Convex Geometry II

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1 Symmetrisations

1.1 Definition [Steiner-Symmetrisation]. *Let $K \in \mathcal{K}^n$, let $H \subset \mathbb{R}^n$ be a hyperplane with normal $\boldsymbol{a} \in S^{n-1}$ and let $K|H$ be the orthogonal projection of K onto H. For $\boldsymbol{x} \in K|H$, let $K \cap (\boldsymbol{x} + \operatorname{lin}\{\boldsymbol{a}\}) = \operatorname{conv}\{\boldsymbol{v}_x, \boldsymbol{w}_x\}$. The set*

$$\operatorname{st}_H(K) = \bigcup_{\boldsymbol{x} \in K|H} \left[\left(\boldsymbol{x} - \frac{1}{2}(\boldsymbol{v}_x + \boldsymbol{w}_x) \right) + \operatorname{conv}\{\boldsymbol{v}_x, \boldsymbol{w}_x\} \right]$$

is called the Steiner-Symmetral *of K with respect to H.*

1.2 Definition [Inradius, Circumradius, Diameter]. *For $K \in \mathcal{K}^n$,*

i) $\mathrm{r}(K) = \max\{r \geq 0 : \boldsymbol{x} + rB_n \subseteq K,\, \boldsymbol{x} \in \mathbb{R}^n\}$ *is the* inradius *of K,*

ii) $\mathrm{R}(K) = \min\{R \geq 0 : K \subseteq \boldsymbol{x} + RB_n,\, \boldsymbol{x} \in \mathbb{R}^n\}$ *is the* circumradius *of K,*

iii) $\mathrm{D}(K) = \max\big\{\|\boldsymbol{x} - \boldsymbol{y}\| : \boldsymbol{x}, \boldsymbol{y} \in K\big\} = \max\{h(K, \boldsymbol{u}) + h(K, -\boldsymbol{u}) : \boldsymbol{u} \in S^{n-1}\}$ *is the* diameter *of K.*

iv) $\mathrm{w}(K) = \min\{h(K, \boldsymbol{u}) + h(K, -\boldsymbol{u}) : \boldsymbol{u} \in S^{n-1}\}$ *is the* minimal width *of K.*

1.3 Proposition. *Let $K, M \in \mathcal{K}^n$ and let H be a hyperplane.*

i) $\operatorname{st}_H(K) \in \mathcal{K}^n$.

ii) $\operatorname{st}_H(\lambda K) = \lambda \operatorname{st}_H(K)$ *for $\lambda \geq 0$, up to translations.*

iii) $\operatorname{st}_H(K) + \operatorname{st}_H(M) \subseteq \operatorname{st}_H(K + M)$*, up to translations.*

iv) *If $K \subseteq M$ then $\operatorname{st}_H(K) \subseteq \operatorname{st}_H(M)$.*

v) $\operatorname{vol}\big(\operatorname{st}_H(K)\big) = \operatorname{vol}(K)$.

vi) $\mathrm{F}\big(\operatorname{st}_H(K)\big) \leq \mathrm{F}(K)$.

vii) $\mathrm{D}\big(\operatorname{st}_H(K)\big) \leq \mathrm{D}(K)$.

viii) $\mathrm{r}\big(\operatorname{st}_H(K)\big) \geq \mathrm{r}(K)$ *and* $\mathrm{R}(\operatorname{st}_H(K)) \leq \mathrm{R}(K)$.

ix) $\operatorname{st}_H : \{K \in \mathcal{K}^n : \dim K = n\} \to \{K \in \mathcal{K}^n : \dim K = n\}$ *is a continuous map. The sequence $K_i = \operatorname{conv}\{(\frac{1}{i}, 0)^\intercal, (0, 1)^\intercal\} \subset \mathbb{R}^2$ shows that the assumption $\dim K = n$ is needed.*

Proof. Without loss of generality we assume that $\mathbf{0} \in H$ and let $L = \operatorname{lin}\{\boldsymbol{a}\}$ be the orthogonal complement of H. ii) and iv) are certainly true.

i) Obviously, $\operatorname{st}_H(K)$ is a compact set (cf. Exercises). For the convexity of $\operatorname{st}_H(K)$, let $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{st}_H(K)$ and consider the convex trapezoid $T = \operatorname{conv}\big\{K \cap (\boldsymbol{x} + L), K \cap (\boldsymbol{y} + L)\big\} \subseteq K$. Clearly $\operatorname{st}_H(T)$ is also a convex trapezoid with $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{st}_H(T)$, and so we have $\operatorname{conv}\{\boldsymbol{x}, \boldsymbol{y}\} \subseteq \operatorname{st}_H(T) \subseteq \operatorname{st}_H(K)$.

iii) Let $\boldsymbol{x} \in \mathrm{st}_H(K)$ and $\boldsymbol{y} \in \mathrm{st}_H(M)$. Then we can write $\boldsymbol{x} = \overline{\boldsymbol{x}} + l_x$ and $\boldsymbol{y} = \overline{\boldsymbol{y}} + l_y$, with $\overline{\boldsymbol{x}} = \boldsymbol{x}|H$, $\overline{\boldsymbol{y}} = \boldsymbol{y}|H$ and $l_x, l_y \in L$. Then

$$\boldsymbol{x} + \boldsymbol{y} = \overline{\boldsymbol{x}} + \overline{\boldsymbol{y}} + (l_x + l_y),$$

and since $\mathbf{0} \in H$ we have $\overline{\boldsymbol{x}} + \overline{\boldsymbol{y}} \in (K + M)|H$. In remains to show that $\|l_x + l_y\| \leq \frac{1}{2}\mathrm{vol}_1\Big((K+M)\cap(\boldsymbol{x}+\boldsymbol{y}+L)\Big)$, which follwos from the oberservation

$$\begin{aligned}
\|l_x + l_y\| \leq \|l_x\| + \|l_y\| &\leq \frac{1}{2}\mathrm{vol}_1\big(K \cap (\boldsymbol{x} + L)\big) + \frac{1}{2}\mathrm{vol}_1\big(M \cap (\boldsymbol{y} + L)\big) \\
&= \frac{1}{2}\mathrm{vol}_1\Big([K \cap (\boldsymbol{x} + L)] + [M \cap (\boldsymbol{y} + L)]\Big) \\
&\leq \frac{1}{2}\mathrm{vol}_1\Big((K + M) \cap (\boldsymbol{x} + \boldsymbol{y} + L)\Big).
\end{aligned}$$

v) It is an immediate consequence of Fubini's theorem:

$$\mathrm{vol}\,(K) = \int_{K|H} \mathrm{vol}_1\big(K \cap (\boldsymbol{x} + L)\big)\, \mathrm{d}\boldsymbol{x} = \mathrm{vol}\,\big(\mathrm{st}_H(K)\big).$$

vi) By (ii) and (iii) we have $\mathrm{st}_H(K) + \lambda B_n = \mathrm{st}_H(K) + \lambda\,\mathrm{st}_H(B_n) \subseteq \mathrm{st}_H(K + \lambda B_n)$ for any $\lambda > 0$, and with Remark 5.31[1] and property (v) we get

$$\begin{aligned}
\mathrm{F}\big(\mathrm{st}_H(K)\big) &= \lim_{\lambda\to 0} \frac{\mathrm{vol}\,\big(\mathrm{st}_H(K) + \lambda B_n\big) - \mathrm{vol}\,\big(\mathrm{st}_H(K)\big)}{\lambda} \\
&\leq \lim_{\lambda\to 0} \frac{\mathrm{vol}\,\big(\mathrm{st}_H(K + \lambda B_n)\big) - \mathrm{vol}\,\big(\mathrm{st}_H(K)\big)}{\lambda} \\
&= \lim_{\lambda\to 0} \frac{\mathrm{vol}\,(K + \lambda B_n) - \mathrm{vol}\,(K)}{\lambda} = \mathrm{F}(K).
\end{aligned}$$

vii) Let $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{st}_H(K)$ and let $T, \mathrm{st}_H(T)$ be the convex trapezoids as in the proof of (i). Then one of the diagonals (or both) of T has length greater than or equal to $\|\boldsymbol{x} - \boldsymbol{y}\|$, which proves the assertion.

viii) Let $\boldsymbol{x} \in \mathbb{R}^n$ be such that $\boldsymbol{x} + \mathrm{r}(K)B_n \subseteq K$. Then, using ii), iii) and iv) we get $\mathrm{st}_H(K) \supseteq \mathrm{st}_H\big(\boldsymbol{x} + \mathrm{r}(K)B_n\big) \supseteq \mathrm{st}_H(\boldsymbol{x}) + \mathrm{r}(K)\,\mathrm{st}_H(B_n) = \mathrm{st}_H(\boldsymbol{x}) + \mathrm{r}(K)B_n$. Hence, $\mathrm{r}\big(\mathrm{st}_H(K)\big) \geq \mathrm{r}(K)$, and similary, $\mathrm{R}(\mathrm{st}_H(K)) \leq \mathrm{R}(K)$.

ix) Let $K_i \in \mathcal{K}^n$, $i \in \mathbb{N}$, be a sequence of full-dimensional convex bodies converging in the Hausdorff metric to $K \in \mathcal{K}^n$, with $\dim K = n$. Without loss of generality let $\mathbf{0} \in \mathrm{int}\, K$. Since $K_i \to K$, for any $\varepsilon > 0$ we have $(1 - \varepsilon)K \subseteq K_i \subseteq (1 + \varepsilon)K$ (see Exercise) for sufficiently large i. Hence, by ii) and iv) we get $(1 - \varepsilon)\mathrm{st}_H(K) \subseteq \mathrm{st}_H(K_i) \subseteq (1 + \varepsilon)\mathrm{st}_H(K)$ for sufficiently large i. Hence $\mathrm{st}_H(K_i) \to \mathrm{st}_H(K)$. □

1.4 Remark. *Regarding the monontonicty of the surface area (Proposition 1.3 vi)) it is also known that* $\mathrm{F}(\mathrm{st}_H(K)) = F(K)$ *if and only if* K *is symmetric to a hyperplane parallel to* H*.*

[1] Skript WS14

1.5 Theorem. *Let $K \in \mathcal{K}^n$ with $\dim K = n$. Let $\mathcal{S}_K$ be the set of all convex bodies which are obtained by finitely many successive Steiner-symmetrisations of K with respect to hyperplanes containing 0. Then there exists a sequence $K_i \in \mathcal{S}_K$, $i \in \mathbb{N}$, such that*

$$\lim_{i\to\infty} K_i = \left(\frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(B_n)}\right)^{1/n} B_n.$$

Proof. For $M \in \mathcal{K}^n$ let $\rho(M) = \min\{R > 0 : M \subseteq RB_n\}$, let $\rho = \inf\{\rho(M) : M \in \mathcal{S}_K\}$ and let $K_i \in \mathcal{S}_K$ with $\rho(K_i) \to \rho$. Since $K_i \subset rho(K_i)B_n \subset \rho(K)B_n$ (cf. Proposition 1.3 iv)), we may assume by Blaschke's selection Theorem 4.9[2] that $K_i \to \overline{K} \in \mathcal{K}^n$, say. In the follwoing we show

$$\overline{K} = \rho\, B_n,$$

which implies the assertion, since then $K_i \to \rho\, B_n$ and $\rho^n \mathrm{vol}\,(B_n) = \mathrm{vol}\,(\overline{K}) = \lim_{i\to\infty} \mathrm{vol}\,(K_i) = \mathrm{vol}\,(K)$. (cf.)

Suppose $\overline{K} \neq \rho\, B_n$. By the continuity of $\rho(\cdot)$ we have $\rho(\overline{K}) = \lim_{i\to\infty} \rho(K_i) = \rho$, and so we have $\overline{K} \subset \rho\, B_n$ (strictly). Hence we might choose $\boldsymbol{x} \in \rho B_n \setminus \overline{K}$ and a hyperplane $H(\boldsymbol{a}, \alpha)$ strictly separating $\overline{K}$ and $\boldsymbol{x}$. Let $\langle \boldsymbol{a}, \boldsymbol{x}\rangle > \alpha$, and let $C = \{\boldsymbol{y} \in \rho S^{n-1} : \langle \boldsymbol{a}, \boldsymbol{y}\rangle \geq \alpha\}$. Let $H_1, \ldots H_k$ be hyperplanes such that the successive reflections of C with respect to these planes, i.e., first we reflect C at H_1 then the image at H_2 and so on, cover ρS^{n-1}. Then for $\widehat{K} = \mathrm{st}_{H_k}\mathrm{st}_{H_{k-1}} \cdots \mathrm{st}_{H_1}(\overline{K})$ we have $\widehat{K} \subset \mathrm{int}\,(\rho B_n)$, i.e., $\rho(\widehat{K}) < \rho$.

On the other hand, by Proposition 1.3 ix) we know

$$\mathrm{st}_{H_k} \cdots \mathrm{st}_{H_1}(K_i) \to \widehat{K},$$

and since the bodies on the left side belong to $\mathcal{S}_K$ and ρ is continuous we arrive at the contradiction $\rho(\widehat{K}) \geq \rho$. □

1.6 Corollary. *Let $K_1, K_2 \in \mathcal{K}^n$ with $\dim K_1 = \dim K_2 = n$. Then there exist Steiner-symmetrisations st_{H_i}, $i \in \mathbb{N}$, such that*

$$\lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K_1) = \left(\frac{\mathrm{vol}\,(K_1)}{\mathrm{vol}\,(B_n)}\right)^{1/n} B_n$$
$$\lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K_2) = \left(\frac{\mathrm{vol}\,(K_2)}{\mathrm{vol}\,(B_n)}\right)^{1/n} B_n.$$

Proof. With out loss of generality we assume $\mathrm{vol}\,(K_1) = \mathrm{vol}\,(K_2) = \mathrm{vol}\,(B_n)$. Let $\varepsilon > 0$. By Theorem 1.5 there exist hyperplanes $H_1, \ldots, H_k$ containing $\mathbf{0}$ such that

$$(1-\varepsilon)B_n \subseteq \mathrm{st}_{H_k} \cdots \mathrm{st}_{H_1}(K_1) \subseteq (1+\varepsilon)B_n.$$

Applying again Theorem 1.5 with respect to $\mathrm{st}_{H_k} \cdots \mathrm{st}_{H_1}(K_2)$, we get hyperplanes $H_{k+1}, \ldots, H_l$ containing the origin such that

$$(1-\varepsilon)B_n \subseteq \mathrm{st}_{H_l} \cdots \mathrm{st}_{H_{k+1}}\big(\mathrm{st}_{H_k} \cdots \mathrm{st}_{H_1}(K_2)\big) \subseteq (1+\varepsilon)B_n.$$

[2] Skript WS14

Due to Proposition 1.3 ii) and iv) we also know that for this sequence of the symmetrisations with respect to K_1 still holds

$$(1-\varepsilon)B_n \subseteq \mathrm{st}_{H_l}\cdots\mathrm{st}_{H_{k+1}}\big(\mathrm{st}_{H_k}\cdots\mathrm{st}_{H_1}(K_1)\big) \subseteq (1+\varepsilon)B_n.$$

Thus applying the same argumentation to $\mathrm{st}_{H_l}\ldots\mathrm{st}_{H_1}(K_1)$ and $\mathrm{st}_{H_l}\ldots\mathrm{st}_{H_1}(K_2)$ with $\varepsilon/2, \varepsilon/3\ldots$ we get the required sequence of hyperplanes. $\square$

1.7 Corollary. *Let* $K, M \in \mathcal{K}^n$, $\dim K = \dim M = n$. *Then*

i) isodiametric inequality: $\mathrm{vol}\,(K) \leq \left(\frac{\mathrm{D}(K)}{2}\right)^n \mathrm{vol}\,(B_n)$ *(and equality holds only for a ball).*

ii) isoperimetric inequality: $\mathrm{F}(K)^n/\mathrm{vol}\,(K)^{n-1} \geq \mathrm{F}(B_n)^n/\mathrm{vol}\,(B_n)^{n-1}$ *and equality holds only for a ball.*

iii) Brunn-Minkowski inequality: $\mathrm{vol}\,(K+M)^{1/n} \geq \mathrm{vol}\,(K)^{1/n} + \mathrm{vol}\,(M)^{1/n}$.

