Convex Geometry II

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1 Symmetrisations

1.1 Definition [Steiner-Symmetrisation]. *Let $K \in \mathcal{K}^n$, let $H \subset \mathbb{R}^n$ be a hyperplane with normal $\boldsymbol{a} \in S^{n-1}$ and let $K|H$ be the orthogonal projection of K onto H. For $\boldsymbol{x} \in K|H$, let $K \cap (\boldsymbol{x} + \operatorname{lin}\{\boldsymbol{a}\}) = \operatorname{conv}\{\boldsymbol{v}_x, \boldsymbol{w}_x\}$. The set*

$$\operatorname{st}_H(K) = \bigcup_{\boldsymbol{x} \in K|H} \left[\left(\boldsymbol{x} - \frac{1}{2}(\boldsymbol{v}_x + \boldsymbol{w}_x) \right) + \operatorname{conv}\{\boldsymbol{v}_x, \boldsymbol{w}_x\} \right]$$

is called the Steiner-Symmetral *of K with respect to H.*

1.2 Definition [Inradius, Circumradius, Diameter]. *For $K \in \mathcal{K}^n$,*

i) $\mathrm{r}(K) = \max\{r \geq 0 : \boldsymbol{x} + rB_n \subseteq K,\ \boldsymbol{x} \in \mathbb{R}^n\}$ *is the* inradius *of K,*

ii) $\mathrm{R}(K) = \min\{R \geq 0 : K \subseteq \boldsymbol{x} + RB_n,\ \boldsymbol{x} \in \mathbb{R}^n\}$ *is the* circumradius *of K,*

iii) $\mathrm{D}(K) = \max\big\{\|\boldsymbol{x} - \boldsymbol{y}\| : \boldsymbol{x}, \boldsymbol{y} \in K\big\} = \max\{h(K, \boldsymbol{u}) + h(K, -\boldsymbol{u}) : \boldsymbol{u} \in S^{n-1}\}$ *is the* diameter *of K.*

iv) $\mathrm{w}(K) = \min\{h(K, \boldsymbol{u}) + h(K, -\boldsymbol{u}) : \boldsymbol{u} \in S^{n-1}\}$ *is the* minimal width *of K.*

1.3 Proposition. *Let $K, M \in \mathcal{K}^n$ and let H be a hyperplane.*

i) $\operatorname{st}_H(K) \in \mathcal{K}^n$.

ii) $\operatorname{st}_H(\lambda K) = \lambda \operatorname{st}_H(K)$ *for $\lambda \geq 0$, up to translations.*

iii) $\operatorname{st}_H(K) + \operatorname{st}_H(M) \subseteq \operatorname{st}_H(K + M)$, *up to translations.*

iv) *If $K \subseteq M$ then $\operatorname{st}_H(K) \subseteq \operatorname{st}_H(M)$.*

v) $\operatorname{vol}\big(\operatorname{st}_H(K)\big) = \operatorname{vol}(K)$.

vi) $\mathrm{F}\big(\operatorname{st}_H(K)\big) \leq \mathrm{F}(K)$.

vii) $\mathrm{D}\big(\operatorname{st}_H(K)\big) \leq \mathrm{D}(K)$.

viii) $\mathrm{r}\big(\operatorname{st}_H(K)\big) \geq \mathrm{r}(K)$ *and* $\mathrm{R}(\operatorname{st}_H(K)) \leq \mathrm{R}(K)$.

ix) $\operatorname{st}_H : \{K \in \mathcal{K}^n : \dim K = n\} \to \{K \in \mathcal{K}^n : \dim K = n\}$ *is a continuous map. The sequence $K_i = \operatorname{conv}\{(\frac{1}{i}, 0)^\intercal, (0, 1)^\intercal\} \subset \mathbb{R}^2$ shows that the assumption $\dim K = n$ is needed.*

Proof. Without loss of generality we assume that $\mathbf{0} \in H$ and let $L = \operatorname{lin}\{\boldsymbol{a}\}$ be the orthogonal complement of H. ii) and iv) are certainly true.

i) Obviously, $\operatorname{st}_H(K)$ is a compact set (cf. Exercises). For the convexity of $\operatorname{st}_H(K)$, let $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{st}_H(K)$ and consider the convex trapezoid $T = \operatorname{conv}\big\{K \cap (\boldsymbol{x} + L), K \cap (\boldsymbol{y} + L)\big\} \subseteq K$. Clearly $\operatorname{st}_H(T)$ is also a convex trapezoid with $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{st}_H(T)$, and so we have $\operatorname{conv}\{\boldsymbol{x}, \boldsymbol{y}\} \subseteq \operatorname{st}_H(T) \subseteq \operatorname{st}_H(K)$.

iii) Let $\boldsymbol{x} \in \mathrm{st}_H(K)$ and $\boldsymbol{y} \in \mathrm{st}_H(M)$. Then we can write $\boldsymbol{x} = \overline{\boldsymbol{x}} + l_x$ and $\boldsymbol{y} = \overline{\boldsymbol{y}} + l_y$, with $\overline{\boldsymbol{x}} = \boldsymbol{x}|H$, $\overline{\boldsymbol{y}} = \boldsymbol{y}|H$ and $l_x, l_y \in L$. Then

$$\boldsymbol{x} + \boldsymbol{y} = \overline{\boldsymbol{x}} + \overline{\boldsymbol{y}} + (l_x + l_y),$$

and since $\mathbf{0} \in H$ we have $\overline{\boldsymbol{x}} + \overline{\boldsymbol{y}} \in (K + M)|H$. In remains to show that $\|l_x + l_y\| \leq \frac{1}{2}\mathrm{vol}_1\Big((K+M)\cap(\boldsymbol{x}+\boldsymbol{y}+L)\Big)$, which follwos from the oberservation

$$\begin{aligned}\|l_x + l_y\| \leq \|l_x\| + \|l_y\| &\leq \frac{1}{2}\mathrm{vol}_1\big(K \cap (\boldsymbol{x} + L)\big) + \frac{1}{2}\mathrm{vol}_1\big(M \cap (\boldsymbol{y} + L)\big)\\ &= \frac{1}{2}\mathrm{vol}_1\Big([K \cap (\boldsymbol{x} + L)] + [M \cap (\boldsymbol{y} + L)]\Big)\\ &\leq \frac{1}{2}\mathrm{vol}_1\Big((K + M) \cap (\boldsymbol{x} + \boldsymbol{y} + L)\Big).\end{aligned}$$

v) It is an immediate consequence of Fubini's theorem:

$$\mathrm{vol}\,(K) = \int_{K|H} \mathrm{vol}_1\big(K \cap (\boldsymbol{x} + L)\big)\,\mathrm{d}\boldsymbol{x} = \mathrm{vol}\,\big(\mathrm{st}_H(K)\big).$$

vi) By (ii) and (iii) we have $\mathrm{st}_H(K)+\lambda B_n = \mathrm{st}_H(K)+\lambda\,\mathrm{st}_H(B_n) \subseteq \mathrm{st}_H(K+\lambda B_n)$ for any $\lambda > 0$, and with Remark 5.31[1] and property (v) we get

$$\begin{aligned}\mathrm{F}\big(\mathrm{st}_H(K)\big) &= \lim_{\lambda\to 0} \frac{\mathrm{vol}\,\big(\mathrm{st}_H(K) + \lambda B_n\big) - \mathrm{vol}\,\big(\mathrm{st}_H(K)\big)}{\lambda}\\ &\leq \lim_{\lambda\to 0} \frac{\mathrm{vol}\,\big(\mathrm{st}_H(K + \lambda B_n)\big) - \mathrm{vol}\,\big(\mathrm{st}_H(K)\big)}{\lambda}\\ &= \lim_{\lambda\to 0} \frac{\mathrm{vol}\,(K + \lambda B_n) - \mathrm{vol}\,(K)}{\lambda} = \mathrm{F}(K).\end{aligned}$$

vii) Let $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{st}_H(K)$ and let $T, \mathrm{st}_H(T)$ be the convex trapezoids as in the proof of (i). Then one of the diagonals (or both) of T has length greater than or equal to $\|\boldsymbol{x} - \boldsymbol{y}\|$, which proves the assertion.

viii) Let $\boldsymbol{x} \in \mathbb{R}^n$ be such that $\boldsymbol{x} + \mathrm{r}(K)B_n \subseteq K$. Then, using ii), iii) and iv) we get $\mathrm{st}_H(K) \supseteq \mathrm{st}_H\big(\boldsymbol{x} + \mathrm{r}(K)B_n\big) \supseteq \mathrm{st}_H(\boldsymbol{x}) + \mathrm{r}(K)\,\mathrm{st}_H(B_n) = \mathrm{st}_H(\boldsymbol{x}) + \mathrm{r}(K)B_n$. Hence, $\mathrm{r}\big(\mathrm{st}_H(K)\big) \geq \mathrm{r}(K)$, and similary, $\mathrm{R}(\mathrm{st}_H(K)) \leq \mathrm{R}(K)$.

ix) Let $K_i \in \mathcal{K}^n$, $i \in \mathbb{N}$, be a sequence of full-dimensional convex bodies converging in the Hausdorff metric to $K \in \mathcal{K}^n$, with $\dim K = n$. Without loss of generality let $\mathbf{0} \in \mathrm{int}\,K$. Since $K_i \to K$, for any $\varepsilon > 0$ we have $(1-\varepsilon)K \subseteq K_i \subseteq (1+\varepsilon)K$ (see Exercise) for sufficiently large i. Hence, by ii) and iv) we get $(1-\varepsilon)\mathrm{st}_H(K) \subseteq \mathrm{st}_H(K_i) \subseteq (1+\varepsilon)\mathrm{st}_H(K)$ for sufficiently large i. Hence $\mathrm{st}_H(K_i) \to \mathrm{st}_H(K)$. □

1.4 Remark. *Regarding the monontonicty of the surface area (Proposition 1.3 vi)) it is also known that* $\mathrm{F}(\mathrm{st}_H(K)) = F(K)$ *if and only if* K *is symmetric to a hyperplane parallel to* H*.*

[1] Skript WS14

1.5 Theorem. *Let $K \in \mathcal{K}^n$ with $\dim K = n$. Let $\mathcal{S}_K$ be the set of all convex bodies which are obtained by finitely many successive Steiner-symmetrisations of K with respect to hyperplanes containing 0. Then there exists a sequence $K_i \in \mathcal{S}_K$, $i \in \mathbb{N}$, such that*

$$\lim_{i\to\infty} K_i = \left(\frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(B_n)}\right)^{1/n} B_n.$$

Proof. For $M \in \mathcal{K}^n$ let $\rho(M) = \min\{R > 0 : M \subseteq RB_n\}$, let $\rho = \inf\{\rho(M) : M \in \mathcal{S}_K\}$ and let $K_i \in \mathcal{S}_K$ with $\rho(K_i) \to \rho$. Since $K_i \subset rho(K_i)B_n \subset \rho(K)B_n$ (cf. Proposition 1.3 iv)), we may assume by Blaschke's selection Theorem 4.9[2] that $K_i \to \overline{K} \in \mathcal{K}^n$, say. In the follwoing we show

$$\overline{K} = \rho\, B_n,$$

which implies the assertion, since then $K_i \to \rho\, B_n$ and $\rho^n \mathrm{vol}\,(B_n) = \mathrm{vol}\,(\overline{K}) = \lim_{i\to\infty} \mathrm{vol}\,(K_i) = \mathrm{vol}\,(K)$. (cf.)

Suppose $\overline{K} \neq \rho\, B_n$. By the continuity of $\rho(\cdot)$ we have $\rho(\overline{K}) = \lim_{i\to\infty} \rho(K_i) = \rho$, and so we have $\overline{K} \subset \rho\, B_n$ (strictly). Hence we might choose $\boldsymbol{x} \in \rho B_n \setminus \overline{K}$ and a hyperplane $H(\boldsymbol{a}, \alpha)$ strictly separating $\overline{K}$ and $\boldsymbol{x}$. Let $\langle \boldsymbol{a}, \boldsymbol{x}\rangle > \alpha$, and let $C = \{\boldsymbol{y} \in \rho S^{n-1} : \langle \boldsymbol{a}, \boldsymbol{y}\rangle \geq \alpha\}$. Let $H_1, \ldots H_k$ be hyperplanes such that the successive reflections of C with respect to these planes, i.e., first we reflect C at H_1 then the image at H_2 and so on, cover ρS^{n-1}. Then for $\widehat{K} = \mathrm{st}_{H_k}\mathrm{st}_{H_{k-1}} \cdots \mathrm{st}_{H_1}(\overline{K})$ we have $\widehat{K} \subset \mathrm{int}\,(\rho B_n)$, i.e., $\rho(\widehat{K}) < \rho$.

On the other hand, by Proposition 1.3 ix) we know

$$\mathrm{st}_{H_k} \cdots \mathrm{st}_{H_1}(K_i) \to \widehat{K},$$

and since the bodies on the left side belong to $\mathcal{S}_K$ and ρ is continuous we arrive at the contradiction $\rho(\widehat{K}) \geq \rho$. □

1.6 Corollary. *Let $K_1, K_2 \in \mathcal{K}^n$ with $\dim K_1 = \dim K_2 = n$. Then there exist Steiner-symmetrisations st_{H_i}, $i \in \mathbb{N}$, such that*

$$\lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K_1) = \left(\frac{\mathrm{vol}\,(K_1)}{\mathrm{vol}\,(B_n)}\right)^{1/n} B_n$$
$$\lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K_2) = \left(\frac{\mathrm{vol}\,(K_2)}{\mathrm{vol}\,(B_n)}\right)^{1/n} B_n.$$

Proof. With out loss of generality we assume $\mathrm{vol}\,(K_1) = \mathrm{vol}\,(K_2) = \mathrm{vol}\,(B_n)$. Let $\varepsilon > 0$. By Theorem 1.5 there exist hyperplanes $H_1, \ldots, H_k$ containing $\mathbf{0}$ such that

$$(1-\varepsilon)B_n \subseteq \mathrm{st}_{H_k} \cdots \mathrm{st}_{H_1}(K_1) \subseteq (1+\varepsilon)B_n.$$

Applying again Theorem 1.5 with respect to $\mathrm{st}_{H_k} \cdots \mathrm{st}_{H_1}(K_2)$, we get hyperplanes $H_{k+1}, \ldots, H_l$ containing the origin such that

$$(1-\varepsilon)B_n \subseteq \mathrm{st}_{H_l} \cdots \mathrm{st}_{H_{k+1}}\big(\mathrm{st}_{H_k} \cdots \mathrm{st}_{H_1}(K_2)\big) \subseteq (1+\varepsilon)B_n.$$

[2] Skript WS14

Due to Proposition 1.3 ii) and iv) we also know that for this sequence of the symmetrisations with respect to K_1 still holds

$$(1-\varepsilon)B_n \subseteq \mathrm{st}_{H_l}\cdots \mathrm{st}_{H_{k+1}}\big(\mathrm{st}_{H_k}\cdots \mathrm{st}_{H_1}(K_1)\big) \subseteq (1+\varepsilon)B_n.$$

Thus applying the same argumentation to $\mathrm{st}_{H_l}\ldots \mathrm{st}_{H_1}(K_1)$ and $\mathrm{st}_{H_l}\ldots \mathrm{st}_{H_1}(K_2)$ with $\varepsilon/2, \varepsilon/3\ldots$ we get the required sequence of hyperplanes. □

1.7 Corollary. *Let* $K, M \in \mathcal{K}^n$, $\dim K = \dim M = n$. *Then*

i) isodiametric inequality: $\mathrm{vol}\,(K) \le \left(\frac{\mathrm{D}(K)}{2}\right)^n \mathrm{vol}\,(B_n)$ *(and equality holds only for a ball).*

ii) isoperimetric inequality: $\mathrm{F}(K)^n/\mathrm{vol}\,(K)^{n-1} \ge \mathrm{F}(B_n)^n/\mathrm{vol}\,(B_n)^{n-1}$ *and equality holds only for a ball.*

iii) Brunn-Minkowski inequality: $\mathrm{vol}\,(K+M)^{1/n} \ge \mathrm{vol}\,(K)^{1/n} + \mathrm{vol}\,(M)^{1/n}$.

