frac{1}{n^3}(1 - \lambda_1) = \lambda_1\left(\frac{1}{n^2} + \frac{1}{n^3}\right) - \frac{1}{n^3} \\
&\geq \frac{1}{n}\left(\frac{1}{n^2} + \frac{1}{n^3}\right) - \frac{1}{n^3} = \frac{1}{n^4},
\end{aligned}$$

which shows Claim I.

Now for $\varepsilon > 0$ let

$$\widetilde{E}_\varepsilon = \left\{ x \in \mathbb{R}^n : \frac{1}{(1+\varepsilon)^2}(x_1 - \varepsilon)^2 + \frac{1}{1+\varepsilon}\sum_{i=2}^n x_i^2 \leq \frac{1}{n^2} \right\}.$$

Claim II. For $\boldsymbol{x} \in \widetilde{E}_\varepsilon$ with $|\boldsymbol{x}| \geq 1/n$ we have $x_1 \geq -1 + \sqrt{1 - 1/n^2}$.
Since $|\boldsymbol{x}| \geq 1/n$ it is $\sum_{i=2}^n x_i^2 \geq 1/n^2 - x_1^2$ and with $\boldsymbol{x} \in \widetilde{E}_\varepsilon$ we get

$$\frac{(1+\varepsilon)^2}{n^2} \geq x_1^2 - 2x_1\varepsilon + \varepsilon^2 + \frac{1+\varepsilon}{n^2} - (1+\varepsilon)x_1^2 = -\varepsilon x_1^2 - 2x_1\varepsilon + \varepsilon^2 + \frac{1+\varepsilon}{n^2},$$

or equivalently,

$$x_1^2 + 2x_1 + \frac{1}{n^2} + \frac{\varepsilon}{n^2} - \varepsilon \geq 0.$$

In particular, $x_1^2 + 2x_1 + 1/n^2 \geq 0$ and hence, either $x_1 \leq -1 - \sqrt{1 - 1/n^2}$ or $x_1 \geq -1 + \sqrt{1 - 1/n^2}$. Since $x \in \widetilde{E}_\varepsilon$, however, we also know $x_1 \geq -1$ and so only the latter solution is valid.
Claim III. $\widetilde{E}_\varepsilon \subset \operatorname{int} K$ for sufficiently small $\varepsilon > 0$.
Since $\mathbf{0} \in K \cap \widetilde{E}_\varepsilon$ it suffices to show

$$\operatorname{bd} K \cap \widetilde{E}_\varepsilon = \emptyset.$$

for sufficiently small $\epsilon > 0$. Suppose the contrary, and let $(\varepsilon_i)_{i \in \mathbb{N}}$ be a sequence with $\varepsilon_i \to 0$ and $\operatorname{bd} K \cap \widetilde{E}_{\varepsilon_i} \neq \emptyset$. Then we can find a convergent sequence of points $\boldsymbol{y}_i \in \operatorname{bd} K \cap \widetilde{E}_{\varepsilon_i}$ with $\boldsymbol{y} = \lim_{i\to\infty} \boldsymbol{y}_i \in \operatorname{bd} K \cap \frac{1}{n} B_n \subset D$.

(4.8.3) and Claim I. imply fro $\boldsymbol{x} \in D$

$$x_1 = \langle \boldsymbol{e}_1, \boldsymbol{x} \rangle = -\frac{1}{\rho} \langle \boldsymbol{u}, \boldsymbol{x} \rangle \leq -\frac{1}{\rho} |\boldsymbol{u}|^2 = -\rho \leq -\frac{1}{n^2}.$$

Hence for $\mu > 0$ there exists an $i(\mu) \in \mathbb{N}$ such that

$$(\boldsymbol{y}_i)_1 \leq -\frac{1}{n^2} + \mu \quad \text{for } i \geq i(\mu). \tag{4.8.4}$$

On the other hand, from $(1/n)B_n \subset K$ we know $|\boldsymbol{y}|_i \geq 1/n$ and by Claim II we conclude $(\boldsymbol{y}_i)_1 \geq -1 + \sqrt{1 - 1/n^2}$ for any i and thus contradicting (4.8.4). This verifies Claim III.

Now we can conclude the proof of the lemma. Since $\widetilde{E}_\varepsilon \subset \operatorname{int} K$ there exists $\delta > 0$ such that $\big((1+\delta)/n\big)(E_\varepsilon - \boldsymbol{u}) + \boldsymbol{u} \subset K \subset E_\varepsilon$. Therefore, $\mathrm{d}_{\mathrm{BM}}(K, B_n) \leq n/(1+\delta) < n$, a contradiction. □

Proof. [Proof of Theorem 4.6] Without loss of generality let B_n be the minimal volume ellipsoid containing K. By Lemma 4.8 we know $\mathbf{0} \in \operatorname{conv}\big((1/n)S^{n-1} \cap \operatorname{bd} K\big)$ and hence there exist $\lambda_i \geq 0$, $\boldsymbol{v}_i \in (1/n)S^{n-1} \cap \operatorname{bd} K$, $1 \leq i \leq m$ with $\sum_{i=1}^m \lambda_i = 1$ and $\mathbf{0} = \sum_{i=1}^m \lambda_i \boldsymbol{v}_i$. By Carathéodory's Theorem 2.7 we may asume $m \leq n+1$. Now let $\boldsymbol{w}_i = -n\boldsymbol{v}_i$, $1 \leq i \leq m$. We will show $K = \operatorname{conv}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$.

The argumentation leading to (4.8.1) and (4.8.2) in the proof of Lemma 4.8 gives $\boldsymbol{w}_i \in K$ and thus $\operatorname{conv}\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_m\} \subseteq K$.

On the other hand, from $\sum_{i=1}^{m} \lambda_i \boldsymbol{w}_i =$ we get for a fixed $k \in \{1, \dots, m\}$ (cf. (4.8.1))

$$\begin{aligned}\mathbf{0} = \sum_{i=1}^{m} \lambda_i \langle \boldsymbol{w}_i, \boldsymbol{w}_k \rangle &= \lambda_k \langle \boldsymbol{w}_k, \boldsymbol{w}_k \rangle + \sum_{i=1, i\neq k}^{m} \lambda_i \langle \boldsymbol{w}_i, \boldsymbol{w}_k \rangle \\ &\geq \lambda_k - \sum_{i=1, i\neq k}^{m} \lambda_i \frac{1}{n} = \lambda_k - \frac{1-\lambda_k}{n} = \frac{n+1}{n}\lambda_k - \frac{1}{n}.\end{aligned}$$

Thus $\lambda_k \leq 1/(n+1)$, $k = 1, \dots, m$, and since $\sum_{i=1}^{m} \lambda_i = 1$, $m \leq n+1$, we must have $m = n+1$ and $\lambda_k = 1/(n+1)$ for $k = 1, \dots, m$. Hence the previous inequality is an equality and we have $\langle \boldsymbol{w}_i, \boldsymbol{w}_j \rangle = -1/n$ for all $i \neq j$, which means that $S = \operatorname{conv}\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_{n+1}\}$ is a (regular) simplex.

In order to show $K \subseteq S$, let $\boldsymbol{x} \notin S$. Let $\mu \in (0,1)$ such that $\mu\boldsymbol{x} \in \operatorname{bd} S$. Then $\mu\boldsymbol{x}$ lies in a facet of S and so we may assume $\mu\boldsymbol{x} = \sum_{i=1}^{n} \mu_i \boldsymbol{w}_i$ with $\mu_i \geq 0$ and $\sum_{i=1}^{n} \mu_i = 1$. Then

$$\begin{aligned}\langle \boldsymbol{w}_{n+1}, \boldsymbol{x} \rangle &= \frac{1}{\mu} \left\langle \boldsymbol{w}_{n+1}, \sum_{i=1}^{n} \mu_i \boldsymbol{w}_i \right\rangle = \frac{1}{\mu} \sum_{i=1}^{n} \mu_i \langle \boldsymbol{w}_{n+1}, \boldsymbol{w}_i \rangle \\ &= -\frac{1}{\mu}\frac{1}{n} \sum_{i=1}^{n} \mu_i = -\frac{1}{\mu n} < -\frac{1}{n}.\end{aligned}$$

By (4.8.1) we have $\langle \boldsymbol{w}_i, \boldsymbol{x} \rangle \geq -1/n$ for any $\boldsymbol{x} \in K$ and thus $\boldsymbol{x} \notin K$. □

4.9 Definition [Spherical cap]. *Let $\boldsymbol{v} \in S^{n-1}$. For $\varepsilon \in [-1,1]$ the set*

$$C(\varepsilon, \boldsymbol{v}) = \left\{\boldsymbol{u} \in S^{n-1} : \langle \boldsymbol{v}, \boldsymbol{u} \rangle \geq \varepsilon \right\} = \left\{\boldsymbol{u} \in S^{n-1} : |\boldsymbol{u} - \boldsymbol{v}| \leq \sqrt{2 - 2\,\varepsilon} \right\}$$

is called an ε-cap or a spherical cap of radius $r = \sqrt{2-2\varepsilon}$, $r \in [0,2]$.

4.10 Notation. *In the following we will denote for a measurable set $A \subset S^{n-1}$ by*

$$\mu_{n-1}(A) = \int_{S^{n-1}} \chi_A(\boldsymbol{u}) \,\mathrm{d}\sigma(\boldsymbol{u})$$

its normalized surface are measure (cf. Proposition 2.8 i)).

4.11 Lemma. *Let $\boldsymbol{v} \in S^{n-1}$ and $\varepsilon \in (0,1)$. Then*

$$\frac{1}{2}\left(\frac{\sqrt{2-2\varepsilon}}{2}\right)^{n-1} \leq \mu_{n-1}\big(C(\varepsilon, \boldsymbol{v})\big) \leq \mathrm{e}^{-n\varepsilon^2/2}.$$

Proof. We set $\overline{C}(\varepsilon,\boldsymbol{v}) = \operatorname{conv}\{\mathbf{0}, C(\varepsilon,\boldsymbol{v})\}$, and since in the following $\boldsymbol{v}$ will be fixed we will just write $C(\varepsilon)$ and $\overline{C}(\varepsilon)$, respectively. We start with the upper bound. Since (cf. Proposition 2.8 ii))

$$\begin{aligned}\operatorname{vol}\big(\overline{C}(\varepsilon)\big) &= \kappa_n \int_{S^{n-1}} r_{\overline{C}(\varepsilon)}(\boldsymbol{u})^n \,\mathrm{d}\sigma(\boldsymbol{u}) = \kappa_n \int_{C(\varepsilon)} \mathrm{d}\sigma(\boldsymbol{u}) \\ &= \kappa_n\, \mu_{n-1}\big(C(\varepsilon)\big),\end{aligned}$$

it suffices to bound $\operatorname{vol}\big(\overline{C}(\varepsilon)\big)$. The diameter of $\overline{C}(\varepsilon)$, denoted by $\mathrm{D}(\overline{C}(\varepsilon))$ is given either by the diameter of the $((n-1))$-dimensional ball determined by the cap $C(\varepsilon)$, i.e., by $2\sqrt{1-\varepsilon^2}$ or by $|\boldsymbol{v}| = 1$. Hence

$$\mathrm{D}(\overline{C}(\varepsilon)) = \begin{cases} 2\sqrt{1-\varepsilon^2}, & \varepsilon \le \sqrt{3}/2, \\ 1, & \text{otherwise.}\end{cases}$$

In the first case we get by the isodiametric inequality (see Proposition [10.7, SkriptWS12])

$$\mu_{n-1}\big(C(\varepsilon)\big) = \frac{\operatorname{vol}\big(\overline{C}(\varepsilon)\big)}{\kappa_n} \le \frac{1}{2^n}\mathrm{D}\big(\overline{C}(\varepsilon)\big)^n = (1-\varepsilon^2)^{n/2} \le \mathrm{e}^{-n\varepsilon^2/2},$$

where for the last inequality we used $\mathrm{e}^{-t} \ge 1-t$. In the same way we also obtain in the second case

$$\mu_{n-1}\big(C(\varepsilon)\big) \le \frac{1}{2^n}\mathrm{D}\big(\overline{C}(\varepsilon)\big)^n = \frac{1}{2^n} \le \mathrm{e}^{-n\varepsilon^2/2},$$

since $\mathrm{e}^{\varepsilon^2/2} \le 2$ when $\varepsilon \le 1$.

For the proof of the lower bound let $H = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{x}, \boldsymbol{v}\rangle = \varepsilon\}$ and $B' = B_n\big(\varepsilon\boldsymbol{v}, \sqrt{1-\varepsilon^2}\big) \cap H$. The double cone $K = \operatorname{conv}\{B', \mathbf{0}, \boldsymbol{v}\}$ is contained in $\overline{C}(\varepsilon)$ and has volume

$$\begin{aligned}\operatorname{vol}(K) &= \operatorname{vol}\big(\operatorname{conv}\{B', \mathbf{0}\}\big) + \operatorname{vol}\big(\operatorname{conv}\{B', \boldsymbol{v}\}\big) \\ &= \frac{1}{n}\big(\sqrt{1-\varepsilon^2}\big)^{n-1}\kappa_{n-1}.\end{aligned}$$

Therefore

$$\mu_{n-1}\big(C(\varepsilon,v)\big) = \frac{\operatorname{vol}\big(\overline{C}(\varepsilon,v)\big)}{\kappa_n} \ge \frac{\operatorname{vol}(K)}{\kappa_n} = \frac{1}{n}\frac{\kappa_{n-1}}{\kappa_n}\big(\sqrt{1-\varepsilon^2}\big)^{n-1},$$

and we have to show

$$\frac{1}{n}\frac{\kappa_{n-1}}{\kappa_n}\big(\sqrt{1-\varepsilon^2}\big)^{n-1} \ge \frac{(\sqrt{2})^{n-1}}{2^n}\big(\sqrt{1-\varepsilon}\big)^{n-1}.$$

For $\varepsilon < 1$ the inequality is equivalent to

$$\frac{n}{2^{(n+1)/2}}\frac{\kappa_n}{\kappa_{n-1}} \le \sqrt{1+\varepsilon}^{\,n-1}, \tag{4.11.1}$$

and it suffices to verify it for $\varepsilon = 0$. Now

$$\frac{n}{2^{(n+1)/2}}\frac{\kappa_n}{\kappa_{n-1}} = \frac{n}{2^{(n+1)/2}}\sqrt{\pi}\frac{\Gamma(n/2+1/2)}{\Gamma(n/2+1)} \leq \frac{n}{2^{(n+1)/2}}\sqrt{\pi} \leq 1,$$

for $n \geq 6$, it remains to prove the lower bound for dimensions 2,3,4,5. For $3 \leq n \leq 5$ (4.11.1) (for $\varepsilon = 0$) can easily verified by a computer, and for $n = 2$ we observe that

$$\mu_1\big(C(\varepsilon)\big) = \frac{\operatorname{vol}\big(\overline{C}(\varepsilon)\big)}{\pi} = \frac{\arccos\varepsilon}{\pi} \geq \frac{\sqrt{2}}{4}\sqrt{1-\varepsilon}.$$[16]

□

The next theorem shows that if a polytope has a small distance to the Euclidean ball, then it must have exponentially many facets.

4.12 Theorem. *Let $P \in \mathcal{P}^n$ be a symmetric polytope with $\mathrm{d_{BM}}(P, B_n) \leq \delta$. Then P has at least $\mathrm{e}^{n/(2\delta^2)}$ facets. On the other hand, there is a polytope $P \in \mathcal{P}^n$ with 4^n facets such that $B_n \subset P \subset 2B_n$, and thus $\mathrm{d_{BM}}(P, B_n) \leq 2$.*

Proof. Let $P = \big\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{v}_i, \boldsymbol{x}\rangle \leq 1, 1 \leq i \leq k\big\}$ be a symmetric polytope with k facets such that $\mathrm{d_{BM}}(P, B_n) = \delta$. We have to show that $k \geq e^{n/(2\delta^2)}$. We may assume that

$$B_n \subset P \subset \delta B_n.$$

Since $B_n \subset P$ we have $\boldsymbol{v}_i/|\boldsymbol{v}_i| \in P$ and thus $1 \geq \langle \boldsymbol{v}_i, \boldsymbol{v}_i/|\boldsymbol{v}_i|\rangle = |\boldsymbol{v}_i|$, $1 \leq i \leq k$. On the other hand since $P \subset \delta B_n$, we know that for $\boldsymbol{x} \in \delta S^{n-1}$ there exists an index $j \in \{1, \ldots, k\}$ with $\langle \boldsymbol{v}_j, \boldsymbol{x}\rangle \geq 1$. Therefore,

$$\left\langle \frac{\boldsymbol{v}_j}{|\boldsymbol{v}_j|}, \frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right\rangle \geq \frac{1}{|\boldsymbol{v}_j|}\frac{1}{\delta} \geq \frac{1}{\delta},$$

i.e., $\boldsymbol{x}/|\boldsymbol{x}| \in C\big(1/\delta, \boldsymbol{v}_j/|\boldsymbol{v}_j|\big)$. Hence

$$S^{n-1} = \bigcup_{i=1}^{k} C\big(1/\delta, \boldsymbol{v}_i/|\boldsymbol{v}_i|\big).$$

By Lemma 4.11 we get $\mu_{n-1}\big(C\big(1/\delta, \boldsymbol{v}_i/|\boldsymbol{v}_i|\big)\big) \leq e^{-n/(2\delta^2)}$, and so

$$1 = \int_{S^{n-1}} \mathrm{d}\sigma(\boldsymbol{u}) \leq \sum_{i=1}^{k}\int_{C\left(1/\delta, \boldsymbol{v}_i/|\boldsymbol{v}_i|\right)} \mathrm{d}\sigma(\boldsymbol{u}) \leq k\, e^{-n/(2\delta^2)},$$

as desired.

In order to prove the second assertion, we construct $k = 4^n$ points $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in S^{n-1}$ such that

$$S^{n-1} = \bigcup_{i=1}^{k} C(1/2, \boldsymbol{v}_i). \tag{4.12.1}$$

[16]Observe that $(\arccos\varepsilon)/\pi - (\sqrt{2}/4)\sqrt{1-\varepsilon}$ in descreasing and 0 for $\varepsilon = 1$.

Then $P = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{v}_i, \boldsymbol{x}\rangle \leq 1,\ 1 \leq i \leq k\}$ is the required polytope. Indeed, by the construction we certainly have $B_n \subset P$. On the other hand, due to (4.12.1) we know for $\boldsymbol{x} \in \mathbb{R}^n$ that there exist a $\boldsymbol{v}_j$ such that

$$\langle \boldsymbol{v}_j, x/|\boldsymbol{x}|\rangle \geq 1/2, \quad \text{i.e.,} \quad |\boldsymbol{x}| \leq 2\langle \boldsymbol{v}_j, \boldsymbol{x}\rangle .$$

Hence, if $\boldsymbol{x} \in P$ we have $|x| \leq 2$ and so $P \subset 2B_n$.

A set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \subset S^{n-1}$ with property (4.12.1) is usually called a 1-net, since (1/2)-caps are caps with radius 1. Obviously, a set $M \subset S^{n-1}$ of maximal cardinality with the property $|\boldsymbol{x} - \boldsymbol{y}| \geq 1$ for any $\boldsymbol{x} \neq \boldsymbol{y} \in M$ is a 1-net, and it remains to show $\#M \leq 4^n$.

The spherical caps of radius 1/2 centered at the points of M do not overlap, and according to Lemma 4.11, there measure is bigger than $1/4^n$. Hence $\#M \leq 4^n$. □

4.13 Corollary. *For the 4^n-dimensional cube C_{4^n} there exists an n-dimensional linear subspace L such that*

$$\mathrm{d_{BM}}\big(C_{4^n} \cap L, B_{4^n} \cap L\big) \leq 2.$$

On the other hand, each linear subspace L fulfilling this bound has dimension at most $c\,n$, where c is an absolute constant. [17]

Proof. First let L be a k-dimensional linear subspace fulfilling the bound. The intersection of C_{4^n} with a k-dimensional linear subspace is a 0-symmetric convex polytope P with at most $2(4^n)$ facets. Since $\mathrm{d_{BM}}(P, B_{4^n} \cap L) \leq 2$, Theorem 4.12 leads to $e^{k/8} \leq 2(4^n)$, and so there exists a constant c such that $k \leq c\,n$.