Proof. i) Let $\overline{K}$ be the convex body which is obtained by applying successive Steiner-symmetrisations to K with respect to the coordinate hyperplanes. Then $\overline{K}$ is symmetric with respect to the origin. Thus $R(\overline{K}) = \mathrm{D}(\overline{K})/2$ and $\overline{K} \subseteq (\mathrm{D}(\overline{K})/2)\,B_n$ and by Proposition 1.3 v), vii) we get

$$\mathrm{vol}\,(K) = \mathrm{vol}\,(\overline{K}) \leq \left(\frac{\mathrm{D}(\overline{K})}{2}\right)^n \mathrm{vol}\,(B_n) \leq \left(\frac{\mathrm{D}(K)}{2}\right)^n \mathrm{vol}\,(B_n).$$

ii) We may assume $\mathrm{vol}\,(K) = \mathrm{vol}\,(B_n)$. By Corollary 1.6 there exist hyperplanes H_i, $i \in \mathbb{N}$, containing the origin, such that

$$\lim_{i\to\infty} \mathrm{st}_{H_i}\cdots\mathrm{st}_{H_1}(K) = B_n.$$

Then by Proposition 1.3 vi) and by the continuity of the surface area F we get

$$\mathrm{F}(B_n) = \lim_{i\to\infty} F(\mathrm{st}_{H_i}\cdots\mathrm{st}_{H_1}(K)) \leq \mathrm{F}(K).$$

In the case of equality we must have $K = B_n$, since otherwise, by Remark 1.4 we can find a Steiner-symmetrisation strictly decreasing the surface area.

iii) By Corollary 1.6 there exist hyperplanes H_i, $i \in \mathbb{N}$, containing the origin, such that

$$\mathrm{st}_{H_i}\cdots\mathrm{st}_{H_1}(K) \to \left(\frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(B_n)}\right)^{1/n} B_n \text{ and } \mathrm{st}_{H_i}\cdots\mathrm{st}_{H_1}(M) \to \left(\frac{\mathrm{vol}\,(M)}{\mathrm{vol}\,(B_n)}\right)^{1/n} B_n.$$

By Proposition 1.3 iii), v) and the continuity of the volume (Lemma 5.12[3]) we

[3] Skript WS14

find

$$\begin{aligned}\operatorname{vol}(K+M) &= \lim_{i\to\infty} \operatorname{vol}\left(\operatorname{st}_{H_i}\cdots\operatorname{st}_{H_1}(K+M)\right)\\ &\geq \lim_{i\to\infty} \operatorname{vol}\left(\operatorname{st}_{H_i}\cdots\operatorname{st}_{H_1}(K)+\operatorname{st}_{H_i}\cdots\operatorname{st}_{H_1}(M)\right)\\ &= \operatorname{vol}\left(\left(\frac{\operatorname{vol}(K)}{\operatorname{vol}(B_n)}\right)^{1/n} B_n + \left(\frac{\operatorname{vol}(M)}{\operatorname{vol}(B_n)}\right)^{1/n} B_n\right)\\ &= \operatorname{vol}\left(\frac{\operatorname{vol}(K)^{1/n}+\operatorname{vol}(M)^{1/n}}{\operatorname{vol}(B_n)^{1/n}} B_n\right)\\ &= \left(\operatorname{vol}(K)^{1/n}+\operatorname{vol}(M)^{1/n}\right)^n .\end{aligned}$$

□

1.8 Lemma. *Let $K \in \mathcal{K}_0^n$ with $\dim K = n$ and let H be a hyperplane containing $\mathbf{0}$. Then*

$$\operatorname{vol}\left(\operatorname{st}_H(K)^\star\right) \geq \operatorname{vol}(K^\star).$$

Proof. Without loss of generality let $H = \{\boldsymbol{x} \in \mathbb{R}^n : x_n = 0\}$. Then

$$\operatorname{st}_H(K) = \left\{\left(\boldsymbol{x}, \frac{1}{2}(a-b)\right)^\intercal \in \mathbb{R}^n : (\boldsymbol{x},0)^\intercal \in K|H,\, (\boldsymbol{x},a)^\intercal, (\boldsymbol{x},b)^\intercal \in K\right\}$$

and we also may write

$$\begin{aligned}K^\star &= \{(\boldsymbol{y},t)^\intercal \in \mathbb{R}^n : \langle \boldsymbol{x},\boldsymbol{y}\rangle + s\,t \leq 1,\, (\boldsymbol{x},0)^\intercal \in K|H,\, (\boldsymbol{x},s)^\intercal \in K\},\\ \operatorname{st}_H(K)^\star &= \Big\{(\boldsymbol{y},t)^\intercal \in \mathbb{R}^n : \langle \boldsymbol{x},\boldsymbol{y}\rangle + \frac{1}{2}(a-b)t \leq 1,\\ &\qquad (\boldsymbol{x},0)^\intercal \in K|H,\, (\boldsymbol{x},a)^\intercal, (\boldsymbol{x},b)^\intercal \in K\Big\}.\end{aligned}$$

For $A \subset \mathbb{R}^n$, let $A(t) = \{\boldsymbol{x} \in \mathbb{R}^{n-1} : (\boldsymbol{x},t)^\intercal \in A\}$. Next we observe that

$$\frac{1}{2}\left(K^\star(t)+K^\star(-t)\right) \subseteq \operatorname{st}_H(K)^\star(t), \tag{1.8.1}$$

i.e., for $\boldsymbol{y}_1 \in K^\star(t)$ and $\boldsymbol{y}_2 \in K^\star(-t)$, we have to show $\left((\boldsymbol{y}_1+\boldsymbol{y}_2)/2, t\right)^\intercal \in \operatorname{st}_H(K)^\star$. To this end let $(\boldsymbol{x},a)^\intercal, (\boldsymbol{x},b)^\intercal \in K$, then

$$\left\langle \boldsymbol{x}, \frac{1}{2}(\boldsymbol{y}_1+\boldsymbol{y}_2)\right\rangle + \frac{1}{2}(a-b)t = \frac{1}{2}\langle \boldsymbol{x},\boldsymbol{y}_1\rangle + \frac{1}{2}\langle \boldsymbol{x},\boldsymbol{y}_2\rangle + \frac{1}{2}at + \frac{1}{2}b(-t) \leq 1,$$

and this shows (1.8.1). Moreover, since K is o-symmetric, $K^\star \in \mathcal{K}_0^n$ and hence $K^\star(t) = -K^\star(-t)$, and so $\operatorname{vol}_{n-1}\left(K^\star(t)\right) = \operatorname{vol}_{n-1}\left(K^\star(-t)\right)$. Thus, (1.8.1) and Brunn-Minkowski inequality imply for all $t \in \mathbb{R}$

$$\begin{aligned}\operatorname{vol}_{n-1}\left(\operatorname{st}_H(K)^\star(t)\right)^{1/(n-1)} &\geq \operatorname{vol}_{n-1}\left(\frac{1}{2}\left(K^\star(t)+K^\star(-t)\right)\right)^{1/(n-1)}\\ &\geq \frac{1}{2}\operatorname{vol}_{n-1}\left(K^\star(t)\right)^{1/(n-1)} + \frac{1}{2}\operatorname{vol}_{n-1}\left(K^\star(-t)\right)^{1/(n-1)}\\ &= \operatorname{vol}_{n-1}\left(K^\star(t)\right)^{1/(n-1)}.\end{aligned}$$

Integrating over t gives $\operatorname{vol}\left(\operatorname{st}_H(K)^\star\right) \geq \operatorname{vol}(K^\star)$. □

1.9 Theorem [Blaschke-Santaló]. [4,5] *Let $K \in \mathcal{K}_0^n$ with $\dim K = n$. Then*

$$\mathrm{vol}\,(K)\,\mathrm{vol}\,(K^\star) \leq \mathrm{vol}\,(B_n)^2,$$

(and equality holds if and only if K is an ellipsoid).

Proof. Let $\mathrm{vol}\,(K) = \mathrm{vol}\,(B_n)$. It is to show $\mathrm{vol}\,(K^\star) \leq \mathrm{vol}\,(B_n)$. By Corollary 1.6 there exist hyperplanes H_i, $i \in \mathbb{N}$, containing the origin such that

$$K_i = \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K) \to B_n.$$

By Lemma 1.8 we know

$$\begin{aligned}\mathrm{vol}\,(K^\star) &\leq \mathrm{vol}\,(\mathrm{st}_{H_1}(K)^\star) = \mathrm{vol}\,(K_1{}^\star)\\ &\leq \mathrm{vol}\,(\mathrm{st}_{H_2}(K_1)^\star) = \mathrm{vol}\,(K_2{}^\star) \leq \ldots \leq \mathrm{vol}\,(K_i{}^\star)\end{aligned}$$

for every $i \in \mathbb{N}$. Since $K_i{}^\star \to B_n$, as well, we get by the continuity of the volume

$$\mathrm{vol}\,(K^\star) \leq \lim_{i\to\infty} \mathrm{vol}\,(K_i{}^\star) = \mathrm{vol}\,(B_n).$$

□

1.10 Proposition. [6,7] *Let $K \in \mathcal{K}_0^n$ with $\dim K = n$. Then*

$$\mathrm{vol}\,(K)\,\mathrm{vol}\,(K^\star) \geq \frac{4^n}{(n!)^2}.$$

Proof. Without loss of generality we may assume that $C_n^\star = \mathrm{conv}\,\{\pm \boldsymbol{e}_i : 1 \leq i \leq n\} \subseteq K$ has maximum volume amongst all $AC_n^\star$ with $A \in \mathrm{GL}(n,\mathbb{R})$. This implies that $K \subseteq C_n$, since otherwise there exits a $\boldsymbol{v} \in K$ with $v_j > 1$, say, and replacing $\pm\boldsymbol{e}_j$ by $\pm\boldsymbol{v}$ yields a crosspolytope in K having larger volume than $C_n{}^\star$. Hence, by polarity we also have $C_n^\star \subseteq K^\star$ and so

$$\mathrm{vol}\,(K)\mathrm{vol}\,(K^\star) \geq \mathrm{vol}\,(C_n^\star)^2 = \left(\frac{2^n}{n!}\right)^2.$$

□

1.11 Definition [Schwarz symmetral]. [8]*Let $K \in \mathcal{K}^n$, let L be a $(n-k)$-dimensional subspace, $1 \leq k \leq n-1$, with orthogonal complement $L^\perp$. For any $\boldsymbol{y} \in K|L$ (orthogonal projection of K onto L) let $B_k(\boldsymbol{y}, r_y) \subset \boldsymbol{y} + L^\perp$ be the k-dimensional ball with center $\boldsymbol{y}$ and radius r_y such that $\mathrm{vol}_k(B_k(\boldsymbol{y}, r_y)) = \mathrm{vol}_k(K \cap (\boldsymbol{y} + L^\perp))$. Then*

$$\mathrm{S}_L(K) = \bigcup_{\boldsymbol{y}\in K|L} B_k(\boldsymbol{y}, r_y)$$

is called the Schwarz symmetral *of K with respect to L.*

[4] Wilhelm Blaschke, 1885–1962.
[5] Luis Santaló, 1911–2001.
[6] The Mahler conjecture states $\mathrm{vol}\,(K)\,\mathrm{vol}\,(K^\star) \geq \frac{4^n}{n!}$.
[7] Kurt Mahler, 1903 – 1988
[8] Hermann Schwarz, 1843 – 1921

1.12 Proposition. $\mathrm{S}_L(K)$ *is a convex body.*

Proof. $\mathrm{S}_L(K)$ is certainly bounded, and in order to show that it is also closed let $\boldsymbol{z}_i \in \mathrm{S}_L(K)$ be a sequence converging to $\boldsymbol{z}$. Let $\boldsymbol{y}_i$ and $\boldsymbol{y}$ be the projections of $\boldsymbol{z}_i$ and $\boldsymbol{z}$ onto L, respectively. Then $\boldsymbol{y}_i \to \boldsymbol{y}$ and to due the continuity of $\mathrm{vol}_k((\boldsymbol{y}_i + L^\perp) \cap K)$ on its support $K|L$ we get

$$\begin{aligned}\|\boldsymbol{z}-\boldsymbol{y}\| &= \lim_{m\to\infty} \|\boldsymbol{z}_i - \boldsymbol{y}_i\| \\ &\leq \limsup_{m\to\infty} \left(\frac{\mathrm{vol}_k((\boldsymbol{y}_i + L^\perp)\cap K)}{\mathrm{vol}_k(B_k)}\right)^{1/k} = \left(\frac{\mathrm{vol}_k((\boldsymbol{y} + L^\perp)\cap K)}{\mathrm{vol}_k(B_k)}\right)^{1/k}.\end{aligned}$$

Hence, $\boldsymbol{z} \in \mathrm{S}_L(K)$. It remains to establish the convexity of $\mathrm{S}_L(K)$. Let $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathrm{S}_L(K)$, and let $\boldsymbol{y}_i = \boldsymbol{z}_i|L$, $i = 1, 2$. Then for $\lambda \in [0, 1]$ we have $(\lambda\boldsymbol{z}_1 + (1 - \lambda)\boldsymbol{z}_2)|L = \lambda\boldsymbol{y}_1 + (1 - \lambda)\boldsymbol{y}_2$ and

$$\begin{aligned}&\|\lambda\boldsymbol{z}_1 + (1-\lambda)\boldsymbol{z}_2 - (\lambda\boldsymbol{y}_1 + (1-\lambda)\boldsymbol{y}_2)\| \leq \lambda\,\|\boldsymbol{z}_1 - \boldsymbol{y}_1\| + (1-\lambda)\,\|\boldsymbol{z}_2 - \boldsymbol{y}_2\| \\ &\leq \lambda\left(\frac{\mathrm{vol}_k((\boldsymbol{y}_1 + L^\perp)\cap K)}{\mathrm{vol}_k(B_k)}\right)^{1/k} + (1-\lambda)\left(\frac{\mathrm{vol}_k((\boldsymbol{y}_2 + L^\perp)\cap K)}{\mathrm{vol}_k(B_k)}\right)^{1/k} \\ &\leq \left(\frac{\mathrm{vol}_k\left(\lambda[(\boldsymbol{y}_1 + L^\perp)\cap K] + (1-\lambda)[(\boldsymbol{y}_2 + L^\perp)\cap K]\right)}{\mathrm{vol}_k(B_k)}\right)^{1/k},\end{aligned}$$

where for the last inequality we use the Brunn-Minkowski theorem. Since $\lambda[(\boldsymbol{y}_1 + L) \cap K] + (1 - \lambda)[(\boldsymbol{y}_2 + L) \cap K] \subseteq (\lambda\boldsymbol{y}_1 + (1 - \lambda)\boldsymbol{y}_2 + L) \cap K$ we are done. □

1.13 Definition [Central symmetrisation]. *Let $K \in \mathcal{K}^n$. The* central symmetrisation *of K is defined as the body $\frac{1}{2}(K - K)$.*

1.14 Proposition. *Let $K \in \mathcal{K}^n$ and $\boldsymbol{u} \in S^{n-1}$. Then*

$$h(K, \boldsymbol{u}) + h(K, -\boldsymbol{u}) = h(\frac{1}{2}(K - K), \boldsymbol{u}) + h(\frac{1}{2}(K - K), -\boldsymbol{u}).$$

In particular, $D(K) = D(\frac{1}{2}(K - K))$ and $\mathrm{w}(K) = \mathrm{w}(\frac{1}{2}(K - K))$.

1.15 Theorem [Rogers&Shephard]. [9] *Let $K \in \mathcal{K}^n$ with $\dim K = n$. Then*

$$\mathrm{vol}\,(K) \leq \mathrm{vol}\left(\frac{1}{2}(K-K)\right) \leq \frac{1}{2^n}\binom{2n}{n}\mathrm{vol}\,(K).$$

The lower bound is attained if and only if K is centrally symmetric, (and the upper bound if and only if K is a simplex).

Proof. The lower bound is an immediate consequence of the Brunn-Minkowski inequality

$$\mathrm{vol}\left(\frac{1}{2}(K-K)\right) \geq \left(\frac{1}{2}\mathrm{vol}\,(K)^{1/n} + \frac{1}{2}\mathrm{vol}\,(-K)^{1/n}\right)^n = \mathrm{vol}\,(K).$$

[9] Geoffrey Shephard, 1927

Since K is full-dimensional, equality holds here if and only if K and $-K$ are homothetic (up to translations), i.e., if and only if $K = -K + \boldsymbol{t}$ for some $\boldsymbol{t} \in \mathbb{R}^n$, i.e., K is symmetric to $\boldsymbol{t}/2$.

For the upper bound let $\chi_K(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be the characteristic function of K. Then

$$\begin{aligned} \mathrm{vol}\,(K)^2 &= \int_{\mathbb{R}^n} \chi_K(\boldsymbol{y})\,\mathrm{vol}\,(K)\;\mathrm{d}\boldsymbol{y} \\ &= \int_{\mathbb{R}^n} \chi_K(\boldsymbol{y}) \left(\int_{\mathbb{R}^n} \chi_K(\boldsymbol{y}-\boldsymbol{x})\;\mathrm{d}\boldsymbol{x}\right)\;\mathrm{d}\boldsymbol{y} \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \chi_K(\boldsymbol{y}-\boldsymbol{x})\chi_K(\boldsymbol{y})\;\mathrm{d}\boldsymbol{y}\right)\;\mathrm{d}\boldsymbol{x} \qquad (1.15.1) \\ &= \int_{\mathbb{R}^n} \mathrm{vol}\,(K \cap (\boldsymbol{x}+K))\;\mathrm{d}\boldsymbol{x} \\ &= \int_{K-K} \mathrm{vol}\,(K \cap (\boldsymbol{x}+K))\;\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Now, for any $\boldsymbol{x} \in K - K$ let $\rho = \rho(\boldsymbol{x}) \in [0,1]$ such that $\boldsymbol{x} \in \rho\,\mathrm{bd}\,(K-K)$, i.e., $\rho = |\boldsymbol{x}|_{K-K}$ is just the norm of $\boldsymbol{x}$ induced by $K - K$. Let $\boldsymbol{x} = \rho(\boldsymbol{v} - \boldsymbol{w})$ with $\boldsymbol{v}, \boldsymbol{w} \in K$. Then

$$(1-\rho)K + \rho\boldsymbol{v} \subseteq K \text{ and } (1-\rho)K + \rho\boldsymbol{v} = (1-\rho)K + \rho\boldsymbol{w} + \boldsymbol{x} \subseteq \boldsymbol{x} + K.$$

Hence $\mathrm{vol}\,(K \cap (\boldsymbol{x}+K)) \geq \mathrm{vol}\,((1-\rho)K + \rho\boldsymbol{v}) = (1-\rho)^n \mathrm{vol}\,(K)$, and so with (1.15.1)

$$\mathrm{vol}\,(K)^2 \geq \int_{K-K} \big(1-\rho(\boldsymbol{x})\big)^n \mathrm{vol}\,(K)\;\mathrm{d}\boldsymbol{x},$$

which leads to

$$\begin{aligned} \mathrm{vol}\,(K) &\geq \int_{K-K} \big(1-\rho(\boldsymbol{x})\big)^n\;\mathrm{d}\boldsymbol{x} \\ &= \int_{K-K} \left(\int_{\rho(\boldsymbol{x})}^1 n(1-t)^{n-1}\;\mathrm{d}t\right)\;\mathrm{d}\boldsymbol{x} \\ &= \int_0^1 \left(\int_{\{\boldsymbol{x}\in K-K:\rho(\boldsymbol{x})\leq t\}} n\,(1-t)^{n-1}\;\mathrm{d}\boldsymbol{x}\right)\;\mathrm{d}t \\ &= \int_0^1 n(1-t)^{n-1}\mathrm{vol}\,\big(t(K-K)\big)\;\mathrm{d}t \\ &= \mathrm{vol}\,(K-K)\int_0^1 n(1-t)^{n-1}t^n\;\mathrm{d}t \\ &= \frac{(n!)^2}{(2n)!}\mathrm{vol}\,(K-K). \end{aligned}$$

□

2 Brunn-Minkowski revisited

2.1 Theorem [Brunn-Minkowski Inequality]. *Let $K, L \subset \mathcal{K}^n$, and $\lambda \in [0,1]$. Then*

$$\mathrm{vol}\left(\lambda K + (1-\lambda) L\right)^{1/n} \geq \lambda\, \mathrm{vol}\,(K)^{1/n} + (1-\lambda)\mathrm{vol}\,(L)^{1/n}, \tag{2.1.1}$$

and for $0 < \lambda < 1$ equality holds if and only if either K and L are homothetic or K and L lie in parallel hyperplanes.