Proof. i) Let $\overline{K}$ be the convex body which is obtained by applying successive Steiner-symmetrisations to K with respect to the coordinate hyperplanes. Then $\overline{K}$ is symmetric with respect to the origin. Thus $R(\overline{K}) = \mathrm{D}(\overline{K})/2$ and $\overline{K} \subseteq (\mathrm{D}(\overline{K})/2)\, B_n$ and by Proposition 1.3 v), vii) we get

$$\mathrm{vol}\,(K) = \mathrm{vol}\,(\overline{K}) \le \left(\frac{\mathrm{D}(\overline{K})}{2}\right)^n \mathrm{vol}\,(B_n) \le \left(\frac{\mathrm{D}(K)}{2}\right)^n \mathrm{vol}\,(B_n).$$

ii) We may assume $\mathrm{vol}\,(K) = \mathrm{vol}\,(B_n)$. By Corollary 1.6 there exist hyperplanes H_i, $i \in \mathbb{N}$, containing the origin, such that

$$\lim_{i\to\infty} \mathrm{st}_{H_i}\cdots \mathrm{st}_{H_1}(K) = B_n.$$

Then by Proposition 1.3 vi) and by the continuity of the surface area F we get

$$\mathrm{F}(B_n) = \lim_{i\to\infty} F(\mathrm{st}_{H_i}\cdots \mathrm{st}_{H_1}(K)) \le \mathrm{F}(K).$$

In the case of equality we must have $K = B_n$, since otherwise, by Remark 1.4 we can find a Steiner-symmetrisation strictly decreasing the surface area.

iii) By Corollary 1.6 there exist hyperplanes H_i, $i \in \mathbb{N}$, containing the origin, such that

$$\mathrm{st}_{H_i}\cdots \mathrm{st}_{H_1}(K) \to \left(\frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(B_n)}\right)^{1/n} B_n \text{ and } \mathrm{st}_{H_i}\cdots \mathrm{st}_{H_1}(M) \to \left(\frac{\mathrm{vol}\,(M)}{\mathrm{vol}\,(B_n)}\right)^{1/n} B_n.$$

By Proposition 1.3 iii), v) and the continuity of the volume (Lemma 5.12[3]) we

[3] Skript WS14

find

$$\begin{aligned}\operatorname{vol}(K+M) &= \lim_{i\to\infty} \operatorname{vol}\left(\operatorname{st}_{H_i}\cdots\operatorname{st}_{H_1}(K+M)\right)\\ &\geq \lim_{i\to\infty} \operatorname{vol}\left(\operatorname{st}_{H_i}\cdots\operatorname{st}_{H_1}(K)+\operatorname{st}_{H_i}\cdots\operatorname{st}_{H_1}(M)\right)\\ &= \operatorname{vol}\left(\left(\frac{\operatorname{vol}(K)}{\operatorname{vol}(B_n)}\right)^{1/n} B_n + \left(\frac{\operatorname{vol}(M)}{\operatorname{vol}(B_n)}\right)^{1/n} B_n\right)\\ &= \operatorname{vol}\left(\frac{\operatorname{vol}(K)^{1/n}+\operatorname{vol}(M)^{1/n}}{\operatorname{vol}(B_n)^{1/n}} B_n\right)\\ &= \left(\operatorname{vol}(K)^{1/n}+\operatorname{vol}(M)^{1/n}\right)^n .\end{aligned}$$

□

1.8 Lemma. *Let $K \in \mathcal{K}_0^n$ with $\dim K = n$ and let H be a hyperplane containing $\mathbf{0}$. Then*

$$\operatorname{vol}\left(\operatorname{st}_H(K)^\star\right) \geq \operatorname{vol}(K^\star).$$

Proof. Without loss of generality let $H = \{\boldsymbol{x} \in \mathbb{R}^n : x_n = 0\}$. Then

$$\operatorname{st}_H(K) = \left\{\left(\boldsymbol{x}, \frac{1}{2}(a-b)\right)^\intercal \in \mathbb{R}^n : (\boldsymbol{x},0)^\intercal \in K|H,\, (\boldsymbol{x},a)^\intercal, (\boldsymbol{x},b)^\intercal \in K\right\}$$

and we also may write

$$\begin{aligned}K^\star &= \{(\boldsymbol{y},t)^\intercal \in \mathbb{R}^n : \langle \boldsymbol{x},\boldsymbol{y}\rangle + s\,t \leq 1,\, (\boldsymbol{x},0)^\intercal \in K|H,\, (\boldsymbol{x},s)^\intercal \in K\},\\ \operatorname{st}_H(K)^\star &= \Big\{(\boldsymbol{y},t)^\intercal \in \mathbb{R}^n : \langle \boldsymbol{x},\boldsymbol{y}\rangle + \frac{1}{2}(a-b)t \leq 1,\\ &\qquad (\boldsymbol{x},0)^\intercal \in K|H,\, (\boldsymbol{x},a)^\intercal, (\boldsymbol{x},b)^\intercal \in K\Big\}.\end{aligned}$$

For $A \subset \mathbb{R}^n$, let $A(t) = \left\{\boldsymbol{x} \in \mathbb{R}^{n-1} : (\boldsymbol{x},t)^\intercal \in A\right\}$. Next we observe that

$$\frac{1}{2}\left(K^\star(t) + K^\star(-t)\right) \subseteq \operatorname{st}_H(K)^\star(t), \tag{1.8.1}$$

i.e., for $\boldsymbol{y}_1 \in K^\star(t)$ and $\boldsymbol{y}_2 \in K^\star(-t)$, we have to show $\left((\boldsymbol{y}_1+\boldsymbol{y}_2)/2, t\right)^\intercal \in \operatorname{st}_H(K)^\star$. To this end let $(\boldsymbol{x},a)^\intercal, (\boldsymbol{x},b)^\intercal \in K$, then

$$\left\langle \boldsymbol{x}, \frac{1}{2}(\boldsymbol{y}_1+\boldsymbol{y}_2)\right\rangle + \frac{1}{2}(a-b)t = \frac{1}{2}\langle \boldsymbol{x},\boldsymbol{y}_1\rangle + \frac{1}{2}\langle \boldsymbol{x},\boldsymbol{y}_2\rangle + \frac{1}{2}at + \frac{1}{2}b(-t) \leq 1,$$

and this shows (1.8.1). Moreover, since K is o-symmetric, $K^\star \in \mathcal{K}_0^n$ and hence $K^\star(t) = -K^\star(-t)$, and so $\operatorname{vol}_{n-1}\left(K^\star(t)\right) = \operatorname{vol}_{n-1}\left(K^\star(-t)\right)$. Thus, (1.8.1) and Brunn-Minkowski inequality imply for all $t \in \mathbb{R}$

$$\begin{aligned}\operatorname{vol}_{n-1}\left(\operatorname{st}_H(K)^\star(t)\right)^{1/(n-1)} &\geq \operatorname{vol}_{n-1}\left(\frac{1}{2}\left(K^\star(t)+K^\star(-t)\right)\right)^{1/(n-1)}\\ &\geq \frac{1}{2}\operatorname{vol}_{n-1}\left(K^\star(t)\right)^{1/(n-1)} + \frac{1}{2}\operatorname{vol}_{n-1}\left(K^\star(-t)\right)^{1/(n-1)}\\ &= \operatorname{vol}_{n-1}\left(K^\star(t)\right)^{1/(n-1)}.\end{aligned}$$

Integrating over t gives $\operatorname{vol}\left(\operatorname{st}_H(K)^\star\right) \geq \operatorname{vol}(K^\star)$. □

1.9 Theorem [Blaschke-Santaló]. [4,5] *Let $K \in \mathcal{K}_0^n$ with $\dim K = n$. Then*

$$\mathrm{vol}\,(K)\,\mathrm{vol}\,(K^\star) \leq \mathrm{vol}\,(B_n)^2,$$

(and equality holds if and only if K is an ellipsoid).

Proof. Let $\mathrm{vol}\,(K) = \mathrm{vol}\,(B_n)$. It is to show $\mathrm{vol}\,(K^\star) \leq \mathrm{vol}\,(B_n)$. By Corollary 1.6 there exist hyperplanes H_i, $i \in \mathbb{N}$, containing the origin such that

$$K_i = \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K) \to B_n.$$

By Lemma 1.8 we know

$$\begin{aligned}\mathrm{vol}\,(K^\star) &\leq \mathrm{vol}\,(\mathrm{st}_{H_1}(K)^\star) = \mathrm{vol}\,({K_1}^\star)\\ &\leq \mathrm{vol}\,(\mathrm{st}_{H_2}(K_1)^\star) = \mathrm{vol}\,({K_2}^\star) \leq \ldots \leq \mathrm{vol}\,({K_i}^\star)\end{aligned}$$

for every $i \in \mathbb{N}$. Since ${K_i}^\star \to B_n$, as well, we get by the continuity of the volume

$$\mathrm{vol}\,(K^\star) \leq \lim_{i\to\infty} \mathrm{vol}\,({K_i}^\star) = \mathrm{vol}\,(B_n).$$

□

1.10 Proposition. [6,7] *Let $K \in \mathcal{K}_0^n$ with $\dim K = n$. Then*

$$\mathrm{vol}\,(K)\,\mathrm{vol}\,(K^\star) \geq \frac{4^n}{(n!)^2}.$$

Proof. Without loss of generality we may assume that $C_n^\star = \mathrm{conv}\,\{\pm \boldsymbol{e}_i : 1 \leq i \leq n\} \subseteq K$ has maximum volume amongst all $AC_n^\star$ with $A \in \mathrm{GL}(n, \mathbb{R})$. This implies that $K \subseteq C_n$, since otherwise there exits a $\boldsymbol{v} \in K$ with $v_j > 1$, say, and replacing $\pm\boldsymbol{e}_j$ by $\pm\boldsymbol{v}$ yields a crosspolytope in K having larger volume than ${C_n}^\star$. Hence, by polarity we also have $C_n^\star \subseteq K^\star$ and so

$$\mathrm{vol}\,(K)\mathrm{vol}\,(K^\star) \geq \mathrm{vol}\,(C_n^\star)^2 = \left(\frac{2^n}{n!}\right)^2.$$

□

1.11 Definition [Schwarz symmetral]. [8]*Let $K \in \mathcal{K}^n$, let L be a $(n-k)$-dimensional subspace, $1 \leq k \leq n-1$, with orthogonal complement $L^\perp$. For any $\boldsymbol{y} \in K|L$ (orthogonal projection of K onto L) let $B_k(\boldsymbol{y}, r_y) \subset \boldsymbol{y} + L^\perp$ be the k-dimensional ball with center $\boldsymbol{y}$ and radius r_y such that $\mathrm{vol}_k(B_k(\boldsymbol{y}, r_y)) = \mathrm{vol}_k(K \cap (\boldsymbol{y} + L^\perp))$. Then*

$$\mathrm{S}_L(K) = \bigcup_{\boldsymbol{y}\in K|L} B_k(\boldsymbol{y}, r_y)$$

is called the Schwarz symmetral *of K with respect to L.*

[4]Wilhelm Blaschke, 1885–1962.
[5]Luis Santaló, 1911–2001.
[6]The Mahler conjecture states $\mathrm{vol}\,(K)\,\mathrm{vol}\,(K^\star) \geq \frac{4^n}{n!}$.
[7]Kurt Mahler, 1903 – 1988
[8]Hermann Schwarz, 1843 – 1921

1.12 Proposition. $\mathrm{S}_L(K)$ *is a convex body.*

Proof. $\mathrm{S}_L(K)$ is certainly bounded, and in order to show that it is also closed let $\boldsymbol{z}_i \in \mathrm{S}_L(K)$ be a sequence converging to $\boldsymbol{z}$. Let $\boldsymbol{y}_i$ and $\boldsymbol{y}$ be the projections of $\boldsymbol{z}_i$ and $\boldsymbol{z}$ onto L, respectively. Then $\boldsymbol{y}_i \to \boldsymbol{y}$ and to due the continuity of $\mathrm{vol}_k((\boldsymbol{y}_i + L^\perp) \cap K)$ on its support $K|L$ we get

$$\begin{aligned} \|\boldsymbol{z} - \boldsymbol{y}\| &= \lim_{m\to\infty} \|\boldsymbol{z}_i - \boldsymbol{y}_i\| \\ &\leq \limsup_{m\to\infty} \left(\frac{\mathrm{vol}_k((\boldsymbol{y}_i + L^\perp) \cap K)}{\mathrm{vol}_k(B_k)}\right)^{1/k} = \left(\frac{\mathrm{vol}_k((\boldsymbol{y} + L^\perp) \cap K)}{\mathrm{vol}_k(B_k)}\right)^{1/k}. \end{aligned}$$

Hence, $\boldsymbol{z} \in \mathrm{S}_L(K)$. It remains to establish the convexity of $\mathrm{S}_L(K)$. Let $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathrm{S}_L(K)$, and let $\boldsymbol{y}_i = \boldsymbol{z}_i|L$, $i = 1, 2$. Then for $\lambda \in [0, 1]$ we have $(\lambda\boldsymbol{z}_1 + (1 - \lambda)\boldsymbol{z}_2)|L = \lambda\boldsymbol{y}_1 + (1 - \lambda)\boldsymbol{y}_2$ and

$$\begin{aligned} &\|\lambda\boldsymbol{z}_1 + (1-\lambda)\boldsymbol{z}_2 - (\lambda\boldsymbol{y}_1 + (1-\lambda)\boldsymbol{y}_2)\| \leq \lambda \|\boldsymbol{z}_1 - \boldsymbol{y}_1\| + (1-\lambda) \|\boldsymbol{z}_2 - \boldsymbol{y}_2\| \\ &\leq \lambda \left(\frac{\mathrm{vol}_k((\boldsymbol{y}_1 + L^\perp) \cap K)}{\mathrm{vol}_k(B_k)}\right)^{1/k} + (1-\lambda) \left(\frac{\mathrm{vol}_k((\boldsymbol{y}_2 + L^\perp) \cap K)}{\mathrm{vol}_k(B_k)}\right)^{1/k} \\ &\leq \left(\frac{\mathrm{vol}_k\left(\lambda[(\boldsymbol{y}_1 + L^\perp) \cap K] + (1-\lambda)[(\boldsymbol{y}_2 + L^\perp) \cap K]\right)}{\mathrm{vol}_k(B_k)}\right)^{1/k}, \end{aligned}$$

where for the last inequality we use the Brunn-Minkowski theorem. Since $\lambda[(\boldsymbol{y}_1 + L) \cap K] + (1 - \lambda)[(\boldsymbol{y}_2 + L) \cap K] \subseteq (\lambda\boldsymbol{y}_1 + (1 - \lambda)\boldsymbol{y}_2 + L) \cap K$ we are done. □

1.13 Definition [Central symmetrisation]. *Let* $K \in \mathcal{K}^n$. *The* central symmetrisation *of* K *is defined as the body* $\frac{1}{2}(K - K)$.

1.14 Proposition. *Let* $K \in \mathcal{K}^n$ *and* $\boldsymbol{u} \in S^{n-1}$. *Then*

$$h(K, \boldsymbol{u}) + h(K, -\boldsymbol{u}) = h(\frac{1}{2}(K - K), \boldsymbol{u}) + h(\frac{1}{2}(K - K), -\boldsymbol{u}).$$

In particular, $D(K) = D(\frac{1}{2}(K - K))$ *and* $\mathrm{w}(K) = \mathrm{w}(\frac{1}{2}(K - K))$.

1.15 Theorem [Rogers&Shephard]. [9] *Let* $K \in \mathcal{K}^n$ *with* $\dim K = n$. *Then*

$$\mathrm{vol}\,(K) \leq \mathrm{vol}\left(\frac{1}{2}(K - K)\right) \leq \frac{1}{2^n}\binom{2n}{n}\mathrm{vol}\,(K).$$

The lower bound is attained if and only if K *is centrally symmetric, (and the upper bound if and only if* K *is a simplex).*

Proof. The lower bound is an immediate consequence of the Brunn-Minkowski inequality

$$\mathrm{vol}\left(\frac{1}{2}(K - K)\right) \geq \left(\frac{1}{2}\mathrm{vol}\,(K)^{1/n} + \frac{1}{2}\mathrm{vol}\,(-K)^{1/n}\right)^n = \mathrm{vol}\,(K).$$

[9] Geoffrey Shephard, 1927

Since K is full-dimensional, equality holds here if and only if K and $-K$ are homothetic (up to translations), i.e., if and only if $K = -K + \boldsymbol{t}$ for some $\boldsymbol{t} \in \mathbb{R}^n$, i.e., K is symmetric to $\boldsymbol{t}/2$.

For the upper bound let $\chi_K(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be the characteristic function of K. Then

$$\begin{aligned}
\mathrm{vol}\,(K)^2 &= \int_{\mathbb{R}^n} \chi_K(\boldsymbol{y})\,\mathrm{vol}\,(K)\;\mathrm{d}\boldsymbol{y} \\
&= \int_{\mathbb{R}^n} \chi_K(\boldsymbol{y}) \left(\int_{\mathbb{R}^n} \chi_K(\boldsymbol{y}-\boldsymbol{x})\;\mathrm{d}\boldsymbol{x} \right) \mathrm{d}\boldsymbol{y} \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \chi_K(\boldsymbol{y}-\boldsymbol{x})\chi_K(\boldsymbol{y})\;\mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \qquad (1.15.1) \\
&= \int_{\mathbb{R}^n} \mathrm{vol}\,(K \cap (\boldsymbol{x}+K))\;\mathrm{d}\boldsymbol{x} \\
&= \int_{K-K} \mathrm{vol}\,(K \cap (\boldsymbol{x}+K))\;\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Now, for any $\boldsymbol{x} \in K - K$ let $\rho = \rho(\boldsymbol{x}) \in [0,1]$ such that $\boldsymbol{x} \in \rho\,\mathrm{bd}\,(K-K)$, i.e., $\rho = |\boldsymbol{x}|_{K-K}$ is just the norm of $\boldsymbol{x}$ induced by $K-K$. Let $\boldsymbol{x} = \rho(\boldsymbol{v}-\boldsymbol{w})$ with $\boldsymbol{v}, \boldsymbol{w} \in K$. Then

$$(1-\rho)K + \rho\boldsymbol{v} \subseteq K \text{ and } (1-\rho)K + \rho\boldsymbol{v} = (1-\rho)K + \rho\boldsymbol{w} + \boldsymbol{x} \subseteq \boldsymbol{x} + K.$$

Hence $\mathrm{vol}\,(K \cap (\boldsymbol{x}+K)) \geq \mathrm{vol}\,((1-\rho)K + \rho\boldsymbol{v}) = (1-\rho)^n \mathrm{vol}\,(K)$, and so with (1.15.1)

$$\mathrm{vol}\,(K)^2 \geq \int_{K-K} \big(1-\rho(\boldsymbol{x})\big)^n \mathrm{vol}\,(K)\;\mathrm{d}\boldsymbol{x},$$

which leads to

$$\begin{aligned}
\mathrm{vol}\,(K) &\geq \int_{K-K} \big(1-\rho(\boldsymbol{x})\big)^n\;\mathrm{d}\boldsymbol{x} \\
&= \int_{K-K} \left(\int_{\rho(\boldsymbol{x})}^1 n(1-t)^{n-1}\;\mathrm{d}t \right) \mathrm{d}\boldsymbol{x} \\
&= \int_0^1 \left(\int_{\left\{\boldsymbol{x}\in K-K : \rho(\boldsymbol{x}) \leq t\right\}} n\,(1-t)^{n-1}\;\mathrm{d}\boldsymbol{x} \right) \mathrm{d}t \\
&= \int_0^1 n(1-t)^{n-1} \mathrm{vol}\,\big(t(K-K)\big)\;\mathrm{d}t \\
&= \mathrm{vol}\,(K-K) \int_0^1 n(1-t)^{n-1} t^n\;\mathrm{d}t \\
&= \frac{(n!)^2}{(2n)!} \mathrm{vol}\,(K-K).
\end{aligned}$$

□

2 Brunn-Minkowski revisited

2.1 Theorem [Brunn-Minkowski Inequality]. *Let $K, L \subset \mathcal{K}^n$, and $\lambda \in [0,1]$. Then*

$$\operatorname{vol}\big(\lambda K + (1-\lambda) L\big)^{1/n} \geq \lambda \operatorname{vol}(K)^{1/n} + (1-\lambda)\operatorname{vol}(L)^{1/n}, \tag{2.1.1}$$

and for $0 < \lambda < 1$ equality holds if and only if either K and L are homothetic or K and L lie in parallel hyperplanes.