On the other hand, let $P = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{v}_i, \boldsymbol{x}\rangle \leq 1,\ 1 \leq i \leq 4^n\} \in \mathcal{P}^n$ be the polytope as in Theorem 4.12 with 4^n facets such that $B_n \subset P \subset 2B_n$. Let $\overline{P} = \{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle| \leq 1,\ 1 \leq i \leq 4^n\}$. Then $\overline{P}$ is o-symmetric and it still holds $B_n \subset \overline{P} \subset 2B_n$. Let $L \subset \mathbb{R}^{4^n}$ be an n-dimensional linear subspace given by $L = \{V\,\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n\}$ where $V \in \mathbb{R}^{4^n \times n}$ with rows $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{4^n}$. Then

$$\begin{aligned} L \cap C_{4^n} &= \{V\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n\} \cap \{\boldsymbol{x} \in \mathbb{R}^{4^n} : |x_i| \leq 1,\ 1 \leq i \leq 4^n\} \\ &= \{V\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n,\ |\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle| \leq 1,\ 1 \leq i \leq 4^n\} \\ &= V\overline{P}. \end{aligned}$$

Thus we have found an n-dimensional section of the cube, namely $V\overline{P} = C_{4^n} \cap L$ with $\mathrm{d_{BM}}(V\overline{P}, B_n) \leq 2$, since $VB_n \subset V\overline{P} \subset 2VB_n$ by our assumption on P. □

4.14 Proposition. *For every $(2n-1)$-dimensional ellipsoid $E \in \mathcal{K}_o^n$ there exists an n-dimensional linear subspace L such that $E \cap L$ is an Euclidean ball.*

[17] This property is also expressed by saying, that the 4^n-dimensional cube contains an almost 2-spherical section of dimension n.

Proof. After a suitable rotation we may assume

$$E = \left\{ \boldsymbol{x} \in \mathbb{R}^{2n-1} : \sum_{i=1}^{2n-1} \frac{x_i^2}{\alpha_i^2} \leq 1 \right\},$$

with $0 < \alpha_1 \leq \cdots \leq \alpha_{2n-1}$. Now let be the subspace

$$L = \left\{ \boldsymbol{x} \in \mathbb{R}^{2n-1} : x_i \sqrt{\frac{1}{\alpha_i^2} - \frac{1}{\alpha_n^2}} = x_{2n-i} \sqrt{\frac{1}{\alpha_n^2} - \frac{1}{\alpha_{2n-i}^2}},\ i = 1, \ldots, n-1 \right\},$$

which has dimension $\dim L = 2n - 1 - (n-1) = n$. For $\boldsymbol{x} \in L$ we have $x_i^2/\alpha_i^2 + x_{2n-i}^2/\alpha_{2n-i}^2 = (x_i^2 + x_{2n-i}^2)/\alpha_n^2$, $1 \leq i \leq n-1$, and hence, for $x \in E \cap L$ we find

$$1 \geq \sum_{i=1}^{2n-1} \frac{x_i^2}{\alpha_i^2} = \sum_{i=1}^{n-1} \left(\frac{x_i^2}{\alpha_i^2} + \frac{x_{2n-i}^2}{\alpha_{2n-i}^2} \right) + \frac{x_n^2}{\alpha_n^2} = \frac{1}{\alpha_n^2} \sum_{i=1}^{2n-1} x_i^2.$$

Therefore, $|\boldsymbol{x}| \leq \alpha_n$ and so $E \cap L \subseteq \alpha_n(B_{2n-1} \cap L)$. The same argument shows that $\alpha_n(B_{2n-1} \cap L) \subseteq E \cap L$, and thus $E \cap L$ is an n-dimensional ball of radius α_n. □

4.15 Theorem. *Let $K \in \mathcal{K}_o^n$ with $B_n \subseteq K$ and $\big(\mathrm{vol}\,(K)/\mathrm{vol}\,(B_n)\big)^{1/n} \leq r$. Then there exists an orthogonal transformation $U \in \mathrm{O}(n)$ such that for all $\boldsymbol{u} \in S^{n-1}$*

$$\frac{|\boldsymbol{u}|_{UK} + |\boldsymbol{u}|_K}{2} \geq \frac{1}{8r^2}.$$

In particular, $K \cap UK \subset (8r^2)B_n$.

Proof. Recall that $K \subseteq L$ if and only if $|\boldsymbol{x}|_L \leq |\boldsymbol{x}|_K$ for all $\boldsymbol{x} \in S^{n-1}$, and $|\boldsymbol{x}|_{\mu L} = |\boldsymbol{x}/\mu|_L = (1/\mu)\,|\boldsymbol{x}|_L$ for $\mu > 0$. Next we observe

$$\begin{aligned} |\boldsymbol{x}|_{UK\cap K} &= \min\{\lambda \geq 0 : \boldsymbol{x} \in \lambda\,(UK \cap K)\} \\ &= \max\big\{|\boldsymbol{x}|_{UK}, |\boldsymbol{x}|_K\big\} \\ &\geq \frac{|\boldsymbol{x}|_{UK} + |\boldsymbol{x}|_K}{2} = \frac{\left|U^{-1}\boldsymbol{x}\right|_K + |\boldsymbol{x}|_K}{2}. \end{aligned}$$

Hence $(|\boldsymbol{u}|_{UK} + |\boldsymbol{u}|_K)/2 \geq (8r^2)^{-1}\,|\boldsymbol{u}|$ for all $\boldsymbol{u} \in S^{n-1}$ implies the inclusion $UK \cap K \subset 8r^2 B_n$.

For a fixed but arbitary $U \in \mathrm{O}(n)$ let $N_U(\boldsymbol{x}) = \big(|U\,\boldsymbol{x}|_K + |\boldsymbol{x}|_K\big)/2$ for $\boldsymbol{x} \in \mathbb{R}^n$. It is easy to see that $N_U(\cdot)$ is a norm in $\mathbb{R}^n$ and since $B_n \subset K$ we have

$$N_U(\boldsymbol{x}) = \frac{|U\boldsymbol{x}|_K + |\boldsymbol{x}|_K}{2} \leq \frac{|U\boldsymbol{x}| + |\boldsymbol{x}|}{2} = |\boldsymbol{x}|\,.$$

Next we are going to prove that there exists an $\overline{U} \in \mathrm{O}(n)$ with

$$\int_{S^{n-1}} \frac{1}{N_{\overline{U}}(\boldsymbol{u})^{2n}}\,\mathrm{d}\sigma(\boldsymbol{u}) \leq r^{2n}. \tag{4.15.1}$$

By the geometric-arithmetic mean ineuality we have $N_{\overline{U}}(\boldsymbol{u})^2 \geq \left|\overline{U}\boldsymbol{u}\right|_K |\boldsymbol{u}|_K$ and so it suffices to prove

$$\int_{S^{n-1}} \frac{1}{\left|\overline{U}\boldsymbol{u}\right|_K^n |\boldsymbol{u}|_K^n} \,\mathrm{d}\sigma(\boldsymbol{u}) \leq r^{2n}.$$

The compact group $\mathrm{O}(n)$ of orthogonal transformations has a unique probability measure (Haar measure) $\theta(\cdot)$, say, which is related to $\mu_{n-1}(\cdot)$ by

$$\theta(A) = \mu_{n-1}(U\,\boldsymbol{v} : U \in A),$$

where $A \subseteq \mathrm{O}(n)$ and $\boldsymbol{v} \in S^{n-1}$ is an arbitrary point. Let now f be a function on S^{n-1} and let $\boldsymbol{v} \in S^{n-1}$. Then the average of $f(U\boldsymbol{v})$ with respect to $U \in \mathrm{O}(n)$, denoted by $\mathrm{ave}_U f(U\boldsymbol{v})$, is given by

$$\mathrm{ave}_U(f(U\boldsymbol{v})) = \int_{S^{n-1}} f(\boldsymbol{u})\,\mathrm{d}\sigma(\boldsymbol{u}).$$

and hence it holds

$$\begin{aligned}\mathrm{ave}_U\left(\int_{S^{n-1}} \frac{1}{|U\boldsymbol{u}|_K^n |\boldsymbol{u}|_K^n}\,\mathrm{d}\sigma(u)\right) &= \int_{S^{n-1}} \left(\mathrm{ave}_U \frac{1}{|U\boldsymbol{u}|_K^n}\right) \frac{1}{|\boldsymbol{u}|_K^n}\,\mathrm{d}\sigma(\boldsymbol{u})\\ &= \int_{S^{n-1}} \left(\int_{S^{n-1}} \frac{1}{|\boldsymbol{v}|_K^n}\,\mathrm{d}\sigma(\boldsymbol{v})\right) \frac{1}{|\boldsymbol{u}|_K^n}\,\mathrm{d}\sigma(\boldsymbol{u})\\ &= \left(\int_{S^{n-1}} \frac{1}{|\boldsymbol{u}|_K^n}\,\mathrm{d}\sigma(\boldsymbol{u})\right)^2 \leq r^{2n},\end{aligned}$$

where for the last inequality we have used $\mathrm{vol}\,(K) = \kappa_n \int_{S^{n-1}} |\boldsymbol{u}|_K^{-n}\,\mathrm{d}\sigma(\boldsymbol{u})$ (cf. Proposition 2.8 iii)) and the assumption. Therefore, there exists a matrix $\overline{U}$ with

$$\int_{S^{n-1}} \frac{1}{\left|\overline{U}\boldsymbol{u}\right|_K^n |\boldsymbol{u}|_K^n}\,\mathrm{d}\sigma(\boldsymbol{u}) \leq r^{2n},$$

which shows (4.15.1).

For the matrix $\overline{U}$, (4.15.1) says that $N_{\overline{U}}(u)$ is relatively large on the average, but we have to show that it is large everywhere. Here the norm property of our function $N_{\overline{U}}(\cdot)$ helps. Let $\boldsymbol{v} \in S^{n-1}$ and let $t = N_{\overline{U}}(\boldsymbol{v}) \leq |\boldsymbol{v}| = 1$. For $\boldsymbol{w} \in S^{n-1}$ with $|\boldsymbol{w} - \boldsymbol{v}| \leq t$ we have $N_{\overline{U}}(\boldsymbol{w}) \leq N_{\overline{U}}(\boldsymbol{v}) + N_{\overline{U}}(\boldsymbol{w} - \boldsymbol{v}) \leq t + |\boldsymbol{w} - \boldsymbol{v}| \leq 2\,t$. Hence $N_{\overline{U}}(\boldsymbol{w}) \leq 2t$ for any $\boldsymbol{w}$ in a spherical cap at $\boldsymbol{v}$ with radius t. According to Lemma 4.11 the spherical normalized volume of such a cap is at least,

$$\frac{1}{2}\left(\frac{t}{2}\right)^{n-1} \geq \left(\frac{t}{2}\right)^n.$$

Hence we have $1/N_{\overline{U}}(\boldsymbol{u})^{2n} \geq 1/(2t)^{2n}$ on a set $A \subset S^{n-1}$ with $\mu_{n-1}(A) \geq (t/2)^n$. Thus

$$\int_{S^{n-1}} \frac{1}{N_{\overline{U}}(\boldsymbol{u})^{2n}}\,\mathrm{d}\sigma(\boldsymbol{u}) \geq \left(\frac{1}{2t}\right)^{2n} \left(\frac{t}{2}\right)^n = \frac{1}{2^{3n}t^n}.$$

Together with (4.15.1) it follows that $1/(2^{3n}t^n) \leq r^{2n}$ and hence $t \geq 1/(8r^2)$. This shows that $N_{\overline{U}}(\boldsymbol{v}) \geq 1/(8r^2)$ for every $\boldsymbol{v} \in S^{n-1}$. □

4.16 Corollary. *For the $2n$-dimensional cross-polytope $C_{2n}^\star$ there exists an n-dimensional linear subspace L such that*

$$\frac{1}{\sqrt{2n}}(B_{2n}\cap L)\subseteq (C_{2n}^\star\cap L)\subset 32\frac{1}{\sqrt{2n}}(B_{2n}\cap L).$$

Proof. For abbreviation we write $|\cdot|_1$ instead of $|\cdot|_{C_n^\star}$, and we recall that $|\boldsymbol{x}|_1=\sum_{i=1}^n |x_i|$. It is $B_n\subset\sqrt{n}\,C_n^\star$ and $\mathrm{vol}\,(\sqrt{n}\,C_n^\star)/\mathrm{vol}\,(B_n)\leq 2^n$. By Theorem 4.15 there exists an orthogonal matrix $U\in\mathrm{O}(n)$ such that for all $\boldsymbol{x}\in\mathbb{R}^n$

$$|U\boldsymbol{x}|_1+|\boldsymbol{x}|_1\geq\frac{\sqrt{n}}{16}\,|\boldsymbol{x}|\,.$$

Let

$$T:\mathbb{R}^n\to\mathbb{R}^{2n}\quad\text{given by}\quad T\boldsymbol{x}=\begin{pmatrix}U\\ \mathrm{I}_n\end{pmatrix}\boldsymbol{x}.$$

First we observe that $|T\boldsymbol{x}|=\sqrt{|U\boldsymbol{x}|^2+|\boldsymbol{x}|^2}=\sqrt{2}\,|\boldsymbol{x}|$ and so we may write

$$|T\boldsymbol{x}|_1=|U\boldsymbol{x}|_1+|\boldsymbol{x}|_1\geq\frac{\sqrt{n}}{16}\,|\boldsymbol{x}|=\frac{\sqrt{n}}{16}\frac{1}{\sqrt{2}}\,|T\boldsymbol{x}|=\frac{\sqrt{2n}}{32}\,|T\boldsymbol{x}|\,.$$

This shows

$$T\mathbb{R}^n\cap C_{2n}^\star\subset\frac{32}{\sqrt{2n}}(T\mathbb{R}^n\cap B_{2n}).$$

Finally, since $|T\boldsymbol{x}|_1\leq\sqrt{2n}\,|T\boldsymbol{x}|$ we also get

$$\frac{1}{\sqrt{2n}}(T\mathbb{R}^n\cap B_{2n})\subset(T\mathbb{R}^n\cap C_{2n}^\star),$$

and with $L=T\mathbb{R}^n$ the corollary is proved. □

4.17 Lemma. *Let $S\subset B_n$ be an n-dimensional simplex, and let $\boldsymbol{x}\in S$. For $k\in\{1,\dots,n\}$ there exists a $(k-1)$-dimensional face F of S such that*

$$\boldsymbol{x}\in F+\left(\sum_{i=k}^n i^{-2}\right)^{1/2}\big(B_n\cap(\mathrm{aff}\,F)^\perp\big).$$

Proof. Let $\rho(n,k)=(\sum_{i=k}^n i^{-2})^{1/2}$. For the given $\boldsymbol{x}\in S$ let $\rho\geq 0$ be the maximum value such that $\boldsymbol{x}+\rho B_n\subset S$. Then $\rho\leq\mathrm{r}(S)\leq(1/n)\,\mathrm{R}(S)\leq 1/n$ (cf. ??). $\boldsymbol{x}+\rho B_n$ touches a facet $\overline{F}$, say, of S in $\overline{\boldsymbol{x}}$. Then $\boldsymbol{x}-\overline{\boldsymbol{x}}\in(\mathrm{aff}\,\overline{F})^\perp$ and $|\boldsymbol{x}-\overline{\boldsymbol{x}}|=\rho\leq 1/n$. [18]

We prove the lemma by induction on $n-k$. For $k=n$ we are done since $\rho(n,n)=1/n\geq\rho$ and by discussion above we have $\boldsymbol{x}\in\overline{F}+\rho\big(B_n\cap(\mathrm{aff}\,\overline{F})^\perp\big)$.

So we may assume that $k\leq n-1$. By induction hypothesis applied to the $(n-1)$-simplex $\overline{F}$ and the point $\overline{\boldsymbol{x}}\in\overline{F}$, there exists a $(k-1)$-face F of

[18] $(\mathrm{aff}\,\overline{F})^\perp=\{\boldsymbol{w}\in\mathbb{R}^n:\langle\boldsymbol{w},\boldsymbol{x}-\boldsymbol{y}\rangle=0\text{ for all }\boldsymbol{x},\boldsymbol{y}\in\mathrm{aff}\,F\}$

$\overline{F}$ such that $\overline{\boldsymbol{x}} \in F + \rho(n-1,k)\big((B_n \cap \operatorname{aff}\overline{F}) \cap (\operatorname{aff} F)^{\perp}\big)$. So let $\boldsymbol{y} \in F$ with $|\overline{\boldsymbol{x}} - \boldsymbol{y}| \le \rho(n-1,k)$ and $\overline{\boldsymbol{x}} - \boldsymbol{y} \in (\operatorname{aff} F)^{\perp} \cap \operatorname{aff}\overline{F}$. Then

$$\begin{aligned} |\boldsymbol{x}-\boldsymbol{y}|^2 &= |\boldsymbol{x}-\overline{\boldsymbol{x}}+\overline{\boldsymbol{x}}-\boldsymbol{y}|^2 = |\boldsymbol{x}-\overline{\boldsymbol{x}}|^2 + |\overline{\boldsymbol{x}}-\boldsymbol{y}|^2 \\ &\le \frac{1}{n^2} + \rho(n-1,k)^2 = \rho(n,k)^2. \end{aligned}$$

Finally, since also $\boldsymbol{x}-\overline{\boldsymbol{x}} \in (\operatorname{aff}\overline{F})^{\perp} \subset \operatorname{aff} F^{\perp}$ we have $\boldsymbol{x}-\boldsymbol{y} \in (\operatorname{aff} F)^{\perp}$. □

4.18 Theorem. *Let $P \subset B_n$ be an n-dimensional polytope with m vertices. Then*

$$\frac{\operatorname{vol}(P)}{\operatorname{vol}(B_n)} \le \left(c\,\frac{\log\left(\frac{m}{n}+1\right)}{n}\right)^{n/2},$$

where c is an absolute constant.

