Basic (and some more) facts from Convexity

In this chapter we will briefly collect some basic results and concepts concerning convex bodies, which may be considered as the main "continuous" component/ingredient of the Geometry of Numbers. For proofs of the results we will refer to the literature, mainly to the books [2, 5, 6, 7, 9, 11, 13, 14], which are also excellent sources for further information on the state of the art concerning convex bodies, polytopes and their applications. As usual we will start with setting up some basic notation; more specific ones will be introduced when needed.

Let $\mathbb{R}^n = \{ x = (x_1, \ldots, x_n)^T : x_i \in \mathbb{R} \}$ be the $n$-dimensional Euclidean space, equipped with the standard inner product $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i, \ x, y \in \mathbb{R}^n$.

For a subset $X \subseteq \mathbb{R}^n$, let $\text{lin } X$ and $\text{aff } X$ be its linear and affine hull, respectively, i.e., with respect to inclusions, the smallest linear or affine subspace containing $X$. If $X = \emptyset$ we set $\text{lin } X = \{ 0 \}$, and $\text{aff } X = \emptyset$. The dimension of a set $X$ is the dimension of its affine hull and will be denoted by $\dim X$. By $X + Y$ we mean the vectorwise addition of subsets $X, Y \subseteq \mathbb{R}^n$, i.e.,

$$X + Y = \{ x + y : x \in X, y \in Y \}.$$  

The multiplication $\lambda X$ of a set $X \subseteq \mathbb{R}^n$ with $\lambda \in \mathbb{R}$ is also declared vectorwise, i.e., $\lambda X = \{ \lambda x : x \in X \}$. We write $-X = (-1) X$, $X - Y = X + (-1) Y$ and $x + Y$ instead of $\{ x \} + Y$.

A subset $C \subseteq \mathbb{R}^n$ is called convex, iff for all $c_1, c_2 \in C$ and for all $\lambda \in [0, 1]$ hold $\lambda c_1 + (1 - \lambda) c_2 \in C$. A convex compact (i.e., bounded and closed) subset $K \subseteq \mathbb{R}^n$ is called a convex body. The set of all convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$. $K \in \mathcal{K}^n$ is called o-symmetric, i.e., symmetric with respect to the origin, if $K = -K$. The set of all o-symmetric convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}_o^n$.

There is a simple one-to-one correspondence between $n$-dimensional o-symmetric convex bodies and norms on $\mathbb{R}^n$: given a norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$
we consider its unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$, which is a $o$-symmetric convex body by the definition of a norm. For the reverse correspondence we need the notion of a distance function. For $K \in \mathcal{K}^n_o$, $\dim K = n$, the function

$$ |\cdot|_K : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \text{ given by } |x|_K := \min\{\rho \in \mathbb{R}_{\geq 0} : x \in \rho K\} $$

is called the distance function of $K$.

By the convexity and symmetry of $K$ it is easily verified that $|\cdot|_K$ is indeed a norm. A well-studied family of $o$-symmetric convex bodies are the unit balls $B^p_n := \{x \in \mathbb{R}^n : |x|_p \leq 1\}$ associated to the $p$-norms

$$ |x|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad p \in [1, \infty), \quad \text{and } |x|_{\infty} := \max_{1 \leq i \leq n} \{|x_i|\}. $$

In the Euclidean case, i.e., $p = 2$, we denote the Euclidean norm just by $|\cdot|$ and its unit ball by $B_n$.

![Figure 1.1: Unit balls of $p$-norms](image)

$B^\infty_n = \{x \in \mathbb{R}^n : -1 \leq x_i \leq 1\}$ is the $o$-symmetric (regular) cube with edge length 2, and $B^1_n = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} |x_i| \leq 1\}$ is called the (regular) crosspolytope.

One big advantage of convex sets $K$ is their property that points not belonging to $K$ can easily be separated from $K$. In order to formulate this precisely and a bit more generally, we denote for $a \in \mathbb{R}^n \setminus \{0\}$, $b \in \mathbb{R}$ by $H(a, b) = \{x \in \mathbb{R}^n : \langle a, x \rangle = b\}$ the hyperplane with normal vector $a$ and right hand side $b$. The associated two closed halfspaces are denoted by $H_\leq(a, b)$ and $H_\geq(a, b)$, respectively, i.e., $H_\leq(a, b) = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq b\}$ and $H_\geq(a, b) = H_\leq(-a, -b)$.