Proof. For $\lambda \in [0,1]$ let $K_\lambda = \lambda K + (1-\lambda)L$. First we check that the theorem holds for convex bodies $K, L \in \mathcal{K}^n$ of which one is lower dimensional. With out loss of generality let $\dim L < n$.

If also $\dim K < n$ then $\operatorname{vol}(K) = \operatorname{vol}(L) = 0$ and (2.1.1) holds trivially. If equality holds for a $\lambda \in (0,1)$ then $\operatorname{vol}(K_\lambda) = 0$ and so K_λ lies in a hyperplane H, say. But this implies that K and L lie in hyperplanes parallel to H. On the other hand , if K and L lie in parallel hyperplanes then also K_λ is contained in a hyperplane and $\operatorname{vol}(K_\lambda) = 0$ for $0 \leq \lambda \leq 1$.

Now suppose $\dim K = n$. Since $K_\lambda \supseteq \lambda K + (1-\lambda)\boldsymbol{x}$ for any $\boldsymbol{x} \in L$, we have $\operatorname{vol}(K_\lambda) \geq \operatorname{vol}\big(\lambda K + (1-\lambda)\boldsymbol{x}\big) = \lambda^n \operatorname{vol}(K)$ with equality if and only if $L = \{\boldsymbol{x}\}$ and then K and L are homothetic.

Thus, in the following let $\dim K = \dim L = n$, and next we observe that it is sufficient to prove (2.1.1) in the particular situation when $\operatorname{vol}(K) = \operatorname{vol}(L) = 1$. The general case can be reduced to this setting via the normalisation $\overline{K} = \operatorname{vol}(K)^{-1/n} K$, $\overline{L} = \operatorname{vol}(L)^{-1/n} L$, and

$$\overline{\lambda} = \frac{\lambda \operatorname{vol}(K)^{1/n}}{\lambda \operatorname{vol}(K)^{1/n} + (1-\lambda)\operatorname{vol}(L)^{1/n}},$$

since

$$\begin{aligned}
&\operatorname{vol}\big(\overline{\lambda}\overline{K} + (1-\overline{\lambda})\overline{L}\big)\\
&= \operatorname{vol}\left(\frac{\lambda}{\lambda \operatorname{vol}(K)^{1/n} + (1-\lambda)\operatorname{vol}(L)^{1/n}} K + \frac{1-\lambda}{\lambda \operatorname{vol}(K)^{1/n} + (1-\lambda)\operatorname{vol}(L)^{1/n}} L\right)\\
&= \frac{1}{\big(\lambda \operatorname{vol}(K)^{1/n} + (1-\lambda)\operatorname{vol}(L)^{1/n}\big)^n} \operatorname{vol}\big(\lambda K + (1-\lambda)L\big).
\end{aligned}$$

So let $\operatorname{vol}(K) = \operatorname{vol}(L) = 1$ and we have to show $\operatorname{vol}(K_\lambda) \geq 1$. This will be done by induction on the dimension. If $n = 1$ the result is certainly true, since K and L are intervals which are homothetic. Let $n \geq 2$. For an arbitrary but fixed $\boldsymbol{u} \in S^{n-1}$ and for $\mu \in \mathbb{R}$ we set

$$\begin{aligned}
v_{K}(\mu) &= \operatorname{vol}_{n-1}\big(K \cap H(\boldsymbol{u}, \mu)\big), \qquad & v_{L}(\mu) &= \operatorname{vol}_{n-1}\big(L \cap H(\boldsymbol{u}, \mu)\big),\\
w_{K}(\mu) &= \operatorname{vol}\big(K \cap H^-(\boldsymbol{u}, \mu)\big), & w_{L}(\mu) &= \operatorname{vol}\big(L \cap H^-(\boldsymbol{u}, \mu)\big).
\end{aligned}$$

Then

$$w_{K}(\mu) = \int_{-h(K,-\boldsymbol{u})}^{\mu} v_{K}(\boldsymbol{s})\,\mathrm{d}\boldsymbol{s} \quad \text{and} \quad w_{L}(\mu) = \int_{-h(L,-\boldsymbol{u})}^{\mu} v_{L}(\boldsymbol{s})\,\mathrm{d}\boldsymbol{s}.$$

The functions $v_{_K}, v_{_L}$ are continuous on the intervals $\big(-h(K,-\boldsymbol{u}), h(K,\boldsymbol{u})\big)$ and $\big(-h(L,-\boldsymbol{u}), h(L,\boldsymbol{u})\big)$, respectively, which ensures that $w_{_K}, w_{_L}$ are differentiable with $w'_{_K}(\mu) = v_{_K}(\mu)$ and $w'_{_L}(\mu) = v_{_L}(\mu)$. Moreover, if we denote by $z_{_K}, z_{_L}$ the inverse function of $w_{_K}, w_{_L}$ respectively, then

$$z'_{_K}(\eta) = \frac{1}{v_{_K}\big(z_{_K}(\eta)\big)}, \quad z'_{_L}(\eta) = \frac{1}{v_{_L}\big(z_{_L}(\eta)\big)}, \quad \text{for } 0 < \eta < 1.$$

Now in order to apply induction we compare we compare $(n-1)$-dimensional sections of the bodies via the function $z_K(\eta)$ and set

$$\begin{aligned} &\tilde{K}_\eta = K \cap H\big(\boldsymbol{u}, z_{_K}(\eta)\big), \qquad \tilde{L}_\eta = L \cap H\big(\boldsymbol{u}, z_{_L}(\eta)\big), \\ &z_\lambda(\eta) = \lambda\, z_{_K}(\eta) + (1-\lambda) z_{_L}(\eta). \end{aligned}$$

Observe that $z_\lambda(0) = \lambda\,(-h(K,-\boldsymbol{u})) + (1-\lambda)\,(-h(L,-\boldsymbol{u})) = -h(K_\lambda,-\boldsymbol{u})$ and in the same way we find $z_\lambda(1) = h(K_\lambda,\boldsymbol{u})$. $\tilde{K}_\eta, \tilde{L}_\eta$ are $(n-1)$-dimensional with $\mathrm{vol}_{n-1}(\tilde{K}_\eta) = v_{_K}\big(z_{_K}(\eta)\big)$ and $\mathrm{vol}_{n-1}(\tilde{L}_\eta) = v_{_L}\big(z_{_L}(\eta)\big)$. Since $\lambda\,\tilde{K}_\eta + (1-\lambda)\tilde{L}_\eta \subseteq K_\lambda \cap H\big(\boldsymbol{u}, z_\lambda(\eta)\big)$ we get in view of our induction hypothesis

$$\begin{aligned} &\mathrm{vol}\,(K_\lambda) = \int_{-h(K_\lambda,-\boldsymbol{u})}^{h(K_\lambda,\boldsymbol{u})} \mathrm{vol}_{n-1}\big(K_\lambda \cap H(\boldsymbol{u},\mu)\big)\;\mathrm{d}\mu \\ &= \int_0^1 \mathrm{vol}_{n-1}\big(K_\lambda \cap H(\boldsymbol{u}, z_\lambda(\eta))\big) z'_\lambda(\eta)\;\mathrm{d}\eta \\ &\geq \int_0^1 \mathrm{vol}_{n-1}\big(\lambda\tilde{K}_\eta + (1-\lambda)\tilde{L}_\eta\big) \left[\frac{\lambda}{v_{_K}\big(z_{_K}(\eta)\big)} + \frac{1-\lambda}{v_{_L}\big(z_{_L}(\eta)\big)}\right]\;\mathrm{d}\eta \\ &\geq \int_0^1 \Big(\lambda\, v_{_K}\big(z_{_K}(\eta)\big)^{\frac{1}{n-1}} + (1-\lambda) v_{_L}\big(z_{_L}(\eta)\big)^{\frac{1}{n-1}}\Big)^{n-1} \left[\frac{\lambda}{v_{_K}\big(z_{_K}(\eta)\big)} + \frac{1-\lambda}{v_{_L}\big(z_{_L}(\eta)\big)}\right]\;\mathrm{d}\eta \\ &\geq \int_0^1 1\;\mathrm{d}\eta = 1, \end{aligned} \tag{2.1.2}$$

where the second to last inequality follows by the (weighted) arithmetic/geometric inequality.

We certainly have equality if K and L are homothetic. Thus, suppose we have equality in (2.1.1). Again we may assume $\mathrm{vol}\,(K) = \mathrm{vol}\,(L) = 1$ and so $\mathrm{vol}\,(K_\lambda) = 1$ for some $\lambda \in (0,1)$. Then we have equality in (2.1.2) and by the equality case of the arithmetic/geometric inequality we conclude $v_{_K}\big(z_{_K}(\eta)\big) = v_{_L}\big(z_{_L}(\eta)\big)$ for $\eta \in [0,1]$ (remember v_K, v_L, z_K, z_L are continuous functions). Hence $z'_{_K}(\eta) = z'_{_L}(\eta)$ for $0 \leq \eta \leq 1$, i.e., $z_{_K}(\eta) - z_{_L}(\eta)$ is constant.

Let g_K, g_L be the centroids of K, L, respectively, and let $g_K = g_L = \mathbf{0}$. Then $\mathbf{0} = \int_K \boldsymbol{x}\;\mathrm{d}\boldsymbol{x}$ and so

$$\begin{aligned} 0 &= \int_K \langle \boldsymbol{x}, \boldsymbol{u}\rangle\;\mathrm{d}\boldsymbol{x} \\ &= \int_{-h(K,-\boldsymbol{u})}^{h(K,\boldsymbol{u})} t\,\mathrm{vol}_{n-1}\big(K \cap H(\boldsymbol{u},t)\big)\;\mathrm{d}t \\ &= \int_0^1 \mathrm{vol}_{n-1}\Big(K \cap H\big(\boldsymbol{u}, z_{_K}(\eta)\big)\Big) z_{_K}(\eta) \frac{1}{v_{_K}\big(z_{_K}(\eta)\big)}\;\mathrm{d}\eta = \int_0^1 z_{_K}(\eta)\;\mathrm{d}\eta. \end{aligned}$$

Analogously we obtain the relation $0 = \int_0^1 z_L(\eta)\,\mathrm{d}\eta$, and thus $z_K(\eta) = z_L(\eta)$ for $0 \le \eta \le 1$. Hence, $h(K, \boldsymbol{u}) = h(L, \boldsymbol{u})$. By the arbitrariety of $\boldsymbol{u} \in S^{n-1}$ we conclude $h(K, \boldsymbol{u}) = h(L, \boldsymbol{u})$ for all $\boldsymbol{u} \in S^{n-1}$ and thus $K = L$. □

2.2 Lemma. *Let $K, L \in \mathcal{K}^n$ and suppose there exists a hyperplane H such that $K|H = L|H$. Then for $\lambda \in [0, 1]$*

$$\mathrm{vol}\big(\lambda K + (1-\lambda)L\big) \ge \lambda\,\mathrm{vol}\,(K) + (1-\lambda)\,\mathrm{vol}\,(L).$$

Proof. Let $H = H(\boldsymbol{a}, 0)$ with $\|\boldsymbol{a}\| = 1$. For abbreviation we write $U = K|H = L|H$, and we set $K_\lambda = \lambda K + (1-\lambda)L$. Then $K_\lambda|H = \lambda(K|H) + (1-\lambda)(L|H) = U$ for all $\lambda \in [0, 1]$, and K_λ can be described as

$$K_\lambda = \big\{\boldsymbol{y} + t\boldsymbol{a} : \boldsymbol{y} \in U, f_\lambda(\boldsymbol{y}) \le t \le g_\lambda(\boldsymbol{y})\big\},$$

where the functions f_λ and g_λ satisfy $f_\lambda \le g_\lambda$, f_λ is convex and g_λ is concave. For $\boldsymbol{y} \in U$ and $t_1, t_2 \in \mathbb{R}$ with $\boldsymbol{y} + t_1\boldsymbol{a} \in K$ and $\boldsymbol{y} + t_2\boldsymbol{a} \in L$ we have

$$\boldsymbol{y} + \big(\lambda t_1 + (1-\lambda)t_2\big)\boldsymbol{a} = \lambda(\boldsymbol{y} + t_1\boldsymbol{a}) + (1-\lambda)(\boldsymbol{y} + t_2\boldsymbol{a}) \in K_\lambda.$$

Hence, $f_\lambda(\boldsymbol{y}) \le \lambda t_1 + (1-\lambda)t_2 \le g_\lambda(\boldsymbol{y})$. For $t_1 = f_1(\boldsymbol{y})$ and $t_2 = f_0(\boldsymbol{y})$ we obtain $f_\lambda(\boldsymbol{y}) \le \lambda f_1(\boldsymbol{y}) + (1-\lambda)f_0(\boldsymbol{y})$, and for $t_1 = g_1(\boldsymbol{y})$ and $t_2 = g_0(\boldsymbol{y})$ we get $g_\lambda(\boldsymbol{y}) \ge \lambda g_1(\boldsymbol{y}) + (1-\lambda)g_0(\boldsymbol{y})$. Therefore

$$\begin{aligned}
\mathrm{vol}\,(K_\lambda) &= \int_U \big[g_\lambda(\boldsymbol{y}) - f_\lambda(\boldsymbol{y})\big]\,\mathrm{d}\boldsymbol{y} \\
&\ge \int_U \big[\lambda g_1(\boldsymbol{y}) + (1-\lambda)g_0(\boldsymbol{y}) - \lambda f_1(\boldsymbol{y}) - (1-\lambda)f_0(\boldsymbol{y})\big]\,\mathrm{d}\boldsymbol{y} \\
&= \lambda \int_U \big[g_1(\boldsymbol{y}) - f_1(\boldsymbol{y})\big]\,\mathrm{d}\boldsymbol{y} + (1-\lambda)\int_U \big[g_0(\boldsymbol{y}) - f_0(\boldsymbol{y})\big]\,\mathrm{d}\boldsymbol{y} \\
&= \lambda\,\mathrm{vol}\,(K) + (1-\lambda)\,\mathrm{vol}\,(L),
\end{aligned}$$

which concludes the proof. □

2.3 Theorem. *Let $K, L \in \mathcal{K}^n$ and suppose there exists a hyperplane H such that $\mathrm{vol}_{n-1}(K|H) = \mathrm{vol}_{n-1}(L|H)$. Then for $\lambda \in [0, 1]$*

$$\mathrm{vol}\big(\lambda K + (1-\lambda)L\big) \ge \lambda\,\mathrm{vol}\,(K) + (1-\lambda)\,\mathrm{vol}\,(L).$$

Proof. Let $H = H(\boldsymbol{a}, 0)$, and let $K' = \mathrm{st}_H(K)$ and $L' = \mathrm{st}_H(L)$ be the Steiner-symmetrals of K and L with respect to H, respectively. By Proposition 1.3[10] we have $\lambda K' + (1-\lambda)L' \subseteq \mathrm{st}_H\big(\lambda K + (1-\lambda)L\big)$ and since the Steiner symmetrisation preserves the volume, it suffices to prove

$$\mathrm{vol}\big(\lambda K' + (1-\lambda)L'\big) \ge \lambda\,\mathrm{vol}\,(K') + (1-\lambda)\,\mathrm{vol}\,(L').$$

[10] Skript WS14

Observe that $K|H = K' \cap H$ and $L|H = L' \cap H$. According to Corollary 1.6[11] we can find hyperplanes $H_i = H(\boldsymbol{a}_i, 0)$, $\boldsymbol{a}_i \in H$, $i \in \mathbb{N}$, such that

$$\begin{aligned}\lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K' \cap H) &= \left(\frac{\mathrm{vol}_{n-1}(K' \cap H)}{\kappa_{n-1}}\right)^{\frac{1}{n-1}} B_{n-1} \\ &= \left(\frac{\mathrm{vol}_{n-1}(L' \cap H)}{\kappa_{n-1}}\right)^{\frac{1}{n-1}} B_{n-1} \\ &= \lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(L' \cap H),\end{aligned} \tag{2.3.1}$$

where B_{n-1} is the ball of radius 1 centered at the origin in H with volume κ_{n-1}. Now let

$$K'' = \lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K') \text{ and } L'' = \lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(L').$$

Since K' is symmetric to H and $\boldsymbol{a}_i \in H$, the symmetrals $\mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K')$ are also symmetric to H and so

$$\begin{aligned}[\mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K')]|H &= [\mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K')] \cap H \\ &= [\mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K' \cap H)] = \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K'|H).\end{aligned}$$

Hence

$$K''|H = \lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K' \cap H) \text{ and } L''|H = \lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(L' \cap H).$$

we get with Lemma 2.2 applied to K'', L''

$$\begin{aligned}\mathrm{vol}\left(\lambda K' + (1-\lambda)L'\right) &= \mathrm{vol}\left(\lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}\left(\lambda K' + (1-\lambda)L'\right)\right) \\ &\geq \mathrm{vol}\left(\lambda K'' + (1-\lambda)L''\right) \geq \lambda\, \mathrm{vol}\,(K'') + (1-\lambda)\, \mathrm{vol}\,(L'') \\ &= \lambda\, \mathrm{vol}\,(K') + (1-\lambda)\, \mathrm{vol}\,(L').\end{aligned}$$

□

2.4 Theorem [Grünbaum]. [12] *Let $K \in \mathcal{K}^n$ with $\dim K = n$ and center of gravity $\boldsymbol{c}(K)$. Let H^+ be a subspace containing $\boldsymbol{c}(K)$. Then*

$$\frac{\mathrm{vol}\,(K \cap H^+)}{\mathrm{vol}\,(K)} \geq \left(\frac{n}{n+1}\right)^n.$$

(Observe that $(n/(n+1))^n \to 1/e$).

Proof. Without loss of generality let $\boldsymbol{c}(K) = \boldsymbol{0}$ and $H^+ = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle = x_1 \geq 0\}$. Let $H^- = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle \leq 0\}$, and for $t \in \mathbb{R}$ let $H_t = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle = t\}$ and $f(t) = \mathrm{vol}_{n-1}(K \cap H_t)$.