Proof. Let $P = \operatorname{conv} V$ with $V = \{\boldsymbol{v}_1,\dots,\boldsymbol{v}_m\}$. By Carathéodory [Theorem 2.7,WS2012] and Lemma 4.17 we get that for every $k \in \{1,\dots,n\}$,

$$\begin{aligned} P &= \operatorname{conv} V \\ &\subset \bigcup_{\substack{J \subset \{\boldsymbol{v}_1,\dots,\boldsymbol{v}_m\} \\ \#J = k}} \Big(\operatorname{conv}\{\boldsymbol{v}_j : j \in J\} + \big(\sum_{i=k}^{n} i^{-2}\big)^{1/2}\Big[B_n \cap \big(\operatorname{aff}\operatorname{conv}\{\boldsymbol{v}_j : j \in J\}\big)^{\perp}\Big]\Big). \end{aligned}$$

Since (cf. Exercise ?) $\operatorname{vol}_{k-1}\big(\operatorname{conv}\{\boldsymbol{v}_j : j \in J\}\big) \le \operatorname{vol}_{k-1}(S_{k-1})$, where S_{k-1} is the regular $(k-1)$-simplex with $S_{k-1} \subset B_{k-1}$ and circumradius 1. Since

$$\operatorname{vol}_{k-1}(S_{k-1}) = \left(\frac{k}{k-1}\right)^{(k-1)/2} \frac{\sqrt{k}}{(k-1)!},$$

and noticing that

$$\rho(n,k)^2 = \sum_{i=k}^{n} \frac{1}{i^2} \le \sum_{i=k}^{n} \frac{1}{i(i-1)} = \sum_{i=k}^{n} \left(\frac{1}{i-1} - \frac{1}{i}\right) = \frac{1}{k-1} - \frac{1}{n} \le \frac{1}{k-1},$$

we get

$$\begin{aligned} \frac{\operatorname{vol}(P)}{\operatorname{vol}(B_n)} &\le \binom{m}{k}\left(\frac{k}{k-1}\right)^{(k-1)/2} \frac{\sqrt{k}}{(k-1)!}\,\rho(n,k)^{n-k+1}\frac{\kappa_{n-k+1}}{\kappa_n} \\ &\le \binom{m}{k}\left(\frac{k}{k-1}\right)^{(k-1)/2} \frac{\sqrt{k}}{(k-1)!}\left(\frac{1}{k-1}\right)^{(n-k+1)/2}\frac{\kappa_{n-k+1}}{\kappa_n}. \end{aligned} \tag{4.18.1}$$

Let $k = \big\lfloor n/\ln\big((m/n)+1\big)\big\rfloor$ and, in the following, we are going to bound the logarithms of all terms in (4.18.1). Since $k\ln k = k\ln n + O(n)$ and $\ln(k!) = k\ln k - k + O(\ln k) = k\ln n + O(n)$ we have

$$\begin{aligned} \ln\binom{m}{k} &\le k\log m - \ln(k!) = k\left(\ln\frac{m}{n} + \ln n\right) - \ln(k!) \\ &= k\ln\frac{m}{n} + k\ln n - k\ln n + O(n) = k\ln\frac{m}{n} + O(n) = n + O(n) \le c\,n. \end{aligned}$$

For the next terms we find

$$\begin{aligned}
\ln\left(\left(\frac{k}{k-1}\right)^{(k-1)/2}\frac{\sqrt{k}}{(k-1)!}\right) &= \frac{k-1}{2}\ln\frac{k}{k-1}+\frac{1}{2}\ln k-\ln\big((k-1)!\big)\\
&= -k\ln n + O(n),\\
\ln\left(\frac{1}{k-1}\right)^{(n-k+1)/2} &\le -(n-k)\ln\sqrt{k-1}\\
&= -n\ln\sqrt{k-1}+k\ln\sqrt{k-1}\\
&= -\frac{n}{2}\ln k+\frac{k}{2}\ln k+O(n)\\
&= \frac{n}{2}\Big(-\ln n+\ln\ln\Big(\frac{m}{n}+1\Big)\Big)+\frac{k}{2}\ln n+O(n),
\end{aligned}$$

$$\ln\frac{\kappa_{n-k+1}}{\kappa_n}\le\ln\left(\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n-k+1}{2}+1\right)}\right)+c\,n\le\ln n^{k/2}+c\,n=\frac{k}{2}\ln n+c\,n.$$

Hence by (4.18.1) we get

$$\ln\left(\frac{\operatorname{vol}P}{\operatorname{vol}B_n}\right)\le -k\ln n+\frac{n}{2}\Big(-\ln n+\ln\ln\Big(\frac{m}{n}+1\Big)\Big)+\frac{k}{2}\ln n+\frac{k}{2}\ln n+c\,n,$$

as required. □

4.19 Theorem [Measure concentration for the sphere]. *Let $A\subseteq S^{n-1}$ with $\mu_{n-1}(A)>0$, $t\in[0,2]$, and let $A_t=\big\{\boldsymbol{u}\in S^{n-1}:\exists\,\boldsymbol{a}\in A$ with $|\boldsymbol{u}-\boldsymbol{a}|\le t\big\}$ be the spherical parallel set of A at distance t. Then*

$$\mu_{n-1}(A_t)\ge 1-\frac{1}{\mu_{n-1}(A)}\,e^{-nt^2/4}.$$

So the measure on the sphere is concentrated near the boundary of "any" set A with $\mu_{n-1}(A)\ge 1/2$, say.

Proof. Let $B=S^{n-1}\backslash A_t=\big\{\boldsymbol{u}\in S^{n-1}:|\boldsymbol{u}-\boldsymbol{a}|\ge t$ for all $\boldsymbol{a}\in A\big\}$. Then $\mu_{n-1}(A_t)=1-\mu_{n-1}(B)$ and is is to show

$$\mu_{n-1}(B)\,\mu_{n-1}(A)\le e^{-nt^2/4}. \qquad (4.19.1)$$

To this end we complete both sets with respect to B_n, i.e., we consider $\widetilde{A}=\big\{\alpha\boldsymbol{x}:\alpha\in[0,1],\,\boldsymbol{x}\in A\big\}$ and analogously we define $\widetilde{B}$. Then $\mu_{n-1}(A)=\operatorname{vol}(\widetilde{A})/\kappa_n$ and $\mu_{n-1}(B)=\operatorname{vol}(\widetilde{B})/\kappa_n$. Next we show

$$\frac{1}{2}(\widetilde{A}+\widetilde{B})\subseteq\left(1-\frac{t^2}{8}\right)B_n. \qquad (4.19.2)$$

Let $\widetilde{\boldsymbol{a}}=\alpha\boldsymbol{a}\in\widetilde{A}$ and $\widetilde{\boldsymbol{b}}=\beta\boldsymbol{b}\in\widetilde{B}$, with $\boldsymbol{a}\in A$, $\boldsymbol{b}\in B$ and $\alpha,\beta\in[0,1]$. Then

$$\begin{aligned}
\left|\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})\right|^2 &= \frac{1}{4}\big(2\,|\boldsymbol{a}|^2+2\,|\boldsymbol{b}|^2-|\boldsymbol{a}-\boldsymbol{b}|^2\big)\\
&\le\frac{1}{4}(4-t^2)=1-\frac{t^2}{4}\le\left(1-\frac{t^2}{8}\right)^2.
\end{aligned}$$

Assume $\alpha \le \beta$. Then with $\gamma = \alpha/\beta \le 1$ we get

$$\left|\frac{1}{2}(\widetilde{\boldsymbol{a}}+\widetilde{\boldsymbol{b}})\right| = \left|\frac{1}{2}(\alpha\boldsymbol{a}+\beta\boldsymbol{b})\right| = \left|\frac{\beta}{2}(\gamma\boldsymbol{a}+\boldsymbol{b})\right| = \beta\left|\gamma\frac{\boldsymbol{a}+\boldsymbol{b}}{2}+(1-\gamma)\frac{\boldsymbol{b}}{2}\right|$$
$$\le \beta\left(\gamma\left|\frac{\boldsymbol{a}+\boldsymbol{b}}{2}\right|+(1-\gamma)\left|\frac{\boldsymbol{b}}{2}\right|\right) \le \gamma\left(1-\frac{t^2}{8}\right)+(1-\gamma)\frac{1}{2} \le 1-\frac{t^2}{8}.$$

If $\alpha \le \beta$ we just replace the role of $\boldsymbol{a}$ and $\boldsymbol{b}$. This shows (4.19.2) and thus

$$\left(1-\frac{t^2}{8}\right)^n \kappa_n \ge \mathrm{vol}\left(\frac{1}{2}(\widetilde{A}+\widetilde{B})\right) \ge \sqrt{\mathrm{vol}\,(\widetilde{A})\mathrm{vol}\,(\widetilde{B})},$$

according to the Brunn-Minkowski inequality. So we get

$$\mu_{n-1}(B)\mu_{n-1}(A) = \frac{\mathrm{vol}\,(\widetilde{B})}{\kappa_n}\frac{\mathrm{vol}\,(\widetilde{A})}{\kappa_n} \le \left(1-\frac{t^2}{8}\right)^{2n} \le e^{-2nt^2/8} = e^{-nt^2/4},$$

i.e., (4.19.1). □

5 A convex body is pretty much sphere-like

5.1 Definition [**Median**]. *Let* $f : S^{n-1} \to \mathbb{R}$. *Then*

$$\mathrm{med}(f) = \sup\left\{t \in \mathbb{R} : \mu_{n-1}(f \le t) \le \frac{1}{2}\right\}$$

is called the median *of* f. *Here* $\mu_{n-1}(f \le t)$ *denotes the normalized spherical measure of the set* $\{\boldsymbol{u} \in S^{n-1} : f(\boldsymbol{u}) \le t\}$, *i.e., the probability that* $f(\boldsymbol{u}) \le t$.

5.2 Proposition. *Let* $f : S^{n-1} \to \mathbb{R}$. *Then* $\mu_{n-1}(f < \mathrm{med}(f)) \le 1/2$, *and* $\mu_{n-1}(f > \mathrm{med}(f)) \le 1/2$.

Proof. By the σ-additivity of the probability measure μ_{n-1} we have

$$\begin{aligned}\mu_{n-1}(f < \mathrm{med}(f)) &= \mu_{n-1}\left(f \le \mathrm{med}(f) - 1\right) \\ &\quad + \sum_{k=1}^{\infty} \mu_{n-1}\left(\mathrm{med}(f) - \frac{1}{k-1} < f \le \mathrm{med}(f) - \frac{1}{k}\right) \\ &= \sup_{k \ge 1} \mu_{n-1}\left(f \le \mathrm{med} f - \frac{1}{k}\right) \le \frac{1}{2}.\end{aligned}$$

□

5.3 Lemma [**Levy's Lemma**]. [19] *Let* $f : S^{n-1} \to \mathbb{R}$ *be a* 1*-Lipschitz function and let* $t > 0$. *Then*

$$\mu_{n-1}(|f - \mathrm{med}(f)| > t) \le C\,\mathrm{e}^{-c\,n\,t^2},$$

for some universal constants $c, C > 0$.

Proof. First we consider the set $U^+ = \{\boldsymbol{u} \in S^{n-1} : f(\boldsymbol{a}) > \mathrm{med}(f) + t\}$. Let $A = \{\boldsymbol{a} \in S^{n-1} : f(\boldsymbol{u}) \le \mathrm{med}(f)\}$ and $A_t = \{\boldsymbol{v} \in S^{n-1} : \exists\, \boldsymbol{a} \in A \text{ with } |\boldsymbol{v} - \boldsymbol{a}| \le t\}$. Then $U^+ \subseteq S^{n-1} \setminus A_t$, since for $\boldsymbol{v} \in A_t$ there exists an $\boldsymbol{a} \in A$ with $|\boldsymbol{v} - \boldsymbol{a}| \le t$ and so by the 1-Lipschitz property

$$f(\boldsymbol{v}) - \mathrm{med}(f) \le f(\boldsymbol{v}) - f(\boldsymbol{a}) \le |\boldsymbol{v} - \boldsymbol{a}| \le t.$$

Hence $\mu_{n-1}(U^+) \le 1 - \mu_{n-1}(A_t)$ and with Theorem 4.19 and Proposition 5.2 we obtain

$$\mu_{n-1}(U^+) \le \frac{1}{\mu_{n-1}(A)}\mathrm{e}^{-n\,t^2/4} \le 2\,\mathrm{e}^{-n\,t^2/4}.$$

The set $U^- = \{\boldsymbol{u} \in S^{n-1} : f(\boldsymbol{a}) < \mathrm{med}(f) - t\}$ can be treated analogously. □

5.4 Corollary. *Let* $f : S^{n-1} \to \mathbb{R}$ *be a* 1*-Lipschitz function,* $t > 0$, *and let* $\mathbb{E}(f) = \int_{S^{n-1}} f(\boldsymbol{u})\,\mathrm{d}\sigma(\boldsymbol{u})$. *Then*

$$\mu_{n-1}(|f - \mathbb{E}(f)| > t) \le c\,\mathrm{e}^{-n\,t^2\,C},$$

for some universal constants $c, C > 0$.

[19] Paul Pierre Lévy, 1886–1971

5.5 Definition [**δ-net**]. *For $\delta > 0$ a subset $N \subseteq S^{n-1}$ is called* δ-net *if for all $\boldsymbol{u} \in S^{n-1}$ there exists a $\boldsymbol{v} \in N$ such that $|\boldsymbol{u} - \boldsymbol{v}| \leq \delta$, i.e., $S^{n-1} = \cup_{\boldsymbol{v} \in N} C((2 - \delta^2)/2, \boldsymbol{v})$.*

5.6 Proposition. *There exists always a δ-net on S^{n-1} consisting of at most $(4/\delta)^n$ points.*

5.7 Remark. *Let $K \in \mathcal{K}_o^n$ with norm $|\cdot|$, let H be an k-dimensional subspace and let $0 < \alpha \leq \beta$. Then $\alpha \leq |\boldsymbol{v}| \leq \beta$ for all $\boldsymbol{v} \in H \cap S^{n-1}$ is equivalent to*

$$\frac{1}{\beta}(H \cap B_n) \subseteq K \cap H \subseteq \frac{1}{\alpha}(H \cap B_n).$$

In particular, $\mathrm{d}_{BM}(K \cap H, B_n \cap H) \leq \beta/\alpha$.

5.8 Lemma. *Let $|\cdot|$ be a norm on $\mathbb{R}^n$, $M, \gamma > 0$, and let N be a δ-net on S^{n-1} such that for all $\boldsymbol{v} \in N$ $M\,(1-\gamma) \leq |\boldsymbol{v}| \leq M\,(1+\gamma)$. Then for all $\boldsymbol{u} \in S^{n-1}$*

$$M\,\frac{(1-\gamma-2\,\delta)}{1-\delta} \leq |\boldsymbol{u}| \leq M\,\frac{1+\gamma}{1-\delta}.$$

5.9 Lemma. *There exists an absolute constant c such that for all norms $|\cdot|$ on $\mathbb{R}^n$ with $|\cdot| \leq \|\cdot\|$, all $\gamma > 0$ and for all*

$$k \leq c\,\frac{\gamma^2}{\ln(4/\delta)} n M^2,$$

where $M = \int_{S^{n-1}} |\boldsymbol{u}|\,\mathrm{d}\sigma(\boldsymbol{u})$, there exists a k-dimensional linear subspace H and a δ-net N in $H \cap S^{n-1}$ such that for $\boldsymbol{v} \in N$

$$M\,(1-\gamma) \leq |\boldsymbol{v}| \leq M\,(1+\gamma).$$

5.10 Theorem. *There exists an absolute constant c such that for all $K \in \mathcal{K}_o^n$, $\epsilon > 0$ and for all*

$$k \leq c\,\frac{\epsilon^2}{\ln(1+\epsilon^{-1})} n M^2,$$

where $M = \int_{S^{n-1}} |\boldsymbol{u}|_K\,\mathrm{d}\sigma(\boldsymbol{u})$, there exists a k-dimensional subspace H with $\mathrm{d}_{BM}(K \cap H, B_n \cap H) \leq 1 + \epsilon$.

5.11 Notation. *Let $\gamma_n(\cdot)$ be the standard Gaussian measure with density $\sqrt{2\pi}^{-n} \mathrm{e}^{-|\boldsymbol{x}|^2/2}$, i.e.,*

$$\gamma_n(A) = \int_{\mathbb{R}^n} \chi_A(\boldsymbol{x}) \mathrm{d}\gamma_n(\boldsymbol{x}) = \frac{1}{\sqrt{2\pi}^n} \int_A \mathrm{e}^{-|\boldsymbol{x}|^2/2} \mathrm{d}\boldsymbol{x}.$$

5.12 Proposition. *Let $K \in \mathcal{K}_o^n$. Then*

$$\frac{1}{\sqrt{n-1}} \int_{\mathbb{R}^n} |\boldsymbol{u}|_K\ \mathrm{d}\gamma_n(\boldsymbol{x}) > \int_{S^{n-1}} |\boldsymbol{u}|_K\ \mathrm{d}\sigma(\boldsymbol{u}) > \frac{1}{\sqrt{n}} \int_{\mathbb{R}^n} |\boldsymbol{u}|_K\ \mathrm{d}\gamma_n(\boldsymbol{x}).$$

Proof. In view of Proposition 2.8 we may write

$$\begin{aligned}\frac{1}{n\kappa_n}\int_{\mathbb{R}^n}|\boldsymbol{x}|_K\,\mathrm{e}^{-|x|^2/2}\,\mathrm{d}\boldsymbol{x} &= \int_0^\infty\int_{S^{n-1}} r^{n-1}\,|r\boldsymbol{u}|_K\,\mathrm{e}^{-|r\boldsymbol{u}|^2/2}\,\mathrm{d}\sigma(\boldsymbol{u})\,\mathrm{d}r\\ &= \int_{S^{n-1}}|\boldsymbol{u}|_K\;\mathrm{d}\sigma(\boldsymbol{u})\int_0^\infty r^n\mathrm{e}^{-|r|^2/2}\,\mathrm{d}r\\ &= \int_{S^{n-1}}|\boldsymbol{u}|_K\;\mathrm{d}\sigma(\boldsymbol{u})\sqrt{2}^{n-1}\int_0^\infty t^{n/2-1/2}\mathrm{e}^{-t}\,\mathrm{d}t\\ &= \sqrt{2}^{n-1}\Gamma((n+1)/2)\int_{S^{n-1}}|\boldsymbol{u}|_K\;\mathrm{d}\sigma(\boldsymbol{u}).\end{aligned}$$

Hence,

$$\begin{aligned}\int_{S^{n-1}}|\boldsymbol{u}|_K\;\mathrm{d}\sigma(\boldsymbol{u}) &= \frac{\Gamma(n/2+1)}{\Gamma((n+1)/2)}\frac{1}{n}\sqrt{2}\int_{\mathbb{R}^n}|\boldsymbol{x}|_K\left(\frac{1}{\sqrt{2\pi}}\right)^n\mathrm{e}^{-|\boldsymbol{x}|^2/2}\,\mathrm{d}\boldsymbol{x}\\ &= \frac{\Gamma(n/2)}{\Gamma((n+1)/2)}\frac{1}{\sqrt{2}}\int_{\mathbb{R}^n}|\boldsymbol{x}|_K\;\mathrm{d}\gamma_n(\boldsymbol{x})\\ &= \frac{1}{\sqrt{2\pi}}\frac{\kappa_{n-1}}{\kappa_{n-2}}\int_{\mathbb{R}^n}|\boldsymbol{x}|_K\;\mathrm{d}\gamma_n(\boldsymbol{x}),\end{aligned}$$

and it remains to show

$$\sqrt{\frac{2\pi}{n}} > \frac{\kappa_n}{\kappa_{n-1}} > \sqrt{\frac{2\pi}{n+1}},$$

which is left as an exercise. □

5.13 Proposition.

i) *Let $K=\sqrt{n}C_n^\star$ be the regular cross-polytope with inradius 1. Then*

$$\int_{S^{n-1}}|\boldsymbol{u}|_{C_n^\star}\;\mathrm{d}\sigma(\boldsymbol{u}) > \sqrt{\frac{2}{\pi}}.$$

ii) *Let $K=C_n$ be the unit cube with inradius 1. There exists an absolute constant c such that for n large*

$$\int_{S^{n-1}}|\boldsymbol{u}|_{C_n}\;\mathrm{d}\sigma(\boldsymbol{u}) > c\sqrt{\frac{\ln n}{n}}.$$

Proof. For ii) let $R>0$ such that $\gamma_n(RC_n)=\gamma_n(|\boldsymbol{x}|_{C_n}\le R)=1/2$ and so also $\gamma_n(|\boldsymbol{x}|_{C_n}\ge R)=1/2$. Hence we know

$$\begin{aligned}\int_{S^{n-1}}|\boldsymbol{u}|_{C_n}\;\mathrm{d}\sigma(\boldsymbol{u}) &> \frac{1}{\sqrt{n}}\int_{\mathbb{R}^n}|\boldsymbol{x}|_{C_n}\,\mathrm{d}\gamma_n(\boldsymbol{x})\\ &\ge \frac{1}{\sqrt{n}}\int_{\mathbb{R}^n,|\boldsymbol{x}|_{C_n}\ge R}|\boldsymbol{x}|_{C_n}\,\mathrm{d}\gamma_n(\boldsymbol{x}) \ge \frac{1}{\sqrt{n}}\frac{R}{2}.\end{aligned}\tag{5.13.1}$$

In order to lower bound R we note that

$$\frac{1}{2} = \gamma_n(|\boldsymbol{x}|_{C_n} \leq R) = \frac{1}{\sqrt{2\pi}^n} \int_{RC_n} \mathrm{e}^{-|\boldsymbol{x}|^2/2} \mathrm{d}\boldsymbol{x} = \left(\frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \mathrm{e}^{-t^2/2} \mathrm{d}t \right)^n .$$

Hence,

$$\frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \mathrm{e}^{-t^2/2} \mathrm{d}t = \mathrm{e}^{-\ln(2)/n} \geq 1 - \frac{\ln(2)}{n}.^{20}$$

For the left hand side we also have

$$\frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \mathrm{e}^{-t^2/2} \mathrm{d}t = 1 - \frac{2}{\sqrt{2\pi}} \int_{R}^{\infty} \mathrm{e}^{-t^2/2} \mathrm{d}t \leq 1 - c_1 \, \mathrm{e}^{-(R+1)^2/2},$$

for a suitable constant c_1. Combining the last two inequalities leads to $R \geq c\sqrt{\ln n}$ and we are done by (5.13.1). □

5.14 Lemma [Dvoretzky-Rogers]. *Let $K \in \mathcal{K}_o^n$ and let B_n be the volume maximal ellipsoid in K. Then there exists an orthonormal basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in S^{n-1}$ such that for $1 \leq i \leq n$*

$$|\boldsymbol{v}_i|_K \geq \frac{1}{\mathrm{e}} \left(1 - \frac{i-1}{n} \right).$$

Proof. We choose recursively the points $\boldsymbol{v}_i$ in the following way: Let $\boldsymbol{v}_1 \in S^{n-1} \subset K$ be a point of maximal norm $|\cdot|_K$. For $i > 1$ let $\boldsymbol{v}_i$ be a point of maximal norm $|\cdot|_K$ in $S^{n-1} \cap \mathrm{lin}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i-1}\}^\perp$. Then $B_n = \{\sum_{i=1}^n \alpha_i \boldsymbol{v}_i : \sum_{i=1}^n (\alpha_i)^2 \leq 1\}$, and by the choice of $\boldsymbol{v}_j$ we have for all $\boldsymbol{x} \in \mathrm{lin}\{\boldsymbol{v}_j, \ldots, \boldsymbol{v}_n\} \cap S^{n-1}$

$$|\boldsymbol{x}|_K \leq |\boldsymbol{v}_j|_K . \tag{5.14.1}$$

For $1 \leq j \leq n$ and some positive $\beta, \gamma > 0$ we consider the ellipsoid

$$\mathcal{E}_j = \left\{ \sum_{i=1}^n \alpha_i \, \boldsymbol{v}_i : \sum_{i=1}^{j-1} \frac{\alpha_i^2}{\gamma^2} + \sum_{i=j}^n \frac{\alpha_i^2}{\beta^2} \leq 1 \right\}$$

of volume $\mathrm{vol}(\mathcal{E}_j) = \gamma^{j-1}\beta^{n-(j-1)}\kappa_n$. Let $\sum_{i=1}^n \alpha_i \, \boldsymbol{v}_i \in \mathcal{E}_j$. Then we certainly have $\sum_{i=1}^{j-1} \alpha_i \boldsymbol{v}_i \in \gamma B_n$ and so, since $B_n \subset K$, $\left|\sum_{i=1}^{j-1} \alpha_i \boldsymbol{v}_i\right|_K \leq \gamma$.