**1.1 Theorem [Separation Theorem].** Let $C_1, C_2 \subset \mathbb{R}^n$ be convex with $C_1 \cap C_2 = \emptyset$.

i) Then there exists a separating hyperplane $H(a, b)$, i.e., $C_1 \subseteq H_\leq(a, b)$ and $C_2 \subseteq H_\geq(a, b)$.

ii) If $C_1$ is compact, i.e., $C_1 \in \mathcal{K}^n_o$ and $C_2$ is closed then there exists a strictly separating hyperplane $H(a, b)$, i.e., $C_1 \subset H_\leq(a, b)$ and $C_2 \subset H_\geq(a, b)$ and $C_1 \cap H(a, b) = \emptyset = C_2 \cap H(a, b)$. 
A hyperplane $H(a, b)$ passing through the boundary of a closed convex set $C$ and containing $C$ entirely in one of the two halfspaces, i.e., $C \cap H(a, b) \neq \emptyset$ and $C \subseteq H_{\leq}(a, b)$, say, is called a supporting hyperplane of $C$. The intersection of $F = C \cap H(a, b)$ is called a face of $C$, or more precisely, a $k$-face if $\dim F = k$. If $k = 0$ then $F$ is called an extreme point of $C$. Faces themselves are closed convex sets, and the intersection of faces of $C$ is again a face of $C$, where we regard the empty set as the $(-1)$-dimensional face of $C$. The boundary of $C$, denoted by $\text{bd} C$, is the union of its faces.

For a convex body $K \subseteq R^n$ and a given normal vector (direction) $a \in R^n \setminus \{0\}$ the supporting hyperplane in direction $a$ is given by $H(a, h_K(a))$ where

$$h_K : R^n \to R \text{ with } h_K(a) := \max\{\langle a, x \rangle : x \in K\}$$

is called the support function of $K$. The support function is positive homogeneous, i.e., $h_K(\lambda a) = \lambda h_K(a)$, $\lambda \in R_{\geq 0}$, and sub-additive, i.e., $h_K(x + y) \leq h_K(x) + h_K(y)$ for all $x, y \in R^n$. Indeed, these two properties characterize support functions of convex bodies among the real-valued functions on $R^n$.

A quite general concept in different branches of mathematics is that of duality and/or polarity. In our setting it is defined as follows: for $X \subseteq R^n$, the set

$$X^* := \{y \in R^n : \langle x, y \rangle \leq 1 \text{ for all } x \in X\}$$

is called the polar set of $X$. As the intersection of convex closed sets containing the origin, the polar set is always a convex closed set containing $0$. Its actual shape, however, depends on its position relative to the origin. For instance, $X^*$ is bounded if and only if $0$ is an interior point of $X$.

On the other hand, as stated in the next proposition, its shape behaves properly with respect to linear transformations and a polar set of an $o$-symmetric convex body is again an $o$-symmetric convex body.

1.2 Proposition.

i) Let $X \subseteq R^n$, and let $M \in \text{GL}(n, R)$. Then $(MX)^* = M^{-1}X^*$. Here $\text{GL}(n, R)$ denotes the general linear group of all invertible real $n \times n$
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Figure 1.3: Polar bodies of $B_2^1$ and $(\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix})$ matrices.

ii) Let $K \subseteq \mathbb{R}^n$ be convex and closed with $0 \in K$. Then $(K^*)^* = K$.

iii) Let $K \in \mathcal{K}_o^n$. Then $K^* \in \mathcal{K}_o^n$ and $|x|_{K^*} = \max_{y \in K} \langle x, y \rangle = h_K(x)$.