Proof. For $\lambda \in [0,1]$ let $K_\lambda = \lambda K + (1-\lambda)L$. First we check that the theorem holds for convex bodies $K, L \in \mathcal{K}^n$ of which one is lower dimensional. With out loss of generality let $\dim L < n$.

If also $\dim K < n$ then $\mathrm{vol}\,(K) = \mathrm{vol}\,(L) = 0$ and (2.1.1) holds trivially. If equality holds for a $\lambda \in (0,1)$ then $\mathrm{vol}\,(K_\lambda) = 0$ and so K_λ lies in a hyperplane H, say. But this implies that K and L lie in hyperplanes parallel to H. On the other hand , if K and L lie in parallel hyperplanes then also K_λ is contained in a hyperplane and $\mathrm{vol}\,(K_\lambda) = 0$ for $0 \leq \lambda \leq 1$.

Now suppose $\dim K = n$. Since $K_\lambda \supseteq \lambda K + (1-\lambda)\boldsymbol{x}$ for any $\boldsymbol{x} \in L$, we have $\mathrm{vol}\,(K_\lambda) \geq \mathrm{vol}\left(\lambda K + (1-\lambda)\boldsymbol{x}\right) = \lambda^n \mathrm{vol}\,(K)$ with equality if and only if $L = \{\boldsymbol{x}\}$ and then K and L are homothetic.

Thus, in the following let $\dim K = \dim L = n$, and next we observe that it is sufficient to prove (2.1.1) in the particular situation when $\mathrm{vol}\,(K) = \mathrm{vol}\,(L) = 1$. The general case can be reduced to this setting via the normalisation $\overline{K} = \mathrm{vol}\,(K)^{-1/n}K$, $\overline{L} = \mathrm{vol}\,(L)^{-1/n}L$, and

$$\overline{\lambda} = \frac{\lambda\, \mathrm{vol}\,(K)^{1/n}}{\lambda\, \mathrm{vol}\,(K)^{1/n} + (1-\lambda)\mathrm{vol}\,(L)^{1/n}},$$

since

$$\begin{aligned}
&\mathrm{vol}\left(\overline{\lambda}\overline{K} + (1-\overline{\lambda})\overline{L}\right) \\
&= \mathrm{vol}\left(\frac{\lambda}{\lambda\, \mathrm{vol}\,(K)^{1/n} + (1-\lambda)\mathrm{vol}\,(L)^{1/n}}K + \frac{1-\lambda}{\lambda\, \mathrm{vol}\,(K)^{1/n} + (1-\lambda)\mathrm{vol}\,(L)^{1/n}}L\right) \\
&= \frac{1}{\left(\lambda\, \mathrm{vol}\,(K)^{1/n} + (1-\lambda)\mathrm{vol}\,(L)^{1/n}\right)^n}\, \mathrm{vol}\left(\lambda K + (1-\lambda)L\right).
\end{aligned}$$

So let $\mathrm{vol}\,(K) = \mathrm{vol}\,(L) = 1$ and we have to show $\mathrm{vol}\,(K_\lambda) \geq 1$. This will be done by induction on the dimension. If $n = 1$ the result is certainly true, since K and L are intervals which are homothetic. Let $n \geq 2$. For an arbitrary but fixed $\boldsymbol{u} \in S^{n-1}$ and for $\mu \in \mathbb{R}$ we set

$$\begin{aligned}
v_K(\mu) &= \mathrm{vol}_{n-1}\left(K \cap H(\boldsymbol{u},\mu)\right), & v_L(\mu) &= \mathrm{vol}_{n-1}\left(L \cap H(\boldsymbol{u},\mu)\right), \\
w_K(\mu) &= \mathrm{vol}\left(K \cap H^-(\boldsymbol{u},\mu)\right), & w_L(\mu) &= \mathrm{vol}\left(L \cap H^-(\boldsymbol{u},\mu)\right).
\end{aligned}$$

Then

$$w_K(\mu) = \int_{-h(K,-\boldsymbol{u})}^{\mu} v_K(\boldsymbol{s})\, \mathrm{d}\boldsymbol{s} \quad \text{and} \quad w_L(\mu) = \int_{-h(L,-\boldsymbol{u})}^{\mu} v_L(\boldsymbol{s})\, \mathrm{d}\boldsymbol{s}.$$

The functions v_K, v_L are continuous on the intervals $\big(-h(K,-\boldsymbol{u}), h(K,\boldsymbol{u})\big)$ and $\big(-h(L,-\boldsymbol{u}), h(L,\boldsymbol{u})\big)$, respectively, which ensures that w_K, w_L are differentiable with $w_K'(\mu) = v_K(\mu)$ and $w_L'(\mu) = v_L(\mu)$. Moreover, if we denote by z_K, z_L the inverse function of w_K, w_L respectively, then

$$z_K'(\eta) = \frac{1}{v_K\big(z_K(\eta)\big)}, \quad z_L'(\eta) = \frac{1}{v_L\big(z_L(\eta)\big)}, \quad \text{for } 0 < \eta < 1.$$

Now in order to apply induction we compare we compare $(n-1)$-dimensional sections of the bodies via the function $z_K(\eta)$ and set

$$\tilde{K}_\eta = K \cap H\big(\boldsymbol{u}, z_K(\eta)\big), \qquad \tilde{L}_\eta = L \cap H\big(\boldsymbol{u}, z_L(\eta)\big),$$
$$z_\lambda(\eta) = \lambda\, z_K(\eta) + (1-\lambda) z_L(\eta).$$

Observe that $z_\lambda(0) = \lambda\,(-h(K,-\boldsymbol{u})) + (1-\lambda)\,(-h(L,-\boldsymbol{u})) = -h(K_\lambda,-\boldsymbol{u})$ and in the same way we find $z_\lambda(1) = h(K_\lambda,\boldsymbol{u})$. $\tilde{K}_\eta, \tilde{L}_\eta$ are $(n-1)$-dimensional with $\operatorname{vol}_{n-1}(\tilde{K}_\eta) = v_K\big(z_K(\eta)\big)$ and $\operatorname{vol}_{n-1}(\tilde{L}_\eta) = v_L\big(z_L(\eta)\big)$. Since $\lambda\,\tilde{K}_\eta + (1-\lambda)\tilde{L}_\eta \subseteq K_\lambda \cap H\big(\boldsymbol{u}, z_\lambda(\eta)\big)$ we get in view of our induction hypothesis

$$\begin{aligned}
\operatorname{vol}(K_\lambda) &= \int_{-h(K_\lambda,-\boldsymbol{u})}^{h(K_\lambda,\boldsymbol{u})} \operatorname{vol}_{n-1}\big(K_\lambda \cap H(\boldsymbol{u},\mu)\big)\, \mathrm{d}\mu \\
&= \int_0^1 \operatorname{vol}_{n-1}\big(K_\lambda \cap H(\boldsymbol{u}, z_\lambda(\eta))\big) z_\lambda'(\eta)\, \mathrm{d}\eta \\
&\geq \int_0^1 \operatorname{vol}_{n-1}\big(\lambda\tilde{K}_\eta + (1-\lambda)\tilde{L}_\eta\big) \left[\frac{\lambda}{v_K\big(z_K(\eta)\big)} + \frac{1-\lambda}{v_L\big(z_L(\eta)\big)}\right] \mathrm{d}\eta \\
&\geq \int_0^1 \Big(\lambda\, v_K\big(z_K(\eta)\big)^{\frac{1}{n-1}} + (1-\lambda) v_L\big(z_L(\eta)\big)^{\frac{1}{n-1}}\Big)^{n-1} \left[\frac{\lambda}{v_K\big(z_K(\eta)\big)} + \frac{1-\lambda}{v_L\big(z_L(\eta)\big)}\right] \mathrm{d}\eta \\
&\geq \int_0^1 1\, \mathrm{d}\eta = 1,
\end{aligned} \tag{2.1.2}$$

where the second to last inequality follows by the (weighted) arithmetic/geometric inequality.

We certainly have equality if K and L are homothetic. Thus, suppose we have equality in (2.1.1). Again we may assume $\operatorname{vol}(K) = \operatorname{vol}(L) = 1$ and so $\operatorname{vol}(K_\lambda) = 1$ for some $\lambda \in (0,1)$. Then we have equality in (2.1.2) and by the equality case of the arithmetic/geometric inequality we conclude $v_K\big(z_K(\eta)\big) = v_L\big(z_L(\eta)\big)$ for $\eta \in [0,1]$ (remember v_K, v_L, z_K, z_L are continuous functions). Hence $z_K'(\eta) = z_L'(\eta)$ for $0 \leq \eta \leq 1$, i.e., $z_K(\eta) - z_L(\eta)$ is constant.

Let g_K, g_L be the centroids of K, L, respectively, and let $g_K = g_L = \mathbf{0}$. Then $\mathbf{0} = \int_K \boldsymbol{x}\, \mathrm{d}\boldsymbol{x}$ and so

$$\begin{aligned}
0 &= \int_K \langle \boldsymbol{x}, \boldsymbol{u}\rangle\, \mathrm{d}\boldsymbol{x} \\
&= \int_{-h(K,-\boldsymbol{u})}^{h(K,\boldsymbol{u})} t \operatorname{vol}_{n-1}\big(K \cap H(\boldsymbol{u}, t)\big)\, \mathrm{d}t \\
&= \int_0^1 \operatorname{vol}_{n-1}\Big(K \cap H\big(\boldsymbol{u}, z_K(\eta)\big)\Big) z_K(\eta) \frac{1}{v_K\big(z_K(\eta)\big)}\, \mathrm{d}\eta = \int_0^1 z_K(\eta)\, \mathrm{d}\eta.
\end{aligned}$$

Analogously we obtain the relation $0 = \int_0^1 z_L(\eta)\, \mathrm{d}\eta$, and thus $z_K(\eta) = z_L(\eta)$ for $0 \le \eta \le 1$. Hence, $h(K, \boldsymbol{u}) = h(L, \boldsymbol{u})$. By the arbitrarity of $\boldsymbol{u} \in S^{n-1}$ we conclude $h(K, \boldsymbol{u}) = h(L, \boldsymbol{u})$ for all $\boldsymbol{u} \in S^{n-1}$ and thus $K = L$. □

2.2 Lemma. *Let $K, L \in \mathcal{K}^n$ and suppose there exists a hyperplane H such that $K|H = L|H$. Then for $\lambda \in [0, 1]$*

$$\mathrm{vol}\,\big(\lambda K + (1-\lambda)L\big) \ge \lambda\,\mathrm{vol}\,(K) + (1-\lambda)\,\mathrm{vol}\,(L).$$

Proof. Let $H = H(\boldsymbol{a}, 0)$ with $\|\boldsymbol{a}\| = 1$. For abbreviation we write $U = K|H = L|H$, and we set $K_\lambda = \lambda K + (1-\lambda)L$. Then $K_\lambda|H = \lambda(K|H) + (1-\lambda)(L|H) = U$ for all $\lambda \in [0, 1]$, and K_λ can be described as

$$K_\lambda = \big\{\boldsymbol{y} + t\boldsymbol{a} : \boldsymbol{y} \in U, f_\lambda(\boldsymbol{y}) \le t \le g_\lambda(\boldsymbol{y})\big\},$$

where the functions f_λ and g_λ satisfy $f_\lambda \le g_\lambda$, f_λ is convex and g_λ is concave. For $\boldsymbol{y} \in U$ and $t_1, t_2 \in \mathbb{R}$ with $\boldsymbol{y} + t_1\boldsymbol{a} \in K$ and $\boldsymbol{y} + t_2\boldsymbol{a} \in L$ we have

$$\boldsymbol{y} + \big(\lambda t_1 + (1-\lambda)t_2\big)\boldsymbol{a} = \lambda(\boldsymbol{y} + t_1\boldsymbol{a}) + (1-\lambda)(\boldsymbol{y} + t_2\boldsymbol{a}) \in K_\lambda.$$

Hence, $f_\lambda(\boldsymbol{y}) \le \lambda t_1 + (1-\lambda)t_2 \le g_\lambda(\boldsymbol{y})$. For $t_1 = f_1(\boldsymbol{y})$ and $t_2 = f_0(\boldsymbol{y})$ we obtain $f_\lambda(\boldsymbol{y}) \le \lambda f_1(\boldsymbol{y}) + (1-\lambda) f_0(\boldsymbol{y})$, and for $t_1 = g_1(\boldsymbol{y})$ and $t_2 = g_0(\boldsymbol{y})$ we get $g_\lambda(\boldsymbol{y}) \ge \lambda g_1(\boldsymbol{y}) + (1-\lambda) g_0(\boldsymbol{y})$. Therefore

$$\begin{aligned}
\mathrm{vol}\,(K_\lambda) &= \int_U \big[g_\lambda(\boldsymbol{y}) - f_\lambda(\boldsymbol{y})\big]\, \mathrm{d}\boldsymbol{y} \\
&\ge \int_U \big[\lambda g_1(\boldsymbol{y}) + (1-\lambda) g_0(\boldsymbol{y}) - \lambda f_1(\boldsymbol{y}) - (1-\lambda) f_0(\boldsymbol{y})\big]\, \mathrm{d}\boldsymbol{y} \\
&= \lambda \int_U \big[g_1(\boldsymbol{y}) - f_1(\boldsymbol{y})\big]\, \mathrm{d}\boldsymbol{y} + (1-\lambda) \int_U \big[g_0(\boldsymbol{y}) - f_0(\boldsymbol{y})\big]\, \mathrm{d}\boldsymbol{y} \\
&= \lambda\,\mathrm{vol}\,(K) + (1-\lambda)\,\mathrm{vol}\,(L),
\end{aligned}$$

which concludes the proof. □

2.3 Theorem. *Let $K, L \in \mathcal{K}^n$ and suppose there exists a hyperplane H such that $\mathrm{vol}_{n-1}(K|H) = \mathrm{vol}_{n-1}(L|H)$. Then for $\lambda \in [0, 1]$*

$$\mathrm{vol}\,\big(\lambda K + (1-\lambda)L\big) \ge \lambda\,\mathrm{vol}\,(K) + (1-\lambda)\,\mathrm{vol}\,(L).$$

Proof. Let $H = H(\boldsymbol{a}, 0)$, and let $K' = \mathrm{st}_H(K)$ and $L' = \mathrm{st}_H(L)$ be the Steiner-symmetrals of K and L with respect to H, respectively. By Proposition 1.3[10] we have $\lambda K' + (1-\lambda)L' \subseteq \mathrm{st}_H\big(\lambda K + (1-\lambda)L\big)$ and since the Steiner symmetrisation preserves the volume, it suffices to prove

$$\mathrm{vol}\,\big(\lambda K' + (1-\lambda)L'\big) \ge \lambda\,\mathrm{vol}\,(K') + (1-\lambda)\,\mathrm{vol}\,(L').$$

[10] Skript WS14

Observe that $K|H = K' \cap H$ and $L|H = L' \cap H$. According to Corollary 1.6[11] we can find hyperplanes $H_i = H(\boldsymbol{a}_i, 0)$, $\boldsymbol{a}_i \in H$, $i \in \mathbb{N}$, such that

$$\begin{aligned}\lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K' \cap H) &= \left(\frac{\mathrm{vol}_{n-1}(K' \cap H)}{\kappa_{n-1}}\right)^{\frac{1}{n-1}} B_{n-1} \\ &= \left(\frac{\mathrm{vol}_{n-1}(L' \cap H)}{\kappa_{n-1}}\right)^{\frac{1}{n-1}} B_{n-1} \\ &= \lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(L' \cap H),\end{aligned} \tag{2.3.1}$$

where B_{n-1} is the ball of radius 1 centered at the origin in H with volume κ_{n-1}. Now let

$$K'' = \lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K') \text{ and } L'' = \lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(L').$$

Since K' is symmetric to H and $\boldsymbol{a}_i \in H$, the symmetrals $\mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K')$ are also symmetric to H and so

$$\begin{aligned}[\mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K')]|H &= [\mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K')] \cap H \\ &= [\mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K' \cap H)] = \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K'|H).\end{aligned}$$

Hence

$$K''|H = \lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(K' \cap H) \text{ and } L''|H = \lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}(L' \cap H).$$

we get with Lemma 2.2 applied to K'', L''

$$\begin{aligned}\mathrm{vol}\left(\lambda K' + (1-\lambda)L'\right) &= \mathrm{vol}\left(\lim_{i\to\infty} \mathrm{st}_{H_i} \cdots \mathrm{st}_{H_1}\left(\lambda K' + (1-\lambda)L'\right)\right) \\ &\geq \mathrm{vol}\left(\lambda K'' + (1-\lambda)L''\right) \geq \lambda\,\mathrm{vol}\left(K''\right) + (1-\lambda)\,\mathrm{vol}\left(L''\right) \\ &= \lambda\,\mathrm{vol}\left(K'\right) + (1-\lambda)\,\mathrm{vol}\left(L'\right).\end{aligned}$$

□

2.4 Theorem [Grünbaum]. [12] *Let $K \in \mathcal{K}^n$ with $\dim K = n$ and center of gravity $\boldsymbol{c}(K)$. Let H^+ be a subspace containing $\boldsymbol{c}(K)$. Then*

$$\frac{\mathrm{vol}\,(K \cap H^+)}{\mathrm{vol}\,(K)} \geq \left(\frac{n}{n+1}\right)^n.$$

(Observe that $(n/(n+1))^n \to 1/e$).