[11] Skript WS14
[12] Branko Grünbaum, 1929

Let $b = h(K, \boldsymbol{e}_1)$ and $a = h(K, -\boldsymbol{e}_1)$, where $h(K, \cdot)$ denotes the support function of K. Since $\boldsymbol{0} = \int_K \boldsymbol{x}\, \mathrm{d}\boldsymbol{x}$ we have

$$0 = \int_K \langle \boldsymbol{x}, \boldsymbol{e}_1 \rangle \, \mathrm{d}\boldsymbol{x} = \int_{-a}^{b} t\, f(t)\, \mathrm{d}t. \tag{2.4.1}$$

First we apply Schwarz-Symmetrization to K with respect to the hyperplane H, i.e., we replace $K \cap H_t$ by an $(n-1)$-dimensional ball of volume $f(t)$, i.e., of radius $(f(t)/\kappa_{n-1})^{1/(n-1)}$ and centred at the point $(t, \boldsymbol{0})^\intercal \in \mathbb{R}^n$. Let $\overline{K}$ be the resulting Schwarz-Symmetral. By construction, $\mathrm{vol}\,(K \cap H_t) = \mathrm{vol}\,(\overline{K} \cap H_t)$ and, in particular, $\mathrm{vol}\,(K) = \mathrm{vol}\,(\overline{K})$, $\mathrm{vol}\,(K \cap H^+) = \mathrm{vol}\,(\overline{K} \cap H^+)$, and we also have $\boldsymbol{c}(\overline{K}) = \boldsymbol{c}(K) = \boldsymbol{0}$. For the last statement we observe that $\overline{K}$ is rotational symmetric with respect to the axis $\{\lambda\, \boldsymbol{e}_1 : \lambda \in \mathbb{R}\}$. Hence $\boldsymbol{c}(\overline{K})$ has to be on this axis, i.e., $\boldsymbol{c}(\overline{K}) = \gamma\, \boldsymbol{e}_1$. Thus, on account of (2.4.1), we get

$$\begin{aligned} \mathrm{vol}\,(\overline{K})\, \langle \gamma\, \boldsymbol{e}_1, \boldsymbol{e}_1 \rangle &= \int_{\overline{K}} \langle \boldsymbol{x}, \boldsymbol{e}_1 \rangle \, \mathrm{d}\boldsymbol{x} \\ &= \int_{-a}^{b} t\, \mathrm{vol}_{\,n-1}(\overline{K} \cap H_t)\, \mathrm{d}t = \int_{-a}^{b} t\, f(t)\, \mathrm{d}t = 0. \end{aligned}$$

Hence $\gamma = 0$, i.e., $\boldsymbol{c}(\overline{K}) = \boldsymbol{0}$ and in the following we may assume $K = \overline{K}$.

Let $K_0 = K \cap H_0$, $K^+ = K \cap H^+$, $K^- = K \cap H^-$, and let $P^+ = \mathrm{conv}\,\{K_0, \beta \boldsymbol{e}_1\}$ be the pyramid with basis K_0 such that $\mathrm{vol}\,(P^+) = \mathrm{vol}\,(K^+)$, i.e., $\beta = n\mathrm{vol}\,(K^+)/\mathrm{vol}_{\,n-1}(K_0)$.

We now extend this pyramid in the other direction such that for $\alpha > 0$ and $P^- = \mathrm{conv}\,\{K_0, -\alpha \boldsymbol{e}_1 + \frac{\alpha+\beta}{\beta} K_0)\}$ holds $\mathrm{vol}\,(P^-) = \mathrm{vol}\,(K^-)$. Let $P = P^+ \cup P^-$, i.e.,

$$P = \mathrm{conv}\left\{-\alpha \boldsymbol{e}_1 + \frac{\alpha+\beta}{\beta} K_0, \beta \boldsymbol{e}_1\right\}$$

is a circular pyramid of height $\alpha + \beta$ and $\mathrm{vol}\,(P) = \mathrm{vol}\,(K)$, and we have

$$\frac{\mathrm{vol}\,(K^+)}{\mathrm{vol}\,(K)} = \frac{\mathrm{vol}\,(P^+)}{\mathrm{vol}\,(P)} = \left(\frac{\beta}{\alpha+\beta}\right)^n.$$

Hence, it remains to show $\beta/(\alpha+\beta) \geq n/(n+1)$. The crucial observation here is that due to the concavity of $f(t)^{1/(n-1)}$ the centroid $c(P)$ is on the non-negative x-axis, i.e., $c(P) = (\gamma, \boldsymbol{0})^\intercal \in \mathbb{R}^n$ with $\gamma \geq 0$ (see the end of the proof). Then with

$$l(t) = \mathrm{vol}_{\,n-1}(P \cap H_t) = \left(\frac{\beta - t}{\beta}\right)^{n-1} \mathrm{vol}_{\,n-1}(K_0)$$

for $t \in [-\alpha, \beta]$ we may write

$$\begin{aligned} 0 \leq \gamma = \mathrm{vol}\,(P)\, \langle \boldsymbol{c}(P), \boldsymbol{e}_1 \rangle &= \int_{-\alpha}^{\beta} t\, l(t)\, \mathrm{d}t \\ &= \frac{\mathrm{vol}_{\,n-1}(K_0)}{\beta^{n-1}} \int_{-\alpha}^{\beta} t\, (\beta - t)^{n-1}\, \mathrm{d}t \\ &= \frac{\mathrm{vol}_{\,n-1}(K_0)}{\beta^{n-1}} \left(-\frac{t}{n}(\beta - t)^n - \frac{(\beta - t)^{n+1}}{n(n+1)} \Big|_{-\alpha}^{\beta} \right) \\ &= \frac{\mathrm{vol}_{\,n-1}(K_0)}{\beta^{n-1}} \frac{(\beta + \alpha)^n}{n} \left(-\alpha + \frac{\alpha + \beta}{n+1} \right). \end{aligned}$$

Hence $\alpha/(\alpha+\beta) \le 1/(n+1)$ which is equivalent to $\beta/(\alpha+\beta) \ge n/(n+1)$.

Here we give a more formal argument why $\langle \boldsymbol{e}_1, \boldsymbol{c}(P)\rangle \ge 0$. Since

$$\operatorname{vol}(P)\ \langle \boldsymbol{e}_1, \boldsymbol{c}(P)\rangle = \int_{-\alpha}^{\beta} t\, l(t)\,\mathrm{d}t = -\int_{-\alpha}^{0} (-t)\, l(t)\,\mathrm{d}t + \int_{0}^{\beta} t\, l(t)\,\mathrm{d}t,$$

it is to show

$$\int_{0}^{\beta} t\, l(t)\,\mathrm{d}t \ge \int_{-\alpha}^{0} (-t)\, l(t)\,\mathrm{d}t.$$

In view of (2.4.1) this will follow from the two inequalities

$$\int_{0}^{\beta} t\, l(t)\,\mathrm{d}t \ge \int_{0}^{b} t f(t)\,\mathrm{d}t \text{ and } \int_{-a}^{0} (-t) f(t)\,\mathrm{d}t \ge \int_{-\alpha}^{0} (-t)\, l(t)\,\mathrm{d}t. \tag{2.4.2}$$

We will just prove the first relation, the second is treated analogously. $P \cap H_t$ and $K \cap H_t$ are $(n-1)$-dimensional balls of radius $(l(t)/\kappa_{n-1})^{1/(n-1)}$ and $(f(t)/\kappa_{n-1})^{1/(n-1)}$, respectively. Moreover, $l(0) = f(0)$, $l(t)^{1/(n-1)}$ is a linear function and according to the Brunn-Minkowski theorem $f(t)^{1/(n-1)}$ is a concave function. Hence we have

$$\gamma = \min\{\langle \boldsymbol{e}_1, \boldsymbol{x}\rangle : \boldsymbol{x} \in P^+ \setminus K^+\} = \max\{\langle \boldsymbol{e}_1, \boldsymbol{x}\rangle : \boldsymbol{x} \in K^+ \setminus P^+\}.$$

We also have $P^+ = P^+ \setminus K^+ \cup (P^+ \cap K^+)$ and $K^+ = K^+ \setminus P^+ \cup (K^+ \cap P^+)$, and so $\operatorname{vol}(P^+ \setminus K^+) = \operatorname{vol}(K^+ \setminus P^+)$. Altogether we may write

$$\begin{aligned}
\int_{0}^{\beta} t\, l(t)\,\mathrm{d}t &= \int_{P^+} \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle\,\mathrm{d}\boldsymbol{x} \\
&= \int_{P^+\setminus K^+} \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle\ \mathrm{d}\boldsymbol{x} + \int_{P^+\cap K^+} \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle\ \mathrm{d}\boldsymbol{x} \\
&\ge \operatorname{vol}(P^+ \setminus K^+)\,\gamma + \int_{P^+\cap K^+} \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle\ \mathrm{d}\boldsymbol{x} \\
&= \operatorname{vol}(K^+ \setminus P^+)\,\gamma + \int_{K^+\cap P^+} \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle\ \mathrm{d}\boldsymbol{x} \\
&\ge \int_{K^+\setminus P^+} \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle\ \mathrm{d}\boldsymbol{x} + \int_{K^+\cap P^+} \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle\ \mathrm{d}\boldsymbol{x} \\
&= \int_{K^+} \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle\,\mathrm{d}\boldsymbol{x} = \int_{0}^{b} t f(t)\,\mathrm{d}t.
\end{aligned}$$

□

We recall from Proposition 5.18[13] that the Brunn-Minkowski inequality is equivalent to its multiplicative version

$$\operatorname{vol}\big(\lambda K + (1-\lambda)L\big) \ge \operatorname{vol}(K)^{\lambda} \operatorname{vol}(L)^{1-\lambda}, \quad \text{for all } \lambda \in [0,1]. \tag{2.4.3}$$

If we denote the characteristic functions of K and L by χ_K and χ_L, respectively, and if for a given $\lambda \in [0,1]$ the characteristic function of $\lambda\, K + (1-\lambda)\, L$ is denoted

[13] Skript WS14

by χ_λ, then we have $\chi_\lambda(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \chi_K(\boldsymbol{x})\,\chi_L(\boldsymbol{y}) = \chi_K(\boldsymbol{x})^\lambda\,\chi_L(\boldsymbol{y})^{1-\lambda}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and (2.4.3) becomes

$$\int_{\mathbb{R}^n} \chi_\lambda(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \geq \left(\int_{\mathbb{R}^n} \chi_K(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^\lambda \left(\int_{\mathbb{R}^n} \chi_L(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{1-\lambda}.$$

2.5 Theorem [Prékopa-Leindler Inequality]. [14] [15] *Let $\lambda \in (0,1)$ and let $f, g, h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (Lebesgue-)measurable functions with*

$$h\big(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}\big) \geq f(\boldsymbol{x})^\lambda g(\boldsymbol{y})^{1-\lambda}, \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \tag{2.5.1}$$

and let f, g be (Lebesgue-)integrable. Then

$$\int_{\mathbb{R}^n} h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \geq \left(\int_{\mathbb{R}^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^\lambda \left(\int_{\mathbb{R}^n} g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{1-\lambda}. \tag{2.5.2}$$

Proof. Without loss of generality let $f, g \neq 0$ and bounded; otherwise, we consider for an arbitrary integer m the function $\min\{f, m\}$ or $\min\{g, m\}$ instead of f or g and apply Beppo Levi's [16] monotone convergence theorem. Since

$$\frac{h\big(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}\big)}{(\sup f)^\lambda (\sup g)^{1-\lambda}} \geq \left(\frac{f(\boldsymbol{x})}{\sup f}\right)^\lambda \left(\frac{g(\boldsymbol{y})}{\sup g}\right)^{1-\lambda}$$

let $\sup f = \sup g = 1$. Finally, since we can approximate absolute integrable functions by continuous functions arbitrary well and due to the Lebesgue dominated convergence theorem we may assume that f, g are continuous.

We show the result by induction on the dimension n. Let $n = 1$. For a measurable function $\psi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ and a $t \geq 0$ we consider the super level set $[\psi \geq t] = \{x \in \mathbb{R} : \psi(x) \geq t\}$, and we write $|\psi \geq t|$ to denote the volume of the set $[\psi \geq t]$. Observe that the super level sets are measurable since ψ is measurable and $\int_{\mathbb{R}} \psi(x)\ \mathrm{d}x = \int_0^\infty |\psi \geq t|\ \mathrm{d}t$.

If $f(x) \geq t$ and $g(y) \geq t$, then by (2.5.1) we also have $h\big(\lambda x + (1-\lambda)y\big) \geq t$ and so

$$\lambda[f \geq t] + (1-\lambda)[g \geq t] \subseteq [h \geq t].$$

For $t \in [0,1)$ the sets on the left-hand side are non-empty, bounded and closed, and thus, the (general) Brunn-Minkowski theorem for compact sets Theorem 2.1[17] in the case $n = 1$ gives $|h \geq t| \geq \lambda|f \geq t| + (1-\lambda)|g \geq t|$ and so we obtain

$$\begin{aligned}
\int_{\mathbb{R}} h(x)\ \mathrm{d}x \geq \int_0^1 |h \geq t|\ \mathrm{d}t &\geq \lambda \int_0^1 |f \geq t|\ \mathrm{d}t + (1-\lambda)\int_0^1 |g \geq t|\ \mathrm{d}t \\
&= \lambda \int_0^\infty f(x)\ \mathrm{d}x + (1-\lambda)\int_0^\infty g(x)\ \mathrm{d}x \\
&\geq \left(\int_{\mathbb{R}} f(x)\ \mathrm{d}x\right)^\lambda \left(\int_{\mathbb{R}} g(x)\ \mathrm{d}x\right)^{1-\lambda},
\end{aligned}$$

[14] András Prékopa, 1929
[15] László Leindler, 1935
[16] Beppo Levi, 1875–1961
[17] Skript WS14

where the last inequality follows from the arithmetic-geometric mean inequality. This proves the case $n = 1$.

Now let $n > 1$. As usual we identify $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \mathbb{R}$ and for $\boldsymbol{z} \in \mathbb{R}^{n-1}$, $s \in \mathbb{R}$ we define $h_s(\boldsymbol{z}) = h(\boldsymbol{z}, s)$, $f_s(\boldsymbol{z}) = f(\boldsymbol{z}, s)$ and $g_s(\boldsymbol{z}) = g(\boldsymbol{z}, s)$ on $\mathbb{R}^{n-1}$. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n-1}$, $\alpha, \beta \in \mathbb{R}$ and $\gamma = \lambda\alpha + (1-\lambda)\beta$. Then

$$\begin{aligned} h_\gamma\big(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}\big) &= h\big(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}, \lambda\alpha + (1-\lambda)\beta\big) \\ &= h\big(\lambda(\boldsymbol{x}, \alpha) + (1-\lambda)(\boldsymbol{y}, \beta)\big) \\ &\geq f(\boldsymbol{x}, \alpha)^\lambda g(\boldsymbol{y}, \beta)^{1-\lambda} = f_\alpha(\boldsymbol{x})^\lambda g_\beta(\boldsymbol{y})^{1-\lambda}. \end{aligned}$$

Thus, by our inductive argument we get

$$\int_{\mathbb{R}^{n-1}} h_\gamma(\boldsymbol{z}) \,\mathrm{d}\boldsymbol{z} \geq \left(\int_{\mathbb{R}^{n-1}} f_\alpha(\boldsymbol{z}) \,\mathrm{d}\boldsymbol{z}\right)^\lambda \left(\int_{\mathbb{R}^{n-1}} g_\beta(\boldsymbol{z}) \,\mathrm{d}\boldsymbol{z}\right)^{1-\lambda}.$$

With

$$H(\gamma) = \int_{\mathbb{R}^{n-1}} h_\gamma(\boldsymbol{z}) \,\mathrm{d}\boldsymbol{z}, \quad F(\alpha) = \int_{\mathbb{R}^{n-1}} f_\alpha(\boldsymbol{z}) \,\mathrm{d}\boldsymbol{z}, \quad G(\beta) = \int_{\mathbb{R}^{n-1}} g_\beta(\boldsymbol{z}) \,\mathrm{d}\boldsymbol{z},$$

this becomes $H\big(\lambda\alpha + (1-\lambda)\beta\big) \geq F(\alpha)^\lambda G(\beta)^{1-\lambda}$. Hence we may apply the case $n = 1$ to these functions and get the desired result:

$$\begin{aligned} \int_{\mathbb{R}^n} h(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} h_\gamma(\boldsymbol{z}) \,\mathrm{d}\boldsymbol{z}\right) \mathrm{d}\gamma \\ &= \int_{\mathbb{R}} H(\gamma) \,\mathrm{d}\gamma \geq \left(\int_{\mathbb{R}} F(\alpha) \,\mathrm{d}\alpha\right)^\lambda \left(\int_{\mathbb{R}} G(\beta) \,\mathrm{d}\beta\right)^{1-\lambda} \\ &= \left(\int_{\mathbb{R}^n} f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}\right)^\lambda \left(\int_{\mathbb{R}^n} g(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}\right)^{1-\lambda}. \end{aligned}$$

□

2.6 Remark. *The Prékopa-Leindler inequality can be extended to m functions: Let $h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $f_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be measurable functions, $1 \leq i \leq m$, and let $\lambda_i > 0$ with $\sum_{i=1}^m \lambda_i = 1$ such that*

$$h\left(\sum_{i=1}^m \lambda_i \boldsymbol{x}_i\right) \geq \prod_{i=1}^m f_i(\boldsymbol{x}_i)^{\lambda_i},$$

and let f_i be integrable, $1 \leq i \leq m$. Then

$$\int_{\mathbb{R}^n} h(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \geq \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}\right)^{\lambda_i}.$$

2.7 Remark. *Hölder's-inequality* [18]

$$\int_{\mathbb{R}^n} f(\boldsymbol{x}) \, g(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \leq \left(\int_{\mathbb{R}^n} f(\boldsymbol{x})^p \,\mathrm{d}\boldsymbol{x}\right)^{1/p} \left(\int_{\mathbb{R}^n} g(\boldsymbol{x})^q \,\mathrm{d}\boldsymbol{x}\right)^{1/q}.$$