By Proposition 1.2 iii) and Hölder’s inequality for sums,

$$\sum_{i=1}^{n} |x_k y_k| \leq |x|_p |y|_{p/(p-1)}, \quad x, y \in \mathbb{R}^n, p \in (1, \infty),$$

with equality if and only if $|x_i| = c |y_i|^{p-1}$ for a constant $c$, the polar bodies of the $p$-norms are given by

$$(B_n^p)^* = B_n^{p^{-1}}, 1 < p < \infty, \text{ and } (B_n^1)^* = B_n^\infty \text{ and thus } (B_n^\infty)^* = B_n^1.$$

Since we are mainly dealing with simple structured bounded convex sets, we define the volume of sets via the Jordan-Peano measure: A subset $D \subset \mathbb{R}^n$ is called (Jordan-Peano) measurable if its indicator function $\chi_D : \mathbb{R}^n \to \{0, 1\}$ given by

$$\chi_D(x) := \begin{cases} 1, & x \in D, \\ 0, & x \not\in D, \end{cases}$$

is Riemann integrable, i.e., the Riemann integral

$$\text{vol}(D) := \int_{\mathbb{R}^n} \chi_D(x) \, dx$$

exists. vol($D$) is called the volume of $D$. So the volume of a set $D$ is based on the principle of exhausting or approximating $D$ by "small" $n$-dimensional boxes \{x \in \mathbb{R}^n : \alpha_i \leq x_i \leq \beta_i, 1 \leq i \leq n\}, \alpha_i < \beta_i \in \mathbb{R},$ for which the volume is defined as $\prod_{i=1}^{n} (\beta_i - \alpha_i)$. In particular, we have $V(X) = 0$ for a measurable set of dim $X < n$, and

1.3 Proposition. Let $X \subset \mathbb{R}^n$ be a measurable set. Then

$$\text{vol}(X) = \lim_{m \to \infty} \frac{\#(X \cap \frac{1}{m} \mathbb{Z}^n)}{m^n},$$
where $\mathbb{Z}^n = \{ z \in \mathbb{R}^n : z_i \in \mathbb{Z} \}$ denotes the set of all points with integral coordinates.

The unit ball $B^\infty_n$, i.e., the cube (box) of edge length 2 centered at the origin has $\text{vol}(B^\infty_n) = 2^n$. In general, the volume of the unit balls $B^p_n$ can be expressed via the $\Gamma$-function $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t} \, dt$ as (cf. Exercise 1.3)

$$\text{vol}(B^p_n) = 2^n \frac{\Gamma(1 + 1/p)^n}{\Gamma(1 + n/p)}, \quad (1.1)$$

where the $p = \infty$ case is covered by the limit of the formula above.

As a function $\text{vol} : \mathcal{K}^n \to \mathbb{R}_{\geq 0}$ on the set of convex bodies, the volume shares the following properties: first of all it is valuation, i.e., it is an additive function on the space of all convex bodies, which means that for $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$ it is $\text{vol}(K \cup L) = \text{vol}(K) + \text{vol}(L) - \text{vol}(K \cap L)$. The volume is translation invariant, i.e., $\text{vol}(t + X) = \text{vol}(X)$ for any measurable set $X$ and $t \in \mathbb{R}^n$, and since $\text{vol}(A X) = |\det A| \text{vol}(X)$ for $A \in \text{GL}(n, \mathbb{R})$ the volume is also invariant with respect to rotations, i.e., $\text{vol}(A X) = \text{vol}(X)$ for any orthogonal matrix $A$ of determinant 1. Moreover, the volume is homogeneous of degree $n$, i.e., $\text{vol}(\lambda K) = |\lambda|^n \text{vol}(K)$ and (strictly) monotone in the sense that for $K \subseteq L \in \mathcal{K}^n$ we have $\text{vol}(K) \leq \text{vol}(L)$ with equality iff $\dim L < n$ or $\dim L = n$ and $K = L$. Finally, we mention that the volume is also a continuous function with respect to the Hausdorff metric induced by the Hausdorff distance which for $K, L \in \mathcal{K}^n$ is defined as

$$d_H(K, L) := \min\{\rho \in \mathbb{R}_{\geq 0} : K \subseteq L + \rho B_n \text{ and } L \subseteq K + \rho B_n\}.$$

For instance, for the convex bodies $B^p_n$ we have (cf. Exercise 1.4)

$$d_H(B^p_n, B^q_n) = \sqrt{n} |n^{-1/p} - n^{-1/q}|.$$

$d_H(\cdot, \cdot)$ defines a metric on $\mathcal{K}^n$ which makes $\mathcal{K}^n$ a complete space. The next theorem, known as Blaschke’s selection theorem, is a very useful tool in guaranteeing the existence of solutions of extremum problems.