Proof. Without loss of generality let $\boldsymbol{c}(K) = \boldsymbol{0}$ and $H^+ = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle = x_1 \geq 0\}$. Let $H^- = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle \leq 0\}$, and for $t \in \mathbb{R}$ let $H_t = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle = t\}$ and $f(t) = \mathrm{vol}_{n-1}(K \cap H_t)$.

[11] Skript WS14
[12] Branko Grünbaum, 1929

Let $b = h(K, \boldsymbol{e}_1)$ and $a = h(K, -\boldsymbol{e}_1)$, where $h(K, \cdot)$ denotes the support function of K. Since $\mathbf{0} = \int_K \boldsymbol{x}\,\mathrm{d}\boldsymbol{x}$ we have

$$0 = \int_K \langle \boldsymbol{x}, \boldsymbol{e}_1\rangle \,\mathrm{d}\boldsymbol{x} = \int_{-a}^{b} t\, f(t)\,\mathrm{d}t. \tag{2.4.1}$$

First we apply Schwarz-Symmetrization to K with respect to the hyperplane H, i.e., we replace $K \cap H_t$ by an $(n-1)$-dimensional ball of volume $f(t)$, i.e., of radius $(f(t)/\kappa_{n-1})^{1/(n-1)}$ and centred at the point $(t, \mathbf{0})^\intercal \in \mathbb{R}^n$. Let $\overline{K}$ be the resulting Schwarz-Symmetral. By construction, $\mathrm{vol}\,(K \cap H_t) = \mathrm{vol}\,(\overline{K} \cap H_t)$ and, in particular, $\mathrm{vol}\,(K) = \mathrm{vol}\,(\overline{K})$, $\mathrm{vol}\,(K \cap H^+) = \mathrm{vol}\,(\overline{K} \cap H^+)$, and we also have $\boldsymbol{c}(\overline{K}) = \boldsymbol{c}(K) = \mathbf{0}$. For the last statement we observe that $\overline{K}$ is rotational symmetric with respect to the axis $\{\lambda\,\boldsymbol{e}_1 : \lambda \in \mathbb{R}\}$. Hence $\boldsymbol{c}(\overline{K})$ has to be on this axis, i.e., $\boldsymbol{c}(\overline{K}) = \gamma\,\boldsymbol{e}_1$. Thus, on account of (2.4.1), we get

$$\begin{aligned}\mathrm{vol}\,(\overline{K})\,\langle \gamma\,\boldsymbol{e}_1, \boldsymbol{e}_1\rangle &= \int_{\overline{K}} \langle \boldsymbol{x}, \boldsymbol{e}_1\rangle\,\mathrm{d}\boldsymbol{x}\\ &= \int_{-a}^{b} t\,\mathrm{vol}_{\,n-1}(\overline{K} \cap H_t)\,\mathrm{d}t = \int_{-a}^{b} t\, f(t)\,\mathrm{d}t = 0.\end{aligned}$$

Hence $\gamma = 0$, i.e., $\boldsymbol{c}(\overline{K}) = \mathbf{0}$ and in the following we may assume $K = \overline{K}$.

Let $K_0 = K \cap H_0$, $K^+ = K \cap H^+$, $K^- = K \cap H^-$, and let $P^+ = \mathrm{conv}\,\{K_0, \beta\boldsymbol{e}_1\}$ be the pyramid with basis K_0 such that $\mathrm{vol}\,(P^+) = \mathrm{vol}\,(K^+)$, i.e., $\beta = n\mathrm{vol}\,(K^+)/\mathrm{vol}_{\,n-1}(K_0)$.

We now extend this pyramid in the other direction such that for $\alpha > 0$ and $P^- = \mathrm{conv}\,\{K_0, -\alpha\boldsymbol{e}_1 + \frac{\alpha+\beta}{\beta}K_0)\}$ holds $\mathrm{vol}\,(P^-) = \mathrm{vol}\,(K^-)$. Let $P = P^+ \cup P^-$, i.e.,

$$P = \mathrm{conv}\left\{-\alpha\boldsymbol{e}_1 + \frac{\alpha+\beta}{\beta}K_0, \beta\boldsymbol{e}_1\right\}$$

is a circular pyramid of height $\alpha + \beta$ and $\mathrm{vol}\,(P) = \mathrm{vol}\,(K)$, and we have

$$\frac{\mathrm{vol}\,(K^+)}{\mathrm{vol}\,(K)} = \frac{\mathrm{vol}\,(P^+)}{\mathrm{vol}\,(P)} = \left(\frac{\beta}{\alpha+\beta}\right)^n.$$

Hence, it remains to show $\beta/(\alpha+\beta) \geq n/(n+1)$. The crucial observation here is that due to the concavity of $f(t)^{1/(n-1)}$ the centroid $c(P)$ is on the non-negative x-axis, i.e., $c(P) = (\gamma, \mathbf{0})^\intercal \in \mathbb{R}^n$ with $\gamma \geq 0$ (see the end of the proof). Then with

$$l(t) = \mathrm{vol}_{\,n-1}(P \cap H_t) = \left(\frac{\beta - t}{\beta}\right)^{n-1} \mathrm{vol}_{\,n-1}(K_0)$$

for $t \in [-\alpha, \beta]$ we may write

$$\begin{aligned}0 \leq \gamma = \mathrm{vol}\,(P)\,\langle \boldsymbol{c}(P), \boldsymbol{e}_1\rangle &= \int_{-\alpha}^{\beta} t\, l(t)\,\mathrm{d}t\\ &= \frac{\mathrm{vol}_{\,n-1}(K_0)}{\beta^{n-1}} \int_{-\alpha}^{\beta} t\,(\beta - t)^{n-1}\,\mathrm{d}t\\ &= \frac{\mathrm{vol}_{\,n-1}(K_0)}{\beta^{n-1}} \left(-\frac{t}{n}(\beta - t)^n - \frac{(\beta - t)^{n+1}}{n(n+1)}\Big|_{-\alpha}^{\beta}\right)\\ &= \frac{\mathrm{vol}_{\,n-1}(K_0)}{\beta^{n-1}} \frac{(\beta + \alpha)^n}{n} \left(-\alpha + \frac{\alpha+\beta}{n+1}\right).\end{aligned}$$

Hence $\alpha/(\alpha+\beta) \leq 1/(n+1)$ which is equivalent to $\beta/(\alpha+\beta) \geq n/(n+1)$.

Here we give a more formal argument why $\langle \boldsymbol{e}_1, \boldsymbol{c}(P)\rangle \geq 0$. Since

$$\mathrm{vol}\,(P)\ \langle \boldsymbol{e}_1, \boldsymbol{c}(P)\rangle = \int_{-\alpha}^{\beta} t\, l(t)\,\mathrm{d}t = -\int_{-\alpha}^{0} (-t)\, l(t)\,\mathrm{d}t + \int_{0}^{\beta} t\, l(t)\,\mathrm{d}t,$$

it is to show

$$\int_{0}^{\beta} t\, l(t)\,\mathrm{d}t \geq \int_{-\alpha}^{0} (-t)\, l(t)\,\mathrm{d}t.$$

In view of (2.4.1) this will follow from the two inequalities

$$\int_{0}^{\beta} t\, l(t)\,\mathrm{d}t \geq \int_{0}^{b} t f(t)\,\mathrm{d}t \text{ and } \int_{-a}^{0} (-t) f(t)\,\mathrm{d}t \geq \int_{-\alpha}^{0} (-t)\, l(t)\,\mathrm{d}t. \tag{2.4.2}$$

We will just prove the first relation, the second is treated analogously. $P \cap H_t$ and $K \cap H_t$ are $(n-1)$-dimensional balls of radius $(l(t)/\kappa_{n-1})^{1/(n-1)}$ and $(f(t)/\kappa_{n-1})^{1/(n-1)}$, respectively. Moreover, $l(0) = f(0)$, $l(t)^{1/(n-1)}$ is a linear function and according to the Brunn-Minkowski theorem $f(t)^{1/(n-1)}$ is a concave function. Hence we have

$$\gamma = \min\{\langle \boldsymbol{e}_1, \boldsymbol{x}\rangle : \boldsymbol{x} \in P^+ \setminus K^+\} = \max\{\langle \boldsymbol{e}_1, \boldsymbol{x}\rangle : \boldsymbol{x} \in K^+ \setminus P^+\}.$$

We also have $P^+ = P^+ \setminus K^+ \cup (P^+ \cap K^+)$ and $K^+ = K^+ \setminus P^+ \cup (K^+ \cap P^+)$, and so $\mathrm{vol}\,(P^+ \setminus K^+) = \mathrm{vol}\,(K^+ \setminus P^+)$. Altogether we may write

$$\begin{aligned}
\int_{0}^{\beta} t\, l(t)\,\mathrm{d}t &= \int_{P^+} \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle\,\mathrm{d}\boldsymbol{x} \\
&= \int_{P^+\setminus K^+} \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle\ \mathrm{d}\boldsymbol{x} + \int_{P^+\cap K^+} \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle\ \mathrm{d}\boldsymbol{x} \\
&\geq \mathrm{vol}\,(P^+ \setminus K^+)\,\gamma + \int_{P^+\cap K^+} \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle\ \mathrm{d}\boldsymbol{x} \\
&= \mathrm{vol}\,(K^+ \setminus P^+)\,\gamma + \int_{K^+\cap P^+} \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle\ \mathrm{d}\boldsymbol{x} \\
&\geq \int_{K^+\setminus P^+} \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle\ \mathrm{d}\boldsymbol{x} + \int_{K^+\cap P^+} \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle\ \mathrm{d}\boldsymbol{x} \\
&= \int_{K^+} \langle \boldsymbol{e}_1, \boldsymbol{x}\rangle\,\mathrm{d}\boldsymbol{x} = \int_{0}^{b} t f(t)\,\mathrm{d}t.
\end{aligned}$$

□

We recall from Proposition 5.18[13] that the Brunn-Minkowski inequality is equivalent to its multiplicative version

$$\mathrm{vol}\,\big(\lambda K + (1-\lambda)L\big) \geq \mathrm{vol}\,(K)^{\lambda}\,\mathrm{vol}\,(L)^{1-\lambda}, \quad \text{for all } \lambda \in [0,1]. \tag{2.4.3}$$

If we denote the characteristic functions of K and L by χ_K and χ_L, respectively, and if for a given $\lambda \in [0,1]$ the characteristic function of $\lambda\, K + (1-\lambda)\, L$ is denoted

[13] Skript WS14

by χ_λ, then we have $\chi_\lambda(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \chi_K(\boldsymbol{x})\,\chi_L(\boldsymbol{y}) = \chi_K(\boldsymbol{x})^\lambda\,\chi_L(\boldsymbol{y})^{1-\lambda}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and (2.4.3) becomes

$$\int_{\mathbb{R}^n} \chi_\lambda(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \geq \left(\int_{\mathbb{R}^n} \chi_K(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^\lambda \left(\int_{\mathbb{R}^n} \chi_L(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{1-\lambda}.$$

2.5 Theorem [Prékopa-Leindler Inequality]. [14] [15] *Let $\lambda \in (0,1)$ and let $f, g, h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (Lebesgue-)measurable functions with*

$$h\big(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}\big) \geq f(\boldsymbol{x})^\lambda g(\boldsymbol{y})^{1-\lambda}, \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \tag{2.5.1}$$

and let f, g be (Lebesgue-)integrable. Then

$$\int_{\mathbb{R}^n} h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \geq \left(\int_{\mathbb{R}^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^\lambda \left(\int_{\mathbb{R}^n} g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{1-\lambda}. \tag{2.5.2}$$

Proof. Without loss of generality let $f, g \neq 0$ and bounded; otherwise, we consider for an arbitrary integer m the function $\min\{f, m\}$ or $\min\{g, m\}$ instead of f or g and apply Beppo Levi's [16] monotone convergence theorem. Since

$$\frac{h\big(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}\big)}{(\sup f)^\lambda(\sup g)^{1-\lambda}} \geq \left(\frac{f(\boldsymbol{x})}{\sup f}\right)^\lambda \left(\frac{g(\boldsymbol{y})}{\sup g}\right)^{1-\lambda}$$

let $\sup f = \sup g = 1$. Finally, since we can approximate absolute integrable functions by continuous functions arbitrary well and due to the Lebesgue dominated convergence theorem we may assume that f, g are continuous.

We show the result by induction on the dimension n. Let $n = 1$. For a measurable function $\psi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ and a $t \geq 0$ we consider the super level set $[\psi \geq t] = \{x \in \mathbb{R} : \psi(x) \geq t\}$, and we write $|\psi \geq t|$ to denote the volume of the set $[\psi \geq t]$. Observe that the super level sets are measurable since ψ is measurable and $\int_{\mathbb{R}} \psi(x)\ \mathrm{d}x = \int_0^\infty |\psi \geq t|\ \mathrm{d}t$.

If $f(x) \geq t$ and $g(y) \geq t$, then by (2.5.1) we also have $h\big(\lambda x + (1-\lambda)y\big) \geq t$ and so

$$\lambda[f \geq t] + (1-\lambda)[g \geq t] \subseteq [h \geq t].$$

For $t \in [0,1)$ the sets on the left-hand side are non-empty, bounded and closed, and thus, the (general) Brunn-Minkowski theorem for compact sets Theorem 2.1[17] in the case $n = 1$ gives $|h \geq t| \geq \lambda|f \geq t| + (1-\lambda)|g \geq t|$ and so we obtain

$$\begin{aligned}\int_{\mathbb{R}} h(x)\ \mathrm{d}x \geq \int_0^1 |h \geq t|\ \mathrm{d}t &\geq \lambda \int_0^1 |f \geq t|\ \mathrm{d}t + (1-\lambda)\int_0^1 |g \geq t|\ \mathrm{d}t \\ &= \lambda \int_0^\infty f(x)\ \mathrm{d}x + (1-\lambda)\int_0^\infty g(x)\ \mathrm{d}x \\ &\geq \left(\int_{\mathbb{R}} f(x)\ \mathrm{d}x\right)^\lambda \left(\int_{\mathbb{R}} g(x)\ \mathrm{d}x\right)^{1-\lambda},\end{aligned}$$

[14] András Prékopa, 1929
[15] László Leindler, 1935
[16] Beppo Levi, 1875–1961
[17] Skript WS14

where the last inequality follows from the arithmetic-geometric mean inequality. This proves the case $n = 1$.