[18] Otto Hölder, 1859 - 1937

for integrable functions $f, g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *and real numbers* $p, q > 1$ *with* $1/p + 1/q = 1$*, may be regarded as a counterpart to the Prékopa-Leindler inequality in the special setting* $p = 1/\lambda$*,* $q = 1/(1-\lambda)$*,* $0 < \lambda < 1$*, and with respect to the functions* $f^{1/p}, g^{1/q}$*:*

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})^{\lambda}\, g(\boldsymbol{x})^{1-\lambda}\, \mathrm{d}\boldsymbol{x} \leq \left(\int_{\mathbb{R}^n} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\right)^{\lambda} \left(\int_{\mathbb{R}^n} g(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\right)^{1-\lambda}.$$

3 Applications of the inequalities of Brascamp-Lieb and Barthe

3.1 Remark. *Let $f_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be integrable functions, $1 \leq i \leq m$, and let $\lambda_i \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$. If we set*

$$h(\boldsymbol{y}) = \sup\left\{\prod_{i=1}^m f_i(\boldsymbol{y}_i)^{\lambda_i} : \boldsymbol{y} = \sum_{i=1}^m \lambda_i\, \boldsymbol{y}_i,\, \boldsymbol{y}_i \in \mathbb{R}^n\right\}$$

then $h\left(\sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i\right) \geq \prod_{i=1}^m f_i(\boldsymbol{x}_i)^{\lambda_i}$, for all $\boldsymbol{x}_i \in \mathbb{R}^n$, $1 \leq i \leq m$. Hence, Prékopa-Leindler inequality (2.5.2) *(cf. Remark 2.6) can also be equivalently written in the form*

$$\int_{\mathbb{R}^n}^{*} \sup\left\{\prod_{i=1}^m f_i(\boldsymbol{x}_i)^{\lambda_i} : \boldsymbol{y} = \sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i,\, \boldsymbol{x}_i \in \mathbb{R}^n\right\} \mathrm{d}\boldsymbol{y} \geq \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{\lambda_i}.$$

Here $\int_{\mathbb{R}^n}^{*} g(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$ denotes the outer integral, which for non-negative functions $g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is defined as

$$\int_{\mathbb{R}^n}^{*} g(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \inf\left\{\int_{\mathbb{R}^n} u(\boldsymbol{y})\mathrm{d}\boldsymbol{y} : u \geq g, u \text{ measureable}\right\}.$$

3.2 Theorem* [Brascamp-Lieb, Barthe]. [19] [20] [21] *Let $c_i > 0$ and $n_i \in \mathbb{N}$, $1 \leq i \leq m$, satisfying $\sum_{i=1}^m c_i\, n_i = n$. Let $M_i \in \mathbb{R}^{n_i \times n}$, $1 \leq i \leq m$, with $\mathrm{rg}(M_i) = n_i$, and let*

$$\alpha = \inf\left\{\frac{\det\left(\sum_{i=1}^m c_i\, M_i^{\intercal} A_i\, M_i\right)}{\prod_{i=1}^m (\det A_i)^{c_i}} : A_i \in \mathbb{R}^{n_i \times n_i} \text{ positive definite}\right\}.$$

Let $f_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ be integrable functions. The Brascamp-Lieb *inequality states*

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(M_i\, \boldsymbol{x})^{c_i}\right) \mathrm{d}\boldsymbol{x} \leq \frac{1}{\sqrt{\alpha}} \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(\boldsymbol{x}_i)\,\mathrm{d}\boldsymbol{x}_i\right)^{c_i}, \tag{BL-I}$$

and the Barthe *inequality states*

$$\int_{\mathbb{R}^n}^{*} \sup\left\{\prod_{i=1}^m f_i(\boldsymbol{x}_i)^{c_i} : \boldsymbol{y} = \sum_{i=1}^m c_i\, M_i^{\intercal} \boldsymbol{x}_i,\, \boldsymbol{x}_i \in \mathbb{R}^{n_i}\right\} \mathrm{d}\boldsymbol{y} \geq \sqrt{\alpha} \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(\boldsymbol{x}_i)\,\mathrm{d}\boldsymbol{x}_i\right)^{c_i}. \tag{B-I}$$

[19] Brascamp-Lieb inequality
[20] Elliott H. Lieb, 1932
[21] Franck Barthe, ??

3.3 Corollary [Hölder and Prékopa-Leindler inequalities]. *Let $f_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be integrable functions and let $\lambda_i > 0$, $1 \leq i \leq m$, with $\sum_{i=1}^m \lambda_i = 1$. Then*

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(\boldsymbol{x})^{\lambda_i} \right) \mathrm{d}\boldsymbol{x} \leq \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right)^{\lambda_i}$$
$$\leq \int_{\mathbb{R}^n}^{*} \sup \left\{ \prod_{i=1}^m f_i(\boldsymbol{x}_i)^{\lambda_i} : \boldsymbol{y} = \sum_{i=1}^m \lambda_i \, \boldsymbol{x}_i, \, \boldsymbol{x}_i \in \mathbb{R}^n \right\} \mathrm{d}\boldsymbol{y}.$$

Proof. We start with Hölder's inequality for which we set $n_i = n$, $c_i = \lambda_i$ and $M_i = \mathrm{I}_n$, $1 \leq i \leq m$. The (BL-I) gives

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(\boldsymbol{x})^{\lambda_i} \right) \mathrm{d}\boldsymbol{x} \leq \frac{1}{\sqrt{\alpha}} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right)^{\lambda_i},$$

and it remains to prove

$$1 \leq \alpha = \inf \left\{ \frac{\det \left(\sum_{i=1}^m \lambda_i A_i \right)}{\prod_{i=1}^m (\det A_i)^{\lambda_i}} : A_i \in \mathbb{R}^{n \times n} \text{ positive definite} \right\}.$$

Applying Minkowski's Theorem 6.4[22] we get

$$\det \left(\sum_{i=1}^m \lambda_i A_i \right)^{1/n} \geq \sum_{i=1}^m \left(\det(\lambda_i A_i) \right)^{1/n} = \sum_{i=1}^m \lambda_i (\det A_i)^{1/n} \geq \prod_{i=1}^m (\det A_i)^{\lambda_i/n},$$

where the last inequality follows from the arithmetic/geometric-mean inequality. Therefore $\alpha \geq 1$.

In fact, taking $A_i = \mathrm{I}_n$ for all $i = 1, \ldots, m$ shows $\alpha = 1$, and Prékopa-Leindler's inequality follows immeaditely from (B-I). □

3.4 Corollary [Young inequality]. [23] *Let $p_i > 0$, $1 \leq i \leq 3$, with $1/p_1 + 1/p_2 + 1/p_3 = 2$, and let $f_i : \mathbb{R}^n \to \mathbb{R}$, $1 \leq i \leq 3$, be integrable functions. Then*

$$\int_{\mathbb{R}^n} f_1(\boldsymbol{x}) \left(\int_{\mathbb{R}^n} f_2(\boldsymbol{x} - \boldsymbol{y}) \, f_3(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \leq c^n \prod_{i=1}^3 \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x})^{p_i} \, \mathrm{d}\boldsymbol{x} \right)^{1/p_i},$$

where $c = c_{p_1} c_{p_2} c_{p_3}$ and $c_p = p^{1/p} / q^{1/q}$, with q such that $1/q + 1/p = 1$.

Proof. We apply (BL-I) to the functions $f_i^{p_i}$, $c_i = 1/p_i$, $n_i = n$ and to the matrices $M_i \in \mathbb{R}^{n \times 2n}$, $1 \leq i \leq 3 = m$, given by $M_1(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}$, $M_2(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x} - \boldsymbol{y}$ and $M_3(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}$ for $(\boldsymbol{x}, \boldsymbol{y})^\intercal \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$. Then $\sum_{i=1}^m c_i n_i = \sum_{i=1}^3 n/p_i = 2n$ and

$$\int_{\mathbb{R}^{2n}} f_1(M_1(\boldsymbol{x}, \boldsymbol{y})) \, f_2(M_2 \, (\boldsymbol{x}, \boldsymbol{y}) \, f_3(M_3(\boldsymbol{x}, \boldsymbol{y})) \, \mathrm{d}(\boldsymbol{x}, \boldsymbol{y}) \leq$$
$$\frac{1}{\sqrt{\alpha}} \prod_{i=1}^3 \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x})^{p_i} \, \mathrm{d}\boldsymbol{x} \right)^{1/p_i}.$$

The computation of the constant α is quite involved and we omit it here. □

[22] Skript WS14
[23] William Henry Young, 1863-1942

3.5 Lemma. *Let* $c_i > 0$ *and let* $\boldsymbol{v}_i \in S^{n-1}$, $1 \le i \le m$, $m \ge n$, *such that* $\sum_{i=1}^m c_i\,(\boldsymbol{v}_i \boldsymbol{v}_i^\intercal) = \mathrm{I}_n$. *Then*

$$\inf\left\{\frac{\det\bigl(\sum_{i=1}^m c_i\,\alpha_i\,(\boldsymbol{v}_i\boldsymbol{v}_i^\intercal)\bigr)}{\prod_{i=1}^m \alpha_i^{c_i}} : \alpha_i \in \mathbb{R}_{>0}\right\} = 1.$$

Proof. Let the infimum on the left hand side be denoted by α. Setting $\alpha_i = 1$, $1 \le i \le m$, shows $\alpha \le 1$, and in the following we prove $\alpha \ge 1$. To this end, let $\boldsymbol{w}_i = \sqrt{c_i}\boldsymbol{v}_i$, $1 \le i \le m$. Then $\mathrm{I}_n = \sum_{i=1}^m c_i\,(\boldsymbol{v}_i\boldsymbol{v}_i^\intercal) = \sum_{i=1}^m (\boldsymbol{w}_i\boldsymbol{w}_i^\intercal) = W\,W^\intercal$, where $W = (\boldsymbol{w}_1, \cdots, \boldsymbol{w}_m) \in \mathbb{R}^{n\times m}$. Hence, the Cauchy-Binet formula[24] gives

$$1 = \det(WW^\intercal) = \sum_{\#I=n} \beta_I, \tag{3.5.1}$$

$\beta_I = \bigl(\det(\boldsymbol{w}_i : i \in I)\bigr)^2$ for each $I \subseteq \{1,\ldots,m\}$ with $\#I = n$. In the same way we see that

$$\begin{aligned}\det\left(\sum_{i=1}^m \alpha_i(\boldsymbol{w}_i\boldsymbol{w}_i^\intercal)\right) &= \sum_{\#I=n} \bigl(\det\bigl(\sqrt{\alpha_i}\,\boldsymbol{w}_i : i \in I\bigr)\bigr)^2 \\ &= \sum_{\#I=n}\left(\prod_{i\in I}\alpha_i\right)\beta_I = \sum_{\#I=n}\alpha_I\beta_I,\end{aligned}$$

with $\alpha_I = \prod_{i\in I}\alpha_i$. In view of (3.5.1) we may apply the arithmetic/geometric-mean inequality and get

$$\det\left(\sum_{i=1}^m \alpha_i(\boldsymbol{w}_i\boldsymbol{w}_i^\intercal)\right) \ge \prod_{\#I=n}\alpha_I^{\beta_I}.$$

The exponent of a fixed α_i in the right-hand side is given by

$$\begin{aligned}\sum_{i\in I,\#I=n}\beta_I &= \sum_{\#I=n}\beta_I - \sum_{i\notin I,\#I=n}\beta_I \\ &= 1 - \det\left(\sum_{j=1,j\neq i}^m \boldsymbol{w}_j\,\boldsymbol{w}_j^\intercal\right) = 1 - \det\bigl(\mathrm{I}_n - \boldsymbol{w}_i\boldsymbol{w}_i^\intercal\bigr) \\ &= 1 - \bigl(1 - \|\boldsymbol{w}_i\|^2\bigr) = c_i,\end{aligned}$$

where for the last step we observe that the eigenvalues of $\mathrm{I}_n - (\boldsymbol{w}_i\boldsymbol{w}_i^\intercal)$ are $1 - \|\boldsymbol{w}_i\|^2$ (with eigenvector $\boldsymbol{w}_i$) and $(n-1)$-times 1 (with eigenvectors orthogonal to $\boldsymbol{w}_i$). Hence $\prod_{\#I=n}\alpha_I^{\beta_I} = \prod_{i=1}^m \alpha_i^{c_i}$ and so

$$\frac{\det\bigl(\sum_{i=1}^m c_i\,\alpha_i\,(\boldsymbol{v}_i\boldsymbol{v}_i^\intercal)\bigr)}{\prod_{i=1}^m\alpha_i^{c_i}} = \frac{\det\bigl(\sum_{i=1}^m \alpha_i\,(\boldsymbol{w}_i\boldsymbol{w}_i^\intercal)\bigr)}{\prod_{i=1}^m\alpha_i^{c_i}} \ge \frac{\prod_{\#I=n}\alpha_I^{\beta_I}}{\prod_{i=1}^m\alpha_i^{c_i}} = 1,$$

which shows that $\alpha \ge 1$, as required. □

[24]In its general form it says, that for $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$ we have $\det AB = \sum_{I\in\binom{m}{n}} \det A^I \det B_I$, where A^I, B_I are the $n\times n$ submatrices of A and B with row and column indices in I, respectively. For $m < n$ it just gives/means $\det AB = 0$.

3.6 Theorem. *Let $f_i : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be measurable functions, and let $c_i > 0$ and $\boldsymbol{v}_i \in S^{n-1}$, $1 \leq i \leq m$, such that $\sum_{i=1}^m c_i\,(\boldsymbol{v}_i\boldsymbol{v}_i^\intercal) = \mathrm{I}_n$. Then*

$$\int_{\mathbb{R}^n}\left(\prod_{i=1}^m f_i\big(\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\big)^{c_i}\right)\mathrm{d}\boldsymbol{x} \leq \prod_{i=1}^m\left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{c_i}$$
$$\leq \int_{\mathbb{R}^n}^{*} \sup\left\{\prod_{i=1}^m f_i(t_i)^{c_i} : \boldsymbol{y} = \sum_{i=1}^m c_i\,t_i\,\boldsymbol{v}_i,\ t_i \in \mathbb{R}\right\}\mathrm{d}\boldsymbol{y}. \tag{3.6.1}$$

For $m = n$ we have equality in both inequalities and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ build an orthonormal basis.

Proof. Since $\mathrm{I}_n = \sum_{i=1}^m c_i\,(\boldsymbol{v}_i\boldsymbol{v}_i^\intercal)$ we may write $\boldsymbol{x} = \sum_{i=1}^m c_i\,\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\,\boldsymbol{v}_i$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and thus, in particular, we have $\mathrm{lin}\,\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\} = \mathbb{R}^n$. First we discuss the case $m = n$. For any $k \in \{1, \ldots, n\}$ we can write $\boldsymbol{v}_k = \sum_{i=1}^n c_i\,\langle \boldsymbol{v}_i, \boldsymbol{v}_k\rangle\,\boldsymbol{v}_i$ and thus we get

$$0 = \sum_{i \neq k} c_i\,\langle \boldsymbol{v}_i, \boldsymbol{v}_k\rangle\,\boldsymbol{v}_i + (c_k - 1)\boldsymbol{v}_k,$$

which implies that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ has to be an orthonormal basis and $c_i = 1$, $1 \leq i \leq n$. Hence we get with $V = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)^\intercal$

$$\int_{\mathbb{R}^n}\left(\prod_{i=1}^n f_i\big(\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\big)\right)\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n}\left(\prod_{i=1}^n f_i\big(\boldsymbol{e}_i^\intercal V\boldsymbol{x}\big)\right)\mathrm{d}\boldsymbol{x}$$
$$= \int_{\mathbb{R}^n}\left(\prod_{i=1}^n f_i(z_i)\right)|\det V|\,\mathrm{d}\boldsymbol{z} = \prod_{i=1}^n\left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right).$$

For the upper bound we observe that for $\boldsymbol{y} = \sum_{i=1}^n t_i\boldsymbol{v}_i$ we have $t_i = \langle \boldsymbol{v}_i, \boldsymbol{y}\rangle$ and hence we get as before

$$\int_{\mathbb{R}^n}^{*} \sup\left\{\prod_{i=1}^n f_i(t_i) : \boldsymbol{y}=\sum_{i=1}^n t_i\,\boldsymbol{v}_i,\ t_i \in \mathbb{R}\right\}\mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^n}\left(\prod_{i=1}^n f_i\big(\langle \boldsymbol{v}_i, \boldsymbol{y}\rangle\big)\right)\mathrm{d}\boldsymbol{y}$$
$$=\prod_{i=1}^n\left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right).$$

The general validity of the inequalities is reduced to the (BL-I) and (B-I). Since

$$n = \mathrm{tr}\,\mathrm{I}_n = \sum_{i=1}^m c_i\,\mathrm{tr}(\boldsymbol{v}_i\boldsymbol{v}_i^\intercal) = \sum_{i=1}^m c_i\,\|\boldsymbol{v}_i\|^2 = \sum_{i=1}^m c_i,$$

we may apply (BL-I) and (B-I) with $n_i = 1$ and $M_i = \boldsymbol{v}_i^\intercal$, for $1 \leq i \leq m$, and get

$$\int_{\mathbb{R}^n}\left(\prod_{i=1}^m f_i\big(\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\big)^{c_i}\right)\mathrm{d}\boldsymbol{x} \leq \frac{1}{\sqrt{\alpha}}\prod_{i=1}^m\left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{c_i} \qquad \text{and}$$
$$\int_{\mathbb{R}^n}^{*} \sup\left\{\prod_{i=1}^m f_i(t_i)^{c_i} : \boldsymbol{y} = \sum_{i=1}^m c_i\,t_i\,\boldsymbol{v}_i,\ t_i \in \mathbb{R}\right\}\mathrm{d}\boldsymbol{y} \geq \sqrt{\alpha}\prod_{i=1}^m\left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{c_i},$$

with

$$\alpha = \inf\left\{ \frac{\det\left(\sum_{i=1}^m c_i\, \alpha_i(\boldsymbol{v}_i \boldsymbol{v}_i^\intercal)\right)}{\prod_{i=1}^m \alpha_i^{c_i}} : \alpha_i > 0 \right\}.$$