We also have $\left|\sum_{i=j}^n \alpha_i \boldsymbol{v}_i\right| \leq \beta$ and since $\left(\sum_{i=j}^n \alpha_i \boldsymbol{v}_i\right) / \left|\sum_{i=j}^n \alpha_i \boldsymbol{v}_i\right| \in \mathrm{lin}\{\boldsymbol{v}_j, \ldots, \boldsymbol{v}_n\} \cap S^{n-1}$ we conclude by (5.14.1) $\left|\sum_{i=j}^n \bar{\alpha}_i \boldsymbol{v}_i\right|_K \leq \beta \, |\boldsymbol{v}_j|_K$. Hence

$$\left| \sum_{i=1}^n \bar{\alpha}_i \, \boldsymbol{v}_i \right|_K \leq \gamma + \beta \, |\boldsymbol{v}_j|_K .$$

Thus, if $\gamma + \beta \, |\boldsymbol{v}_j|_K \leq 1$ we have $\mathcal{E}_j \subseteq K$ and by the volume maximality of B_n we get for $j = 1, \ldots, n$

$$\gamma^{j-1}\beta^{n-(j-1)} \leq 1 \text{ for all } \gamma, \beta > 0 \text{ with } \gamma + \beta \, |\boldsymbol{v}_j|_K \leq 1.$$

[20] $\mathrm{e}^{-t} \geq 1 - t$

Hence, setting $\beta = (1-\gamma)/\left|\boldsymbol{v}_j\right|_K$ yields

$$\left|\boldsymbol{v}_j\right|_K^{n-j+1} \geq (1-\gamma)^{n-j+1}\gamma^{j-1},$$

which for $\gamma = \frac{j-1}{n}$ leads to

$$\left|\boldsymbol{v}_j\right|_K \geq \left(1-\frac{j-1}{n}\right)\left(\frac{j-1}{n}\right)^{\frac{j-1}{n-j+1}} \geq \left(1-\frac{j-1}{n}\right)\frac{1}{\mathrm{e}}.$$

□

5.15 Theorem [Dvoretzky]. *There exists an absolute constant c such that for all $K \in \mathcal{K}_o^n$ and $\epsilon > 0$ and for*

$$k \leq c\frac{\epsilon^2}{\ln(1+\epsilon^{-1})}\ln(n)$$

there exists a k-dimensional subspace H with $\mathrm{d}_{BM}(K\cap H, B_n \cap H) \leq 1+\epsilon$.

Proof. We may assume that B_n is the volume maximal ellipsoid in K. In view of Theorem 5.10 it is to show that

$$M = \int_{S^{n-1}} |\boldsymbol{u}|_K \,\mathrm{d}\sigma(\boldsymbol{u}) \geq c\sqrt{\frac{\ln n}{n}}.$$

Let $\boldsymbol{v}_1,\dots,\boldsymbol{v}_n$ be an orthonormal basis according to Lemma 5.14. Then $S^{n-1} = \{\sum_{i=1}^n \alpha_i\boldsymbol{v}_i : \sum_{i=1}^n \alpha_i^2 = 1\}$ and for $i \leq n/2$, say, we, in particular, have

$$|\boldsymbol{v}_i|_K \geq \bar{c},$$

for a certain constant $\bar{c}$. Hence, we may write

$$\begin{aligned}
\int_{S^{n-1}} |\boldsymbol{u}|_K \,\mathrm{d}\sigma(\boldsymbol{u}) &= \int_{S^{n-1}} \left|\sum_{i=1}^n \alpha_i\boldsymbol{v}_i\right|_K \mathrm{d}\sigma(\alpha) \\
&= \int_{S^{n-1}} \frac{1}{2}\left(\left|\sum_{i=1}^{n-1}\alpha_i\boldsymbol{v}_i + \alpha_n\boldsymbol{v}_n\right|_K + \left|\sum_{i=1}^{n-1}\alpha_i\boldsymbol{v}_i - \alpha_n\boldsymbol{v}_n\right|_K\right)\mathrm{d}\sigma(\alpha) \\
&\geq \int_{S^{n-1}} \max\left\{\left|\sum_{i=1}^{n-1}\alpha_i\boldsymbol{v}_i\right|_K, |\alpha_n\boldsymbol{v}_n|_K\right\}\mathrm{d}\sigma(\alpha),
\end{aligned}$$

where we have used the triangle inequality $2\,|a|_K \leq |a+b+a-b|_K \leq |a+b|_K + |a-b|_K$. Repeating in this way we obtain

$$\begin{aligned}
\int_{S^{n-1}} |\boldsymbol{u}|_K \,\mathrm{d}\sigma(\boldsymbol{u}) &\geq \int_{S^{n-1}} \max\{|\alpha_1 \boldsymbol{v}_1|_K, \cdots, |\alpha_n \boldsymbol{v}_n|_K\} \,\mathrm{d}\sigma(\alpha) \\
&\geq \overline{c} \int_{S^{n-1}} \max\left\{|\alpha_1|, \cdots, \left|\alpha_{n/2}\right|\right\} \mathrm{d}\sigma(\alpha) \\
&\geq \frac{\overline{c}}{\sqrt{n}} \int_{\mathbb{R}^n} \max\left\{|x_1|, \cdots, \left|x_{n/2}\right|\right\} \mathrm{d}\gamma_n(\boldsymbol{x}) \\
&= \frac{\overline{c}}{\sqrt{n}} \int_{\mathbb{R}^{n/2}} \max\left\{|x_1|, \cdots, \left|x_{n/2}\right|\right\} \mathrm{d}\gamma_{n/2}(\boldsymbol{x}) \\
&= \frac{\overline{c}}{\sqrt{n}} \int_{\mathbb{R}^{n/2}} |\boldsymbol{x}|_{C_{n/2}} \,\mathrm{d}\gamma_{n/2}(\boldsymbol{x}) \\
&\geq c \int_{S^{n/2-1}} |\boldsymbol{u}|_{C_{n/2}} \,\mathrm{d}\sigma(\boldsymbol{u}) \geq c\sqrt{\frac{\ln n}{n}},
\end{aligned}$$

where we have applied Propositions 5.12 and 5.13. □

6 Polytopal aspects of the log-Minkowski problem

6.1 Definition [Facet data]. *Let $P \subset \mathbb{R}^n$, $\dim P = n$, be a polytope with m facets $F_1, \dots, F_m$ Let $\boldsymbol{u}_i \in S^{n-1}$ be the outer unit normal vector of the facet F_i, and let $\phi_i = \mathrm{vol}_{n-1}(F_i)$. The data $\{(\boldsymbol{u}_i, \phi_i) \in S^{n-1} \times \mathbb{R}^n_{>0} : 1 \leq i \leq m\}$ are called the* facet data *of P.*

6.2 Theorem [Minkowski, 1903]. *$\{(\boldsymbol{u}_i, \phi_i) \in S^{n-1} \times \mathbb{R}^n_{>0} : 1 \leq i \leq m\}$ are the facet data of an n-dimensional polytope if and only if $\boldsymbol{u}_i \neq \boldsymbol{u}_j$, $i \neq j$, $\mathrm{lin}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_m\} = \mathbb{R}^n$ and*

$$\sum_{i=1}^{m} \phi_i \, \boldsymbol{u}_i = \boldsymbol{0}.$$

Moreover, P is by its facet data uniquely determined (up to translations).

Proof. For the necessity we observe that the volume is a translation-invariant functional, and so for $\boldsymbol{t} \in \mathbb{R}^n$

$$\begin{aligned}\mathrm{vol}(P) = \mathrm{vol}(\boldsymbol{t} + P) &= \frac{1}{n} \sum_{i=1}^{m} h(\boldsymbol{t} + P, \boldsymbol{u}_i)\, \phi_i \\ &= \frac{1}{n} \sum_{i=1}^{m} (h(P, \boldsymbol{u}_i) + \langle \boldsymbol{t}, \boldsymbol{u}_i \rangle)\, \phi_i = \frac{1}{n} \sum_{i=1}^{m} h(P, \boldsymbol{u}_i)\, \phi_i + \frac{1}{n} \sum_{i=1}^{m} \langle \boldsymbol{t}, \boldsymbol{u}_i \rangle \, \phi_i \\ &= \mathrm{vol}(P) + \left\langle \boldsymbol{t}, \sum_{i=1}^{m} \phi_i \, \boldsymbol{u}_i \right\rangle,\end{aligned}$$

where $h(P, \cdot)$ is the support function of P. Since this holds true for any $\boldsymbol{t} \in \mathbb{R}^n$ we must have $\sum_{i=1}^{m} \phi_i \, \boldsymbol{u}_i = \boldsymbol{0}$. Since P is bounded we have $\mathrm{lin}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_m\} = \mathbb{R}^n$, and of course, the outer normals are pairwise different.

In order to prove the sufficiency we consider for $\boldsymbol{b} \in \mathbb{R}^m$

$$P(\boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}_i, \boldsymbol{x} \rangle \leq b_i, \, 1 \leq i \leq m\}.$$

Observe, due to our assumption we actually know that $\mathrm{pos}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_m\} = \mathbb{R}^n$, and hence $P(\boldsymbol{b})$ is a bounded set, i.e., a (possible empty) polytope. The outer unit normals of the facets of $P(\boldsymbol{b})$ are a subset of $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_m\}$, and so let for $1 \leq i \leq m$

$$\beta_i(\boldsymbol{b}) = \mathrm{vol}_{n-1}\{\text{facet of } P(\boldsymbol{b}) \text{ with outer normal } \boldsymbol{u}_i\},$$

where we set $\beta_i(\boldsymbol{b}) = 0$ if $P(\boldsymbol{b})$ has no facet with outer normal $\boldsymbol{u}_i$. Observe that we may uniquely identify the facets of $P(\boldsymbol{b})$ via the vectors $\boldsymbol{u}_i$ since they are pairwise different. So we have

$$\mathrm{vol}(P(\boldsymbol{b})) = \frac{1}{n} \sum_{i=1}^{m} b_i \, \beta_i(\boldsymbol{b}).$$

Next we consider the linear function $\Psi : \mathbb{R}^m \to \mathbb{R}$ with

$$\Psi(\boldsymbol{b}) = \frac{1}{n} \sum_{i=1}^{m} b_i \, \phi_i.$$

Let $S = \{\boldsymbol{b} \in \mathbb{R}^m : \boldsymbol{b} \geq \boldsymbol{0} \text{ and } \mathrm{vol}\,(P(\boldsymbol{b})) = 1\}$ and we set

$$\Psi^\star = \inf\{\Psi(\boldsymbol{b}) : \boldsymbol{b} \in S\}.$$

Let $\boldsymbol{b}_i \in S$, $i \in \mathbb{N}$, with $\Psi^\star = \lim_{i\to\infty} \Psi(\boldsymbol{b}_i)$. By the non-negativity of $\boldsymbol{b}_i$ and the values ϕ, the sequence $\boldsymbol{b}_i \in S$, $i \in \mathbb{N}$, is bounded. Hence we may assume that it converges to $\boldsymbol{b}^\star \in \mathbb{R}^m_{\geq 0}$. Since $\mathrm{vol}\,(\cdot)$ is a continuous we also have $\mathrm{vol}\,(P(\boldsymbol{b}^\star)) = 1$, i.e., $\boldsymbol{b}^\star \in S$, and so $\Psi(\boldsymbol{b}^\star) \leq \Psi(\boldsymbol{b})$ for all $\boldsymbol{b} \in S$. In fact, we have

$$\Psi(\boldsymbol{b}^\star) \leq \Psi(\boldsymbol{b}) \text{ for all } \boldsymbol{b} \in \mathbb{R}^m \text{ with } \mathrm{vol}\,(P(\boldsymbol{b})) = 1. \tag{6.2.1}$$

To this end we observe, that for any $\boldsymbol{b} \in \mathbb{R}^m$ there exists a $\boldsymbol{t} \in \mathbb{R}^n$ such that $b_i + \langle \boldsymbol{t}, \boldsymbol{u}_i \rangle \geq 0$, $1 \leq i \leq m$, because we just translate $P(\boldsymbol{b})$ such that $\boldsymbol{t} + P(\boldsymbol{b})$ contains the origin. Assuming $\mathrm{vol}\,(P(\boldsymbol{b})) = 1$ we also have $1 = \mathrm{vol}\,(\boldsymbol{t} + P(\boldsymbol{b})) = \mathrm{vol}\,(P(\tilde{\boldsymbol{b}}))$ with $\tilde{b}_i = b_i + \langle \boldsymbol{t}, \boldsymbol{u}_i \rangle$, $1 \leq i \leq m$. Then $\tilde{\boldsymbol{b}} \in S$ and in view of our assumption we find

$$\begin{aligned} \Psi(\boldsymbol{b}) &= \frac{1}{n}\sum_{i=1}^m b_i\,\phi_i = \frac{1}{n}\sum_{i=1}^m b_i\,\phi_i + \frac{1}{n}\sum_{i=1}^m \langle \boldsymbol{t}, \phi_i \boldsymbol{u}_i \rangle \\ &= \frac{1}{n}\sum_{i=1}^m \tilde{b}_i\,\phi_i = \Psi(\tilde{\boldsymbol{b}}) \geq \Psi(\boldsymbol{b}^\star). \end{aligned}$$

Let $\boldsymbol{\beta}^\star = (\beta_1(\boldsymbol{b}^\star), \ldots, \beta_m(\boldsymbol{b}^\star))^\intercal$ be the vector with facet areas of $P(\boldsymbol{b}^\star)$, and let $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_m)^\intercal$ be the vector with the desired facet areas. Next we show that these vectors are proportional and hence, $P(\boldsymbol{b}^\star)$ is a homothetic copy of the polytope we are looking for. To this we will prove that

$$\left\{\boldsymbol{b} \in \mathbb{R}^m : \frac{1}{n}\langle \boldsymbol{\phi}, \boldsymbol{b} \rangle = \Psi(\boldsymbol{b}^\star)\right\} = \left\{\boldsymbol{b} \in \mathbb{R}^m : \frac{1}{n}\langle \boldsymbol{\beta}^\star, \boldsymbol{b} \rangle = 1\right\}.$$

In order to prove it, we first observe that $\boldsymbol{b}^\star$ is contained in both sets. Let U be an open neighbourhood of $\boldsymbol{b}^\star$ in the left hand set. It suffices to prove

$$U \subset \left\{\boldsymbol{b} \in \mathbb{R}^m : \frac{1}{n}\langle \boldsymbol{\beta}^\star, \boldsymbol{b} \rangle \leq 1\right\}. \tag{6.2.2}$$

Next we choose the neighbourhood U of $\boldsymbol{b}^\star$ such that for all $\boldsymbol{b} \in U$

$$\beta_i(\boldsymbol{b}) \neq 0 \text{ if } \beta_i(\boldsymbol{b}^\star) \neq 0. \tag{6.2.3}$$

Let $\boldsymbol{c} \in U$. Then $\Psi(\boldsymbol{c}) = \Psi(\boldsymbol{b}^\star)$, and if $\mathrm{vol}\,(P(\boldsymbol{c})) = \mu > 1$ then we get $\mathrm{vol}\,(P(\mu^{-1/n}\boldsymbol{c})) = 1$ and $\Psi(\mu^{-1/n}\boldsymbol{c}) < \Psi(\boldsymbol{b}^\star)$ violating the minimality of $\boldsymbol{b}^\star$. Hence we have $\mathrm{vol}\,(P(\boldsymbol{c})) \leq 1$ and for $\lambda \in [0,1]$ let

$$P_\lambda = P(\lambda \boldsymbol{b}^\star + (1-\lambda)\boldsymbol{c}).$$

By the linearity of Ψ we have $\Psi(P_\lambda) = \Psi(\boldsymbol{b}^\star)$ and as above we conclude $\mathrm{vol}\,(P_\lambda) \leq 1$ for $0 \leq \lambda \leq 1$. On the other hand we have $P(\lambda \boldsymbol{b}^\star) + P((1-\lambda)\boldsymbol{c}) \subseteq$

P_λ and so we get

$$\begin{aligned}1 &\geq \mathrm{vol}\,(P_\lambda)\\ &\geq \mathrm{vol}\,(P(\lambda \boldsymbol{b}^\star) + P((1-\lambda)\boldsymbol{c})) = \mathrm{vol}\,(\lambda\, P(\boldsymbol{b}^\star) + (1-\lambda)\, P(\boldsymbol{c}))\\ &= \sum_{i=0}^{n} \binom{n}{i} \mathrm{V}_{n-i}(P(\boldsymbol{b}^\star), P(\boldsymbol{c}))\lambda^i (1-\lambda)^{n-i},\end{aligned}$$

where on the right hand side we have the so called Steiner polynomial whose coefficients are the mixed volumes (cf. [Skript WS12, Theorem 5.20]). Since the Steiner polynomial above is 1 at $\lambda = 1$, the left hand side derivative of the polynomial at $\lambda = 1$ has to be non-negative which gives

$$n\mathrm{V}_0(P(\boldsymbol{b}^\star), P(\boldsymbol{c})) - n\mathrm{V}_1(P(\boldsymbol{b}^\star), P(\boldsymbol{c})) \geq 0.$$

Since $\mathrm{V}_0(P(\boldsymbol{b}^\star), P(\boldsymbol{c})) = \mathrm{vol}\,(P(\boldsymbol{b}^\star)) = 1$ we get (cf. [Skript WS12, Theorem 5.26])

$$\begin{aligned}1 \geq \mathrm{V}_1(P(\boldsymbol{b}^\star), P(\boldsymbol{c})) &= \mathrm{V}(\underbrace{P(\boldsymbol{b}^\star), \ldots, P(\boldsymbol{b}^\star)}_{n-1}, P(\boldsymbol{c}))\\ &= \frac{1}{n}\sum_{i=1}^{m} h(P(\boldsymbol{c}), \boldsymbol{u}_i)\, \beta_i(\boldsymbol{b}^\star) = \frac{1}{n}\langle \boldsymbol{c}, \boldsymbol{\beta}^\star\rangle\,,\end{aligned}$$

where for the last equation we have used (6.2.3). Hence we have shown (6.2.2).