Now let $n > 1$. As usual we identify $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \mathbb{R}$ and for $\boldsymbol{z} \in \mathbb{R}^{n-1}$, $s \in \mathbb{R}$ we define $h_s(\boldsymbol{z}) = h(\boldsymbol{z}, s)$, $f_s(\boldsymbol{z}) = f(\boldsymbol{z}, s)$ and $g_s(\boldsymbol{z}) = g(\boldsymbol{z}, s)$ on $\mathbb{R}^{n-1}$. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n-1}$, $\alpha, \beta \in \mathbb{R}$ and $\gamma = \lambda\alpha + (1-\lambda)\beta$. Then

$$\begin{aligned} h_\gamma\big(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}\big) &= h\big(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}, \lambda\alpha + (1-\lambda)\beta\big) \\ &= h\big(\lambda(\boldsymbol{x}, \alpha) + (1-\lambda)(\boldsymbol{y}, \beta)\big) \\ &\geq f(\boldsymbol{x}, \alpha)^\lambda g(\boldsymbol{y}, \beta)^{1-\lambda} = f_\alpha(\boldsymbol{x})^\lambda g_\beta(\boldsymbol{y})^{1-\lambda}. \end{aligned}$$

Thus, by our inductive argument we get

$$\int_{\mathbb{R}^{n-1}} h_\gamma(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} \geq \left(\int_{\mathbb{R}^{n-1}} f_\alpha(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\right)^\lambda \left(\int_{\mathbb{R}^{n-1}} g_\beta(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\right)^{1-\lambda}.$$

With

$$H(\gamma) = \int_{\mathbb{R}^{n-1}} h_\gamma(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}, \quad F(\alpha) = \int_{\mathbb{R}^{n-1}} f_\alpha(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}, \quad G(\beta) = \int_{\mathbb{R}^{n-1}} g_\beta(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z},$$

this becomes $H\big(\lambda\alpha + (1-\lambda)\beta\big) \geq F(\alpha)^\lambda G(\beta)^{1-\lambda}$. Hence we may apply the case $n = 1$ to these functions and get the desired result:

$$\begin{aligned} \int_{\mathbb{R}^n} h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} h_\gamma(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\right) \mathrm{d}\gamma \\ &= \int_{\mathbb{R}} H(\gamma)\,\mathrm{d}\gamma \geq \left(\int_{\mathbb{R}} F(\alpha)\,\mathrm{d}\alpha\right)^\lambda \left(\int_{\mathbb{R}} G(\beta)\,\mathrm{d}\beta\right)^{1-\lambda} \\ &= \left(\int_{\mathbb{R}^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^\lambda \left(\int_{\mathbb{R}^n} g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{1-\lambda}. \end{aligned}$$

□

2.6 Remark. *The Prékopa-Leindler inequality can be extended to m functions: Let $h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $f_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be measurable functions, $1 \leq i \leq m$, and let $\lambda_i > 0$ with $\sum_{i=1}^m \lambda_i = 1$ such that*

$$h\left(\sum_{i=1}^m \lambda_i\,\boldsymbol{x}_i\right) \geq \prod_{i=1}^m f_i(\boldsymbol{x}_i)^{\lambda_i},$$

and let f_i be integrable, $1 \leq i \leq m$. Then

$$\int_{\mathbb{R}^n} h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \geq \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{\lambda_i}.$$

2.7 Remark. *Hölder's-inequality* [18]

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})\,g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \leq \left(\int_{\mathbb{R}^n} f(\boldsymbol{x})^p\,\mathrm{d}\boldsymbol{x}\right)^{1/p} \left(\int_{\mathbb{R}^n} g(\boldsymbol{x})^q\,\mathrm{d}\boldsymbol{x}\right)^{1/q}.$$

[18] Otto Hölder, 1859 - 1937

for integrable functions $f, g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *and real numbers* $p, q > 1$ *with* $1/p + 1/q = 1$*, may be regarded as a counterpart to the Prékopa-Leindler inequality in the special setting* $p = 1/\lambda$*,* $q = 1/(1-\lambda)$*,* $0 < \lambda < 1$*, and with respect to the functions* $f^{1/p}, g^{1/q}$*:*

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})^{\lambda}\, g(\boldsymbol{x})^{1-\lambda}\,\mathrm{d}\boldsymbol{x} \leq \left(\int_{\mathbb{R}^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{\lambda} \left(\int_{\mathbb{R}^n} g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{1-\lambda}.$$

3 Applications of the inequalities of Brascamp-Lieb and Barthe

3.1 Remark. *Let* $f_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *be integrable functions,* $1 \leq i \leq m$*, and let* $\lambda_i \geq 0$ *with* $\sum_{i=1}^m \lambda_i = 1$*. If we set*

$$h(\boldsymbol{y}) = \sup\left\{\prod_{i=1}^m f_i(\boldsymbol{y}_i)^{\lambda_i} : \boldsymbol{y} = \sum_{i=1}^m \lambda_i\, \boldsymbol{y}_i,\ \boldsymbol{y}_i \in \mathbb{R}^n\right\}$$

then $h\left(\sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i\right) \geq \prod_{i=1}^m f_i(\boldsymbol{x}_i)^{\lambda_i}$*, for all* $\boldsymbol{x}_i \in \mathbb{R}^n$*,* $1 \leq i \leq m$*. Hence, Prékopa-Leindler inequality* (2.5.2) *(cf. Remark* 2.6*) can also be equivalently written in the form*

$$\int_{\mathbb{R}^n}^* \sup\left\{\prod_{i=1}^m f_i(\boldsymbol{x}_i)^{\lambda_i} : \boldsymbol{y} = \sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i,\ \boldsymbol{x}_i \in \mathbb{R}^n\right\} \mathrm{d}\boldsymbol{y} \geq \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\right)^{\lambda_i}.$$

Here $\int_{\mathbb{R}^n}^* g(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$ *denotes the outer integral, which for non-negative functions* $g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *is defined as*

$$\int_{\mathbb{R}^n}^* g(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \inf\left\{\int_{\mathbb{R}^n} u(\boldsymbol{y})\mathrm{d}\boldsymbol{y} : u \geq g, u \text{ measureable}\right\}.$$

3.2 Theorem* [Brascamp-Lieb, Barthe]. [19] [20] [21] *Let* $c_i > 0$ *and* $n_i \in \mathbb{N}$*,* $1 \leq i \leq m$*, satisfying* $\sum_{i=1}^m c_i\, n_i = n$*. Let* $M_i \in \mathbb{R}^{n_i \times n}$*,* $1 \leq i \leq m$*, with* $\mathrm{rg}(M_i) = n_i$*, and let*

$$\alpha = \inf\left\{\frac{\det\left(\sum_{i=1}^m c_i\, M_i^\intercal A_i\, M_i\right)}{\prod_{i=1}^m (\det A_i)^{c_i}} : A_i \in \mathbb{R}^{n_i \times n_i} \text{ positive definite}\right\}.$$

Let $f_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ *be integrable functions. The* BRASCAMP-LIEB *inequality states*

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(M_i\, \boldsymbol{x})^{c_i}\right) \mathrm{d}\boldsymbol{x} \leq \frac{1}{\sqrt{\alpha}} \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(\boldsymbol{x}_i)\, \mathrm{d}\boldsymbol{x}_i\right)^{c_i}, \tag{BL-I}$$

and the BARTHE *inequality states*

$$\int_{\mathbb{R}^n}^* \sup\left\{\prod_{i=1}^m f_i(\boldsymbol{x}_i)^{c_i} : \boldsymbol{y} = \sum_{i=1}^m c_i\, M_i^\intercal \boldsymbol{x}_i,\ \boldsymbol{x}_i \in \mathbb{R}^{n_i}\right\} \mathrm{d}\boldsymbol{y} \geq \sqrt{\alpha} \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(\boldsymbol{x}_i)\, \mathrm{d}\boldsymbol{x}_i\right)^{c_i}. \tag{B-I}$$

[19] Brascamp-Lieb inequality
[20] Elliott H. Lieb, 1932
[21] Franck Barthe, ??

3.3 Corollary [Hölder and Prékopa-Leindler inequalities]. *Let $f_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be integrable functions and let $\lambda_i > 0$, $1 \leq i \leq m$, with $\sum_{i=1}^m \lambda_i = 1$. Then*

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(\boldsymbol{x})^{\lambda_i} \right) \mathrm{d}\boldsymbol{x} \leq \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \right)^{\lambda_i}$$
$$\leq \int_{\mathbb{R}^n}^{*} \sup \left\{ \prod_{i=1}^m f_i(\boldsymbol{x}_i)^{\lambda_i} : \boldsymbol{y} = \sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i,\, \boldsymbol{x}_i \in \mathbb{R}^n \right\} \mathrm{d}\boldsymbol{y}.$$

Proof. We start with Hölder's inequality for which we set $n_i = n$, $c_i = \lambda_i$ and $M_i = \mathrm{I}_n$, $1 \leq i \leq m$. The (BL-I) gives

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(\boldsymbol{x})^{\lambda_i} \right) \mathrm{d}\boldsymbol{x} \leq \frac{1}{\sqrt{\alpha}} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \right)^{\lambda_i},$$

and it remains to prove

$$1 \leq \alpha = \inf \left\{ \frac{\det\left(\sum_{i=1}^m \lambda_i A_i\right)}{\prod_{i=1}^m (\det A_i)^{\lambda_i}} : A_i \in \mathbb{R}^{n \times n} \text{ positive definite} \right\}.$$

Applying Minkowski's Theorem 6.4[22] we get

$$\det \left(\sum_{i=1}^m \lambda_i A_i \right)^{1/n} \geq \sum_{i=1}^m (\det(\lambda_i A_i))^{1/n} = \sum_{i=1}^m \lambda_i (\det A_i)^{1/n} \geq \prod_{i=1}^m (\det A_i)^{\lambda_i/n},$$

where the last inequality follows from the arithmetic/geometric-mean inequality. Therefore $\alpha \geq 1$.

In fact, taking $A_i = \mathrm{I}_n$ for all $i = 1, \ldots, m$ shows $\alpha = 1$, and Prékopa-Leindler's inequality follows immeaditely from (B-I). □

3.4 Corollary [Young inequality].[23] *Let $p_i > 0$, $1 \leq i \leq 3$, with $1/p_1 + 1/p_2 + 1/p_3 = 2$, and let $f_i : \mathbb{R}^n \to \mathbb{R}$, $1 \leq i \leq 3$, be integrable functions. Then*

$$\int_{\mathbb{R}^n} f_1(\boldsymbol{x}) \left(\int_{\mathbb{R}^n} f_2(\boldsymbol{x} - \boldsymbol{y})\, f_3(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \leq c^n \prod_{i=1}^3 \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x})^{p_i}\, \mathrm{d}\boldsymbol{x} \right)^{1/p_i},$$

where $c = c_{p_1} c_{p_2} c_{p_3}$ and $c_p = p^{1/p}/q^{1/q}$, with q such that $1/q + 1/p = 1$.

Proof. We apply (BL-I) to the functions $f_i^{p_i}$, $c_i = 1/p_i$, $n_i = n$ and to the matrices $M_i \in \mathbb{R}^{n \times 2n}$, $1 \leq i \leq 3 = m$, given by $M_1(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}$, $M_2(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x} - \boldsymbol{y}$ and $M_3(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}$ for $(\boldsymbol{x}, \boldsymbol{y})^\intercal \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$. Then $\sum_{i=1}^m c_i n_i = \sum_{i=1}^3 n/p_i = 2n$ and

$$\int_{\mathbb{R}^{2n}} f_1(M_1(\boldsymbol{x}, \boldsymbol{y}))\, f_2(M_2\,(\boldsymbol{x}, \boldsymbol{y})\, f_3(M_3(\boldsymbol{x}, \boldsymbol{y}))\, \mathrm{d}(\boldsymbol{x}, \boldsymbol{y}) \leq$$
$$\frac{1}{\sqrt{\alpha}} \prod_{i=1}^3 \left(\int_{\mathbb{R}^n} f_i(\boldsymbol{x})^{p_i}\, \mathrm{d}\boldsymbol{x} \right)^{1/p_i}.$$

The computation of the constant α is quite involved and we omit it here. □

[22] Skript WS14
[23] William Henry Young, 1863-1942

3.5 Lemma. *Let* $c_i > 0$ *and let* $\boldsymbol{v}_i \in S^{n-1}$, $1 \leq i \leq m$, $m \geq n$, *such that* $\sum_{i=1}^m c_i\,(\boldsymbol{v}_i\boldsymbol{v}_i^\intercal) = \mathrm{I}_n$. *Then*

$$\inf\left\{\frac{\det\bigl(\sum_{i=1}^m c_i\,\alpha_i\,(\boldsymbol{v}_i\boldsymbol{v}_i^\intercal)\bigr)}{\prod_{i=1}^m \alpha_i^{c_i}} : \alpha_i \in \mathbb{R}_{>0}\right\} = 1.$$

Proof. Let the infimum on the left hand side be denoted by α. Setting $\alpha_i = 1$, $1 \leq i \leq m$, shows $\alpha \leq 1$, and in the following we prove $\alpha \geq 1$. To this end, let $\boldsymbol{w}_i = \sqrt{c_i}\boldsymbol{v}_i$, $1 \leq i \leq m$. Then $\mathrm{I}_n = \sum_{i=1}^m c_i\,(\boldsymbol{v}_i\boldsymbol{v}_i^\intercal) = \sum_{i=1}^m(\boldsymbol{w}_i\boldsymbol{w}_i^\intercal) = W\,W^\intercal$, where $W = (\boldsymbol{w}_1,\cdots,\boldsymbol{w}_m) \in \mathbb{R}^{n\times m}$. Hence, the Cauchy-Binet formula[24] gives

$$1 = \det(WW^\intercal) = \sum_{\#I=n} \beta_I, \tag{3.5.1}$$

$\beta_I = \bigl(\det(\boldsymbol{w}_i : i \in I)\bigr)^2$ for each $I \subseteq \{1,\ldots,m\}$ with $\#I = n$. In the same way we see that

$$\begin{aligned}\det\left(\sum_{i=1}^m \alpha_i(\boldsymbol{w}_i\boldsymbol{w}_i^\intercal)\right) &= \sum_{\#I=n}\bigl(\det\bigl(\sqrt{\alpha_i}\,\boldsymbol{w}_i : i \in I\bigr)\bigr)^2\\ &= \sum_{\#I=n}\left(\prod_{i\in I}\alpha_i\right)\beta_I = \sum_{\#I=n}\alpha_I\beta_I,\end{aligned}$$

with $\alpha_I = \prod_{i\in I}\alpha_i$. In view of (3.5.1) we may apply the arithmetic/geometric-mean inequality and get

$$\det\left(\sum_{i=1}^m \alpha_i(\boldsymbol{w}_i\boldsymbol{w}_i^\intercal)\right) \geq \prod_{\#I=n}\alpha_I^{\beta_I}.$$

The exponent of a fixed α_i in the right-hand side is given by

$$\begin{aligned}\sum_{i\in I,\#I=n}\beta_I &= \sum_{\#I=n}\beta_I - \sum_{i\notin I,\#I=n}\beta_I\\ &= 1 - \det\left(\sum_{j=1,j\neq i}^m \boldsymbol{w}_j\,\boldsymbol{w}_j^\intercal\right) = 1 - \det\bigl(\mathrm{I}_n - \boldsymbol{w}_i\boldsymbol{w}_i^\intercal\bigr)\\ &= 1 - \bigl(1 - \|\boldsymbol{w}_i\|^2\bigr) = c_i,\end{aligned}$$

where for the last step we observe that the eigenvalues of $\mathrm{I}_n - (\boldsymbol{w}_i\boldsymbol{w}_i^\intercal)$ are $1 - \|\boldsymbol{w}_i\|^2$ (with eigenvector $\boldsymbol{w}_i$) and $(n-1)$-times 1 (with eigenvectors orthogonal to $\boldsymbol{w}_i$). Hence $\prod_{\#I=n}\alpha_I^{\beta_I} = \prod_{i=1}^m \alpha_i^{c_i}$ and so

$$\frac{\det\bigl(\sum_{i=1}^m c_i\,\alpha_i\,(\boldsymbol{v}_i\boldsymbol{v}_i^\intercal)\bigr)}{\prod_{i=1}^m\alpha_i^{c_i}} = \frac{\det\bigl(\sum_{i=1}^m \alpha_i\,(\boldsymbol{w}_i\boldsymbol{w}_i^\intercal)\bigr)}{\prod_{i=1}^m\alpha_i^{c_i}} \geq \frac{\prod_{\#I=n}\alpha_I^{\beta_I}}{\prod_{i=1}^m\alpha_i^{c_i}} = 1,$$

which shows that $\alpha \geq 1$, as required. □

[24] In its general form it says, that for $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$ we have $\det AB = \sum_{I\in\binom{m}{n}}\det A^I\det B_I$, where A^I, B_I are the $n\times n$ submatrices of A and B with row and column indices in I, respectively. For $m < n$ it just gives/means $\det AB = 0$.

3.6 Theorem. *Let $f_i : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be measurable functions, and let $c_i > 0$ and $\boldsymbol{v}_i \in S^{n-1}$, $1 \leq i \leq m$, such that $\sum_{i=1}^m c_i\,(\boldsymbol{v}_i \boldsymbol{v}_i^\intercal) = \mathrm{I}_n$. Then*

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i\big(\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\big)^{c_i} \right) \mathrm{d}\boldsymbol{x} \leq \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t \right)^{c_i}$$
$$\leq \int_{\mathbb{R}^n}^{*} \sup \left\{ \prod_{i=1}^m f_i(t_i)^{c_i} : \boldsymbol{y} = \sum_{i=1}^m c_i\, t_i\, \boldsymbol{v}_i,\ t_i \in \mathbb{R} \right\} \mathrm{d}\boldsymbol{y}. \tag{3.6.1}$$

For $m = n$ we have equality in both inequalities and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ build an orthonormal basis.