Lemma 3.5 shows that $\alpha \geq 1$. □

3.7 Theorem*. *Let $n \geq 2$, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in S^{n-1}$ be pairwise different and let $c_i > 0$ such that $\sum_{i=1}^m c_i\, \boldsymbol{v}_i \boldsymbol{v}_i^\intercal = \mathrm{I}_n$. Let $f_i : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be measureable function which are neither identical 0 nor Gaussian, i.e., of the type $c\,\mathrm{e}^{-\gamma x}$ for positive numbers c, γ. If we have equality in one of the inequalities (3.6.1), then $m = n$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ build an orthonormal basis.*

3.8 Proposition.

i) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function. Via spherical coordinates we get*

$$\begin{aligned}\int_{\mathbb{R}^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= \int_0^\infty \int_{S^{n-1}} r^{n-1} f(r\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}r \\ &= n\kappa_n \int_0^\infty \int_{S^{n-1}} r^{n-1} f(r\boldsymbol{u})\,\mathrm{d}\sigma(\boldsymbol{u})\,\mathrm{d}r.\end{aligned}$$

Here $\mathrm{d}\boldsymbol{u}$ denotes the rotational invariant area measure on the sphere of total mass $\mathrm{F}(S^{n-1}) = n\mathrm{vol}\,(B_n)$, and $\mathrm{d}\sigma(\boldsymbol{u})$ its normalization to a probability measure, i.e., $\int_{S^{n-1}} 1\,\mathrm{d}\sigma(\boldsymbol{u}) = 1$.

ii) *Let $K \in \mathcal{K}^n$ with $\mathbf{0} \in K$. For $\boldsymbol{u} \in S^{n-1}$ let $r_K(\boldsymbol{u}) = \max\{\rho \geq 0 : \rho\,\boldsymbol{u} \in K\}$ be its* radial function*. Then*

$$\mathrm{vol}\,(K) = \kappa_n \int_{S^{n-1}} r_K(\boldsymbol{u})^n\,\mathrm{d}\sigma(\boldsymbol{u}).$$

iii) *Let $K \in \mathcal{K}_0^n$. Then*

$$\mathrm{vol}\,(K) = \kappa_n \int_{S^{n-1}} |\boldsymbol{u}|_K^{-n}\,\mathrm{d}\sigma(\boldsymbol{u}).$$

iv) [25] *Let $K \in \mathcal{K}_0^n$ and $1 \leq p < \infty$. Then*

$$\mathrm{vol}\,(K) = \frac{1}{\Gamma\left(\frac{n}{p}+1\right)} \int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}|_K^p}\,\mathrm{d}\boldsymbol{x}.$$

v)

$$\kappa_n = \mathrm{vol}\,(B_n) = \frac{\pi^{n/2}}{\Gamma\big((n/2)+1\big)} \approx \left(\frac{2\pi e}{n}\right)^{n/2} \frac{1}{\sqrt{\pi n}}.$$

[25] It is $\Gamma(x) = \int_0^\infty \mathrm{e}^{-t} t^{x-1} \mathrm{d}t$. In particular, $\Gamma(t+1) = t\Gamma(t)$, $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$.

Proof. i) is just a coordinate transformation. For ii) we apply i) to the characteristic function χ_K of K

$$\begin{aligned}\mathrm{vol}\,(K) &= \int_{\mathbb{R}^n} \chi_K(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = n\kappa_n \int_0^\infty \int_{S^{n-1}} \chi_K(r\boldsymbol{u}) r^{n-1}\,\mathrm{d}\sigma(\boldsymbol{u})\,\mathrm{d}r \\ &= n\kappa_n \int_{S^{n-1}} \left(\int_0^{r_K(\boldsymbol{u})} r^{n-1}\,\mathrm{d}r \right) \mathrm{d}\sigma(\boldsymbol{u}) = \kappa_n \int_{S^{n-1}} r_K(\boldsymbol{u})^n\,\mathrm{d}\sigma(\boldsymbol{u}).\end{aligned}$$

In iii) K is 0-symmetric, and so we have $r_K(\boldsymbol{u}) = 1/\,|\boldsymbol{u}|_K$ (it was an exercise). Hence, in this case iii) is just a reformulation of ii).

For iv) we observe that

$$\begin{aligned}\int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}|_K^p}\,\mathrm{d}\boldsymbol{x} &= n\,\kappa_n \int_0^\infty \int_{S^{n-1}} r^{n-1}\mathrm{e}^{-|r\boldsymbol{u}|_K^p}\,\mathrm{d}\sigma(\boldsymbol{u})\,\mathrm{d}r \\ &= n\,\kappa_n \int_{S^{n-1}} \int_0^\infty r^{n-1}\mathrm{e}^{-|r\boldsymbol{u}|_K^p}\,\mathrm{d}r\,\mathrm{d}\sigma(\boldsymbol{u}) \\ &\overset{s=r|\boldsymbol{u}|_K}{=} n\,\kappa_n \left(\int_{S^{n-1}} |u|_K^{-n}\,\mathrm{d}\sigma(u) \right) \left(\int_0^\infty \mathrm{e}^{-s^p} s^{n-1}\,\mathrm{d}s \right) \\ &\overset{\text{ii)}}{=} n\,\mathrm{vol}\,(K) \int_0^\infty \mathrm{e}^{-s^p} s^{n-1}\,\mathrm{d}s \\ &\overset{t=s^p}{=} \frac{n}{p}\,\mathrm{vol}\,(K) \int_0^\infty \mathrm{e}^{-t}\, t^{n/p-1}\,\mathrm{d}t \\ &= \frac{n}{p}\,\mathrm{vol}\,(K)\,\Gamma\left(\frac{n}{p}\right) = \mathrm{vol}\,(K)\,\Gamma\left(\frac{n}{p}+1\right).\end{aligned}$$

For v) let $f : \mathbb{R}^n \to \mathbb{R}$ be the function $f(\boldsymbol{x}) = \mathrm{e}^{-\|\boldsymbol{x}\|^2}$. Then by iv) we have

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \kappa_n\,\Gamma\left(\frac{n}{2}+1\right).$$

On the other hand we may write

$$\begin{aligned}\int_{\mathbb{R}^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^n} \mathrm{e}^{\sum_{i=1}^n x_i^2}\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} \left(\prod_{i=1}^n \mathrm{e}^{-x_i^2} \right) \mathrm{d}\boldsymbol{x} = \prod_{i=1}^n \int_{-\infty}^\infty \mathrm{e}^{-x_i^2}\,\mathrm{d}x_i \\ &= 2^n \prod_{i=1}^n \int_0^\infty \mathrm{e}^{-x_i^2}\,\mathrm{d}x_i = \prod_{i=1}^n \int_0^\infty \mathrm{e}^{-t} t^{-1/2}\,\mathrm{d}t \\ &= \Gamma\left(\frac{1}{2}\right)^n = \left(\sqrt{\pi}\right)^n.\end{aligned}$$

Hence together with *Stirling's formula* $n! \approx \sqrt{2\pi n}(n/e)^n$ we get

$$\mathrm{vol}\,(B_n) = \kappa_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \approx \frac{\pi^{n/2}}{\sqrt{2\pi}\sqrt{\frac{n}{2}}\left(\frac{n}{2e}\right)^{n/2}} = \left(\frac{2\pi e}{n}\right)^{n/2} \frac{1}{\sqrt{\pi n}}.$$

□

3.9 Remark.

i) *The radius of an n-dimensional ball of volume 1 is about* $\sqrt{\frac{n}{2\pi \mathrm{e}}}$.

ii) *Let* $C_n = [-1,1]^n$ *the cube of volume* 2^n. *From*

$$2^n = \mathrm{vol}\,(C_n) = \kappa_n \int_{S^{n-1}} r_{C_n}(\boldsymbol{u})^n \,\mathrm{d}\sigma(\boldsymbol{u})$$

we get that the average value of the radial function of C_n *is* $2/\kappa_n^{1/n} \approx \sqrt{2\,n/(\pi e)}$. *Observe, that* $1 \le r_{C_n}(\boldsymbol{u}) \le \sqrt{n}$.

iii) *For the crosspolytope* ${C_n}^\star = \mathrm{conv}\,\{\pm \boldsymbol{e}_i : 1 \le i \le n\}$ *we find*

$$\frac{2^n}{n!} = \mathrm{vol}\ (C_n^\star) = \kappa_n \int_{S^{n-1}} r_{C_n^\star}(\boldsymbol{u})^n \,\mathrm{d}\sigma(\boldsymbol{u})$$

and hence the average value of the radial function of the crosspolytope ${C_n}^\star$ *is*

$$\frac{2}{\left(\kappa_n\, n!\right)^{1/n}} \approx 2 \frac{\sqrt{n}\, e}{\sqrt{2\pi e}\left(\sqrt{2\pi n}\right)^{1/n} n} \approx \sqrt{\frac{2e/\pi}{n}}.$$

Observe, that $\frac{1}{\sqrt{n}} \le r_{{C_n}^\star}(\boldsymbol{u}) \le 1$.

3.10 Lemma. *Let* $c_i > 0$ *and* $\boldsymbol{v}_i \in S^{n-1}$, $1 \le i \le m$, *such that* $\sum_{i=1}^m c_i\, \boldsymbol{v}_i \boldsymbol{v}_i^\intercal = \mathrm{I}_n$. *Moreover, let* $\alpha_i > 0$, $1 \le i \le m$ *and* $1 \le p < \infty$.

i) *Let* $K \in \mathcal{K}_0^n$ *be the 0-symmetric convex body with gauge function*

$$|\boldsymbol{x}|_K = \left(\sum_{i=1}^m \alpha_i \left|\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\right|^p \right)^{1/p}.$$

Then

$$\mathrm{vol}\,(K) \le 2^n \frac{\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} \prod_{i=1}^m \left(\frac{c_i}{\alpha_i}\right)^{c_i/p}.$$

For the l_p*-balls* $B_n^p = \{\boldsymbol{x} \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \le 1\}$ *we have* $\mathrm{vol}\,(B_n^p) = 2^n\, \Gamma\left(1+\frac{1}{p}\right)^n / \Gamma\left(1+\frac{n}{p}\right)$.

ii) *Let* Z *be the zonotope* $Z = \sum_{i=1}^m \mathrm{conv}\,\{-\alpha_i \boldsymbol{v}_i, \alpha_i \boldsymbol{v}_i\}$.[26] *Then*

$$\mathrm{vol}\,(Z) \ge 2^n \prod_{i=1}^m \left(\frac{\alpha_i}{c_i}\right)^{c_i}.$$

Equality holds if and only if $m = n$, $c_i = 1$, *and* $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ *form an orthonormal basis of* $\mathbb{R}^n$, *i.e.,* Z *is an orthogonal box.*

[26] The volume of such a zonotope Z is $\mathrm{vol}\,(Z) = 2^n \sum_{I \in \binom{m}{n}} |\det(v_{j_i} : i \in I)|$.

Proof. By Proposition 3.8 iv) we have

$$\int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}|_K^p}\,\mathrm{d}\boldsymbol{x} = \mathrm{vol}\,(K)\,\Gamma\left(\frac{n}{p}+1\right). \tag{3.10.1}$$

On the other hand, due to the definition of the gauge function we may evaluate the integral as

$$\int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}|_K^p}\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n}\left(\prod_{i=1}^m \mathrm{e}^{-\alpha_i|\langle \boldsymbol{v}_i,\boldsymbol{x}\rangle|^p}\right)\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n}\left(\prod_{i=1}^m f_i(\langle \boldsymbol{v}_i,\boldsymbol{x}\rangle)^{c_i}\right)\mathrm{d}\boldsymbol{x},$$

with $f_i(t) = \mathrm{e}^{-(\alpha_i/c_i)|t|^p}$. Now we apply the upper bound of Theorem 3.6 and get[27]

$$\begin{aligned}\int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}|_K^p}\,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^n}\left(\prod_{i=1}^m f_i(\langle \boldsymbol{v}_i,\boldsymbol{x}\rangle)^{c_i}\right)\mathrm{d}\boldsymbol{x} \leq \prod_{i=1}^m\left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{c_i} \\ &= \prod_{i=1}^m\left(2\int_0^\infty \mathrm{e}^{-\frac{\alpha_i}{c_i}t^p}\,\mathrm{d}t\right)^{c_i} = \prod_{i=1}^m\left(2\left(\frac{c_i}{\alpha_i}\right)^{1/p}\Gamma\left(\frac{1}{p}+1\right)\right)^{c_i} \\ &= 2^n\Gamma\left(\frac{1}{p}+1\right)^n\prod_{i=1}^m\left(\frac{c_i}{\alpha_i}\right)^{c_i/p},\end{aligned}$$

since, by assumption, $\sum_{i=1}^m c_i = \sum_{i=1}^m c_i\,\|\boldsymbol{v}_i\|^2 = \sum_{i=1}^m c_i\,\mathrm{tr}(\boldsymbol{v}_i\boldsymbol{v}_i^\intercal) = \mathrm{tr}\,\mathrm{I}_n = n$. Together with (3.10.1) we get the result. Observe, that vor $m=n$, $c_i=\alpha_i=1$, $\boldsymbol{v}_i=\boldsymbol{e}_i$ we have equality (see Theorem 3.6).

ii) For $1\leq i\leq m$ let $f_i(t)=\chi_{[-\alpha_i/c_i,\alpha_i/c_i]}(t)$ be the characteristic functions of the intervals $[-\alpha_i/c_i,\alpha_i/c_i]$, i.e., $f_i(t)=1$ if and only if $c_it\in[-\alpha_i,\alpha_i]$. Hence, if $\boldsymbol{z}\in\mathbb{R}^n$ satisfies

$$\sup\left\{\prod_{i=1}^m f_i(t_i)^{c_i} : \boldsymbol{z}=\sum_{i=1}^m c_i\,t_i\,\boldsymbol{v}_i,\ t_i\in\mathbb{R}\right\}=1,$$

then $\boldsymbol{z}\in Z=\sum_{i=1}^m \mathrm{conv}\,\{-\alpha_i\boldsymbol{v}_i,\alpha_i\boldsymbol{v}_i\}$. With the upper bound in Theorem 3.6 we get

$$\begin{aligned}\mathrm{vol}\,(Z) &= \int_{\mathbb{R}^n}\chi_Z(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \geq \int_{\mathbb{R}^n}^{*}\sup\left\{\prod_{i=1}^m f_i(t_i)^{c_i} : \boldsymbol{z}=\sum_{i=1}^m c_i\,t_i\,\boldsymbol{v}_i,\ t_i\in\mathbb{R}\right\}\mathrm{d}\boldsymbol{z} \\ &\geq \prod_{i=1}^m\left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{c_i} = \prod_{i=1}^m\left(2\frac{\alpha_i}{c_i}\right)^{c_i} = 2^n\prod_{i=1}^m\left(\frac{\alpha_i}{c_i}\right)^{c_i},\end{aligned}$$

since, again, $\sum_{i=1}^m c_i = n$. The characterization of the equality case follows from Theorem 3.7. □

[27] Observe that $\int_0^\infty \mathrm{e}^{-\gamma t^p}\,\mathrm{d}t = \gamma^{-1/p}(1/p)\int_0^\infty \mathrm{e}^{-s}s^{1/p-1}\,\mathrm{d}s = \gamma^{-1/p}\Gamma(1/p+1)$ for $\gamma>0$.

3.11 Theorem. *Let $K \in \mathcal{K}_0^n$ and let B_n be the maximum volume ellipsoid contained in K.*[28] *Then*

$$\operatorname{vol}(K) \le \operatorname{vol}(C_n) = 2^n.$$

Moreover equality holds if and only if K is a cube of edge length 2, i.e., $K = \big\{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle| \le 1,\, 1 \le i \le n\big\}$, where $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ form an orthonormal basis.

Proof. Since B_n is the maximum volume ellipsoid contained in K, we know by Theorem 6.11[30] that there exist $\boldsymbol{v}_i \in S^{n-1} \cap \operatorname{bd} K$ and $\lambda_i > 0$, $1 \le i \le m$, with $\sum_{i=1}^m \lambda_i = n$ and $\mathrm{I}_n = \sum_{i=1}^m \lambda_i\, (\boldsymbol{v}_i \boldsymbol{v}_i^\intercal)$. Let $U = \big\{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle| \le 1,\, 1 \le i \le m\big\}$. Clearly $K \subseteq U$ and so $\operatorname{vol}(K) \le \operatorname{vol}(U)$. With $f_i = \chi_{[-1,1]}$, $1 \le i \le m$, we can write

$$U = \left\{\boldsymbol{x} \in \mathbb{R}^n : \prod_{i=1}^m f_i\big(\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\big)^{\lambda_i} = 1\right\}$$

and with the lower bound in Theorem 3.6 we get

$$\operatorname{vol}(U) = \int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i\big(\langle v_i, x\rangle\big)^{\lambda_i}\right) \mathrm{d}x \le \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t)\, \mathrm{d}t\right)^{\lambda_i} = 2^{\sum_{i=1}^m \lambda_i} = 2^n.$$

The characterization of the equality case follows from Theorem 3.7. □

3.12 Theorem. *Let $K \in \mathcal{K}^n$ and let B_n be the maximum volume ellipsoid contained in K. Then*

$$\operatorname{vol}(K) \le \operatorname{vol}(T_n) = \frac{(\sqrt{n})^n(\sqrt{n+1})^{n+1}}{n!},$$

where T_n is a regular simplex with inradius 1. Moreover, equality holds if and only if K is a regular simplex with inradius 1, i.e., $K = \big\{x \in \mathbb{R}^n : \langle \boldsymbol{v}_i, \boldsymbol{x}\rangle \le 1,\, 1 \le i \le n+1\big\}$ with $\boldsymbol{v}_i \in S^{n-1}$ and $\langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle = -\frac{1}{n}$ for $i \ne j$.