So we know $\boldsymbol{\beta}^\star = \gamma\, \boldsymbol{\phi}$ for some $\gamma > 0$. From $n = \langle \boldsymbol{\beta}^\star, \boldsymbol{b}^\star\rangle = \gamma\,\langle \boldsymbol{\phi}, \boldsymbol{b}^\star\rangle = \gamma\, n\Psi(\boldsymbol{b}^\star)$ we conclude $\boldsymbol{\beta}^\star = \frac{1}{\Psi(\boldsymbol{b}^\star)}\boldsymbol{\phi}$ and $P(\Psi(\boldsymbol{b}^\star)^{\frac{1}{n-1}}\boldsymbol{b}^\star)$ is a polytope whose facet data are $\{(\boldsymbol{u}_i, \phi_i) : 1 \leq i \leq m\}$.

Now suppose that there are two polytopes P, Q having the same facet data $\{(\boldsymbol{u}_i, \phi_i) : 1 \leq i \leq m\}$. Then we have

$$\mathrm{V}(Q, \ldots, Q, P) = \frac{1}{n}\sum_{i=1}^{m} h_P(\boldsymbol{u}_i)\, \phi_i = \mathrm{vol}\,(P),$$

which gives with Minkowski's first inequality (cf. [Skript WS12, Theorem 5.32 i)])

$$\mathrm{vol}\,(P)^n = \mathrm{V}(Q, \ldots, Q, P)^n \geq \mathrm{vol}\,(Q)^{n-1}\mathrm{vol}\,(P). \tag{6.2.4}$$

Hence $\mathrm{vol}\,(P) \geq \mathrm{vol}\,(Q)$ and interchanging the role of P and Q gives $\mathrm{vol}\,(P) = \mathrm{vol}\,(Q)$. Thus we have equality in (6.2.4) and P and Q must be homothetic, which implies P and Q are translates of each other. □

6.3 Definition [Essential boundary]. *Let $K \in \mathcal{K}^n$ with $\dim K = n$. The* essential boundary *of K, denoted by $\mathrm{bd}'K$, is the set of all boundary points $\boldsymbol{x} \in \mathrm{bd}\,K$ having an unique outer normal vector, denoted by $\eta_K(\boldsymbol{x}) \in S^{n-1}$.*

6.4 Remark. *The* Gauss *map $\eta_K(\cdot) : \mathrm{bd}'K \to S^{n-1}$ is well-defined, and for each Borel set $B \subseteq S^{n-1}$ the set $\eta_K^{-1}(B)$ is a measurable set with respect to the $(n-1)$-Hausdorff measure $\mathcal{H}_{n-1}(\cdot)$ on $\mathrm{bd}\,K$.*

6.5 Definition [Area measure]. *Let $K \in \mathcal{K}^n$ with $\dim K = n$. Let $\mathcal{B}(S^{n-1})$ be the set of all Borel sets of S^{n-1}. The measure $\mathrm{S}_1(K,\cdot) : \mathcal{B}(S^{n-1}) \to \mathbb{R}_{\geq 0}$ given by*

$$\mathrm{S}_1(K,B) = \mathcal{H}_{n-1}(\eta_K^{-1}(B)) = \int_{\eta_K^{-1}(B)} \mathrm{d}\mathcal{H}_{n-1}(\boldsymbol{u})$$

is called area measure *of K.*

6.6 Remark. *Observe for a polytope P with outer normals $\boldsymbol{u}_i$ and with facet areas α_i, $1 \leq i \leq m$, it is $\mathrm{S}_1(P,\{\boldsymbol{u}_i\}) = \alpha_i$, and so*

$$\mathrm{S}_1(P,B) = \sum_{\boldsymbol{u}_i \in B} \alpha_i.$$

6.7 Theorem* [Alexandrov, Fenchel&Jessen, Minkowski]. *Let σ be a Borel measure on S^{n-1} which is not concentrated in a great circle. There exists an n-dimensional convex body $K \subset \mathbb{R}^n$ with area measure σ if and only if*

$$\int_{S^{n-1}} \boldsymbol{u}\, \mathrm{d}\sigma(\boldsymbol{u}) = \mathbf{0}.$$

6.8 Definition [Cone data]. *Let $P \subset \mathbb{R}^n$, $\dim P = n$, be a polytope with m facets $F_1,\dots,F_m$ and $\mathbf{0} \in \operatorname{int} P$. Let $\boldsymbol{u}_i \in S^{n-1}$ be the outer unit normal vector of the facet F_i, and let $\gamma_i = \mathrm{vol}\,(\mathrm{conv}\,\{\mathbf{0},F_i\})$, $1 \leq i \leq m$. The data $\{(\boldsymbol{u}_i,\gamma_i) \in S^{n-1} \times \mathbb{R}^n_{>0} : 1 \leq i \leq m\}$ are called the* cone data *of P.*

6.9 Definition [Subspace concentration condition]. *The data $\{(\boldsymbol{u}_i,\gamma_i) \in S^{n-1} \times \mathbb{R}^n_{>0} : 1 \leq i \leq m\}$ satisfy the so called* subspace concentration condition *if for every subspace $L \subseteq \mathbb{R}^n$*

$$\sum_{\boldsymbol{u}_i \in L} \gamma_i \leq \frac{\dim L}{n} \mathrm{vol}\,(P), \tag{6.9.1}$$

and equality holds for a subspace L if and only if there exists a subspace $\overline{L}$, complementary to L, such that

$$A = (A \cap L) \cup (A \cap \overline{L}), \tag{6.9.2}$$

where $A = \{\boldsymbol{u}_1,\dots,\boldsymbol{u}_m\}$.

6.10 Theorem. *The cone data of a polytope with centroid at the origin satisfy the subspace concentration condition.*

The proof is based on several lemmas which we prove first.

6.11 Proposition. *Let $f : C \to \mathbb{R}_{>0}$ be a positive log-concave function on an open convex set $C \subseteq \mathbb{R}^n$. For all $\boldsymbol{x},\boldsymbol{y} \in C$ there exists a subgradient $g(\boldsymbol{y}) \in \mathbb{R}^n$ such that*

$$\ln f(\boldsymbol{x}) - \ln f(\boldsymbol{y}) \leq \langle g(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle .$$

If f is differentiable at y, the subgradient is the gradient of $\ln f$ at $\boldsymbol{y}$, i.e., $g(\boldsymbol{y}) = \nabla \ln f(\boldsymbol{y}) = \frac{1}{f(\boldsymbol{y})} \nabla f(\boldsymbol{y})$.

6.12 Lemma. *Let $C \in \mathcal{K}_o^n$, and let $f : \operatorname{int} C \to \mathbb{R}_{>0}$ be a log-concave function with $\int_C f(\boldsymbol{x})\, \boldsymbol{x} \,\mathrm{d}\boldsymbol{x} = \mathbf{0}$. Furthermore, assume that $\nabla f(\boldsymbol{x})$ exists almost everywhere on $\operatorname{int} C$, and that also $\int_C \langle \boldsymbol{x}, \nabla f(\boldsymbol{x})\rangle \,\mathrm{d}\boldsymbol{x}$ exists. Then*

$$\int_C \langle \boldsymbol{x}, \nabla f(\boldsymbol{x})\rangle \,\mathrm{d}\boldsymbol{x} \leq 0,$$

with equality if and only if there exist $\boldsymbol{c} \in \mathbb{R}^n$, $\gamma \in \mathbb{R}_{>0}$ with $f(\boldsymbol{x}) = \gamma\, \mathrm{e}^{\langle \boldsymbol{c}, \boldsymbol{x}\rangle}$.

Proof. By Proposition 6.11 we have for all $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{int} C$

$$\ln f(\boldsymbol{x}) - \ln f(\boldsymbol{y}) \leq \langle g(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y}\rangle, \tag{6.12.1}$$

where $g(\boldsymbol{y})$ is a subgradient at $\boldsymbol{y}$. Interchanging the role of $\boldsymbol{x}$ and $\boldsymbol{y}$ and adding leads to

$$0 \leq \langle g(\boldsymbol{y}) - g(\boldsymbol{x}), \boldsymbol{x} - \boldsymbol{y}\rangle .$$

Setting $\boldsymbol{y} = \mathbf{0}$ leads to $\langle g(\boldsymbol{x}), \boldsymbol{x}\rangle \leq \langle g(\mathbf{0}), \boldsymbol{x}\rangle$. For points $\boldsymbol{x} \in C'$, where $C' \subseteq C$ is the set where $\nabla f(\boldsymbol{x})$ exists, this gives

$$\langle \nabla f(\boldsymbol{x}), \boldsymbol{x}\rangle \leq \langle g(\mathbf{0}), f(\boldsymbol{x})\, \boldsymbol{x}\rangle .$$

Hence in view of our assumption on ∇f and on the first moment of f on C we get

$$\begin{aligned}\int_C \langle \boldsymbol{x}, \nabla f(\boldsymbol{x})\rangle \,\mathrm{d}\boldsymbol{x} &= \int_{C'} \langle \boldsymbol{x}, \nabla f(\boldsymbol{x})\rangle \,\mathrm{d}\boldsymbol{x} \\ &\leq \int_{C'} \langle g(\mathbf{0}), f(\boldsymbol{x})\, \boldsymbol{x}\rangle \,\mathrm{d}\boldsymbol{x} \\ &= \left\langle g(\mathbf{0}), \int_C f(\boldsymbol{x})\, \boldsymbol{x} \,\mathrm{d}\boldsymbol{x} \right\rangle = 0.\end{aligned}$$

If the inequality holds with equality, we must have almost everywhere equality in (6.12.1) for $\boldsymbol{y} = \mathbf{0}$. Hence, $\ln f(\boldsymbol{x})$ is an affine function. Together with the positivity of f on $\operatorname{int} C$ there exist $\boldsymbol{c} \in \mathbb{R}^n$, $\gamma \in \mathbb{R}_{>0}$ with $f(\boldsymbol{x}) = \gamma\, \mathrm{e}^{\langle \boldsymbol{c}, \boldsymbol{x}\rangle}$. On the other hand, if f is of this form then $\nabla f(\boldsymbol{x}) = f(\boldsymbol{x})\, \boldsymbol{c}$ and so

$$\int_C \langle \boldsymbol{x}, \nabla f(\boldsymbol{x})\rangle \,\mathrm{d}\boldsymbol{x} = \left\langle \boldsymbol{c}, \int_C f(\boldsymbol{x})\boldsymbol{x} \,\mathrm{d}\boldsymbol{x} \right\rangle = 0.$$

□

6.13 Notation. *For a convex body $K \in \mathcal{K}_o^n$ and a k-dimensional subspace L, $0 \leq k \leq n$, let*

$$f_L : K|L \to \mathbb{R}_{\geq 0} \text{ with } \boldsymbol{x} \mapsto \mathrm{V}_{n-k}(K \cap (\boldsymbol{x} + L^{\perp})), \tag{6.13.1}$$

where $L^{\perp}$ is the orthogonal complement of L, and $\mathrm{V}_{n-k}(\cdot)$ denotes the $(n-k)$-dimensional volume

6.14 Remark. *f_L is on the interior of $K|L$ a log-concave function, and*

$$\int_{K|L} f_L(\boldsymbol{x})\,\boldsymbol{x}\;\mathrm{d}\boldsymbol{x} = \mathbf{0}.$$

Proof. Left as an exercise. □

6.15 Remark. *Let $P \in \mathcal{P}^n$ be an n-dimensional polytope and let L be a k-dimensional subspace, $0 \le k \le n-1$. The union of all k-faces of P is called the* k-skeleton *of P. The orthogonal projection of the $(k-1)$-skeleton onto a k-dimensional plane L induces a polytopal subdivision $\mathcal{D}(P)_L$ of $P|L$,* i.e., *$\mathcal{D}(P)_L$ is a collection of k-dimensional polytopes having pairwise disjoint interiors, the intersection of any two of them is a face of both, the union covers $P|L$ and the preimage of the boundary of a polytope in $\mathcal{D}(P)_L$ is contained in a $(k-1)$-face of P.*

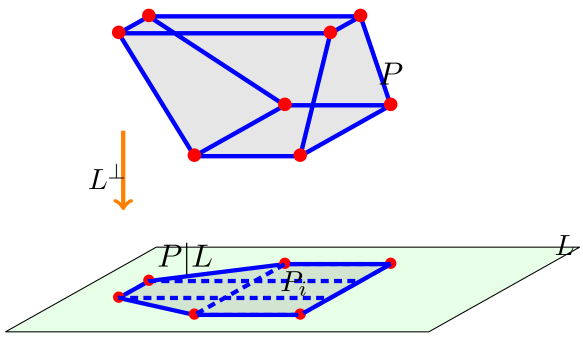

Figure 1: The polytopal subdivision induced by the orthogonal projection of the $(k-1)$-skeleton of P onto L

6.16 Proposition*. *Let L be a k-dimensional subspace, $0 < k < n$. The function $f_L : P|L \to \mathbb{R}_{\ge 0}$ is a piecewise polynomial function of degree at most $n-k$; more precisely, on every k-dimensional polytope of the subdivision $\mathcal{D}(P)_L$ it is a polynomial of degree at most $n-k$.*

6.17 Lemma. *Let $P \in \mathcal{K}^n_o$ be a polytope, let A be the set of its outer unit normals, and let $L \subset \mathbb{R}^n$ be a k-dimensional subspace, $0 < k < n$. Then $f_L : P|L \to \mathbb{R}_{\ge 0}$ is a constant function if and only if there exists a subspace $\overline{L}$, complementary to L, such that*

$$A = (A \cap L) \cup (A \cap \overline{L}).$$

Proof. Suppose $f_L(\boldsymbol{x}) = f_L(\mathbf{0})$ for all $\boldsymbol{x} \in P|L$. Then, in particular,

$$f_L((1-\lambda)\boldsymbol{x} + \lambda\,\mathbf{0})^{1/(n-k)} = (1-\lambda) f_L(\boldsymbol{x})^{1/(n-k)} + \lambda\, f_L(\mathbf{0})^{1/(n-k)}$$

for all $\lambda \in [0,1]$ and $\boldsymbol{x} \in P|L$. Hence we have equality in the Brunn-Minkowski inequality [Skript WS12, 5.16] applied in the space $L^\perp$ and thus, for every $\boldsymbol{x} \in P|L$ there exists a uniquely determined $\boldsymbol{t}(\boldsymbol{x}) \in \mathbb{R}^n$ such that

$$(\boldsymbol{x} + L^\perp) \cap P = \boldsymbol{t}(\boldsymbol{x}) + (L^\perp \cap P).$$

Let $\boldsymbol{t} : P|L \to \mathbb{R}^n$ be the associated map sending $\boldsymbol{x} \mapsto \boldsymbol{t}(\boldsymbol{x})$. Then $\boldsymbol{t}(\cdot)$ is injective and convex linear, i.e., $\boldsymbol{t}((1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y}) = (1-\lambda)\boldsymbol{t}(\boldsymbol{x}) + \lambda\boldsymbol{t}(\boldsymbol{y})$ for $\boldsymbol{x}, \boldsymbol{y} \in P|L$ and $\lambda \in [0,1]$, since by convexity and definition of $\boldsymbol{t}(\cdot)$ we have

$$(1-\lambda)\boldsymbol{t}(\boldsymbol{x}) + \lambda\boldsymbol{t}(\boldsymbol{y}) + (L^\perp \cap P) \subseteq ([(1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y}] + L^\perp) \cap P.$$

Together with constant volume property we have equality. Thus it is an affine function, and since $t(\boldsymbol{0}) = \boldsymbol{0}$ we conclude that $\boldsymbol{t}(\cdot)$ is linear. Hence, $\widetilde{L} = \operatorname{lin} \boldsymbol{t}(P|L)$, i.e., the linear hull of $\boldsymbol{t}(P|L)$, is a k-dimensional linear subspace and we have

$$P = (P \cap \widetilde{L}) + (P \cap L^\perp).$$

Observe, $\boldsymbol{t}(P|L) = P \cap \widetilde{L}$. Since $P \cap \widetilde{L}$ is a k-dimensional polytope and $(P \cap L^\perp)$ an $(n-k)$-dimensional polytope, the facets of P are given by

$$\tilde{F} + (P \cap L^\perp) \text{ or } F + (P \cap \widetilde{L}),$$

where $\tilde{F}$ is a facet, i.e., a $(k-1)$-face of $P \cap \widetilde{L}$ and F is a facet, i.e., a $(n-k-1)$-face of $P \cap L^\perp$. In the first case the outer unit normal of such a facet is contained in $(L^\perp)^\perp = L$, and in the latter case in $\widetilde{L}^\perp$. Hence $A = (A \cap L) \cup (A \cap \widetilde{L}^\perp)$, and since P is bounded we also know that $\widetilde{L}^\perp$ is complementary to L; otherwise, A would be contained in an $(n-1)$-dimensional subspace.

On the other hand, if we have $A = (A \cap L) \cup (A \cap \overline{L})$ for complementary subspaces $L, \overline{L}$, then it is easy to see that

$$P = (P \cap L^\perp) + (P \cap \overline{L}^\perp). \tag{6.17.1}$$

For if, assume that $\boldsymbol{u}_i \in L$, $1 \le i \le l$, say, and $\boldsymbol{u}_i \in \overline{L}$, $l+1 \le i \le m$, is a partition of the normals of the polytope $P = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}_i, \boldsymbol{x} \rangle \le b_i,\, 1 \le i \le m\}$. Then

$$\begin{aligned} P \cap L^\perp &= \{\boldsymbol{x} \in L^\perp : \langle \boldsymbol{u}_i, \boldsymbol{x} \rangle \le b_i,\, l+1 \le i \le m\}, \\ P \cap \overline{L}^\perp &= \{\boldsymbol{x} \in \overline{L}^\perp : \langle \boldsymbol{u}_i, \boldsymbol{x} \rangle \le b_i,\, 1 \le i \le l\}. \end{aligned}$$

Now let $\boldsymbol{y} \in P$. Then it can uniquely be written as $\boldsymbol{y} = \boldsymbol{v} + \boldsymbol{w}$ with $\boldsymbol{v} \in L^\perp$ and $\boldsymbol{w} \in \overline{L}^\perp$, since $L, \overline{L}$ are complementary subspaces. Moreover,

$$\begin{aligned} \langle \boldsymbol{u}_i, \boldsymbol{y} \rangle &= \langle \boldsymbol{u}_i, \boldsymbol{w} \rangle,\ 1 \le i \le l, \\ \langle \boldsymbol{u}_i, \boldsymbol{y} \rangle &= \langle \boldsymbol{u}_i, \boldsymbol{v} \rangle,\ l+1 \le i \le m. \end{aligned} \tag{6.17.2}$$

Hence, $\boldsymbol{v} \in P \cap L^\perp$ and $\boldsymbol{w} \in P \cap \overline{L}^\perp$. On the other hand, given those $\boldsymbol{v}, \boldsymbol{w}$ we also see by (6.17.2) that $\boldsymbol{v} + \boldsymbol{w} \in P$.