Proof. Since $\mathrm{I}_n = \sum_{i=1}^m c_i\,(\boldsymbol{v}_i \boldsymbol{v}_i^\intercal)$ we may write $\boldsymbol{x} = \sum_{i=1}^m c_i\,\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\,\boldsymbol{v}_i$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and thus, in particular, we have $\mathrm{lin}\,\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\} = \mathbb{R}^n$. First we discuss the case $m = n$. For any $k \in \{1, \ldots, n\}$ we can write $\boldsymbol{v}_k = \sum_{i=1}^n c_i\,\langle \boldsymbol{v}_i, \boldsymbol{v}_k\rangle\,\boldsymbol{v}_i$ and thus we get

$$0 = \sum_{i \neq k} c_i\,\langle \boldsymbol{v}_i, \boldsymbol{v}_k\rangle\,\boldsymbol{v}_i + (c_k - 1)\boldsymbol{v}_k,$$

which implies that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ has to be an orthonormal basis and $c_i = 1$, $1 \leq i \leq n$. Hence we get with $V = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)^\intercal$

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^n f_i\big(\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\big) \right) \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} \left(\prod_{i=1}^n f_i(\boldsymbol{e}_i^\intercal V \boldsymbol{x}) \right) \mathrm{d}\boldsymbol{x}$$
$$= \int_{\mathbb{R}^n} \left(\prod_{i=1}^n f_i(z_i) \right) |\det V|\,\mathrm{d}\boldsymbol{z} = \prod_{i=1}^n \left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t \right).$$

For the upper bound we observe that for $\boldsymbol{y} = \sum_{i=1}^n t_i \boldsymbol{v}_i$ we have $t_i = \langle \boldsymbol{v}_i, \boldsymbol{y}\rangle$ and hence we get as before

$$\int_{\mathbb{R}^n}^{*} \sup \left\{ \prod_{i=1}^n f_i(t_i) : \boldsymbol{y} = \sum_{i=1}^n t_i\, \boldsymbol{v}_i,\ t_i \in \mathbb{R} \right\} \mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^n} \left(\prod_{i=1}^n f_i\big(\langle \boldsymbol{v}_i, \boldsymbol{y}\rangle\big) \right) \mathrm{d}\boldsymbol{y}$$
$$= \prod_{i=1}^n \left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t \right).$$

The general validity of the inequalities is reduced to the (BL-I) and (B-I). Since

$$n = \mathrm{tr}\,\mathrm{I}_n = \sum_{i=1}^m c_i\,\mathrm{tr}(\boldsymbol{v}_i \boldsymbol{v}_i^\intercal) = \sum_{i=1}^m c_i\,\|\boldsymbol{v}_i\|^2 = \sum_{i=1}^m c_i,$$

we may apply (BL-I) and (B-I) with $n_i = 1$ and $M_i = \boldsymbol{v}_i^\intercal$, for $1 \leq i \leq m$, and get

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i\big(\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\big)^{c_i} \right) \mathrm{d}\boldsymbol{x} \leq \frac{1}{\sqrt{\alpha}} \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t \right)^{c_i} \qquad \text{and}$$
$$\int_{\mathbb{R}^n}^{*} \sup \left\{ \prod_{i=1}^m f_i(t_i)^{c_i} : \boldsymbol{y} = \sum_{i=1}^m c_i\, t_i\, \boldsymbol{v}_i,\ t_i \in \mathbb{R} \right\} \mathrm{d}\boldsymbol{y} \geq \sqrt{\alpha} \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t \right)^{c_i},$$

with

$$\alpha = \inf\left\{\frac{\det\left(\sum_{i=1}^m c_i\,\alpha_i(\boldsymbol{v}_i\boldsymbol{v}_i^\intercal)\right)}{\prod_{i=1}^m \alpha_i^{c_i}} : \alpha_i > 0\right\}.$$

Lemma 3.5 shows that $\alpha \geq 1$. □

3.7 Theorem*. *Let $n \geq 2$, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in S^{n-1}$ be pairwise different and let $c_i > 0$ such that $\sum_{i=1}^m c_i\,\boldsymbol{v}_i\boldsymbol{v}_i^\intercal = \mathrm{I}_n$. Let $f_i : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be measureable function which are neither identical 0 nor Gaussian, i.e., of the type $c\,\mathrm{e}^{-\gamma x}$ for positive numbers c, γ. If we have equality in one of the inequalities (3.6.1), then $m = n$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ build an orthonormal basis.*

3.8 Proposition.

i) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function. Via spherical coordinates we get*

$$\begin{aligned}\int_{\mathbb{R}^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= \int_0^\infty \int_{S^{n-1}} r^{n-1} f(r\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}r\\ &= n\kappa_n \int_0^\infty \int_{S^{n-1}} r^{n-1} f(r\boldsymbol{u})\,\mathrm{d}\sigma(\boldsymbol{u})\,\mathrm{d}r.\end{aligned}$$

Here $\mathrm{d}\boldsymbol{u}$ denotes the rotational invariant area measure on the sphere of total mass $\mathrm{F}(S^{n-1}) = n\mathrm{vol}\,(B_n)$, and $\mathrm{d}\sigma(\boldsymbol{u})$ its normalization to a probability measure, i.e., $\int_{S^{n-1}} 1\,\mathrm{d}\sigma(\boldsymbol{u}) = 1$.

ii) *Let $K \in \mathcal{K}^n$ with $\mathbf{0} \in K$. For $\boldsymbol{u} \in S^{n-1}$ let $r_K(\boldsymbol{u}) = \max\{\rho \geq 0 : \rho\,\boldsymbol{u} \in K\}$ be its* radial function*. Then*

$$\mathrm{vol}\,(K) = \kappa_n \int_{S^{n-1}} r_K(\boldsymbol{u})^n\,\mathrm{d}\sigma(\boldsymbol{u}).$$

iii) *Let $K \in \mathcal{K}_0^n$. Then*

$$\mathrm{vol}\,(K) = \kappa_n \int_{S^{n-1}} |\boldsymbol{u}|_K^{-n}\,\mathrm{d}\sigma(\boldsymbol{u}).$$

iv) [25] *Let $K \in \mathcal{K}_0^n$ and $1 \leq p < \infty$. Then*

$$\mathrm{vol}\,(K) = \frac{1}{\Gamma\left(\frac{n}{p}+1\right)} \int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}|_K^p}\,\mathrm{d}\boldsymbol{x}.$$

v)

$$\kappa_n = \mathrm{vol}\,(B_n) = \frac{\pi^{n/2}}{\Gamma\big((n/2)+1\big)} \approx \left(\frac{2\pi e}{n}\right)^{n/2} \frac{1}{\sqrt{\pi n}}.$$

[25] It is $\Gamma(x) = \int_0^\infty \mathrm{e}^{-t}t^{x-1}\mathrm{d}t$. In particular, $\Gamma(t+1) = t\Gamma(t)$, $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$.

Proof. i) is just a coordinate transformation. For ii) we apply i) to the characteristic function χ_K of K

$$\begin{aligned}\operatorname{vol}(K) &= \int_{\mathbb{R}^n} \chi_K(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = n\kappa_n \int_0^\infty \int_{S^{n-1}} \chi_K(r\boldsymbol{u}) r^{n-1}\,\mathrm{d}\sigma(\boldsymbol{u})\,\mathrm{d}r \\ &= n\kappa_n \int_{S^{n-1}} \left(\int_0^{r_K(\boldsymbol{u})} r^{n-1}\,\mathrm{d}r\right)\mathrm{d}\sigma(\boldsymbol{u}) = \kappa_n \int_{S^{n-1}} r_K(\boldsymbol{u})^n\,\mathrm{d}\sigma(\boldsymbol{u}).\end{aligned}$$

In iii) K is 0-symmetric, and so we have $r_K(\boldsymbol{u}) = 1/\left|\boldsymbol{u}\right|_K$ (it was an exercise). Hence, in this case iii) is just a reformulation of ii).

For iv) we observe that

$$\begin{aligned}\int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}|_K^p}\,\mathrm{d}\boldsymbol{x} &= n\,\kappa_n \int_0^\infty \int_{S^{n-1}} r^{n-1}\mathrm{e}^{-|r\boldsymbol{u}|_K^p}\,\mathrm{d}\sigma(\boldsymbol{u})\,\mathrm{d}r \\ &= n\,\kappa_n \int_{S^{n-1}} \int_0^\infty r^{n-1}\mathrm{e}^{-|r\boldsymbol{u}|_K^p}\,\mathrm{d}r\,\mathrm{d}\sigma(\boldsymbol{u}) \\ &\overset{s=r|\boldsymbol{u}|_K}{=} n\,\kappa_n \left(\int_{S^{n-1}} |u|_K^{-n}\,\mathrm{d}\sigma(u)\right)\left(\int_0^\infty \mathrm{e}^{-s^p} s^{n-1}\,\mathrm{d}s\right) \\ &\overset{\text{ii)}}{=} n\operatorname{vol}(K)\int_0^\infty \mathrm{e}^{-s^p} s^{n-1}\,\mathrm{d}s \\ &\overset{t=s^p}{=} \frac{n}{p}\operatorname{vol}(K)\int_0^\infty \mathrm{e}^{-t}\,t^{n/p-1}\,\mathrm{d}t \\ &= \frac{n}{p}\operatorname{vol}(K)\,\Gamma\left(\frac{n}{p}\right) = \operatorname{vol}(K)\,\Gamma\left(\frac{n}{p}+1\right).\end{aligned}$$

For v) let $f:\mathbb{R}^n \to \mathbb{R}$ be the function $f(\boldsymbol{x}) = \mathrm{e}^{-\|\boldsymbol{x}\|^2}$. Then by iv) we have

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \kappa_n\,\Gamma\left(\frac{n}{2}+1\right).$$

On the other hand we may write

$$\begin{aligned}\int_{\mathbb{R}^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^n} \mathrm{e}^{\sum_{i=1}^n x_i^2}\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n}\left(\prod_{i=1}^n \mathrm{e}^{-x_i^2}\right)\mathrm{d}\boldsymbol{x} = \prod_{i=1}^n \int_{-\infty}^\infty \mathrm{e}^{-x_i^2}\,\mathrm{d}x_i \\ &= 2^n \prod_{i=1}^n \int_0^\infty \mathrm{e}^{-x_i^2}\,\mathrm{d}x_i = \prod_{i=1}^n \int_0^\infty \mathrm{e}^{-t} t^{-1/2}\,\mathrm{d}t \\ &= \Gamma\left(\frac{1}{2}\right)^n = \left(\sqrt{\pi}\right)^n.\end{aligned}$$

Hence together with *Stirling's formula* $n! \approx \sqrt{2\pi n}(n/e)^n$ we get

$$\operatorname{vol}(B_n) = \kappa_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \approx \frac{\pi^{n/2}}{\sqrt{2\pi}\sqrt{\frac{n}{2}}\left(\frac{n}{2e}\right)^{n/2}} = \left(\frac{2\pi e}{n}\right)^{n/2}\frac{1}{\sqrt{\pi n}}.$$

□

3.9 Remark.

i) *The radius of an n-dimensional ball of volume 1 is about* $\sqrt{\frac{n}{2\pi \mathrm{e}}}$.

ii) *Let* $C_n = [-1,1]^n$ *the cube of volume* 2^n*. From*

$$2^n = \mathrm{vol}\,(C_n) = \kappa_n \int_{S^{n-1}} r_{C_n}(\boldsymbol{u})^n \,\mathrm{d}\sigma(\boldsymbol{u})$$

we get that the average value of the radial function of C_n *is* $2/\kappa_n^{1/n} \approx \sqrt{2\,n/(\pi e)}$*. Observe, that* $1 \leq r_{C_n}(\boldsymbol{u}) \leq \sqrt{n}$*.*

iii) *For the crosspolytope* $C_n{}^\star = \mathrm{conv}\,\{\pm \boldsymbol{e}_i : 1 \leq i \leq n\}$ *we find*

$$\frac{2^n}{n!} = \mathrm{vol}\,(C_n^\star) = \kappa_n \int_{S^{n-1}} r_{C_n^\star}(\boldsymbol{u})^n \,\mathrm{d}\sigma(\boldsymbol{u})$$

and hence the average value of the radial function of the crosspolytope $C_n{}^\star$ *is*

$$\frac{2}{\left(\kappa_n\, n!\right)^{1/n}} \approx 2\frac{\sqrt{n}\, e}{\sqrt{2\pi e}\left(\sqrt{2\pi n}\right)^{1/n} n} \approx \sqrt{\frac{2e/\pi}{n}}.$$

Observe, that $\frac{1}{\sqrt{n}} \leq r_{C_n{}^\star}(\boldsymbol{u}) \leq 1$*.*

3.10 Lemma. *Let* $c_i > 0$ *and* $\boldsymbol{v}_i \in S^{n-1}$*,* $1 \leq i \leq m$*, such that* $\sum_{i=1}^m c_i\, \boldsymbol{v}_i \boldsymbol{v}_i^\intercal = \mathrm{I}_n$*. Moreover, let* $\alpha_i > 0$*,* $1 \leq i \leq m$ *and* $1 \leq p < \infty$*.*

i) *Let* $K \in \mathcal{K}_0^n$ *be the* 0*-symmetric convex body with gauge function*

$$|\boldsymbol{x}|_K = \left(\sum_{i=1}^m \alpha_i \big|\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\big|^p\right)^{1/p}.$$

Then

$$\mathrm{vol}\,(K) \leq 2^n \frac{\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} \prod_{i=1}^m \left(\frac{c_i}{\alpha_i}\right)^{c_i/p}.$$

For the l_p*-balls* $B_n^p = \{\boldsymbol{x} \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \leq 1\}$ *we have* $\mathrm{vol}\,(B_n^p) = 2^n\,\Gamma\left(1+\frac{1}{p}\right)^n / \Gamma\left(1+\frac{n}{p}\right)$*.*

ii) *Let* Z *be the zonotope* $Z = \sum_{i=1}^m \mathrm{conv}\,\{-\alpha_i \boldsymbol{v}_i, \alpha_i \boldsymbol{v}_i\}$*.*[26] *Then*

$$\mathrm{vol}\,(Z) \geq 2^n \prod_{i=1}^m \left(\frac{\alpha_i}{c_i}\right)^{c_i}.$$

Equality holds if and only if $m = n$*,* $c_i = 1$*, and* $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ *form an orthonormal basis of* $\mathbb{R}^n$*, i.e.,* Z *is an orthogonal box.*

[26] The volume of such a zonotope Z is $\mathrm{vol}\,(Z) = 2^n \sum_{I \in \binom{m}{n}} |\det(v_{j_i} : i \in I)|$.

Proof. By Proposition 3.8 iv) we have

$$\int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}|_K^p}\,\mathrm{d}\boldsymbol{x} = \mathrm{vol}\,(K)\,\Gamma\left(\frac{n}{p}+1\right). \tag{3.10.1}$$

On the other hand, due to the definition of the gauge function we may evaluate the integral as

$$\int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}|_K^p}\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n}\left(\prod_{i=1}^m \mathrm{e}^{-\alpha_i|\langle \boldsymbol{v}_i,\boldsymbol{x}\rangle|^p}\right)\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n}\left(\prod_{i=1}^m f_i\big(\langle \boldsymbol{v}_i,\boldsymbol{x}\rangle\big)^{c_i}\right)\mathrm{d}\boldsymbol{x},$$

with $f_i(t) = \mathrm{e}^{-(\alpha_i/c_i)|t|^p}$. Now we apply the upper bound of Theorem 3.6 and get[27]

$$\begin{aligned}\int_{\mathbb{R}^n} \mathrm{e}^{-|\boldsymbol{x}|_K^p}\,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^n}\left(\prod_{i=1}^m f_i\big(\langle \boldsymbol{v}_i,\boldsymbol{x}\rangle\big)^{c_i}\right)\mathrm{d}\boldsymbol{x} \le \prod_{i=1}^m\left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{c_i}\\ &= \prod_{i=1}^m\left(2\int_0^\infty \mathrm{e}^{-\frac{\alpha_i}{c_i}t^p}\,\mathrm{d}t\right)^{c_i} = \prod_{i=1}^m\left(2\left(\frac{c_i}{\alpha_i}\right)^{1/p}\Gamma\left(\frac{1}{p}+1\right)\right)^{c_i}\\ &= 2^n\Gamma\left(\frac{1}{p}+1\right)^n\prod_{i=1}^m\left(\frac{c_i}{\alpha_i}\right)^{c_i/p},\end{aligned}$$

since, by assumption, $\sum_{i=1}^m c_i = \sum_{i=1}^m c_i\,\|\boldsymbol{v}_i\|^2 = \sum_{i=1}^m c_i\,\mathrm{tr}(\boldsymbol{v}_i\boldsymbol{v}_i^\intercal) = \mathrm{tr}\,\mathrm{I}_n = n$. Together with (3.10.1) we get the result. Observe, that vor $m=n$, $c_i=\alpha_i=1$, $\boldsymbol{v}_i=\boldsymbol{e}_i$ we have equality (see Theorem 3.6).

ii) For $1\le i\le m$ let $f_i(t) = \chi_{[-\alpha_i/c_i,\alpha_i/c_i]}(t)$ be the characteristic functions of the intervals $[-\alpha_i/c_i,\alpha_i/c_i]$, i.e., $f_i(t)=1$ if and only if $c_i t\in[-\alpha_i,\alpha_i]$. Hence, if $\boldsymbol{z}\in\mathbb{R}^n$ satisfies

$$\sup\left\{\prod_{i=1}^m f_i(t_i)^{c_i} : \boldsymbol{z} = \sum_{i=1}^m c_i\,t_i\,\boldsymbol{v}_i,\ t_i\in\mathbb{R}\right\} = 1,$$

then $\boldsymbol{z}\in Z = \sum_{i=1}^m \mathrm{conv}\,\{-\alpha_i\boldsymbol{v}_i,\alpha_i\boldsymbol{v}_i\}$. With the upper bound in Theorem 3.6 we get

$$\begin{aligned}\mathrm{vol}\,(Z) &= \int_{\mathbb{R}^n}\chi_Z(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \ge \int_{\mathbb{R}^n}^{*}\sup\left\{\prod_{i=1}^m f_i(t_i)^{c_i} : \boldsymbol{z} = \sum_{i=1}^m c_i\,t_i\,\boldsymbol{v}_i,\ t_i\in\mathbb{R}\right\}\mathrm{d}\boldsymbol{z}\\ &\ge \prod_{i=1}^m\left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{c_i} = \prod_{i=1}^m\left(2\frac{\alpha_i}{c_i}\right)^{c_i} = 2^n\prod_{i=1}^m\left(\frac{\alpha_i}{c_i}\right)^{c_i},\end{aligned}$$

since, again, $\sum_{i=1}^m c_i = n$. The characterization of the equality case follows from Theorem 3.7. □

[27] Observe that $\int_0^\infty \mathrm{e}^{-\gamma t^p}\,\mathrm{d}t = \gamma^{-1/p}(1/p)\int_0^\infty \mathrm{e}^{-s}s^{1/p-1}\,\mathrm{d}s = \gamma^{-1/p}\Gamma(1/p+1)$ for $\gamma>0$.