Proof. Since B_n is the maximum volume ellipsoid contained in K, we know by Theorem 6.14[31] that there exist $\boldsymbol{v}_i \in S^{n-1} \cap \operatorname{bd} K$ and $\lambda_i > 0$, $1 \le i \le m$ and $m \ge n+1$, with $\sum_{i=1}^m \lambda_i = n$, $\mathrm{I}_n = \sum_{i=1}^m \lambda_i\, (\boldsymbol{v}_i \boldsymbol{v}_i^\intercal)$ and $\sum_{i=1}^m \lambda_i\, \boldsymbol{v}_i = 0$. For $U = \big\{x \in \mathbb{R}^n : \langle \boldsymbol{v}_i, \boldsymbol{x}\rangle \le 1,\, 1 \le i \le m\big\}$ it holds $K \subseteq U$ and so $\operatorname{vol}(K) \le \operatorname{vol}(U)$. Now let

$$\boldsymbol{w}_i = \sqrt{\frac{n}{n+1}} \begin{pmatrix} -\boldsymbol{v}_i \\ \frac{1}{\sqrt{n}} \end{pmatrix} \in \mathbb{R}^{n+1} \quad \text{and} \quad c_i = \frac{n+1}{n}\lambda_i, \quad 1 \le i \le m.$$

Then $\boldsymbol{w}_i \in S^n$ for all $1 \le i \le m$, $\sum_{i=1}^m c_i = n+1$ and

$$\sum_{i=1}^m c_i(\boldsymbol{w}_i \boldsymbol{w}_i^\intercal) = \sum_{i=1}^m \lambda_i \begin{pmatrix} (\boldsymbol{v}_i \boldsymbol{v}_i^\intercal) & -\frac{1}{\sqrt{n}}\boldsymbol{v}_i \\ -\frac{1}{\sqrt{n}}\boldsymbol{v}_i^\intercal & \frac{1}{n} \end{pmatrix} = \mathrm{I}_{n+1}.$$

[28] According to Theorem 6.11[29] B_n is the uniquely determined ellipsoid of maximum volume in K if and only if there exist $\boldsymbol{u}_i \in S^{n-1} \cap \operatorname{bd} K$ and $\lambda_i > 0$, $1 \le i \le m$, with $n \le m \le n(n+1)/2$ such that $\mathrm{I}_n = \sum_{i=1}^m \lambda_i\, (\boldsymbol{u}_i \boldsymbol{u}_i^\intercal)$.

[30] Skript WS14

[31] Skript WS14

Now let

$$f_i : \mathbb{R} \to \mathbb{R}_{\geq 0} \quad \text{given by} \quad f_i(t) = \begin{cases} \mathrm{e}^{-t} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases}$$

for all $1 \leq i \leq m$. With $F(\boldsymbol{x}) = \prod_{i=1}^{m} f_i(\langle \boldsymbol{w}_i, \boldsymbol{x}\rangle)^{c_i}$ and Theorem 3.6 we obtain

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} F(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^{n+1}} \left(\prod_{i=1}^{m} f_i(\langle \boldsymbol{w}_i, \boldsymbol{x}\rangle)^{c_i}\right) \mathrm{d}\boldsymbol{x} &\leq \prod_{i=1}^{m} \left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{c_i} \\ &= \prod_{i=1}^{m} \left(\int_0^{\infty} \mathrm{e}^{-t}\,\mathrm{d}t\right)^{c_i} = 1.\end{aligned} \tag{3.12.1}$$

Next for $\boldsymbol{x} \in \mathbb{R}^{n+1}$ we write $\boldsymbol{x} = \binom{\boldsymbol{y}}{r} \in \mathbb{R}^{n+1}$ with $\boldsymbol{y} \in \mathbb{R}^n$, $r \in \mathbb{R}$. Then Then

$$\langle \boldsymbol{w}_i, \boldsymbol{x}\rangle = \frac{r}{\sqrt{n+1}} - \sqrt{\frac{n}{n+1}}\,\langle \boldsymbol{v}_i, \boldsymbol{y}\rangle\,.$$

an so

$$F(\boldsymbol{x}) = \mathrm{e}^{-\sum_{i=1}^{m} c_i \langle \boldsymbol{w}_i, \boldsymbol{x}\rangle} = \mathrm{e}^{-\sum_{i=1}^{m} c_i \frac{r}{\sqrt{n+1}} + \sum_{i=1}^{m} c_i \sqrt{\frac{n}{n+1}}\langle v_i, y\rangle} = \mathrm{e}^{-r\sqrt{n+1}}.$$

Due to definition of $F(\boldsymbol{x})$ we have $F(\boldsymbol{x}) \neq 0$, if and only if

$$\langle \boldsymbol{v}_i, \boldsymbol{y}\rangle \leq \frac{r}{\sqrt{n}}, \quad 1 \leq i \leq m. \tag{3.12.2}$$

Since $\sum_{i=1}^{m} \lambda_i \boldsymbol{v}_i = 0$, $\lambda_i > 0$, we know that for each $\boldsymbol{y} \in \mathbb{R}^n \backslash \{0\}$ there exists $j \in \{1, \ldots, m\}$ such that $\langle \boldsymbol{v}_j, \boldsymbol{y}\rangle \geq 0$, and hence, if $r < 0$ (3.12.2) is never fulfilled. In the case $r \geq 0$, (3.12.2) is equivalent to $\boldsymbol{y} \in \frac{r}{\sqrt{n}} U$, and for such a $\boldsymbol{y} \in \frac{r}{\sqrt{n}} U$, $r \geq 0$, and $\boldsymbol{x} = \binom{\boldsymbol{y}}{r}$, we have

Together with Fubini's theorem we get

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} F(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= \int_0^{\infty} \int_{\frac{r}{\sqrt{n}}U} F(\boldsymbol{y}, r)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}r = \int_0^{\infty} \int_{\frac{r}{\sqrt{n}}U} \mathrm{e}^{-r\sqrt{n+1}}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}r \\ &= \int_0^{\infty} \mathrm{e}^{-r\sqrt{n+1}} \left(\frac{r}{\sqrt{n}}\right)^n \mathrm{vol}\,(U)\,\mathrm{d}r \\ &= \mathrm{vol}\,(U) \left(\frac{1}{\sqrt{n}}\right)^n \int_0^{\infty} \mathrm{e}^{-t} \left(\frac{t}{\sqrt{n+1}}\right)^n \frac{1}{\sqrt{n+1}}\,\mathrm{d}t \\ &= \frac{n!\,\mathrm{vol}\,(U)}{(\sqrt{n})^n(\sqrt{n+1})^{n+1}}.\end{aligned}$$

Together with (3.12.1), the result follows. The characterization of the equality case follows from Theorem 3.7; observe, if we have equality we have $m = n+1$ and the vectors $\boldsymbol{w}_i$ build an orthonormal basis in $\mathbb{R}^{n+1}$. Hence $\langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle = -\frac{1}{n}$ for $i \neq j$, which shows that U and thus K is a regular simplex. □

3.13 Corollary. *For a convex body $K \in \mathcal{K}^n$ the* volume ratio $\mathrm{vr}(K)$ *is defined as*

$$\mathrm{vr}(K) = \inf_{E \subseteq K,\, E \text{ ellipsoid}} \left(\frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(E)}\right)^{\frac{1}{n}}.$$

i) $\mathrm{vr}(K) \le \sqrt{n}\,\sqrt{n+1}^{\,1+1/n}/n!^{1/n}$ *with equality if and only if* K *is a simplex.*

ii) *For* $K \in \mathcal{K}_o^n$ *it is* $\mathrm{vr}(K) \le 2$ *with equality if and only if* K *is (an affine image of) the cube* C_n.

Proof. Immediate consequence of the Theorems 3.11, 3.12. □

3.14 Theorem [Reverse Isoperimetric Inequality]. *Let* $K \in \mathcal{K}^n$, $\dim K = n$. *There exists a regular affine transformation* $T : \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$\frac{\mathrm{F}(T\,K)^n}{\mathrm{vol}\,(T\,K)^{n-1}} \le \frac{\mathrm{F}(T_n)^n}{\mathrm{vol}\,(T_n)^{n-1}} = n^n \frac{(\sqrt{n})^n(\sqrt{n+1})^{n+1}}{n!},$$

where T_n *is a regular simplex. If* K *is o-symmetric then*

$$\frac{\mathrm{F}(TK)^n}{\mathrm{vol}\,(TK)^{n-1}} \le \frac{\mathrm{F}(C_n)^n}{\mathrm{vol}\,(C_n)^{n-1}} = (2\,n)^n,$$

where C_n *is a cube. Both inequalities are best possible.*

Proof. For the general case let the affine transformation T be chosen such that B_n is the maximum volume ellipsoid contained in TK. By the formula for the surface area given in Remark 5.31 iii)[32] , we obtain

$$\begin{aligned}
\mathrm{F}(TK) &= \lim_{\lambda\to 0} \frac{\mathrm{vol}\,(TK+\lambda B_n) - \mathrm{vol}\,(TK)}{\lambda} \le \lim_{\lambda\to 0} \frac{\mathrm{vol}\,(TK+\lambda TK) - \mathrm{vol}\,(TK)}{\lambda}\\
&= \lim_{\lambda\to 0} \frac{(1+\lambda)^n - 1}{\lambda}\,\mathrm{vol}\,(TK) = n\,\mathrm{vol}\,(TK)\\
&= n\,\mathrm{vol}\,(TK)^{1/n}\mathrm{vol}\,(TK)^{(n-1)/n} \le n\,\mathrm{vol}\,(T_n)^{1/n}\mathrm{vol}\,(TK)^{(n-1)/n},
\end{aligned}$$

where the last inequality follows from Theorem 3.12 and T_n is here a regular simplex with inradius 1. Hence $\mathrm{F}(T_n) = n\mathrm{vol}\,(T_n)$ and we get

$$\frac{\mathrm{F}(TK)^n}{\mathrm{vol}\,(TK)^{n-1}} \le n^n \mathrm{vol}\,(T_n) = \frac{\mathrm{F}(T_n)^n}{\mathrm{vol}\,(T_n)^{n-1}}.$$

Now suppose that S is an n-simplex. Then we have $r(S)\,\mathrm{F}(S) = n\mathrm{vol}\,(S)$ and hence we have

$$\frac{\mathrm{F}(TS)^n}{\mathrm{vol}\,(TS)^{n-1}} = n^n \frac{\mathrm{vol}\,(TS)}{r(TS)^n}$$

for any regular affine transformation T. Now for a given inradius the regular simplex has smallest volume and hence we conclude

$$\frac{\mathrm{F}(TS)^n}{\mathrm{vol}\,(TS)^{n-1}} = n^n \frac{\mathrm{vol}\,(TS)}{r(TS)^n} \ge n^n \mathrm{vol}\,(T_n) = \frac{\mathrm{F}(T_n)^n}{\mathrm{vol}\,(T_n)^{n-1}}.$$

Hence for a simplex the inequality can be improved.

The proof for symmetric convex bodies is analogous, but now we apply Theorem 3.11. □

[32] Skript WS14

3.15 Theorem. *Let $K \in \mathcal{K}^n$, and let $c_i > 0$ and $\boldsymbol{v}_i \in S^{n-1}$, for $1 \le i \le m$, be such that $\sum_{i=1}^m c_i\,(\boldsymbol{v}_i \boldsymbol{v}_i^\intercal) = \mathrm{I}_n$. Then*

$$\mathrm{vol}\,(K)^{n-1} \le \prod_{i=1}^m \mathrm{vol}_{\,n-1}(K|\boldsymbol{v}_i^\perp)^{c_i}.$$

Here $K|\boldsymbol{v}_i^\perp$ denotes the orthogonal projection of K onto the orthogonal complement of $\boldsymbol{v}_i$, i.e., the hyperplane $H(\boldsymbol{v}_i, 0)$.

Moreover, equality holds if and only if $m = n$, $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are an orthonormal basis and K is an orthogonal box with respect to $\boldsymbol{v}_i$, i.e., $K = \{\boldsymbol{x} \in \mathbb{R}^n : \alpha_i \le \langle \boldsymbol{v}_i, \boldsymbol{x}\rangle \le \beta_i, 1 \le i \le n\}$ and $\alpha_i < \beta_i \in \mathbb{R}$.

Proof. Let $\alpha_i = c_i/\mathrm{vol}_{\,n-1}(K|\boldsymbol{v}_i^\perp)$, $1 \le i \le m$, and let Z be the zonotope $Z = \sum_{i=1}^m \mathrm{conv}\,\{-\alpha_i \boldsymbol{v}_i, \alpha_i \boldsymbol{v}_i\}$. Applying first Minkowski's inequality Theorem 5.32[33] , then Corollary 5.27[34] and finally using the linearity of the mixed volumes Lemma 5.23 iv)[35] , we get

$$\begin{aligned}\mathrm{vol}\,(Z)^{1/n}\mathrm{vol}\,(K)^{(n-1)/n} &\le \mathrm{V}(K, n-1; Z, 1)\\ &= \sum_{i=1}^m \mathrm{V}\big(K, n-1; \mathrm{conv}\,\{-\alpha_i\,\boldsymbol{v}_i, \alpha_i\,\boldsymbol{v}_i\}, 1\big)\\ &= \frac{2}{n}\sum_{i=1}^m \alpha_i \mathrm{vol}_{\,n-1}(K|\boldsymbol{v}_i^\perp) = \frac{2}{n}\sum_{i=1}^m c_i = 2.\end{aligned}$$

Therefore, $\mathrm{vol}\,(K)^{n-1} \le 2^n/\mathrm{vol}\,(Z)$. Finally, from Lemma 3.10 we get

$$\mathrm{vol}\,(Z) \ge 2^n \prod_{i=1}^m \left(\frac{\alpha_i}{c_i}\right)^{c_i} = 2^n \prod_{i=1}^m \left(\frac{1}{\mathrm{vol}_{\,n-1}(K|\boldsymbol{v}_i^\perp)}\right)^{c_i},$$

and the result follows.

Now, if we have equality then by the equality case of the Minkowski inequality, K and Z are homothetic and from Lemma 3.10 we conclude $m = n$, $c_i = 1$, and $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are an orthonormal basis. Hence K is an orthogonal box with edge directions $\boldsymbol{v}_i$. □

3.16 Corollary [Loomis-Whitney inequality]. *Let $K \in \mathcal{K}^n$. Then*

$$\mathrm{vol}\,(K)^{n-1} \le \prod_{i=1}^n \mathrm{vol}_{\,n-1}(K|\boldsymbol{e}_i^\perp),$$

and equality holds if and only if K is an orthogonal box with facets parallel to the coordinate axes.

Proof. Apply Theorem 3.15 with $m = n$, $\boldsymbol{v}_i = \boldsymbol{e}_i$ and $c_i = 1$, $1 \le i \le n$. □

[33] Skript WS14
[34] Skript WS14
[35] Skript WS14

3.17 Theorem. *Let S be a k-cover of $[n]=\{1,\dots n\}$, i.e., $S=\{S_1,S_2,\dots,S_l\}$ with $S_i\subset[n]$, $1\le i\le l$, and each $j\in[n]$ is contained in exactly k sets S_i. Let $K\in\mathcal{K}^n$ and let $K|S_i$ be the projection onto* $\operatorname{lin}\{\boldsymbol{e}_j : j\in S_i\}$. *Then*

$$\operatorname{vol}(K)^k\ge\prod_{S_i\in S}\operatorname{vol}_{|S_i|}(K|S_i).$$

Proof. To simplify notation, in the following $\operatorname{vol}(A)$ will always denote the volume measured with respect to aff A. We use induction on n. For $n=1$ it is certainly true. So let $n\ge 2$ and let □

3.18 Theorem [Meyer-Inequality]. *Let $K\in\mathcal{K}^n$ and let $H_i=\{\boldsymbol{x}\in\mathbb{R}^n : \langle\boldsymbol{e}_i,\boldsymbol{x}\rangle=0\}$, $1\le i\le n$. Then*

$$\operatorname{vol}(K)^{n-1}\ge\frac{(n-1)!}{n^{n-1}}\prod_{i=1}^{n}\operatorname{vol}_{n-1}(K\cap H_i),$$

and equality holds if and only if K is a generalized cross-polytope, i.e., $K=$ $\operatorname{conv}\{-\alpha_i\boldsymbol{e}_i,\beta_i\boldsymbol{e}_i : \alpha_i,\beta_i>0, 1\le i\le n\}$.

Proof. Applying Steiner-Symmetrization to K with respect to the hyperplane H_1 yields a convex body K_1, say, with $\operatorname{vol}(K_1)=\operatorname{vol}(K)$, $\operatorname{vol}_{n-1}(K\cap H_1)\le\operatorname{vol}_{n-1}(K_1\cap H_1)=\operatorname{vol}_{n-1}(K|\boldsymbol{e}_1^{\perp})$ and $\operatorname{vol}_{n-1}(K\cap H_i)=\operatorname{vol}_{n-1}(K_1\cap H_i)$ for all $i>1$. Hence it suffices to prove the inequality for K_1 and repeating this argument to all coordinate hyperplanes H_i shows that it suffices to prove the inequality for a body L which is symmetric to all coordinate hyperplanes, i.e., for $\boldsymbol{x}\in L$ we also have $(\pm x_1,\dots,\pm x_n)^{\intercal}\in L$. Such a convex body is also called an unconditional convex body. In particular, we have $L\cap H_i=L|\boldsymbol{e}_i^{\perp}$ for such a body.