Thus we have shown (6.17.1). In particular, by the complementarity of the subspaces we know that for every $\boldsymbol{x} \in P|L$ there exists an unique $\boldsymbol{t}(\boldsymbol{x}) \in P \cap \overline{L}^\perp$ with $\boldsymbol{t}(\boldsymbol{x})|L = \boldsymbol{x}$. Hence, $P \cap (\boldsymbol{x} + L^\perp) = \boldsymbol{t}(\boldsymbol{x}) + (P \cap L^\perp)$ for every $\boldsymbol{x} \in P|L$, which shows $f_L(\boldsymbol{x}) = f_L(\boldsymbol{0})$. $\square$

6.18 Notation. *For n linearly independent unit vectors $V = \{v_1, \dots, v_n\}$ and a k-subset $I \subset \{1, \dots, n\}$, $0 < k < n$, we denote by $L_I(V) = \operatorname{lin}\{v_j : j \in I\}$ the k-dimensional subspace generated by this selection of vectors.*

6.19 Lemma*. *Let $P \in \mathcal{K}_o^n$ be a polytope and let $0 < k < n$. There exist n linearly independent unit vectors $V = \{v_1, \dots, v_n\}$ such that the function $f_{L_I(V)} : P|L_I(V) \to \mathbb{R}_{\geq 0}$ is constant for every k-subset $I \subset \{1, \dots, n\}$ if and only if P is a parallelotope.*

6.20 Notation. *Let $F_L : P|L \to L$ be the vector field given by*

$$F_L(\boldsymbol{x}) = f_L(\boldsymbol{x})\,\boldsymbol{x}. \tag{6.20.1}$$

6.21 Lemma. *Let P be a polytope, and let $L \subset \mathbb{R}^n$ be a k-dimensional linear subspace, $0 < k < n$. Let $\mathcal{D}(P)_L = \{P_1, \dots, P_r\}$ be the polytopal subdivision induced by the orthogonal projection of the $(k-1)$-skeleton S of P onto L. Let $F : P|L \to L$ be a vector field which is a polynomial in each component and on each polytope $P_i \in \mathcal{D}(P)_L$. Then* [21]

$$\int_{(P|L)\setminus(S|L)} \operatorname{div} F(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} = \int_{\operatorname{bd}'(P|L)} \langle F(\boldsymbol{x}), \boldsymbol{u}(\boldsymbol{x})\rangle\ \mathrm{d}\mathcal{H}^{k-1}(\boldsymbol{x}).$$

Here $\boldsymbol{u}(\boldsymbol{x}) \in L$ is the unique outer normal of the polytope $P|L$ in the boundary point $\boldsymbol{x} \in \operatorname{bd}'(P|L)$.

Proof. Since F is a polynomial vector field on each P_i, F can canonically be extended to an open neighborhood of P_i, and hence we can use the divergence theorem of Gauss and get

$$\int_{P_i} \operatorname{div} F(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} = \int_{\operatorname{bd}'(P_i)} \langle F(\boldsymbol{x}), \boldsymbol{u}_i(\boldsymbol{x})\rangle\ \mathrm{d}\mathcal{H}^{k-1}(\boldsymbol{x}),$$

with $\boldsymbol{u}_i(x) \in L$ being the unique outer normal of P_i in $\boldsymbol{x} \in \operatorname{bd}'(P_i)$. Thus, in particular, $\int_{P_i\setminus(S|L)} \operatorname{div} F(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x}$ is well defined and since $S|L$ is a set of measure 0, we have

$$\int_{(P|L)\setminus(S|L)} \operatorname{div} F(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} = \sum_{i=1}^{r} \int_{\operatorname{bd}'(P_i)} \langle F(\boldsymbol{x}), \boldsymbol{u}_i(\boldsymbol{x})\rangle\ \mathrm{d}\mathcal{H}^{k-1}(\boldsymbol{x}).$$

Every $\boldsymbol{x} \in \operatorname{bd}'(P_i) \setminus \operatorname{bd}(P|L)$ is contained in exactly one more P_j, $j \neq i$, and $\boldsymbol{u}_j(\boldsymbol{x}) = -\boldsymbol{u}_i(\boldsymbol{x})$. Hence, with $\boldsymbol{u}_i(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x})$ for $\boldsymbol{x} \in \operatorname{bd}'(P_i) \cap \operatorname{bd}'(P|L)$ we get

$$\sum_{i=1}^{r} \int_{\operatorname{bd}'(P_i)} \langle F(\boldsymbol{x}), \boldsymbol{u}_i(\boldsymbol{x})\rangle\ \mathrm{d}\mathcal{H}^{k-1}(\boldsymbol{x}) = \int_{\operatorname{bd}'(P|L)} \langle F(\boldsymbol{x}), \boldsymbol{u}(\boldsymbol{x})\rangle\ \mathrm{d}\mathcal{H}^{k-1}(\boldsymbol{x}),$$

which finishes the proof. □

Proof. [Proof of Theorem 6.10] Let $P \in \mathcal{K}_o^n$ be a polytope with centroid at the origin. Let $F_1, \dots, F_m$ be the facets of P, and let $\boldsymbol{u}_i \in S^{n-1}$ be the outer unit

[21] $\operatorname{div} F(\boldsymbol{x}) = \sum_{i=1}^{k} \partial F_i(\boldsymbol{x})/\partial x_i$

normal of the facet F_i, $1 \leq i \leq m$. Let $A = \{\boldsymbol{u}_1, \dots, \boldsymbol{u}_m\}$, and let $b_i > 0$ be the distance of the facet F_i from the origin. Let L be a k-dimensional subspace with $0 < k < n$. We have to show (cf. Definition 6.9)

$$\sum_{\boldsymbol{u}_i \in L} \mathrm{V}_{n-1}(F_i)\, b_i \leq k\, \mathrm{V}(P). \tag{6.21.1}$$

with equality if and only if there exists a subspace $\overline{L}$, complementary to L, so that $A = (A \cap L) \cup (A \cap \overline{L})$.

According to Proposition 6.16, the vector field $F_L(\boldsymbol{x}) = f_L(\boldsymbol{x})\, \boldsymbol{x}$ satisfies the assumptions of Lemma 6.21 and on account of

$$\mathrm{div} F_L(\boldsymbol{x}) = k\, f_L(\boldsymbol{x}) + \langle \nabla f_L(\boldsymbol{x}), \boldsymbol{x} \rangle$$

we get

$$\begin{aligned}
&\int_{\mathrm{bd}'(P|L)} f_L(\boldsymbol{x})\, \langle \boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}) \rangle \, \mathrm{d}\mathcal{H}^{k-1}(\boldsymbol{x}) \\
&\qquad = k \int_{(P|L)\setminus(S|L)} f_L(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \int_{(P|L)\setminus(S|L)} \langle \nabla f_L(\boldsymbol{x}), \boldsymbol{x} \rangle \, \mathrm{d}\boldsymbol{x} \\
&\qquad = k\, \mathrm{V}(P) + \int_{(P|L)\setminus(S|L)} \langle \nabla f_L(\boldsymbol{x}), \boldsymbol{x} \rangle \, \mathrm{d}\boldsymbol{x},
\end{aligned} \tag{6.21.2}$$

where in the last step we used again that $S|L$ is a set of measure 0.

Now let $\tilde{\boldsymbol{u}}_1, \dots, \tilde{\boldsymbol{u}}_l$ be the outer unit normals of the facets of $P|L$, i.e., the $(k-1)$-faces of $P|L$, having distance $\tilde{b}_i$ to the origin. Let $\tilde{F}_i = P \cap \{\boldsymbol{x} \in \mathbb{R}^n : \langle \tilde{\boldsymbol{u}}_i, \boldsymbol{x} \rangle = \tilde{b}_i\}$, $1 \leq i \leq l$, be the faces of P projected onto the facets of $P|L$. Taking into account that f_L measures the $(n-k)$-dimensional volume we have

$$\int_{\mathrm{bd}'(P|L)} f_L(\boldsymbol{x})\, \langle \boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}) \rangle \, \mathrm{d}\mathcal{H}^{k-1}(\boldsymbol{x}) = \sum_{i=1}^{l} \mathrm{V}_{n-1}(\tilde{F}_i)\, \tilde{b}_i.$$

Hence $\tilde{F}_i$ contributes to the above sum only when it is a facet of P, i.e., $F_j = \tilde{F}_i$, $\boldsymbol{u}_j = \tilde{\boldsymbol{u}}_i \in L$ and $b_j = \tilde{b}_i$ for a certain $j \in \{1, \dots, m\}$. Thus we may write (cf. (6.21.2))

$$\sum_{\boldsymbol{u}_i \in L} \mathrm{V}_{n-1}(F_i)\, b_i = k\, \mathrm{V}(P) + \int_{(P|L)\setminus(S|L)} \langle \nabla f_L(\boldsymbol{x}), \boldsymbol{x} \rangle \, \mathrm{d}\boldsymbol{x}. \tag{6.21.3}$$

Since 0 is the centroid of P we have (cf. Remark 6.14)

$$\int_{P|L} f_L(\boldsymbol{x})\, \boldsymbol{x} \, \mathrm{d}\boldsymbol{x} = \mathbf{0},$$

and since $S|L$ is a set of measure 0 we may apply Lemma 6.12 to f_L. Thus

$$\int_{(P|L)\setminus(S|L)} \langle \nabla f_L(\boldsymbol{x}), \boldsymbol{x} \rangle \, \mathrm{d}\boldsymbol{x} \leq 0, \tag{6.21.4}$$

which yields (6.21.1) by (6.21.3).

Now suppose we have equality in (6.21.1). Then we also have equality in (6.21.4) and by Lemma 6.12 there exist $\gamma > 0$, $\boldsymbol{c} \in \mathbb{R}^n$ such that $f_L(\boldsymbol{x}) = \gamma\, \mathrm{e}^{\langle \boldsymbol{c}, \boldsymbol{x} \rangle}$. Since the $(n-k)$-th root of $f_L(\boldsymbol{x})$ is concave we must have $\boldsymbol{c} = \boldsymbol{0}$, i.e., $f_L(\boldsymbol{x})$ is a constant function. Thus, by Lemma 6.17, there exists a complementary subspace $\overline{L}$ with $A = (A \cap L) \cup (A \cap \overline{L})$.

On the other hand, if we have such a partition of A into complementary subspaces L and $\overline{L}$, $\dim L = k$ and $\dim \overline{L} = n-k$, then we may either apply Lemma 6.17 and then (6.21.3), or we just observe that in this case we may write

$$\begin{aligned} k\,\mathrm{V}(P) + (n-k)\,\mathrm{V}(P) &= n\mathrm{V}(P) \\ &= \sum_{\boldsymbol{u}_i \in A \cap L} \mathrm{V}_{n-1}(F_i)\, b_i + \sum_{\boldsymbol{u}_i \in A \cap \overline{L}} \mathrm{V}_{n-1}(F_i)\, b_i. \end{aligned}$$

Hence, in view of the validity of the inequality (6.21.1) for L and $\overline{L}$, we have actually equality in (6.21.1) for L and $\overline{L}$. □

6.22 Definition [Cone volume measure]. *Let $K \in \mathcal{K}^n$ with $\dim K = n$. The measure $\mathrm{S}_0(K, \cdot) : \mathcal{B}(S^{n-1}) \to \mathbb{R}_{\geq 0}$ given by*

$$\mathrm{S}_0(K, B) = \int_{\eta_K^{-1}(B)} \langle \boldsymbol{u}, \eta_K(\boldsymbol{u}) \rangle \; \mathrm{d}\mathcal{H}_{n-1}(\boldsymbol{u})$$

is called the cone volume measure *of K.*

6.23 Remark. *If P is a polytope with cone data $\{(\boldsymbol{u}_i, \gamma_i) : 1 \leq i \leq m\}$ then*

$$\mathrm{S}_0(P, B) = n \sum_{\boldsymbol{u}_i \in B} \gamma_i.$$

6.24 Definition [Subspace concentration condition for measures]. *A finite Borel measure μ on S^{n-1} is said to satisfy the subspace concentration condition if for every subspace L*

$$\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n} \mu(S^{n-1})$$

and equality holds for a subspace L if and only if there exists a subspace $\overline{L}$, complementary to L, such that

$$\mu(S^{n-1}) = \mu(L \cap S^{n-1}) + \mu(\overline{L} \cap S^{n-1}).$$

6.25 Theorem* [Lutwak, Yang, Zhang, 2013]. *An even finite Borel measure is the cone volume measure of an o-symmetric convex body if and only if it satisfies the subspace concentration condition.*

7 Packing in Infinity

7.1 Definition [l_p**-space**]. *For $p \in [1, \infty]$ let*

$$l_P = \{\boldsymbol{x} = (x_1, x_2, x_3, \dots) : x_i \in \mathbb{R} \text{ and } |\boldsymbol{x}|_p < \infty\},$$

the space of all convergent series with respect to the p-norm

$$|\boldsymbol{x}|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}.$$

In particular, for $p = \infty$ we have $|\boldsymbol{x}|_\infty = \sup\{|x_i| : i \in \mathbb{N}\}$.

7.2 Definition [Ball]. *Let $(V, |\cdot|)$ be a normed vectorspace and let $\boldsymbol{z} \in V$, $r \in \mathbb{R}_{\geq 0}$. The set*

$$B(\boldsymbol{z}, r) = \{\boldsymbol{x} \in V : |\boldsymbol{x} - \boldsymbol{z}| \leq r\}$$

*is called a V-*ball of radius *r. For $\boldsymbol{z} = \boldsymbol{0}$ and $r = 1$ we just write B, and it is called V-*unit ball.

7.3 Proposition. *Let $(V, |\cdot|)$ be a normed vectorspace and let $\boldsymbol{y}, \boldsymbol{z} \in V$, $r, s \in \mathbb{R}_{\geq 0}$. Then*

i) $B(\boldsymbol{z}, r) \subseteq B(\boldsymbol{y}, s)$ *if and only if* $|\boldsymbol{y} - \boldsymbol{z}| \leq s - r$.

ii) $\operatorname{int} B(\boldsymbol{z}, r) \cap \operatorname{int} B(\boldsymbol{y}, s) = \emptyset$ *if and only if* $|\boldsymbol{y} - \boldsymbol{z}| \geq s + r$.

7.4 Definition [Packings set]. *Let $(V, |\cdot|)$ be a normed vectorspace and let $r > 0$. $D \subset B_p(\boldsymbol{0}, 1)$ is called a r-*packing set *if $|\boldsymbol{x} - \boldsymbol{y}| \geq 2r$ for all $\boldsymbol{x} \neq \boldsymbol{y} \in D$, and $B(\boldsymbol{z}, r) \subset B_p(\boldsymbol{0}, 1)$ for all $\boldsymbol{z} \in D$.*

7.5 Example. *For each $m \in \mathbb{N}$ we define recursively the follwing m vectors $\boldsymbol{h}_i^{(m)} \in l_p$, $1 \leq i \leq m$. For $m = 1$ we set $\boldsymbol{h}_1^{(1)} = (1, 0, 0, \dots)$. So let $m > 1$. For $1 \leq i \leq m - 1$ we define*

$$\left(\boldsymbol{h}_i^{(m)}\right)_j = \begin{cases} \left(\boldsymbol{h}_i^{(m-1)}\right)_j & , 1 \leq j \leq 2^{m-2}, \\ \left(\boldsymbol{h}_i^{(m-1)}\right)_{j-2^{m-2}} & , 2^{m-2} + 1 \leq j \leq 2^{m-1}, \\ 0 & , j > 2^{m-1}, \end{cases}$$

and for $i = m$ we set

$$\left(\boldsymbol{h}_m^{(m)}\right)_j = \begin{cases} -1 & , 1 \leq j \leq 2^{m-2}, \\ 1 & , 2^{m-2} + 1 \leq j \leq 2^{m-1}, \\ 0 & , j > 2^{m-1}. \end{cases}$$

Hence each vector $\boldsymbol{h}_i^m$ has 2^{n-1} non-zero entries which are either 1 or -1, and two different vectors $\boldsymbol{h}_i^m, \boldsymbol{h}_j^m$, $i \neq j$, differ in exactly 2^{n-2} entries.

Let $p \geq 1$, $0 < r \leq \left(1 + 2^{1/p}\right)^{-1}$ *and let* $r_m = 2^{(1-m)/p}\,(1-r)$ *and let* $\boldsymbol{y}_i^{(m)} = r_m \boldsymbol{h}_i^m$. *Then*

$$\left|\boldsymbol{y}_i^{(m)}\right|_p = r_m 2^{(m-1)/p} = 1 - r.$$

Hence, $B(\boldsymbol{y}_i^{(m)}, r) \subset B$. *Moreover, for* $i \neq j$ *we have*

$$\left|\boldsymbol{y}_i^{(m)} - \boldsymbol{y}_j^{(m)}\right|_p = r_m\, 2\, 2^{(n-2)/p} = 2^{1-1/p}(1-r) \geq 2\, r,$$

by the choice of r. *Thus, for each* $m \in \mathbb{N}$, *the set* $\{\boldsymbol{y}_i^{(m)} : 1 \leq i \leq m\} \subset (1-r)\, B$ *is a* r*-packing set, i.e., for any* $r \leq \left(1 + 2^{1/p}\right)^{-1}$ *we can pack any finite number of* l_p*-balls of radius* r *inside the* l_p*-unit ball.*

7.6 Lemma. *Let* $1 \leq p \leq 2$, $m \in \mathbb{N}$, $x_1, \dots, x_m \in \mathbb{R}_{\geq 0}$ *and let* $A = (a_{i\,j})$ *be the matrix with entries*

$$a_{i\,j} = (x_i + x_j)^p - x_i^p - x_j^p, \quad 1 \leq i, j \leq m.$$

Then A *is positive semi-definite.*

Moreover, for $1 < p \leq 2$ *and* $\boldsymbol{v} = (v_1, \dots, v_m)^\intercal \in \mathbb{R}^m$ *we have* $\boldsymbol{v}^\intercal A\, \boldsymbol{v} = 0$ *if and only if there exists a partitioning* $A_0 \cup A_1 \cup \dots \cup A_k = \{1, \dots, m\}$ *and pairwise disjoint numbers* α_i, $1 \leq i \leq k$, *such that i)* $x_i = 0$, $i \in A_0$, $x_i = \alpha_j$, $i \in A_j$, *and ii)* $\sum_{i \in A_j} v_i = 0$.

7.7 Corollary. *Let* $1 \leq p \leq 2$, $m \in \mathbb{N}$, *and let* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in l_p$ *be non-negative. Let* A *be the* $(m \times m)$*-matrix with entries* $a_{i\,j} = \|\boldsymbol{x}_i + \boldsymbol{x}_j\|_p^p - \|\boldsymbol{x}_i\|_p^p - \|\boldsymbol{x}_j\|_p^p$. *Then* A *is positive semi-definite.*

7.8 Theorem. *Let* $1 \leq p \leq 2$, $m \in \mathbb{N}$, $x_1, \dots, x_m \in \mathbb{R}$, $\alpha_1, \dots, \alpha_m \in \mathbb{R}_{\geq 0}$ *with* $\sum_{i=1}^m \alpha_i = 1$. *Then*

$$\sum_{i,j=1}^m \alpha_i\, \alpha_j\, |x_i - x_j|^p \leq 2 \sum_{i=1}^m \alpha_i\, |x_i|^p + (2^p - 4) \sum_{i=1}^m \alpha_i^2\, |x_i|^p.$$

If $1 < p < 2$ *and* $\alpha_i > 0$, $1 \leq i \leq m$, *equality holds if and only if* $x_i = 0$, $1 \leq i \leq m$, *or there exist* $k, l \in \{1, \dots, m\}$ *such that i)* $x_i = 0$ *for* $i \notin \{k, l\}$, *ii)* $x_k = -x_l$ *and iii)* $\alpha_k = \alpha_l$.

7.9 Corollary. *Let* $1 \leq p \leq 2$, $m \in \mathbb{N}$, $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in l_p$, $\alpha_1, \dots, \alpha_m \in \mathbb{R}_{\geq 0}$ *with* $\sum_{i=1}^m \alpha_i = 1$. *Then*

$$\sum_{i,j=1}^m \alpha_i\, \alpha_j\, \|\boldsymbol{x}_i - \boldsymbol{x}_j\|_p^p \leq 2 \sum_{i=1}^m \alpha_i\, \|\boldsymbol{x}_i\|_p^p + (2^p - 4) \sum_{i=1}^m \alpha_i^2\, \|\boldsymbol{x}_i\|_p^p.$$

7.10 Theorem. *Let* $1 \leq p \leq 2$, $m \in \mathbb{N}$, *and let* $r_p(m)$ *be the maximum radius of a* r*-packing set* $D \subset B_p(\mathbf{0}, 1)$ *of cardinality* m. *Then*

$$r_p(m) = \left[1 + 2\left(\frac{m-1}{2\,m + 2^p - 4}\right)^{\frac{1}{p}}\right]^{-1}.$$

7.11 Corollary. *Let $1 \le p \le 2$. There exists an infinite r-packing set $D \subset B_p(\mathbf{0},1)$ if and only if*

$$r \le \left[1 + 2^{1-1/p}\right]^{-1}.$$

For $1 \ge r > \left[1 + 2^{1-1/p}\right]^{-1}$ let $m_p(r)$ be the maximum cardinality of r-packing set $D \subset B(\mathbf{0},1)$. Then

$$m_p(r) \le 1 + \left(\frac{1-r}{r}\right)^p \left(\frac{1 - 2^{1-p}}{1 - 2^{1-p}\left(\frac{1-r}{r}\right)^p}\right).$$

7.12 Definition [Dual vectorspace]. *Let $1 \le p \le \infty$. The* dual vectorspace *of l_p, i.e., the space of all continuous linear functionals on l_p, is denoted by $l_p^\star$. For $p < \infty$*[22]*, the dual vector space may be identified for with l_q where $1/p + 1/q = 1$ and for $p = 1$ we set $q = \infty$. For $\boldsymbol{x} = (x_1, x_2, \dots) \in l_p$ let $\phi_{\boldsymbol{x}} = ((\phi_{\boldsymbol{x}})_1, (\phi_{\boldsymbol{x}})_2, \dots) \in l_p^\star$ be defined by*

$$(\phi_{\boldsymbol{x}})_k = |\boldsymbol{x}_k|^{p-1} \mathrm{sgn}\,(\boldsymbol{x}_k),\ k \in \mathbb{N}.$$

For $\boldsymbol{x} \in l_p$ and $\phi \in l_p^\star$ let $\phi(\boldsymbol{x}) = \phi \cdot \boldsymbol{x} = \sum_k \phi_k \boldsymbol{x}_k$.