3.11 Theorem. *Let $K \in \mathcal{K}_0^n$ and let B_n be the maximum volume ellipsoid contained in K.*[28] *Then*

$$\mathrm{vol}\,(K) \leq \mathrm{vol}\,(C_n) = 2^n.$$

Moreover equality holds if and only if K is a cube of edge length 2, i.e., $K = \big\{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle| \leq 1,\, 1 \leq i \leq n\big\}$, where $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ form an orthonormal basis.

Proof. Since B_n is the maximum volume ellipsoid contained in K, we know by Theorem 6.11[30] that there exist $\boldsymbol{v}_i \in S^{n-1} \cap \mathrm{bd}\, K$ and $\lambda_i > 0$, $1 \leq i \leq m$, with $\sum_{i=1}^m \lambda_i = n$ and $\mathrm{I}_n = \sum_{i=1}^m \lambda_i\,(\boldsymbol{v}_i \boldsymbol{v}_i^\intercal)$. Let $U = \big\{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle| \leq 1,\, 1 \leq i \leq m\big\}$. Clearly $K \subseteq U$ and so $\mathrm{vol}\,(K) \leq \mathrm{vol}\,(U)$. With $f_i = \chi_{[-1,1]}$, $1 \leq i \leq m$, we can write

$$U = \left\{\boldsymbol{x} \in \mathbb{R}^n : \prod_{i=1}^m f_i\big(\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\big)^{\lambda_i} = 1\right\}$$

and with the lower bound in Theorem 3.6 we get

$$\mathrm{vol}\,(U) = \int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i\big(\langle v_i, x\rangle\big)^{\lambda_i}\right) \mathrm{d}x \leq \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{\lambda_i} = 2^{\sum_{i=1}^m \lambda_i} = 2^n.$$

The characterization of the equality case follows from Theorem 3.7. □

3.12 Theorem. *Let $K \in \mathcal{K}^n$ and let B_n be the maximum volume ellipsoid contained in K. Then*

$$\mathrm{vol}\,(K) \leq \mathrm{vol}\,(T_n) = \frac{(\sqrt{n})^n(\sqrt{n+1})^{n+1}}{n!},$$

where T_n is a regular simplex with inradius 1. Moreover, equality holds if and only if K is a regular simplex with inradius 1, i.e., $K = \big\{x \in \mathbb{R}^n : \langle \boldsymbol{v}_i, \boldsymbol{x}\rangle \leq 1,\, 1 \leq i \leq n+1\big\}$ with $\boldsymbol{v}_i \in S^{n-1}$ and $\langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle = -\frac{1}{n}$ for $i \neq j$.

Proof. Since B_n is the maximum volume ellipsoid contained in K, we know by Theorem 6.14[31] that there exist $\boldsymbol{v}_i \in S^{n-1} \cap \mathrm{bd}\, K$ and $\lambda_i > 0$, $1 \leq i \leq m$ and $m \geq n+1$, with $\sum_{i=1}^m \lambda_i = n$, $\mathrm{I}_n = \sum_{i=1}^m \lambda_i\,(\boldsymbol{v}_i \boldsymbol{v}_i^\intercal)$ and $\sum_{i=1}^m \lambda_i\, \boldsymbol{v}_i = 0$. For $U = \big\{x \in \mathbb{R}^n : \langle \boldsymbol{v}_i, \boldsymbol{x}\rangle \leq 1,\, 1 \leq i \leq m\big\}$ it holds $K \subseteq U$ and so $\mathrm{vol}\,(K) \leq \mathrm{vol}\,(U)$. Now let

$$\boldsymbol{w}_i = \sqrt{\frac{n}{n+1}} \begin{pmatrix} -\boldsymbol{v}_i \\ \frac{1}{\sqrt{n}} \end{pmatrix} \in \mathbb{R}^{n+1} \quad \text{and} \quad c_i = \frac{n+1}{n}\lambda_i, \quad 1 \leq i \leq m.$$

Then $\boldsymbol{w}_i \in S^n$ for all $1 \leq i \leq m$, $\sum_{i=1}^m c_i = n+1$ and

$$\sum_{i=1}^m c_i(\boldsymbol{w}_i \boldsymbol{w}_i^\intercal) = \sum_{i=1}^m \lambda_i \begin{pmatrix} (\boldsymbol{v}_i \boldsymbol{v}_i^\intercal) & -\frac{1}{\sqrt{n}}\boldsymbol{v}_i \\ -\frac{1}{\sqrt{n}}\boldsymbol{v}_i^\intercal & \frac{1}{n} \end{pmatrix} = \mathrm{I}_{n+1}.$$

[28] According to Theorem 6.11[29] B_n is the uniquely determined ellipsoid of maximum volume in K if and only if there exist $\boldsymbol{u}_i \in S^{n-1} \cap \mathrm{bd}\, K$ and $\lambda_i > 0$, $1 \leq i \leq m$, with $n \leq m \leq n(n+1)/2$ such that $\mathrm{I}_n = \sum_{i=1}^m \lambda_i\,(\boldsymbol{u}_i \boldsymbol{u}_i^\intercal)$.

[30] Skript WS14

[31] Skript WS14

Now let

$$f_i : \mathbb{R} \to \mathbb{R}_{\geq 0} \quad \text{given by} \quad f_i(t) = \begin{cases} \mathrm{e}^{-t} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases}$$

for all $1 \leq i \leq m$. With $F(\boldsymbol{x}) = \prod_{i=1}^m f_i\big(\langle \boldsymbol{w}_i, \boldsymbol{x}\rangle\big)^{c_i}$ and Theorem 3.6 we obtain

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} F(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^{n+1}} \left(\prod_{i=1}^m f_i\big(\langle \boldsymbol{w}_i, \boldsymbol{x}\rangle\big)^{c_i}\right) \mathrm{d}\boldsymbol{x} &\leq \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t)\,\mathrm{d}t\right)^{c_i} \\ &= \prod_{i=1}^m \left(\int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t\right)^{c_i} = 1. \end{aligned} \tag{3.12.1}$$

Next for $\boldsymbol{x} \in \mathbb{R}^{n+1}$ we write $\boldsymbol{x} = \binom{\boldsymbol{y}}{r} \in \mathbb{R}^{n+1}$ with $\boldsymbol{y} \in \mathbb{R}^n$, $r \in \mathbb{R}$. Then Then

$$\langle \boldsymbol{w}_i, \boldsymbol{x}\rangle = \frac{r}{\sqrt{n+1}} - \sqrt{\frac{n}{n+1}}\,\langle \boldsymbol{v}_i, \boldsymbol{y}\rangle\,.$$

an so

$$F(\boldsymbol{x}) = \mathrm{e}^{-\sum_{i=1}^m c_i \langle \boldsymbol{w}_i, \boldsymbol{x}\rangle} = \mathrm{e}^{-\sum_{i=1}^m c_i \frac{r}{\sqrt{n+1}} + \sum_{i=1}^m c_i \sqrt{\frac{n}{n+1}}\langle v_i, y\rangle} = \mathrm{e}^{-r\sqrt{n+1}}.$$

Due to definition of $F(\boldsymbol{x})$ we have $F(\boldsymbol{x}) \neq 0$, if and only if

$$\langle \boldsymbol{v}_i, \boldsymbol{y}\rangle \leq \frac{r}{\sqrt{n}}, \quad 1 \leq i \leq m. \tag{3.12.2}$$

Since $\sum_{i=1}^m \lambda_i \boldsymbol{v}_i = 0$, $\lambda_i > 0$, we know that for each $\boldsymbol{y} \in \mathbb{R}^n \backslash \{0\}$ there exists $j \in \{1, \ldots, m\}$ such that $\langle \boldsymbol{v}_j, \boldsymbol{y}\rangle \geq 0$, and hence, if $r < 0$ (3.12.2) is never fulfilled. In the case $r \geq 0$, (3.12.2) is equivalent to $\boldsymbol{y} \in \frac{r}{\sqrt{n}} U$, and for such a $\boldsymbol{y} \in \frac{r}{\sqrt{n}} U$, $r \geq 0$, and $\boldsymbol{x} = \binom{\boldsymbol{y}}{r}$, we have

Together with Fubini's theorem we get

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} F(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= \int_0^\infty \int_{\frac{r}{\sqrt{n}}U} F(\boldsymbol{y}, r)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}r = \int_0^\infty \int_{\frac{r}{\sqrt{n}}U} \mathrm{e}^{-r\sqrt{n+1}}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}r \\ &= \int_0^\infty \mathrm{e}^{-r\sqrt{n+1}} \left(\frac{r}{\sqrt{n}}\right)^n \mathrm{vol}\,(U)\,\mathrm{d}r \\ &= \mathrm{vol}\,(U) \left(\frac{1}{\sqrt{n}}\right)^n \int_0^\infty \mathrm{e}^{-t} \left(\frac{t}{\sqrt{n+1}}\right)^n \frac{1}{\sqrt{n+1}}\,\mathrm{d}t \\ &= \frac{n!\,\mathrm{vol}\,(U)}{(\sqrt{n})^n(\sqrt{n+1})^{n+1}}. \end{aligned}$$

Together with (3.12.1), the result follows. The characterization of the equality case follows from Theorem 3.7; observe, if we have equality we have $m = n+1$ and the vectors $\boldsymbol{w}_i$ build an orthonormal basis in $\mathbb{R}^{n+1}$. Hence $\langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle = -\frac{1}{n}$ for $i \neq j$, which shows that U and thus K is a regular simplex. □

3.13 Corollary. *For a convex body $K \in \mathcal{K}^n$ the* volume ratio $\mathrm{vr}(K)$ *is defined as*

$$\mathrm{vr}(K) = \inf_{E \subseteq K,\, E \text{ ellipsoid}} \left(\frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(E)}\right)^{\frac{1}{n}}.$$

i) $\mathrm{vr}(K) \leq \sqrt{n}\,\sqrt{n+1}^{\,1+1/n}/n!^{1/n}$ *with equality if and only if* K *is a simplex.*

ii) *For* $K \in \mathcal{K}_o^n$ *it is* $\mathrm{vr}(K) \leq 2$ *with equality if and only if* K *is (an affine image of) the cube* C_n*.*

Proof. Immediate consequence of the Theorems 3.11, 3.12. □

3.14 Theorem [Reverse Isoperimetric Inequality]. *Let* $K \in \mathcal{K}^n$, $\dim K = n$*. There exists a regular affine transformation* $T : \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$\frac{\mathrm{F}(T\,K)^n}{\mathrm{vol}\,(T\,K)^{n-1}} \leq \frac{\mathrm{F}(T_n)^n}{\mathrm{vol}\,(T_n)^{n-1}} = n^n \frac{(\sqrt{n})^n(\sqrt{n+1})^{n+1}}{n!},$$

where T_n *is a regular simplex. If* K *is o-symmetric then*

$$\frac{\mathrm{F}(TK)^n}{\mathrm{vol}\,(TK)^{n-1}} \leq \frac{\mathrm{F}(C_n)^n}{\mathrm{vol}\,(C_n)^{n-1}} = (2\,n)^n,$$

where C_n *is a cube. Both inequalities are best possible.*

Proof. For the general case let the affine transformation T be chosen such that B_n is the maximum volume ellipsoid contained in TK. By the formula for the surface area given in Remark 5.31 iii)[32] , we obtain

$$\begin{aligned}
\mathrm{F}(TK) &= \lim_{\lambda\to 0}\frac{\mathrm{vol}\,(TK+\lambda B_n)-\mathrm{vol}\,(TK)}{\lambda} \leq \lim_{\lambda\to 0}\frac{\mathrm{vol}\,(TK+\lambda TK)-\mathrm{vol}\,(TK)}{\lambda}\\
&= \lim_{\lambda\to 0}\frac{(1+\lambda)^n-1}{\lambda}\,\mathrm{vol}\,(TK) = n\,\mathrm{vol}\,(TK)\\
&= n\,\mathrm{vol}\,(TK)^{1/n}\mathrm{vol}\,(TK)^{(n-1)/n} \leq n\,\mathrm{vol}\,(T_n)^{1/n}\mathrm{vol}\,(TK)^{(n-1)/n},
\end{aligned}$$

where the last inequality follows from Theorem 3.12 and T_n is here a regular simplex with inradius 1. Hence $\mathrm{F}(T_n) = n\mathrm{vol}\,(T_n)$ and we get

$$\frac{\mathrm{F}(TK)^n}{\mathrm{vol}\,(TK)^{n-1}} \leq n^n\mathrm{vol}\,(T_n) = \frac{\mathrm{F}(T_n)^n}{\mathrm{vol}\,(T_n)^{n-1}}.$$

Now suppose that S is an n-simplex. Then we have $r(S)\,\mathrm{F}(S) = n\mathrm{vol}\,(S)$ and hence we have

$$\frac{\mathrm{F}(TS)^n}{\mathrm{vol}\,(TS)^{n-1}} = n^n\frac{\mathrm{vol}\,(TS)}{r(TS)^n}$$

for any regular affine transformation T. Now for a given inradius the regular simplex has smallest volume and hence we conclude

$$\frac{\mathrm{F}(TS)^n}{\mathrm{vol}\,(TS)^{n-1}} = n^n\frac{\mathrm{vol}\,(TS)}{r(TS)^n} \geq n^n\mathrm{vol}\,(T_n) = \frac{\mathrm{F}(T_n)^n}{\mathrm{vol}\,(T_n)^{n-1}}.$$

Hence for a simplex the inequality can be improved.

The proof for symmetric convex bodies is analogous, but now we apply Theorem 3.11. □

[32] Skript WS14

3.15 Theorem. *Let $K \in \mathcal{K}^n$, and let $c_i > 0$ and $\boldsymbol{v}_i \in S^{n-1}$, for $1 \le i \le m$, be such that $\sum_{i=1}^m c_i\,(\boldsymbol{v}_i \boldsymbol{v}_i^\intercal) = \mathrm{I}_n$. Then*

$$\mathrm{vol}\,(K)^{n-1} \le \prod_{i=1}^m \mathrm{vol}_{\,n-1}(K|\boldsymbol{v}_i^\perp)^{c_i}.$$

Here $K|\boldsymbol{v}_i^\perp$ denotes the orthogonal projection of K onto the orthogonal complement of $\boldsymbol{v}_i$, i.e., the hyperplane $H(\boldsymbol{v}_i, 0)$.

Moreover, equality holds if and only if $m = n$, $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are an orthonormal basis and K is an orthogonal box with respect to $\boldsymbol{v}_i$, i.e., $K = \{\boldsymbol{x} \in \mathbb{R}^n : \alpha_i \le \langle \boldsymbol{v}_i, \boldsymbol{x}\rangle \le \beta_i, 1 \le i \le n\}$ and $\alpha_i < \beta_i \in \mathbb{R}$.

Proof. Let $\alpha_i = c_i/\mathrm{vol}_{\,n-1}(K|\boldsymbol{v}_i^\perp)$, $1 \le i \le m$, and let Z be the zonotope $Z = \sum_{i=1}^m \mathrm{conv}\,\{-\alpha_i \boldsymbol{v}_i, \alpha_i \boldsymbol{v}_i\}$. Applying first Minkowski's inequality Theorem 5.32[33] , then Corollary 5.27[34] and finally using the linearity of the mixed volumes Lemma 5.23 iv)[35] , we get

$$\begin{aligned}\mathrm{vol}\,(Z)^{1/n}\mathrm{vol}\,(K)^{(n-1)/n} &\le \mathrm{V}(K, n-1; Z, 1)\\ &= \sum_{i=1}^m \mathrm{V}\big(K, n-1; \mathrm{conv}\,\{-\alpha_i\,\boldsymbol{v}_i, \alpha_i\,\boldsymbol{v}_i\}, 1\big)\\ &= \frac{2}{n}\sum_{i=1}^m \alpha_i \mathrm{vol}_{\,n-1}(K|\boldsymbol{v}_i^\perp) = \frac{2}{n}\sum_{i=1}^m c_i = 2.\end{aligned}$$

Therefore, $\mathrm{vol}\,(K)^{n-1} \le 2^n/\mathrm{vol}\,(Z)$. Finally, from Lemma 3.10 we get

$$\mathrm{vol}\,(Z) \ge 2^n \prod_{i=1}^m \left(\frac{\alpha_i}{c_i}\right)^{c_i} = 2^n \prod_{i=1}^m \left(\frac{1}{\mathrm{vol}_{\,n-1}(K|\boldsymbol{v}_i^\perp)}\right)^{c_i},$$

and the result follows.

Now, if we have equality then by the equality case of the Minkowski inequality, K and Z are homothetic and from Lemma 3.10 we conclude $m = n$, $c_i = 1$, and $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are an orthonormal basis. Hence K is an orthogonal box with edge directions $\boldsymbol{v}_i$. □

3.16 Corollary [Loomis-Whitney inequality]. *Let $K \in \mathcal{K}^n$. Then*

$$\mathrm{vol}\,(K)^{n-1} \le \prod_{i=1}^n \mathrm{vol}_{\,n-1}(K|\boldsymbol{e}_i^\perp),$$

and equality holds if and only if K is an orthogonal box with facets parallel to the coordinate axes.