Let $L^s=L\cap\mathbb{R}^n_{\ge0}$ and $L^s_i=L^s\cap H_i$. By the symmetries we have $\operatorname{vol}(L)=2^n\operatorname{vol}(L^s)$ and $\operatorname{vol}(L\cap H_i)=2^{n-1}\operatorname{vol}_{n-1}(L^s_i)$. Hence it suffices to verify the inequality for L^s and its sections L^s_i, for which we rewrite the inequality as

$$\operatorname{vol}(L^s)\le\frac{n^{n-1}}{(n-1)!}\prod_{i=1}^{n}\frac{\operatorname{vol}(L^s)}{\operatorname{vol}(L^s_i)}. \tag{3.18.1}$$

For $\boldsymbol{y}\in L^s$ the intersection of the pyramid $\operatorname{conv}\{L^s_i,\boldsymbol{y}\}$ with another pyramid $\operatorname{conv}\{L^s_j,\boldsymbol{y}\}$, $j\ne i$, is contained in an $(n-1)$-dimensional subspace. Thus we have

$$\operatorname{vol}(L^s)\ge\sum_{i=1}^{n}\operatorname{vol}(\operatorname{conv}\{L^s_i,\boldsymbol{y}\})=\frac{1}{n}\sum_{i=1}^{n}y_i\operatorname{vol}_{n-1}(L^s_i)$$

for every $\boldsymbol{y}\in L^s$, and so

$$L^s\subseteq\left\{\boldsymbol{y}\in\mathbb{R}^n_{\ge0} : \sum_{i=1}^{n}y_i\operatorname{vol}_{n-1}(L^s_i)\le n\operatorname{vol}(L^s)\right\}.$$

Now the set on the left hand side is an n-simplex with vertices $\mathbf{0}$, $\frac{n\operatorname{vol}(L^s)}{\operatorname{vol}_{n-1}(L^s_i)}\boldsymbol{e}_i$, $1\le i\le n$, and of volume

$$\frac{1}{n!}\prod_{i=1}^{n}\frac{n\operatorname{vol}(L^s)}{\operatorname{vol}_{n-1}(L^s_i)}.$$

and (3.18.1) is proven.

Now if we equality for K then the proof above shows that we also have equality for all the bodies K_i created by successively Steiner-symmetrizations at the planes H_i. In particular, we have equality for the unconditional body $K_n = L$, i.e., we have equality in (3.18.1). Hence L^s is a simplex with vertices $\mathbf{0}$ and at the coordinate axis, and thus $L = \operatorname{conv}\{\pm\gamma_i\, e_i : 1 \leq i \leq n\}$, with $\gamma_i \in \mathbb{R}_{>0}$, and $\operatorname{vol}(L) = \operatorname{vol}(K)$.

By the equality assumption we have $K_{n-1} \cap H_n = K_{n-1}|H_n = L \cap H_n$ and so we conclude $\operatorname{conv}\{L \cap H_n, -\alpha_n \boldsymbol{e}_n, \beta_n \boldsymbol{e}_n\} \subseteq K_{n-1}$ with $\alpha_n + \beta_n = 2\gamma_n$. Comparing the volumes of the two sets shows that we have indeed $K_{n-1} = \operatorname{conv}\{L \cap H_n, -\alpha_n \boldsymbol{e}_n, \beta_n \boldsymbol{e}_n\}$ and $\alpha_n, \beta_n > 0$. Hence $-\alpha_n \boldsymbol{e}_n, \beta_n \boldsymbol{e}_n \in K$. Repeating backwards this argument shows that K has to be a generalized crosspolytope. □

3.19 Theorem. *Let L be a k-dimensional linear subspace of $\mathbb{R}^n$. Then*

$$\operatorname{vol}_k(C_n \cap L) \leq 2^k \left(\frac{n}{k}\right)^{k/2}.$$

If k is a divisor of n then the inequality is best possible.

Proof. Here we want to apply Theorem 3.6 in the k-dimensional space L. To this end let P be the orthogonal projection of $\mathbb{R}^n$ onto L. Let $\boldsymbol{u}_i = P\boldsymbol{e}_i$ and $c_i = \|\boldsymbol{u}_i\|^2$, $1 \leq i \leq n$. If $c_j = 0$ then $L \subseteq \boldsymbol{e}_j^\perp$ and $C_n \cap L = (C_n \cap \boldsymbol{e}_j^\perp) \cap L$. Hence the problem is reduced to a cube of dimension one less and the result follows inductively.

So we assume $c_i > 0$ for $1 \leq i \leq n$. For $\boldsymbol{x} \in \mathbb{R}^n$ we have

$$P\boldsymbol{x} = P\left(\sum_{i=1}^n \langle \boldsymbol{x}, \boldsymbol{e}_i\rangle\, \boldsymbol{e}_i\right) = \sum_{i=1}^n \langle \boldsymbol{x}, \boldsymbol{e}_i\rangle\, P\boldsymbol{e}_i = \sum_{i=1}^n \langle \boldsymbol{x}, \boldsymbol{e}_i\rangle\, \boldsymbol{u}_i,$$

and since for $\boldsymbol{x} \in L$ we also have $\langle \boldsymbol{x}, \boldsymbol{e}_i\rangle = \langle \boldsymbol{x}, \boldsymbol{u}_i\rangle$ we get

$$\boldsymbol{x} = P\boldsymbol{x} = \sum_{i=1}^n \langle \boldsymbol{x}, \boldsymbol{u}_i\rangle\, \boldsymbol{u}_i = \sum_{i=1}^n (\boldsymbol{u}_i \boldsymbol{u}_i^\intercal)\boldsymbol{x} = \sum_{i=1}^n c_i\, (\boldsymbol{v}_i \boldsymbol{v}_i^\intercal)\boldsymbol{x},$$

with $\boldsymbol{v}_i = \boldsymbol{u}_i/\sqrt{c_i} \in L \cap S^{n-1}$. Hence, in L the unit vectors $\boldsymbol{v}_i$ give a decomposition of the identity and since $\operatorname{tr} P = k$ we get $\sum_{i=1}^n c_i = k$. Moreover we have

$$\begin{aligned} C_n \cap L &= \left\{x \in L : \big|\langle \boldsymbol{x}, \boldsymbol{e}_i\rangle\big| \leq 1,\, 1 \leq i \leq n\right\} \\ &= \left\{x \in L : \big|\langle \boldsymbol{x}, \boldsymbol{v}_i\rangle\big| \leq \frac{1}{\sqrt{c_i}},\, 1 \leq i \leq n\right\}, \end{aligned}$$

and with $f_i = \chi_{[-1/\sqrt{c_i}, 1/\sqrt{c_i}]}$ we get by Theorem 3.6 (applied in L)

$$\begin{aligned} \operatorname{vol}_k(C_n \cap L) &= \int_L \left(\prod_{i=1}^n f_i(\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle)^{c_i}\right) \mathrm{d}\boldsymbol{x} \leq \prod_{i=1}^n \left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{c_i} \\ &= \prod_{i=1}^n \left(\frac{2}{\sqrt{c_i}}\right)^{c_i} = 2^k \left(\prod_{i=1}^n c_i^{c_i}\right)^{-1/2}. \end{aligned}$$

The continuous function $\prod_{i=1}^n x_i^{x_i}$ attains a minimum on the compact set $\{\boldsymbol{x} \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^n x_i = k\}$. Inductively it can be shown that such a minimum satisfies $x_1 = \cdots = x_n$. Therefore, exercise

$$\prod_{i=1}^n c_i^{c_i} \geq \prod_{i=1}^n \left(\frac{\sum_{i=1}^n c_i}{n}\right)^{\sum_{i=1}^n c_i/n} = \left(\frac{k}{n}\right)^k.$$

Thus $\operatorname{vol}_k(C_n \cap L) \leq 2^k (n/k)^{k/2}$.

Now suppose that k is a divisor of n, and let $m = n/k$. For each $1 \leq i \leq k$, let $\boldsymbol{u}_i = \sum_{j=1}^m \boldsymbol{e}_{(i-1)m+j}$. These vectors are pairwise orthogonal and with $L = \operatorname{lin}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ we get that

$$Q = \left\{\sum_{i=1}^k \lambda_i \, \boldsymbol{u}_i : \lambda_i \in [-1,1]\right\} \subset C_n \cap L.$$

Since Q is a k-dimensional cube with edge-length $\sqrt{m} = \sqrt{n/k}$ we have equality in this case. □

3.20 Theorem*. *Let L be a k-dimensional linear subspace of $\mathbb{R}^n$. Then*

$$\operatorname{vol}_k(C_n \cap L) \leq 2^k \, 2^{(n-k)/2}.$$

If $k \geq n/2$ the inequality is best possible.

3.21 Theorem [Vaaler]. *Let L be an $(n-1)$-dimensional linear subspace of $\mathbb{R}^n$. Then*

$$\operatorname{vol}_{n-1}(C_n \cap L) \geq 2^{n-1},$$

and equality holds if and only if L is a coordinate hyperplane.

Proof. For an arbitrary but fixed $\boldsymbol{u} \in S^{n-1}$ let $H(t) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{x}, \boldsymbol{u}\rangle = t\}$, $t \in \mathbb{R}$, and let $f(t) = \operatorname{vol}_{n-1}\big(C_n \cap H(t)\big)$. Since C_n is a 0-symmetric convex body, we get by the Brunn-Minkowski theorem that $f(t) \leq f(0)$ for all $t \in \mathbb{R}$.

$$F(t) = \int_0^t f(s)\,\mathrm{d}s,$$

we know $F(t) \leq t\, f(0)$ for $t \geq 0$ where for $t > 0$ equality holds if and only if $f(t) = f(0)$ for all $t > 0$ with $f(t) \neq 0$; also observe that $F(t)' = f(t)$. Therefore we find

$$\begin{aligned}
\int_{C_n} \langle \boldsymbol{x}, \boldsymbol{u}\rangle^2 \,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}} t^2 f(t)\,\mathrm{d}t = \frac{2}{f(0)^2}\int_0^\infty \big(t\, f(0)\big)^2 f(t)\,\mathrm{d}t \\
&\geq \frac{2}{f(0)^2}\int_0^\infty F(t)^2 f(t)\,\mathrm{d}t = \frac{2}{3\, f(0)^2}\int_0^\infty \big[F(t)^3\big]'\,\mathrm{d}t \\
&= \frac{2}{3\, f(0)^2}\big[F(\infty)^3 - F(0)^3\big] = \frac{2}{3 f(0)^2}\left(\frac{\operatorname{vol}(C_n)}{2}\right)^3 \\
&= \frac{2}{3\, f(0)^2}\left(2^{n-1}\right)^3,
\end{aligned}$$

with equality if and only if $F(t) = t\,f(0)$ for all $t \in \mathbb{R}$ with $f(t) > 0$, i.e., if and only if $f(t) = f(0)$ for all $t \in \mathbb{R}$ with $f(t) > 0$. By the continuity of $f(t)$ on its support we conclude that this is equivalent to $\boldsymbol{u} \in \{\pm \boldsymbol{e}_i,\, 1 \le i \le n\}$.

On the other hand, we may evaluate the left hand side integral by

$$\begin{aligned}\int_{C_n} \langle \boldsymbol{x}, \boldsymbol{u}\rangle^2 \,\mathrm{d}\boldsymbol{x} &= \int_{C_n} \left(\sum_{i=1}^n u_i\, x_i\right)^2 \mathrm{d}\boldsymbol{x} \\ &= \int_{C_n} \left(\sum_{i=1}^n u_i^2 x_i^2 + 2 \sum_{1\le i<j\le n} u_i u_j x_i x_j\right) \mathrm{d}\boldsymbol{x} \\ &= \sum_{i=1}^n \int_{C_n} u_i^2 x_i^2 \,\mathrm{d}\boldsymbol{x} + 2 \int_{C_n} \left(\sum_{1\le i<j\le n} u_i x_i\, u_j x_j\right) \mathrm{d}\boldsymbol{x} \qquad (3.21.1)\\ &= \sum_{i=1}^n \left(\int_{-1}^1 \cdots \int_{-1}^1 u_i^2 x_i^2\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n \\ &= \frac{2}{3}\, 2^{n-1} \sum_{i=1}^n u_i^2 = \frac{2}{3}\, 2^{n-1}.\end{aligned}$$

Comparing the two integrals gives $f(0) \ge 2^{n-1}$ with equality if and only if $\boldsymbol{u} \in \{\pm \boldsymbol{e}_i,\, 1 \le i \le n\}$. □

3.22 Remark [Busemann-Petty problem]. *The Busemann-Petty problem was the question whether for two 0-symmetric convex bodies $K, L \in \mathcal{K}_0^n$, the inequalities*

$$\mathrm{vol}_{n-1}\big(K \cap H(\boldsymbol{u}, 0)\big) \ge \mathrm{vol}_{n-1}\big(L \cap H(\boldsymbol{u}, 0)\big), \qquad \textit{for all } \boldsymbol{u} \in S^{n-1},$$

imply $\mathrm{vol}\,(K) \ge \mathrm{vol}\,(L)$? Taking $K = C_n$ and L a ball of volume 2^n, Theorem 3.20 gives

$$\mathrm{vol}\,(K \cap H(\boldsymbol{u}, 0)) \le 2^{n-1}\sqrt{2} < \mathrm{vol}\,(L \cap H(\boldsymbol{u}, 0)) = 2^{n-1} \frac{\Gamma(\frac{n}{2}+1)^{\frac{n-1}{n}}}{\Gamma(\frac{n-1}{2}+1)} \to 2^{n-1}\sqrt{\mathrm{e}}.$$

And so the answer is No *for $n \ge 10$. In the meantime the problem has been completely solved: the answer is affirmative for $n \le 4$ and negative for $n \ge 5$.*

3.23 Definition. *Let $K \in \mathcal{K}^n$. The set*

$$\Pi(K) = \big\{x \in \mathbb{R}^n : |\langle \boldsymbol{u}, \boldsymbol{x}\rangle| \le \mathrm{vol}_{n-1}(K|\boldsymbol{u}^\perp),\ \textit{for all } \boldsymbol{u} \in S^{n-1}\big\}$$

is called the projection body *of K.*

3.24 Proposition. *Let $K \in \mathcal{K}^n$. Then*

i) $h\big(\Pi(K), \boldsymbol{u}\big) = \mathrm{vol}_{n-1}(K|\boldsymbol{u}^\perp)$.

ii) *$\Pi(K)$ is o-symmetric.*

iii) $\Pi(A\,K) = |\det A|\, A^{-\intercal}\Pi(K)$ *for* $A \in \mathrm{GL}(n, \mathbb{R})$, *and* $\Pi(\boldsymbol{t} + K) = \Pi(K)$ *for* $\boldsymbol{t} \in \mathbb{R}^n$.

3.25 Theorem. *Let* $K \in \mathcal{K}^n$. *There exists a regular affine transformation* $T : \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$\mathrm{vol}_{\,n-1}\big((TK)|\boldsymbol{u}^{\perp}\big)^n \geq \mathrm{vol}\,(TK)^{n-1}, \quad \textit{for all } \boldsymbol{u} \in S^{n-1}.$$

Proof. According to Proposition 3.24 iii), we may find a linear transformation $A \in \mathrm{GL}(n, \mathbb{R})$ such that B_n is the maximum volume ellipsoid contained in $A^{-\intercal}\Pi(K) = |\det A|^{-1}\Pi(A\,K)$. In particular, $|\det A| B_n \subseteq \Pi(A\,K)$, and hence, with Proposition 3.24 ii) we get

$$\mathrm{vol}_{\,n-1}((A\,K)|\boldsymbol{u}^{\perp}) = h\big(\Pi(A\,K), \boldsymbol{u}\big) \geq |\det A|,$$

for all $\boldsymbol{u} \in S^{n-1}$.

By Theorem 6.11[36] there exist $\boldsymbol{u}_i \in S^{n-1} \cap \mathrm{bd}\,\big(|\det A|^{-1}\Pi(AK)\big)$ and $\lambda_i > 0$, $1 \leq i \leq m$, such that $\mathrm{I}_n = \sum_{i=1}^{n} \lambda_i\,(\boldsymbol{u}_i\boldsymbol{u}_i^{\intercal})$ and so $\sum_{i=1}^{m} \lambda_i = n$. In particular,

$$\mathrm{vol}_{\,n-1}((AK)|\boldsymbol{u}_i^{\perp}) = h\big(\Pi(AK), \boldsymbol{u}_i\big) = |\det A|.$$

Together with Theorem 3.15 we get

$$\begin{aligned}\mathrm{vol}\,(AK)^{n-1} &\leq \prod_{i=1}^{m} \mathrm{vol}_{\,n-1}(AK|\boldsymbol{u}_i^{\perp})^{\lambda_i} = \prod_{i=1}^{m} |\det A|^{\lambda_i} \\ &= |\det A|^{\sum_{i=1}^{m}\lambda_i} = |\det A|^n \leq \mathrm{vol}_{\,n-1}\big(AK|\boldsymbol{u}^{\perp}\big)^n,\end{aligned}$$

for all $\boldsymbol{u} \in S^{n-1}$. □

3.26 Theorem* [Vaaler, 1979]. *Let* L *be a* k*-dimensional linear subspace. Then* $\mathrm{vol}_{\,k}(C_n \cap L) \geq 2^k$.

3.27 Theorem*. *Let* $H = \{\boldsymbol{x} \in \mathbb{R}^n : \langle a, \boldsymbol{x}\rangle = b\}$, $\boldsymbol{a} \in S^{n-1}$, $b \in \mathbb{R}$, *and assume* $a_i \neq 0$, $1 \leq i \leq n$. *Then*

$$\mathrm{vol}\,(C_n \cap H) = \frac{1}{2\,(n-1)!}\left(\prod_{i=1}^{n} a_i\right)^{-1} \sum_{\boldsymbol{v} \in \mathrm{vert}\,C_n} (\langle \boldsymbol{a}, \boldsymbol{v}\rangle + b)^{n-1}\mathrm{sgn}\,(\langle \boldsymbol{a}, \boldsymbol{v}\rangle + b) \prod_{i=1}^{n} \boldsymbol{v}_i.$$

3.28 Theorem* [Chakerian& Filliman, 1986]. *Let* L *be a* k*-dimensional linear subspace. Then*

$$\mathrm{vol}_{\,k}(C_n|L) \leq 2^k \min\left\{\frac{\kappa_{k-1}^k}{\kappa_k^{k-1}}\binom{n}{k}^{\frac{k}{2}}, \sqrt{\binom{n}{k}}\right\}.$$

3.29 Theorem [McMullen, 1984]. *Let* L *be a* k*-dimensional linear subspace with orthogonal complement* $L^{\perp}$. *Then* $\mathrm{vol}_{\,k}(C_n|L) = \mathrm{vol}_{\,n-k}(C_n|L^{\perp})$.

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Proof. Exercise. The main tool is here the so-called Jacobi's determinat identity which for a regular matrix $A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$ and its inverse $A^{-1} = \begin{pmatrix} B' & C' \\ D' & E' \end{pmatrix}$ where B, B' are $(k \times k)$ matrices says that

$$\det B = \det E' \det A.$$

□

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