**1.4 Theorem [Blaschke selection theorem]**. Any bounded sequence of convex bodies in $\mathcal{K}^n$ contains a convergent subsequence.
Another theorem with an enormous aftermath is about the so-called Löwner-John ellipsoids. In its basic form it says

1.5 Theorem [Löwner-John ellipsoids]. For each convex body $K \in \mathcal{K}^n$, \( \dim K = n \), there is an unique ellipsoid $a + A B_n$, $A \in \text{GL}(n, \mathbb{R})$ of maximal volume contained in $K$ and it holds

$$a + A B_n \subseteq K \subseteq a + n A B_n.$$ 

If $K \in \mathcal{K}_0^n$ then we have $a = 0$ and the factor $n$ can be replaced by $\sqrt{n}$.

The dilatation factors $n$ or $\sqrt{n}$ are also the smallest possible. For instance, in the symmetric case the bound is attained by the cube $B_\infty^n$ or the crosspoylope $B_1^n$ whose maximal volume ellipsoid is a ball. In the general setting a simplex (see Figure 1.8) is an extremal case.

In some cases the Löwner-John ellipsoids provide an easy way to generalize inequalities valid for ellipsoids to arbitrary o-symmetric convex bodies. As an example we consider the so-called Mahler volume $M(K) := \text{vol}(K) \text{vol}(K^*)$ for $K \in \mathcal{K}_0^n$, $\dim K = n$. In view of Proposition 1.2 i) we know that the Mahler volume is invariant with respect to linear transformations, i.e., $M(AK) = M(K)$ for any $A \in \text{GL}(n, \mathbb{R})$, and hence, in order to bound $M(K)$ we may assume that $B_n$ is the volume maximal ellipsoid contained in $K$. By Theorem 1.5 we have $B_n \subseteq K \subseteq \sqrt{n}B_n$ and thus $(1/\sqrt{n})B_n \subseteq K^* \subseteq B_n$ (cf. Exercise 1.1). So by the monotonicity of the volume we get $\text{vol}(K) \text{vol}(K^*) \leq \text{vol}(\sqrt{n}B_n)\text{vol}(B_n)$ and $\text{vol}(K) \text{vol}(K^*) \geq \text{vol}(B_n)\text{vol}((1/\sqrt{n})B_n)$, or

$$\sqrt{n}^{-n} M(B_n) \leq M(K) \leq \sqrt{n}^n M(B_n).$$

Of course, these bounds are not the best possible, but they show that $M(K)$ is bounded. In order to improve these bounds, we firstly observe that convex bodies mini- and maximizing $M(K)$ really exist. This, however, is a consequence of Theorem 1.4 and the fact that it suffices to consider convex bodies $K \in \mathcal{K}_0^n$ fulfilling $B_n \subseteq K \subseteq \sqrt{n}B_n$. 
It was shown by Blaschke and Santaló that the maximum of $M(K)$ is only attained for ellipsoids, and so we have $M(K) \leq M(B_n) = \text{vol}(B_n)^2$. The lower bound leads to the so called Mahler conjecture claiming that

$$M(K) \geq M(B_{n}^\infty) = \text{vol}(B_{n}^\infty) \text{vol}(B_{n}^1) = \frac{4^n}{n!} = \left( \frac{4}{2\pi} + o(1) \right)^n \text{vol}(B_n).$$

This conjecture is open for $n \geq 3$, and it is also known that there is a whole family of (affinely inequivalent) convex bodies with $M(K) = M(B_{n}^\infty)$. The best known lower bounds are of the type $(\text{constant}^n) \cdot M(B_n)$, and the first who established such a bound were Bourgain and Milman.

1.6 Theorem [Bourgain-Milman]. There exists a positive absolute constant $C$ such that $\text{vol}(K) \text{vol}(K^*) \geq C^n \text{vol}(B_n)^2$ for any convex body $K \in \mathbb{K}_n^o$, $\text{dim} K = n$.