7.13 Proposition. *Let $1 \le p < \infty$ and $\boldsymbol{x} \in l_p$. Then*

$$\text{i) } \phi_{\boldsymbol{x}} \cdot \boldsymbol{x} = \|\phi_{\boldsymbol{x}}\|_q \cdot \|\boldsymbol{x}\|_p = \|\boldsymbol{x}\|_p^p \quad \text{and} \quad \text{ii) } \phi_{\phi_{\boldsymbol{x}}} = \boldsymbol{x}.$$

7.14 Lemma. *Let $2 \le p < \infty$, $m \in \mathbb{N}$, $x_1, \dots, x_m \in \mathbb{R}$ and $\alpha_1, \dots, \alpha_m \in \mathbb{R}_{\ge 0}$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(\boldsymbol{x}) = \sum_{i=1}^m \alpha_i \, |\boldsymbol{x}_i - \boldsymbol{x}|^p.$$

Then f has a global minimum, and $\overline{x} \in \mathbb{R}$ is the global minimum if and only if

$$\sum_{i=1}^m \alpha_i |x_i - \overline{x}|^{p-1} \,\mathrm{sgn}\,(x_i - \overline{x}) = 0.$$

7.15 Corollary. *Let $2 \le p < \infty$, $m \in \mathbb{N}$, $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in l_p$ and $\alpha_1, \dots, \alpha_m \in \mathbb{R}_{\ge 0}$. Let $f : l_p \to \mathbb{R}$ be given by*

$$f(\boldsymbol{x}) = \sum_{i=1}^m \alpha_i \, \|\boldsymbol{x}_i - \boldsymbol{x}\|_p^p.$$

Then f has a global minimum, and $\overline{\boldsymbol{x}} \in l_p$ is the global minimum if and only if

$$\sum_{i=1}^m \alpha_i \, \phi_{\boldsymbol{x}_i} \cdot \overline{\boldsymbol{x}} = \mathbf{0}.$$

[22]It is $l_\infty \ne l_1^\star$.

7.16 Lemma. *Let $1 \le p < \infty$ and $x, y \in \mathbb{R}$. Then*

$$\left|\frac{1}{2}\left(|x|^{p-1}\operatorname{sgn}(x) + |y|^{p-1}\operatorname{sgn}(y)\right)\right| \le \left|\frac{1}{2}(x+y)\right|^{p-1}, \quad p \le 2,$$
$$\left|\frac{1}{2}\left(|x|^{p-1}\operatorname{sgn}(x) + |y|^{p-1}\operatorname{sgn}(y)\right)\right| \ge \left|\frac{1}{2}(x+y)\right|^{p-1}, \quad p \ge 2.$$

7.17 Theorem. *Let $1 \le p < \infty$, $m \in \mathbb{N}$, $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in l_p$, $\alpha_1, \ldots, \alpha_m \in \mathbb{R}_{\ge 0}$ with $\sum_{i=1}^m \alpha_i = 1$. Moreover, we assume that $\sum_{i,j=1}^m \alpha_i \alpha_j (\phi_{\boldsymbol{x}_i} \boldsymbol{x}_j) = 0$. Then*

$$\sum_{i,j=1}^m \alpha_i\, \alpha_j\, \|\boldsymbol{x}_i - \boldsymbol{x}_j\|_p^p \ge 2^{p-1} \sum_{i=1}^m \alpha_i\, \|\boldsymbol{x}_i\|_p^p, \quad p \le 2,$$
$$\sum_{i,j=1}^m \alpha_i\, \alpha_j\, \|\boldsymbol{x}_i - \boldsymbol{x}_j\|_p^p \le 2^{p-1} \sum_{i=1}^m \alpha_i\, \|\boldsymbol{x}_i\|_p^p, \quad p \ge 2.$$

7.18 Corollary. *Let $2 \le p < \infty$, $m \in \mathbb{N}$, $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in l_p$, $\alpha_1, \ldots, \alpha_m \in \mathbb{R}_{\ge 0}$ with $\sum_{i=1}^m \alpha_i = 1$. Then*

$$\sum_{i,j=1}^m \alpha_i\, \alpha_j\, \|\boldsymbol{x}_i - \boldsymbol{x}_j\|_p^p \le 2^{p-1} \sum_{i=1}^m \alpha_i\, \|\boldsymbol{x}_i\|_p^p.$$

7.19 Theorem. *Let $2 \le p < \infty$, $m \in \mathbb{N}$, and let $r_p(m)$ be the maximum radius of a r-packing set $D \subset B_p(\mathbf{0}, 1)$ of cardinality m. Then*

$$r_p(m) \le \left[1 + 2^{\frac{1}{p}} \left(\frac{m-1}{m}\right)^{\frac{1}{p}}\right]^{-1}.$$

For m even, the bound best possible.

7.20 Theorem. *Let $2 \le p < \infty$. There exists an infinite r-packing set $D \subset B_p(\mathbf{0}, 1)$ if and only if*

$$r \le \left[1 + 2^{1-1/p}\right]^{-1}.$$

For $\left[1 + 2^{1-1/p}\right]^{-1} < r \le \left[1 + 2^{1/p}\right]^{-1}$ any number of l_p-balls $B(\boldsymbol{z}, r)$ can be packed into the unit l_p-ball $B(\mathbf{0}, 1)$.

For $1 \ge r > \left[1 + 2^{1/p}\right]^{-1}$ let $m_p(r)$ be the maximum cardinality of a r-packing set $D \subset B(\mathbf{0}, 1)$. Then

$$m_p(r) \le \left(1 - \frac{1}{2}\left(\frac{1-r}{r}\right)^p\right)^{-1}.$$

7.21 Proposition. *For $r > 1/2$ only one ball $B_\infty(\boldsymbol{z}, r)$ can be packed into the unit ball $B_\infty(\mathbf{0}, 1)$, whereas for $r \le 1/2$ infinitely many balls $B_\infty(\boldsymbol{z}, r)$ can be packed into $B_\infty(\mathbf{0}, 1)$.*

8 A bit measure concentration

8.1 Notation. *Let*

$$\gamma_n(A) = \frac{1}{(2\pi)^{n/2}} \int_A \mathrm{e}^{-|\boldsymbol{x}|^2/2}\, \mathrm{d}\boldsymbol{x}$$

be the standard Gaussian measure *on* $\mathbb{R}^n$ *for a (Lebesgue)-measurable set* $A \subseteq \mathbb{R}^n$.

8.2 Proposition.

i) *Let* $\delta > 0$, *then*

$$\gamma_n \left\{ \boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}|^2 \geq n + \delta \right\} \leq \left(\frac{n+\delta}{n} \right)^{n/2} \mathrm{e}^{-\delta/2},$$

ii) *Let* $0 < \delta \leq n$, *then*

$$\gamma_n \left\{ \boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}|^2 \leq n - \delta \right\} \leq \left(\frac{n}{n-\delta} \right)^{-n/2} \mathrm{e}^{\delta/2}.$$

8.3 Corollary. *Let* $0 < \epsilon < 1$. *Then*

$$\gamma_n \left\{ \boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}|^2 \geq \frac{n}{1-\epsilon} \right\} \leq \mathrm{e}^{-\epsilon^2 n/4},$$
$$\gamma_n \left\{ \boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}|^2 \leq (1-\epsilon)\, n \right\} \leq \mathrm{e}^{-\epsilon^2 n/4}.$$

8.4 Remark. *Let* $p : \mathbb{R}^n \to L$ *be the orthogonal projection of* $\mathbb{R}^n$ *onto a* k*-dimensional subspace* L. *Then the "push-forward" measure* $p(\gamma_n)$ *on* Y *is just the* k*-dimensinal standard Gaussian measure with density* $\frac{1}{(2\pi)^{k/2}} \mathrm{e}^{-|\boldsymbol{x}|^2/2}$ *for* $\boldsymbol{x} \in L$.

8.5 Lemma. *Let* $L \subseteq \mathbb{R}^n$ *be a* k*-dimensional subspace, and for* $\boldsymbol{x} \in \mathbb{R}^n$ *let* $\boldsymbol{x}_L$ *be the orthogonal projection onto* L. *Then, for* $0 < \epsilon < 1$, *we have*

$$\gamma_n \left\{ \boldsymbol{x} \in \mathbb{R}^n : \sqrt{\frac{n}{k}}\, |\boldsymbol{x}_L| \geq (1-\epsilon)^{-1} |\boldsymbol{x}| \right\} \leq \mathrm{e}^{-\epsilon^2 k/4} + \mathrm{e}^{-\epsilon^2 n/4},$$
$$\gamma_n \left\{ \boldsymbol{x} \in \mathbb{R}^n : \sqrt{\frac{n}{k}}\, |\boldsymbol{x}_L| \leq (1-\epsilon) |\boldsymbol{x}| \right\} \leq \mathrm{e}^{-\epsilon^2 k/4} + \mathrm{e}^{-\epsilon^2 n/4}.$$

8.6 Notation. *Let* μ_n *be the uniform measure on* S^{n-1}, *e.g., for* $A \subset S^{n-1}$ *measurable we may define*

$$\mu_n(A) = \frac{1}{\kappa_n} \mathrm{vol}_n(\mathrm{conv}\, \{\mathbf{0}, A\}).$$

8.7 Remark. *Let* $\phi : \mathbb{R}^n \setminus \{\mathbf{0}\} \to S^{n-1}$ *be the radial projection given by* $\phi(\boldsymbol{x}) = \boldsymbol{x} / |\boldsymbol{x}|$. *Then, with respect to* ϕ, μ_n *is the push-forward measure of* γ_n.

8.8 Corollary. *Let $L \subseteq \mathbb{R}^n$ be a k-dimensional subspace, and for $\boldsymbol{x} \in S^{n-1}$ let $\boldsymbol{x}_L$ be the orthogonal projection onto L. Then, for $0 < \epsilon < 1$, we have*

$$\mu_n \left\{ \boldsymbol{x} \in S^{n-1} : |\boldsymbol{x}_L| \geq (1-\epsilon)^{-1} \sqrt{\frac{k}{n}} \right\} \leq \mathrm{e}^{-\epsilon^2 k/4} + \mathrm{e}^{-\epsilon^2 n/4},$$

$$\mu_n \left\{ \boldsymbol{x} \in S^{n-1} : |\boldsymbol{x}_L| \leq (1-\epsilon) \sqrt{\frac{k}{n}} \right\} \leq \mathrm{e}^{-\epsilon^2 k/4} + \mathrm{e}^{-\epsilon^2 n/4}.$$

8.9 Remark.

i) *Let* $\mathrm{G}_k(\mathbb{R}^n) = \{L \text{ linear subspace of } \mathbb{R}^n \text{ with } \dim L = k\}$ *be the* Grassmannian *of k-dimensional subspaces. This can be made a metric space and the orthognal group O_n acts transitively on $\mathrm{G}_k(\mathbb{R}^n)$. Hence there exists an unique Borel probability measure $\mu_{n,k}$ on $\mathrm{G}_k(\mathbb{R}^n)$. We will just describe it as the push forward measure of $\gamma_n \times \cdots \times \gamma_n$ on $\mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n$ under the map $\phi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) = \mathrm{lin}\,\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$.*

ii) *Also the orthogonal group O_n can be made a metric space, and so, as compact metric space it has an unique Borel probability measure ν_n, which we just describe as the push-forward measure of $\gamma_n \times \cdots \times \gamma_n$ on $\mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n$ under the map* $\phi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) =$ Gram-Schmidt orthonormalized basis of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$.

8.10 Lemma. *Let $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ and for $L \in \mathrm{G}_k(\mathbb{R}^n)$ let $\boldsymbol{x}_L$ be the orthogonal projection of $\boldsymbol{x}$ onto L. Then for $0 < \epsilon < 1$*

$$\mu_{n,k} \left\{ L \in \mathrm{G}_k(\mathbb{R}^n) : |\boldsymbol{x}_L| \geq (1-\epsilon)^{-1} \sqrt{\frac{k}{n}}\, |\boldsymbol{x}| \right\} \leq \mathrm{e}^{-\epsilon^2 k/4} + \mathrm{e}^{-\epsilon^2 n/4},$$

$$\mu_{n,k} \left\{ L \in \mathrm{G}_k(\mathbb{R}^n) : |\boldsymbol{x}_L| \leq (1-\epsilon) \sqrt{\frac{k}{n}}\, |\boldsymbol{x}| \right\} \leq \mathrm{e}^{-\epsilon^2 k/4} + \mathrm{e}^{-\epsilon^2 n/4}.$$

8.11 Theorem. *Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N \in \mathbb{R}^n$, and let $\epsilon > 0$. Let $k \in \mathbb{N}$ such that*

$$N(N-1)(\mathrm{e}^{-\epsilon^2 k/4} + \mathrm{e}^{-\epsilon^2 n/4}) \leq \frac{1}{3}.$$

For instance, $k \geq 4\epsilon^{-2} \ln(6N^2)$. For a randomly chosen $L \in \mathrm{G}_k(\mathbb{R}^n)$ let $\boldsymbol{a}_i|L$ be the orthogonal projection of $\boldsymbol{a}_i$ onto L. Then with probability at least $2/3$ we have

$$(1-\epsilon)\,|\boldsymbol{a}_i - \boldsymbol{a}_j| \leq \sqrt{\frac{n}{k}}\, |\boldsymbol{a}_i|L - \boldsymbol{a}_j|L| \leq (1-\epsilon)^{-1}\, |\boldsymbol{a}_i - \boldsymbol{a}_j|\,.$$

8.12 Lemma [Johnson–Lindenstrauss Lemma]. *Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N \in \mathbb{R}^n$, and let $\epsilon > 0$. Let $k \geq 4\epsilon^{-2} \ln(6N^2)$. Then there exists a linear map such $f : \mathbb{R}^n \to \mathbb{R}^k$ such that*

$$(1-\epsilon)\,|\boldsymbol{a}_i - \boldsymbol{a}_j| \leq |f(\boldsymbol{a}_i) - f(\boldsymbol{a}_j)| \leq (1+\epsilon)\,|\boldsymbol{a}_i - \boldsymbol{a}_j|\,.$$

8.13 Definition. *Let* $I_n = \{0,1\}^n$ *be the* Boolean cube *equipped with the* Hamming distance $\operatorname{dist}(\boldsymbol{x},\boldsymbol{y}) = \#\{i : \boldsymbol{x}_i \neq \boldsymbol{y}_i\}$. *Furthermore, let* ρ_n *be the counting probability measure* $\rho_n(\{\boldsymbol{x}\}) = 1/2^n$. *Then, for a function* $f : I_n \to \mathbb{R}$ *we have*

$$\int_{I_n} f(\boldsymbol{x})\,\mathrm{d}\rho_n = \frac{1}{2^n}\sum_{\boldsymbol{x}\in I_n} f(\boldsymbol{x}).$$

8.14 Theorem. *Let* $A \subset I_n$, $A \neq \emptyset$, *and let* $f : I_n \to \mathbb{R}$ *be given by* $f(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, A)$. *Then for* $t > 0$ *we have*

$$\int_{I_n} \mathrm{e}^{t\,f(\boldsymbol{x})}\,\mathrm{d}\rho_n \leq \frac{1}{\rho_n(A)}\left(\frac{1}{2} + \frac{\mathrm{e}^t + \mathrm{e}^{-t}}{4}\right)^n.$$

8.15 Corollary. *Let* $A \subset I_n$, $A \neq \emptyset$, *and let* $f : I_n \to \mathbb{R}$ *be given by* $f(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, A)$. *Then for* $t > 0$ *we have*

$$\int_{I_n} \mathrm{e}^{t\,f(\boldsymbol{x})}\,\mathrm{d}\rho_n \leq \frac{1}{\rho_n(A)}\mathrm{e}^{t^2 n/4}.$$

8.16 Corollary. *Let* $A \subset I_n$, $A \neq \emptyset$. *Then for* $\epsilon > 0$ *we have*

$$\rho_n\{\boldsymbol{x} \in I_n : \operatorname{dist}(\boldsymbol{x}, A) \geq \epsilon\sqrt{n}\} \leq \frac{1}{\rho_n(A)}\mathrm{e}^{-\epsilon^2}.$$

8.17 Definition [Median]. *The* median *of* $f : I_n \to \mathbb{R}$ *is the number* m_f *satisfying*

$$\rho_n\{\boldsymbol{x} \in I_n : f(\boldsymbol{x}) \geq m_f\} \geq \frac{1}{2} \text{ and } \rho_n\{\boldsymbol{x} \in I_n : f(\boldsymbol{x}) \leq m_f\} \geq \frac{1}{2}.$$

8.18 Theorem. *Let* $f : I_n \to \mathbb{R}$ *be 1-Lipschitz, i.e.,* $f|(\boldsymbol{x}) - f(\boldsymbol{y})| \leq |\boldsymbol{x} - \boldsymbol{y}|$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in I_n$, *with median* m_f. *Let* $\epsilon > 0$. *Then*

$$\rho_n\{\boldsymbol{x} \in I_n : |f(\boldsymbol{x}) - m_f| \geq \epsilon\sqrt{n}\} \leq 4\mathrm{e}^{-\epsilon^2}.$$

8.19 Proposition. *Let* $A \subseteq I_n$, $A \neq \emptyset$, *and let* $f : I_n \to \mathbb{R}$ *be given by* $f(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, A)$.

i) *Let* m_f *be the median of* f *and* a_f *its average, i.e.* $a_f = \int_{I_n} f(\boldsymbol{x})\mathrm{d}\rho_n$. *Then*

$$|m_f - a_f| \leq \sqrt{\frac{n\ln n}{2}} + 4\sqrt{n}.$$

ii)

$$\int_{I_n} f(\boldsymbol{x})\mathrm{d}\rho_n \leq \frac{n}{2}.$$

8.20 Lemma. *Let* $A \subseteq I_n$, $A \neq \emptyset$, *and let* $t \geq 0$. *Then*

$$\ln|A| + t\int_{I_n} \operatorname{dist}(\boldsymbol{x}, A)\mathrm{d}\rho_n \leq n\ln(\mathrm{e}^{t/2} + \mathrm{e}^{-t/2}).$$

8.21 Definition [Entropy]. *The function $H : [0,1] \to \mathbb{R}$ with*

$$H(\rho) = \rho \ln\left(\frac{1}{\rho}\right) + (1-\rho)\ln\left(\frac{1}{1-\rho}\right)$$

is called entropy*.*

8.22 Theorem. *Let $A \subseteq I_n$, $A \neq \emptyset$, and let*

$$\rho = \frac{1}{2} - \frac{1}{n}\int_{I_n} \operatorname{dist}(\boldsymbol{x}, A)\mathrm{d}\rho_n.$$

Then $\rho \in [0, 1/2]$ and

$$\frac{|\ln A|}{n} \leq H(\rho).$$

8.23 Definition [Hamming ball]. *For an integer $r \in \mathbb{N}_0$ the set*

$$B(r) = \{\boldsymbol{x} \in I_n : \operatorname{dist}(\boldsymbol{x}, \mathbf{0}) \leq r\}$$

is called Hamming ball *of radius r. Observe that $|B(r)| = \sum_{k=0}^{r} \binom{n}{k}$.*