Most of the (classical) results in Geometry of Numbers are dealing with $o$-symmetric bodies because of their interpretation as unit balls of norms. On the other hand, most of the underlying geometric problems and structures can be formulated for arbitrary convex bodies and so there are many problems which have a symmetric and non-symmetric side. In particular, since there are already many results for $o$-symmetric bodies, one is interested in transformations making non-symmetric bodies symmetric such that the geometric sizes in question, e.g., volume, keeps controllable. A standard procedure in this respect is building the central symmetral $\text{cs}(K)$ of $K \in \mathbb{K}_n$ defined by

$$\text{cs} : \mathbb{K}_n \to \mathbb{K}_o^o$$

$$\text{cs} (K) := \frac{1}{2} (K - K).$$

Of course, $K \in \mathbb{K}_o^o$ remains invariant under this map and $\text{cs} (t + K) = \text{cs} (K)$ for $t \in \mathbb{R}^n$. The following classical theorem by Rogers and Shephard completely describes the behavior of the volume with respect to this map.
1.7 Theorem [Rogers-Shephard]. Let $K \in \mathcal{K}^n$. Then
\[
\text{vol}(K) \leq \text{vol}(cs(K)) \leq \frac{1}{2^n} \left( \frac{2^n}{n} \right) \text{vol}(K).
\]
In the lower bound equality is attained iff $K \in \mathcal{K}_0^n$ and in the upper bound iff $K$ is a simplex.

In fact, the lower bound is just an application of the famous Brunn-Minkowski theorem

1.8 Theorem [Brunn-Minkowski]. Let $K, L \in \mathcal{K}^n$ and $\lambda \in [0, 1]$. Then
\[
\text{vol}(\lambda K + (1 - \lambda)L)^{1/n} \geq \lambda \text{vol}(K)^{1/n} + (1 - \lambda) \text{vol}(L)^{1/n}
\]  
(1.2)
with equality iff $K$ and $L$ are contained in parallel hyperplanes, or $K$ and $L$ are homothetic, i.e., there exists a $\mu \in \mathbb{R}_{\geq 0}$ and $t \in \mathbb{R}^n$ with $L = t + \mu K$.

In words, the $n$-th root of the volume is a concave function on $\mathcal{K}^n$. Due to the (weighted) arithmetic-geometric mean inequality (1.2) also implies
\[
\text{vol}(\lambda K + (1 - \lambda)L) \geq \text{vol}(K)^{\lambda} \text{vol}(L)^{1-\lambda},
\]  
(1.3)
i.e., the volume is a log-concave function. In fact, if (1.3) holds for all $K, L \in \mathcal{K}^n$ then it also implies (1.2).

Next we turn to some more discrete aspects of convex sets. Let $x_1, \ldots, x_m \in \mathbb{R}^n$ and let $\lambda_i \in \mathbb{R}_{\geq 0}$, $1 \leq i \leq m$, with $\sum_{i=1}^m \lambda_i = 1$. Then $x = \sum_{i=1}^m \lambda_i x_i$ is called a convex combination of $x_1, \ldots, x_m$. As linear or affine sets are closed with respect to taking finite linear or affine combinations, convex sets are closed with respect to taking finite convex combinations. More precisely (cf. Exercise 1.5), $K \subseteq \mathbb{R}^n$ is convex if and only if
\[
K = \left\{ \sum_{i=1}^m \lambda_i x_i : m \in \mathbb{N}, x_i \in K, \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1 \right\}.
\]

In analogy to linear and affine hulls of a subset $X \subseteq \mathbb{R}^n$ we define the convex hull, $\text{conv } X$, of $X$ as the smallest – with respect to inclusions – convex set containing $X$, i.e.,
\[
\text{conv } X := \bigcap_{\substack{C \subseteq \mathbb{R}^n, C \text{ convex}, \quad \forall X \subseteq C}} C.
\]

As in the linear and affine setting, the convex hull of a set $X$ is the set of all finite convex combinations which can be formed by elements of $X$. In fact, due to a classical result of Carathéodory it suffices to consider convex combinations consisting of at most $\dim X + 1$ elements (cf. Exercise 1.6).
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1.9 Theorem [Carathéodory]. Let $X \subseteq \mathbb{R}^n$. Then

$$\text{conv} \ X = \left\{ \sum_{i=1}^{\dim X+1} \lambda_i x_i : x_i \in X, \lambda_i \geq 0, \sum_{i=1}^{\dim X+1} \lambda_i = 1 \right\}.$$ 

By Caratheodory’s theorem one can easily argue that the convex hull of a compact set $X$ is again compact, but observe, convex hulls of closed sets do not necessarily need to be closed. We also note that for $X, Y \subseteq \mathbb{R}^n$

$$\text{conv} \ X + \text{conv} \ Y = \text{conv} \ (X + Y).$$

If $\# X < \infty$, i.e., $X \subseteq \mathbb{R}^n$ is finite, the convex body $\text{conv} \ X$ is called a polytope, and we shall denote the set of all polytopes and all $o$-symmetric polytopes by $\mathcal{P}^n$ and $\mathcal{P}_o^n$, respectively. In order to emphasize the dimension of a polytope $P \in \mathcal{P}^n$, $P$ will be called an $m$-polytope if $\dim P = m$. In the case of a polytope $P \in \mathcal{P}^n$, the extreme points, i.e., the 0-dimensional faces are called vertices, the 1-dimensional faces are called edges and facets are $(\dim P - 1)$-dimensional faces. Each polytope is the convex hull of its vertices, in fact, each convex body is the convex hull of its extreme points (cf. Exercise 1.9). The vertices of the cube $B_3^\infty$, for instance, are given by the $2^n$ vectors whose coordinate are plus or minus 1, and so we have

$$B_n^\infty = \text{conv} \ \{(\epsilon_1, \ldots, \epsilon_n)^T, \epsilon_i \in \{-1, 1\}\}. \quad (1.4)$$

Given a polytope $P = \text{conv} \ \{x_1, \ldots, x_m\} \in \mathcal{P}^n$, the polar set of $P$ is given by the system of inequalities $P^* = \{y \in \mathbb{R}^n : \langle x_i, y \rangle \leq 1, 1 \leq i \leq m \}$. In general, such a set, i.e., a set given by finitely many linear (weak) inequalities is called a polyhedron. It was firstly shown by Minkowski and Weyl that bounded polyhedra are polytopes and vice versa.

1.10 Theorem [Minkowski-Weyl]. $C \subseteq \mathbb{R}^n$ is a convex polytope if and only if it is a bounded polyhedron.
So we always have two representations of polytopes, either as the convex hull of some points, e.g., the vertices or as the intersection of finitely many linear inequalities, e.g., the inequalities corresponding to facet defining supporting hyperplanes. For instance, the cube has also the presentation

$$B^\infty_n = \{ \mathbf{x} \in \mathbb{R}^n : (\pm \mathbf{e}_i, \mathbf{x}) \leq 1, 1 \leq i \leq n \},$$

(1.5)

where $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ with the 1 at the $i$-th position denotes the $i$-th unit vector. In comparison with the representation as the convex hull of $2^n$ points we only need $2n$ linear inequalities here for describing the cube. The convex hull of $n + 1$ affinely independent points in $\mathbb{R}^n$ is called an $n$-simplex or just a simplex, e.g., $T_n = \text{conv} \{ \mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n \}$ is the so-called standard simplex. By definition, any other simplex is an affine transformation of $T_n$. If the distance between any two points of a simplex is equal, the simplex will be called regular. Denoting by $f_i(P)$ the number of $i$-faces of a polytope $P \in \mathcal{P}^n$, we find for an $n$-simplex that $f_i(T_n) = \binom{n+1}{i+1}$, $i = 0, \ldots, n - 1$, and for the cube $B^\infty_n$ and the crosspolytope $B^1_n$ we have

$$f_{n-1-i}(B^1_n) = f_i(B^\infty_n) = 2^{n-i} \binom{n}{i}, i = 0, \ldots, n - 1.$$

Here the first equality follows from the general fact that the polarity map induces an inclusion reversing bijection between the $i$-faces of a polytope $P$ and the $(n - 1 - i)$-faces of $P^*$, where we assume that $\mathbf{0} \in \text{int} P$, i.e., the origin is an interior point of $P$.