8.24 Lemma. *Let $\lambda \in [0,1]$, and let $B_n = B(\lfloor \lambda n \rfloor)$. Then*

$$\text{i) } \ln|B_n| \leq n\,H(\lambda) \text{ and ii) } \lim_{n\to\infty} \frac{\ln|B_n|}{n} = H(\lambda).$$

8.25 Definition [Simplicial order]. *For $\boldsymbol{x}, \boldsymbol{y} \in I_n$ we write $\boldsymbol{x} < \boldsymbol{y}$ if $\sum_{i=1}^{n} x_i < \sum_{i=1}^{n} y_i$ or if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ then for the first coordinate i with $x_i \neq y_i$ we have $x_i = 1$ and $y_i = 0$.*

8.26 Notation. *Let $A \subseteq I_n$, $A \neq \emptyset$. For $t \in \mathbb{N}_0$ the set*

$$A(t) = \{\boldsymbol{x} \in I_n : \operatorname{dist}(\boldsymbol{x}, A) \leq t\}$$

*is called t-*neighbourhood *of A.*

8.27 Proposition. *Let $r, t \in \mathbb{N}_0$.*

i) *$B(r)(t) = B(r+t)$.*

ii) *$B(r) = \{\boldsymbol{x} \in I_n : \boldsymbol{x} \leq (0, \ldots, 0, 1, \ldots, 1)^{\mathsf{T}}\}$.*

iii) *For some $\boldsymbol{y} \in I_n$ let $A = \{\boldsymbol{x} \in I_n : \boldsymbol{x} \leq \boldsymbol{y}\}$. Then there exists a $\boldsymbol{z} \in I_n$ such that $A(t) = \{\boldsymbol{x} \in I_n : \boldsymbol{x} \leq \boldsymbol{z}\}$.*

8.28 Theorem* [Harper]. *Let $A \subseteq I_n$, $A \neq \emptyset$, and let B be the first $|A|$ elements of I_n with respect to the simplicial order. Then $|A(1)| \geq |B(1)|$.*

8.29 Corollary. *Let $A \subseteq I_n$ such that $|A| \geq \sum_{k=0}^{r} \binom{n}{k}$, and let $t \in \mathbb{N}_0$. Then*

$$|A(t)| \geq \sum_{k=0}^{r+t} \binom{n}{k}.$$

9 Algebra of polyhedra and volume of polytopes

9.1 Notation. *In the following let V be a normed finite-dimensional real vectors space. For a subset $A \subseteq V$ we denote here its characteristic function by $[A] : V \to \{0,1\}$, i.e.,*

$$[A](x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

As usual a polyhedra $P \subseteq V$ is a set which admits a representation as

$$P = \{\boldsymbol{x} \in V : l_i(x) \leq \alpha_i,\, i \in I\},$$

where $l_i : V \to \mathbb{R}$ is a linear form, and I is a finite index set. The set of all polyhedra is denoted by $\mathcal{P}(V)$, the set of all closed convex set in V by $\mathcal{K}(V)$, and the set of all closed convex and bounded set by $\mathcal{K}_b(V)$.

9.2 Definition [Algebra of Polyhedra/closed convex (bounded) sets].

i) $[\mathcal{P}(V)] = \operatorname{lin}\{[P] : P \in \mathcal{P}(V)\}$ *is called the* algebra of polyhedra.

ii) $[\mathcal{K}(V)] = \operatorname{lin}\{[K] : K \in \mathcal{K}(V)\}$ *is called the* algebra of closed convex sets.

iii) $[\mathcal{K}_b(V)] = \operatorname{lin}\{[K] : K \in \mathcal{K}_b(V)\}$ *is called the* algebra of closed convex bounded sets.

9.3 Remark.

i) *I addition to their vector space structure, all these sets have a natural algebra structure via the pointwise multiplication of functions.*

ii) $\{[P] : P \in \mathcal{P}(V)\}$ *is not a basis of $\mathcal{P}(V)$.*

iii) $[\mathcal{P}(V)]$ *contains also the indicator functions of open (halfopen) polyhedra.*

9.4 Definition [Valuation]. *Let W be a vector space. A linear map $T : [\mathcal{P}(V)] \to W$ (or for $[\mathcal{K}(V)], [\mathcal{K}_b(V)]$) is called* a valuation.

9.5 Theorem [Euler-Characteristic]. *There exists a unique valuation $\chi : [\mathcal{K}(V)] \to \mathbb{R}$ such that $\chi([K]) = 1$ for all $K \in \mathcal{K}(V)$. It is called* Euler-characteristic.

9.6 Corollary. *Let $l : V \to \mathbb{R}$ be linear, $l \not\equiv 0$, and for $\tau \in \mathbb{R}$ let $H_\tau = \{\boldsymbol{x} \in V : l(\boldsymbol{x}) = \tau\}$. Let $f \in [\mathcal{K}_b(V)]$ and let $f_\tau = f \cdot [H_\tau]$ be the restriction of f onto H_τ. Then*

$$\chi(f) = \sum_{\tau \in \mathbb{R}} \left(\chi(f_\tau) - \lim_{\epsilon \to 0+} \chi(f_{\tau-\epsilon}) \right).$$

9.7 Lemma [Fourier-Motzkin elimination]. *Let $T : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection forgetting the last coordinate, and let $P \in \mathcal{P}(\mathbb{R}^n)$. Then $T(P) \in \mathcal{P}(\mathbb{R}^{n-1})$.*

9.8 Theorem. *Let V, W be finite dimensional normed real linear vector spaces, and let $T : V \to W$ be a linear map. Then*

i) *For $P \in \mathcal{P}(V)$ it is $T(P) \in \mathcal{P}(W)$.*

ii) *There exists a unique linear map $\widetilde{T} : [\mathcal{P}(V)] \to [\mathcal{P}(W)]$ such that for every $P \in \mathcal{P}(V)$*
$$\widetilde{T}([P]) = [T(P)].$$

9.9 Theorem.

i) *Let $P_1, P_2 \in \mathcal{P}(V)$ be non-empty. Then $P_1 + P_2 \in \mathcal{P}(V)$.*

ii) *There exists a unique bilinear map $\star : [\mathcal{P}(V)] \times [\mathcal{P}(V)] \to [\mathcal{P}(V)]$, called* convolution, *such that $[P_1] \star [P_2] = [P_1 + P_2]$ for all non-empty $P_1, P_2 \in \mathcal{P}(V)$.*

9.10 Theorem. *Let $P \in \mathcal{P}(V)$, $P \neq \emptyset$. P has a vertex if and only if P does not contain a line.*

9.11 Definition [Polyhedral cone]. *$P \in \mathcal{P}(V)$ is called* a polyhedral cone *if for all $\boldsymbol{x} \in P$ and for all $\lambda \geq 0$ it holds $\lambda\, \boldsymbol{x} \in P$.*

9.12 Theorem. *Let $P \in \mathcal{P}(V)$, $P \neq \emptyset$. Then we can write $P = M + C + L$, where M is a polytope, C a polyhedral cone and L a linear subspace.*

9.13 Notation. *In the following we additionally assume that V has an inner product $\langle , \rangle$, and as usual, for $A \subset V$ let*
$$A^\star = \{\boldsymbol{y} \in V : \langle \boldsymbol{y}, \boldsymbol{a} \rangle \leq 1,\ \textit{for all } \boldsymbol{a} \in A\}$$
be the polar set of A.

9.14 Theorem.

i) *For $P \in \mathcal{P}(V)$ we have $P^\star \in \mathcal{P}(V)$ as well.*

ii) *There exists a unique linear map $D : [\mathcal{P}(V)] \to [\mathcal{P}(V)]$ such that $D([P]) = [P^\star]$ for all non-empty $P \in \mathcal{P}(V)$.*

9.15 Definition [Tangent cone / Cone of feasible directions]. *Let $P \in \mathcal{P}(V)$ be non-empty, and let $\boldsymbol{v} \in P$. The* tangent cone $\mathrm{tcone}(P, \boldsymbol{v})$ *is the set*
$$\mathrm{tcone}(P, \boldsymbol{v}) = \{\boldsymbol{v} + \boldsymbol{y} : \exists\, \epsilon > 0 \textit{ s.t. } \boldsymbol{v} + \epsilon \boldsymbol{y} \in P\}.$$
The cone $vv + \mathrm{tcone}(P, \boldsymbol{v})$, i.e.,
$$\mathrm{fcone}(P, \boldsymbol{v}) = \{\boldsymbol{y} : \boldsymbol{v} + \boldsymbol{y} \in \mathrm{tcone}(P, \boldsymbol{v})\}$$
is called the cone of feasible directions.

9.16 Lemma. *Let $T : V \to W$ be a linear map between finite-dimensional real vector spaces, and let $P \in \mathcal{P}(V)$ and $\boldsymbol{v} \in P$. Then $T(\boldsymbol{v}) \in T(P) \in \mathcal{P}(W)$ and*

$$T(\text{tcone}(P, \boldsymbol{v})) = \text{tcone}(T(P), T(\boldsymbol{v})).$$

9.17 Definition. *Let $f, g \in [\mathcal{P}(V)]$.*

i) *We write*

$$f \equiv g \bmod \textit{polyhedra with lines}$$

if $f - g \in [\mathcal{P}(V)]$ is a linear combination of characteristic functions of polyhedra containing lines. For short we will write $f \equiv g \bmod \textit{lines}$.

ii) *We write*

$$f \equiv g \bmod \textit{lower-dimensional polyhedra}$$

if $f - g \in [\mathcal{P}(V)]$ is a linear combination of characteristic functions of lower-dimensional polyhedra. For short we will write $f \equiv g \bmod \textit{dim}$.

9.18 Theorem. *Let $P \in \mathcal{P}(V)$ with vertices $\text{vert}\,(P)$. Then*

$$[P] \equiv \sum_{\boldsymbol{v} \in \text{vert}\,(P)} \text{tcone}(P, \boldsymbol{v}) \bmod \textit{lines}.$$

9.19 Corollary. *Let $P_i \in \mathcal{P}(V)$ with $\mathbf{0} \in P_i$ and $\alpha_i \in \mathbb{R}$ for $i \in I$, I finite. Then*

$$\sum_{i \in I} \alpha_i [P_i] \equiv 0 \bmod \textit{lines} \Leftrightarrow \sum_{i \in I} \alpha_i [P_i] \equiv 0 \bmod \textit{dim}.$$

9.20 Theorem. *Let $P \in \mathcal{P}(V)$, $P \neq \emptyset$, and let K_P be the recession of P. Then we have*

$$\sum_{\boldsymbol{v} \in \text{vert}\,(P)} [\text{fcone}(P, \boldsymbol{v})] \equiv [K_P] \bmod \textit{lines}.$$

9.21 Theorem. *Let $P \in \mathcal{P}(V)$ be a polytope. Then $\chi(\text{relint}\, P) = (-1)^{\dim P}$.*

9.22 Corollary [Euler-Poincare formula]. *Let $P \in \mathcal{P}(V)$ be an m-dimensional polytope, and for $k \in \{0, \ldots, m\}$ let f_k be the number of k-faces of P, where $f_m(P) = 1$. Then*

$$\sum_{k=0}^{m} (-1)^k f_k = 1.$$

9.23 Theorem. *Let $K \in \mathcal{P}(V)$ be an n-dimensional polyhedral cone. Then*

$$[K] \equiv (-1)^n \, [-\text{int}\, K] \bmod \textit{lines}.$$

9.24 Theorem. *Let $P \in \mathcal{P}(V)$ be a polytope. Then*

$$[\text{int}\, P] \equiv \sum_{\boldsymbol{v} \in \text{vert}\,(P)} [\text{int}\, \text{tcone}(P, \boldsymbol{v})] \bmod \textit{lines}.$$

9.25 Remark. *Let* $\dim V = n$ *with basis* $U = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)$*, and let* $K = \operatorname{pos}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$*. Then for all* $\boldsymbol{c} \in \operatorname{int} K^\star$ *we have*

$$\int_K \mathrm{e}^{\langle \boldsymbol{c}, \boldsymbol{x} \rangle} \, \mathrm{d}\boldsymbol{x} = |\det U| \prod_{i=1}^{n} \frac{-1}{\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle},$$

where the (improper) integral converges absolutely and compactly (,i.e., uniformly on compact subsets) on the interior of $K^\star$*.*

9.26 Lemma. *Let* $K = \operatorname{pos}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\} \in \mathcal{P}(V)$ *be a pointed cone. For every* $\boldsymbol{c} \in \operatorname{int} K^\star$ *the integral* $\int_K \mathrm{e}^{\langle \boldsymbol{c}, \boldsymbol{x} \rangle} \, \mathrm{d}\boldsymbol{x}$ *converges absolutely and compactly on* $\operatorname{int} K^\star$ *to a rational function* $\phi(K, \boldsymbol{c})$ *of the type*

$$\phi(K, \boldsymbol{c}) = \sum_{I \subset [m], \# I = n} \alpha_I \prod_{i \in I} \frac{-1}{\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle},$$

where $\alpha_I \geq 0$ *are numbers.*

9.27 Definition. *For the given vector space let* $\mathcal{M}(V)$ *be space of functions on* V *spanned by functions of the type*

$$f(\boldsymbol{c}) = \mathrm{e}^{\langle \boldsymbol{c}, \boldsymbol{v} \rangle} \prod_{i=1}^{n} \frac{-1}{\langle \boldsymbol{c}, \boldsymbol{u}_i \rangle},$$

where $\boldsymbol{v} \in V$ *and* $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ *are a basis of* V*.*

9.28 Theorem [Khovanskii&Pukhlikov, 1992; Lawrence, 1991]. *Let* V *be an* n*-dimensional Euclidean vector space. There exists a linear transformation (valuation)*

$$\Phi : [\mathcal{P}(V)] \to \mathcal{M}(V)$$

such that for a polyhedron $P \in \mathcal{P}(V)$*,* $P \neq \emptyset$*, holds*

i) *If* P *contains no lines, then for all* $c \in \operatorname{int} K^\star$*, where* K *is the recession cone of* P*, the integral* $\int_P \mathrm{e}^{\langle \boldsymbol{c}, \boldsymbol{x} \rangle} \, \mathrm{d}\boldsymbol{x}$ *converges absolutely and compactly on* $\operatorname{int} K^\star$ *to the function* $\phi(P, \boldsymbol{c}) = \Phi([P]) \in \mathcal{M}(V)$*.*

ii) *If* P *contains a line then* $\Phi([P]) = 0$*.*

9.29 Corollary [Brion's theorem, 1988]. *Let* $P \in \mathcal{P}(V)$*,* $P \neq \emptyset$*, without lines. Then*

$$\Phi([P]) = \sum_{\boldsymbol{v} \in \operatorname{vert}(P)} \Phi([\operatorname{tcone}(P, \boldsymbol{v})]) = \sum_{\boldsymbol{v} \in \operatorname{vert}(P)} \mathrm{e}^{\langle \boldsymbol{c}, \boldsymbol{v} \rangle} f_{\boldsymbol{v}}(\boldsymbol{c}),$$

where $f_{\boldsymbol{v}}(\boldsymbol{c})$ *is a homogeneous rational function in* $\boldsymbol{c}$ *of degree* $-n$*, and it depends only on* $\operatorname{fcone}(P, \boldsymbol{v})$*.*

9.30 Definition [Strongly combinatorially isomorphic]. *$P_1, P_2 \in \mathcal{P}(V)$ are called* strongly combinatorially isomorphic *if for any non-zero linear functional $l : V \to \mathbb{R}$ we have*

$$b_1(l) = \sup_{\boldsymbol{x} \in P_1} l(\boldsymbol{x}) < \infty \Leftrightarrow b_2(l) = \sup_{\boldsymbol{x} \in P_2} l(\boldsymbol{x}) < \infty$$

and

$$\dim\{\boldsymbol{x} \in P_1 : l(\boldsymbol{x}) = b_1(l)\} = \dim\{\boldsymbol{x} \in P_2 : l(\boldsymbol{x}) = b_2(l)\}.$$

9.31 Remark.

i) *If P_1, P_2 are strongly combinatorially isomorphic then* $\mathrm{fcone}(P, \boldsymbol{v}_1) = \mathrm{fcone}(P_2, \boldsymbol{v}_2)$ *at the corresponding vertices. Hence, having a family $P_\tau \subset \mathcal{P}(V)$ of strongly combinatorially isomorphic polytopes with vertices $\boldsymbol{v}_1(\tau)$, $\dots, \boldsymbol{v}_m(\tau)$ we have*

$$\Phi([P_\tau]) = \sum_{i=1}^{m} \mathrm{e}^{\langle \boldsymbol{c}, \boldsymbol{v}_i(\tau) \rangle} f_i(\boldsymbol{c}),$$

where $f_i(\boldsymbol{c})$ are independent of the particular polyhedron P_τ.

ii) *For $\tau \in [0,1]$ let $\boldsymbol{v}(\tau) = (1-\tau)\boldsymbol{v}_0 + \tau\, \boldsymbol{v}_1$. Then*

$$\int_0^1 \mathrm{e}^{\langle \boldsymbol{c}, \boldsymbol{v}(\tau) \rangle} \, \mathrm{d}\tau = \begin{cases} \frac{1}{\langle \boldsymbol{c}, \boldsymbol{v}_0 - \boldsymbol{v}_1 \rangle} \left(\mathrm{e}^{\langle \boldsymbol{c}, \boldsymbol{v}_0 \rangle} - \mathrm{e}^{\langle \boldsymbol{c}, \boldsymbol{v}_1 \rangle} \right), & \langle \boldsymbol{c}, \boldsymbol{v}_0 - \boldsymbol{v}_1 \rangle \neq 0, \\ \mathrm{e}^{\langle \boldsymbol{c}, \boldsymbol{v}_0 \rangle} = \mathrm{e}^{\langle \boldsymbol{c}, \boldsymbol{v}_1 \rangle}, & \langle \boldsymbol{c}, \boldsymbol{v}_0 - \boldsymbol{v}_1 \rangle = 0. \end{cases}$$

9.32 Theorem. *Let $P \in \mathcal{P}(V)$ be a polytope, and let $\bar{\boldsymbol{c}}$ generic, i.e., it is not a pole of $f_{\boldsymbol{v}}(\bar{\boldsymbol{c}}) = \int_{\mathrm{fcone}(P, \boldsymbol{v})} \mathrm{e}^{\langle \bar{\boldsymbol{c}}, \boldsymbol{x} \rangle} \, \mathrm{d}\boldsymbol{x}$. Then*

$$\mathrm{vol}\,(P) = \sum_{\boldsymbol{v} \in \mathrm{vert}\, P} \frac{\langle \bar{c}, \boldsymbol{v} \rangle^n}{n!} f_{\boldsymbol{v}}(\boldsymbol{c}).$$

9.33 Corollary.

i) *Let $\{P_\tau\}$ be a family of strongly combinatorially isomorphic polytopes with vertices $\boldsymbol{v}_1(\tau), \dots, \boldsymbol{v}_m(\tau)$. Then* $\mathrm{vol}\,(P_\tau)$ *is a polynomial of degree n in the vertices.*

ii) *Let P be a simple n-dimensional polytope with* $\mathrm{fcone}(P, \boldsymbol{v}) = \mathrm{pos}\,\{\boldsymbol{u}_1(\boldsymbol{v}), \dots, \boldsymbol{u}_n(\boldsymbol{v})\}$ *for $\boldsymbol{v} \in \mathrm{vert}\, P$. Then*

$$\mathrm{vol}\,(P) = \sum_{\boldsymbol{v} \in \mathrm{vert}\, P} \frac{\langle \bar{c}, \boldsymbol{v} \rangle^n}{n!} |\det(\boldsymbol{u}_1(\boldsymbol{v}), \dots, \boldsymbol{u}_n(\boldsymbol{v}))| \prod_{i=1}^{n} \frac{-1}{\langle \bar{\boldsymbol{c}}, \boldsymbol{u}_i(\boldsymbol{v}) \rangle},$$

for a generic $\bar{\boldsymbol{c}}$.

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