In particular, for those $n$-polytopes $P$ we have: if $P = \text{conv} \{ \mathbf{v}_1, \ldots, \mathbf{v}_m \}$ with vertices $\mathbf{v}_i$, $1 \leq i \leq m$, then $P^* = \{ \mathbf{x} \in \mathbb{R}^n : \langle \mathbf{v}_i, \mathbf{x} \rangle \leq 1, 1 \leq i \leq m \}$
with facets $P_i \cap H(v_i, 1), 1 \leq i \leq m$, and vice versa. Thus, in view of (1.5) and (1.4) we get the representations for the crosspolytope

$$B_n^1 = \text{conv} \{\pm e_i : 1 \leq i \leq n\}$$

$$= \{x \in \mathbb{R}^n : \langle(\epsilon_1, \ldots, \epsilon_n)^T, x\rangle \leq 1, \epsilon_i \in \{-1, 1\}\}.$$ 

In principle, it is an easy task to calculate the volume of a polytope. To this end we note that the volume of an $n$-dimensional pyramid $P = \text{conv} \{Q, w\}$ over an $(n-1)$-dimensional polytope $Q \subset H(a, b)$, say, and with apex $w \notin H(a, b)$ is given by $\text{vol}(P) = (1/n) \text{vol}_{n-1}(Q) \text{dist}(Q, w)$. Here $\text{vol}_{n-1}(\cdot)$ denotes the $(n-1)$-dimensional volume and $\text{dist}(Q, w) = ||\langle a, v \rangle - b||/|a|$ is the distance between $w$ and $\text{aff } Q = H(a, b)$.

![Figure 1.10: Pyramid over $Q$ with apex $w$](image)

Hence, since the volume is additive we can dissect (without gaps) a polytope $P$ into pyramids over its facets $F_1, \ldots, F_m$, and a common apex $w \in P$, say, and get $\text{vol}(P) = (1/n) \sum_{i=1}^{m} \text{vol}_{n-1}(F_i) \text{dist}(F_i, w)$. Since the volume is translation invariant the apex $w$ can also lie outside of $P$, we just have to take the signed distances. For instance, let $\text{aff } F_i = H(a_i, b_i)$ with $|a_i| = 1$ and $P \subset H_{\leq}(a_i, b_i)$, then with respect to the point $w = 0$ we get

$$\text{vol}(P) = \frac{1}{n} \sum_{i=1}^{m} \text{vol}_{n-1}(F_i) b_i = \frac{1}{n} \sum_{i=1}^{m} \text{vol}_{n-1}(F_i) h_P(a_i). \quad (1.6)$$

This recursive formula, however, is apparently inapplicable for practical computations. In fact, it is known that computing the volume of a polytope is a $\#P$-problem. Another ad hoc approach for computing the volume of a polytope is to dissect the polytope into simplices, since the volume of an $n$-simplex $S = \text{conv} \{v_0, \ldots, v_n\}$ can easily be calculated as

$$\text{vol}(S) = \frac{1}{n!} |\det(v_1 - v_0, \ldots, v_n - v_0)| = \frac{1}{n!} \left| \det \left( \begin{array}{c} v_0 \\ 1 \\ \vdots \\ v_{n+1} \\ 1 \end{array} \right) \right|. $$
1. Basic (and some more) facts from Convexity

Here, however, the question is how to dissect $P$ in as few as possible simplices, which is not as easy as it might look. For instance, "even" for the cube $B_n$ the minimal number of simplices in a dissection is only known in small dimensions and the best bounds differ by an exponential factor.

A dissection of a polytope $P \in \mathcal{P}^n$, dim $P = n$, into $n$-simplices $S_i$, $1 \leq i \leq m$, i.e., $P = \cup_{i=1}^m S_i$ and int $(S_i) \cap \text{int} (S_j) = \emptyset$, $i \neq j$, is called a triangulation if $S_i \cap S_j$ is a face of both, $S_i$ and $S_j$. The theorem of Carathéodory (Theorem 1.9) implies that a polytope can be covered by the simplices generated by its vertices, the next theorem strengthens this statement in the sense that we can even triangulate it with its vertices.

1.11 Theorem. $P \in \mathcal{P}^n$, dim $P = n$, can be triangulated into simplices without introducing new vertices, i.e., the vertices of each simplex of the triangulation are a subset of the vertices of $P$.

Figure 1.12: A triangulation of the 3-cube into 5 simplices