Proof. Apply Theorem 3.15 with $m = n$, $\boldsymbol{v}_i = \boldsymbol{e}_i$ and $c_i = 1$, $1 \le i \le n$. □

[33] Skript WS14
[34] Skript WS14
[35] Skript WS14

3.17 Theorem. *Let S be a k-cover of $[n] = \{1, \dots n\}$, i.e., $S = \{S_1, S_2, \dots, S_l\}$ with $S_i \subset [n]$, $1 \le i \le l$, and each $j \in [n]$ is contained in exactly k sets S_i. Let $K \in \mathcal{K}^n$ and let $K|S_i$ be the projection onto* $\operatorname{lin}\{\boldsymbol{e}_j : j \in S_i\}$. *Then*

$$\operatorname{vol}(K)^k \ge \prod_{S_i \in S} \operatorname{vol}_{|S_i|}(K|S_i).$$

Proof. To simplify notation, in the following $\operatorname{vol}(A)$ will always denote the volume measured with respect to aff A. We use induction on n. For $n = 1$ it is certainly true. So let $n \ge 2$ and let □

3.18 Theorem [Meyer-Inequality]. *Let $K \in \mathcal{K}^n$ and let $H_i = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{e}_i, \boldsymbol{x} \rangle = 0\}$, $1 \le i \le n$. Then*

$$\operatorname{vol}(K)^{n-1} \ge \frac{(n-1)!}{n^{n-1}} \prod_{i=1}^{n} \operatorname{vol}_{n-1}(K \cap H_i),$$

and equality holds if and only if K is a generalized cross-polytope, i.e., $K =$ $\operatorname{conv}\{-\alpha_i \boldsymbol{e}_i, \beta_i \boldsymbol{e}_i : \alpha_i, \beta_i > 0, 1 \le i \le n\}$.

Proof. Applying Steiner-Symmetrization to K with respect to the hyperplane H_1 yields a convex body K_1, say, with $\operatorname{vol}(K_1) = \operatorname{vol}(K)$, $\operatorname{vol}_{n-1}(K \cap H_1) \le \operatorname{vol}_{n-1}(K_1 \cap H_1) = \operatorname{vol}_{n-1}(K|\boldsymbol{e}_1^{\perp})$ and $\operatorname{vol}_{n-1}(K \cap H_i) = \operatorname{vol}_{n-1}(K_1 \cap H_i)$ for all $i > 1$. Hence it suffices to prove the inequality for K_1 and repeating this argument to all coordinate hyperplanes H_i shows that it suffices to prove the inequality for a body L which is symmetric to all coordinate hyperplanes, i.e., for $\boldsymbol{x} \in L$ we also have $(\pm x_1, \dots, \pm x_n)^{\intercal} \in L$. Such a convex body is also called an unconditional convex body. In particular, we have $L \cap H_i = L|\boldsymbol{e}_i^{\perp}$ for such a body.

Let $L^s = L \cap \mathbb{R}^n_{\ge 0}$ and $L^s_i = L^s \cap H_i$. By the symmetries we have $\operatorname{vol}(L) = 2^n \operatorname{vol}(L^s)$ and $\operatorname{vol}(L \cap H_i) = 2^{n-1} \operatorname{vol}_{n-1}(L^s_i)$. Hence it suffices to verify the inequality for L^s and its sections L^s_i, for which we rewrite the inequality as

$$\operatorname{vol}(L^s) \le \frac{n^{n-1}}{(n-1)!} \prod_{i=1}^{n} \frac{\operatorname{vol}(L^s)}{\operatorname{vol}(L^s_i)}. \tag{3.18.1}$$

For $\boldsymbol{y} \in L^s$ the intersection of the pyramid $\operatorname{conv}\{L^s_i, \boldsymbol{y}\}$ with another pyramid $\operatorname{conv}\{L^s_j, \boldsymbol{y}\}$, $j \ne i$, is contained in an $(n-1)$-dimensional subspace. Thus we have

$$\operatorname{vol}(L^s) \ge \sum_{i=1}^{n} \operatorname{vol}(\operatorname{conv}\{L^s_i, \boldsymbol{y}\}) = \frac{1}{n} \sum_{i=1}^{n} y_i \operatorname{vol}_{n-1}(L^s_i)$$

for every $\boldsymbol{y} \in L^s$, and so

$$L^s \subseteq \left\{ \boldsymbol{y} \in \mathbb{R}^n_{\ge 0} : \sum_{i=1}^{n} y_i \operatorname{vol}_{n-1}(L^s_i) \le n \operatorname{vol}(L^s) \right\}.$$

Now the set on the left hand side is an n-simplex with vertices $\boldsymbol{0}$, $\frac{n \operatorname{vol}(L^s)}{\operatorname{vol}_{n-1}(L^s_i)} \boldsymbol{e}_i$, $1 \le i \le n$, and of volume

$$\frac{1}{n!} \prod_{i=1}^{n} \frac{n \operatorname{vol}(L^s)}{\operatorname{vol}_{n-1}(L^s_i)}.$$

and (3.18.1) is proven.

Now if we equality for K then the proof above shows that we also have equality for all the bodies K_i created by successively Steiner-symmetrizations at the planes H_i. In particular, we have equality for the unconditional body $K_n = L$, i.e., we have equality in (3.18.1). Hence L^s is a simplex with vertices $\mathbf{0}$ and at the coordinate axis, and thus $L = \operatorname{conv}\{\pm\gamma_i\, e_i : 1 \leq i \leq n\}$, with $\gamma_i \in \mathbb{R}_{>0}$, and $\operatorname{vol}(L) = \operatorname{vol}(K)$.

By the equality assumption we have $K_{n-1} \cap H_n = K_{n-1}|H_n = L \cap H_n$ and so we conclude $\operatorname{conv}\{L \cap H_n, -\alpha_n \boldsymbol{e}_n, \beta_n \boldsymbol{e}_n\} \subseteq K_{n-1}$ with $\alpha_n + \beta_n = 2\gamma_n$. Comparing the volumes of the two sets shows that we have indeed $K_{n-1} = \operatorname{conv}\{L \cap H_n, -\alpha_n \boldsymbol{e}_n, \beta_n \boldsymbol{e}_n\}$ and $\alpha_n, \beta_n > 0$. Hence $-\alpha_n \boldsymbol{e}_n, \beta_n \boldsymbol{e}_n \in K$. Repeating backwards this argument shows that K has to be a generalized crosspolytope. □

3.19 Theorem. *Let L be a k-dimensional linear subspace of $\mathbb{R}^n$. Then*

$$\operatorname{vol}_k(C_n \cap L) \leq 2^k \left(\frac{n}{k}\right)^{k/2}.$$

If k is a divisor of n then the inequality is best possible.

Proof. Here we want to apply Theorem 3.6 in the k-dimensional space L. To this end let P be the orthogonal projection of $\mathbb{R}^n$ onto L. Let $\boldsymbol{u}_i = P\boldsymbol{e}_i$ and $c_i = \|\boldsymbol{u}_i\|^2$, $1 \leq i \leq n$. If $c_j = 0$ then $L \subseteq \boldsymbol{e}_j^\perp$ and $C_n \cap L = (C_n \cap \boldsymbol{e}_j^\perp) \cap L$. Hence the problem is reduced to a cube of dimension one less and the result follows inductively.

So we assume $c_i > 0$ for $1 \leq i \leq n$. For $\boldsymbol{x} \in \mathbb{R}^n$ we have

$$P\boldsymbol{x} = P\left(\sum_{i=1}^n \langle \boldsymbol{x}, \boldsymbol{e}_i\rangle\, \boldsymbol{e}_i\right) = \sum_{i=1}^n \langle \boldsymbol{x}, \boldsymbol{e}_i\rangle\, P\boldsymbol{e}_i = \sum_{i=1}^n \langle \boldsymbol{x}, \boldsymbol{e}_i\rangle\, \boldsymbol{u}_i,$$

and since for $\boldsymbol{x} \in L$ we also have $\langle \boldsymbol{x}, \boldsymbol{e}_i\rangle = \langle \boldsymbol{x}, \boldsymbol{u}_i\rangle$ we get

$$\boldsymbol{x} = P\boldsymbol{x} = \sum_{i=1}^n \langle \boldsymbol{x}, \boldsymbol{u}_i\rangle\, \boldsymbol{u}_i = \sum_{i=1}^n (\boldsymbol{u}_i \boldsymbol{u}_i^\intercal)\boldsymbol{x} = \sum_{i=1}^n c_i\, (\boldsymbol{v}_i \boldsymbol{v}_i^\intercal)\boldsymbol{x},$$

with $\boldsymbol{v}_i = \boldsymbol{u}_i/\sqrt{c_i} \in L \cap S^{n-1}$. Hence, in L the unit vectors $\boldsymbol{v}_i$ give a decomposition of the identity and since $\operatorname{tr} P = k$ we get $\sum_{i=1}^n c_i = k$. Moreover we have

$$\begin{aligned} C_n \cap L &= \left\{x \in L : \left|\langle \boldsymbol{x}, \boldsymbol{e}_i\rangle\right| \leq 1,\, 1 \leq i \leq n\right\} \\ &= \left\{x \in L : \left|\langle \boldsymbol{x}, \boldsymbol{v}_i\rangle\right| \leq \frac{1}{\sqrt{c_i}},\, 1 \leq i \leq n\right\}, \end{aligned}$$

and with $f_i = \chi_{[-1/\sqrt{c_i}, 1/\sqrt{c_i}]}$ we get by Theorem 3.6 (applied in L)

$$\begin{aligned} \operatorname{vol}_k(C_n \cap L) &= \int_L \left(\prod_{i=1}^n f_i(\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle)^{c_i}\right) \mathrm{d}\boldsymbol{x} \leq \prod_{i=1}^n \left(\int_{\mathbb{R}} f_i(t)\, \mathrm{d}t\right)^{c_i} \\ &= \prod_{i=1}^n \left(\frac{2}{\sqrt{c_i}}\right)^{c_i} = 2^k \left(\prod_{i=1}^n c_i^{c_i}\right)^{-1/2}. \end{aligned}$$

The continuous function $\prod_{i=1}^n x_i^{x_i}$ attains a minimum on the compact set $\{\boldsymbol{x} \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^n x_i = k\}$. Inductively it can be shown that such a minimum satisfies $x_1 = \cdots = x_n$. Therefore, exercise

$$\prod_{i=1}^n c_i^{c_i} \geq \prod_{i=1}^n \left(\frac{\sum_{i=1}^n c_i}{n}\right)^{\sum_{i=1}^n c_i/n} = \left(\frac{k}{n}\right)^k.$$

Thus $\mathrm{vol}_k(C_n \cap L) \leq 2^k (n/k)^{k/2}$.

Now suppose that k is a divisor of n, and let $m = n/k$. For each $1 \leq i \leq k$, let $\boldsymbol{u}_i = \sum_{j=1}^m \boldsymbol{e}_{(i-1)m+j}$. These vectors are pairwise orthogonal and with $L = \mathrm{lin}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ we get that

$$Q = \left\{\sum_{i=1}^k \lambda_i \, \boldsymbol{u}_i : \lambda_i \in [-1, 1]\right\} \subset C_n \cap L.$$

Since Q is a k-dimensional cube with edge-length $\sqrt{m} = \sqrt{n/k}$ we have equality in this case. □

3.20 Theorem*. *Let L be a k-dimensional linear subspace of $\mathbb{R}^n$. Then*

$$\mathrm{vol}_k(C_n \cap L) \leq 2^k \, 2^{(n-k)/2}.$$

If $k \geq n/2$ the inequality is best possible.

3.21 Theorem [Vaaler]. *Let L be an $(n-1)$-dimensional linear subspace of $\mathbb{R}^n$. Then*

$$\mathrm{vol}_{n-1}(C_n \cap L) \geq 2^{n-1},$$

and equality holds if and only if L is a coordinate hyperplane.

Proof. For an arbitrary but fixed $\boldsymbol{u} \in S^{n-1}$ let $H(t) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{x}, \boldsymbol{u}\rangle = t\}$, $t \in \mathbb{R}$, and let $f(t) = \mathrm{vol}_{n-1}\big(C_n \cap H(t)\big)$. Since C_n is a 0-symmetric convex body, we get by the Brunn-Minkowski theorem that $f(t) \leq f(0)$ for all $t \in \mathbb{R}$.

$$F(t) = \int_0^t f(s)\,\mathrm{d}s,$$

we know $F(t) \leq t\, f(0)$ for $t \geq 0$ where for $t > 0$ equality holds if and only if $f(t) = f(0)$ for all $t > 0$ with $f(t) \neq 0$; also observe that $F(t)' = f(t)$. Therefore we find

$$\begin{aligned}
\int_{C_n} \langle \boldsymbol{x}, \boldsymbol{u}\rangle^2 \,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}} t^2 f(t)\,\mathrm{d}t = \frac{2}{f(0)^2}\int_0^\infty \big(t\, f(0)\big)^2 f(t)\,\mathrm{d}t \\
&\geq \frac{2}{f(0)^2}\int_0^\infty F(t)^2 f(t)\,\mathrm{d}t = \frac{2}{3\, f(0)^2}\int_0^\infty \big[F(t)^3\big]'\,\mathrm{d}t \\
&= \frac{2}{3\, f(0)^2}\big[F(\infty)^3 - F(0)^3\big] = \frac{2}{3 f(0)^2}\left(\frac{\mathrm{vol}\,(C_n)}{2}\right)^3 \\
&= \frac{2}{3\, f(0)^2}\left(2^{n-1}\right)^3,
\end{aligned}$$

with equality if and only if $F(t) = t\, f(0)$ for all $t \in \mathbb{R}$ with $f(t) > 0$, i.e., if and only if $f(t) = f(0)$ for all $t \in \mathbb{R}$ with $f(t) > 0$. By the continuity of $f(t)$ on its support we conclude that this is equivalent to $\boldsymbol{u} \in \{\pm \boldsymbol{e}_i,\, 1 \le i \le n\}$.

On the other hand, we may evaluate the left hand side integral by

$$\begin{aligned}
\int_{C_n} \langle \boldsymbol{x}, \boldsymbol{u}\rangle^2 \,\mathrm{d}\boldsymbol{x} &= \int_{C_n} \left(\sum_{i=1}^n u_i\, x_i\right)^2 \mathrm{d}\boldsymbol{x} \\
&= \int_{C_n} \left(\sum_{i=1}^n u_i^2 x_i^2 + 2 \sum_{1\le i<j\le n} u_i u_j x_i x_j\right) \mathrm{d}\boldsymbol{x} \\
&= \sum_{i=1}^n \int_{C_n} u_i^2 x_i^2 \,\mathrm{d}\boldsymbol{x} + 2\int_{C_n} \left(\sum_{1\le i<j\le n} u_i x_i\, u_j x_j\right) \mathrm{d}\boldsymbol{x} \qquad (3.21.1) \\
&= \sum_{i=1}^n \left(\int_{-1}^1 \cdots \int_{-1}^1 u_i^2 x_i^2\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n \\
&= \frac{2}{3}\, 2^{n-1} \sum_{i=1}^n u_i^2 = \frac{2}{3}\, 2^{n-1}.
\end{aligned}$$

Comparing the two integrals gives $f(0) \ge 2^{n-1}$ with equality if and only if $\boldsymbol{u} \in \{\pm \boldsymbol{e}_i,\, 1 \le i \le n\}$. □

3.22 Remark [Busemann-Petty problem]. *The Busemann-Petty problem was the question whether for two 0-symmetric convex bodies $K, L \in \mathcal{K}_0^n$, the inequalities*

$$\mathrm{vol}_{n-1}\big(K \cap H(\boldsymbol{u},0)\big) \ge \mathrm{vol}_{n-1}\big(L \cap H(\boldsymbol{u},0)\big), \qquad \textit{for all } \boldsymbol{u} \in S^{n-1},$$

imply $\mathrm{vol}\,(K) \ge \mathrm{vol}\,(L)$*? Taking $K = C_n$ and L a ball of volume 2^n, Theorem 3.20 gives*

$$\mathrm{vol}\,(K\cap H(\boldsymbol{u},0)) \le 2^{n-1}\sqrt{2} < \mathrm{vol}\,(L\cap H(\boldsymbol{u},0)) = 2^{n-1}\frac{\Gamma(\frac{n}{2}+1)^{\frac{n-1}{n}}}{\Gamma(\frac{n-1}{2}+1)} \to 2^{n-1}\sqrt{\mathrm{e}}.$$

And so the answer is No *for $n \ge 10$. In the meantime the problem has been completely solved: the answer is affirmative for $n \le 4$ and negative for $n \ge 5$.*

3.23 Definition. *Let $K \in \mathcal{K}^n$. The set*

$$\Pi(K) = \big\{x \in \mathbb{R}^n : |\langle \boldsymbol{u}, \boldsymbol{x}\rangle| \le \mathrm{vol}_{n-1}(K|\boldsymbol{u}^\perp),\ \textit{for all } \boldsymbol{u} \in S^{n-1}\big\}$$

is called the projection body *of K.*

3.24 Proposition. *Let $K \in \mathcal{K}^n$. Then*

i) $h\big(\Pi(K), \boldsymbol{u}\big) = \mathrm{vol}_{n-1}(K|\boldsymbol{u}^\perp)$.

ii) $\Pi(K)$ *is o-symmetric.*

iii) $\Pi(A\,K) = |\det A|\, A^{-\intercal}\Pi(K)$ *for* $A \in \mathrm{GL}(n,\mathbb{R})$, *and* $\Pi(\boldsymbol{t}+K) = \Pi(K)$ *for* $\boldsymbol{t} \in \mathbb{R}^n$.

3.25 Theorem. *Let* $K \in \mathcal{K}^n$. *There exists a regular affine transformation* $T : \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$\mathrm{vol}_{n-1}\big((TK)|\boldsymbol{u}^\perp\big)^n \geq \mathrm{vol}\,(TK)^{n-1}, \quad \textit{for all } \boldsymbol{u} \in S^{n-1}.$$

Proof. According to Proposition 3.24 iii), we may find a linear transformation $A \in \mathrm{GL}(n,\mathbb{R})$ such that B_n is the maximum volume ellipsoid contained in $A^{-\intercal}\Pi(K) = |\det A|^{-1}\Pi(A\,K)$. In particular, $|\det A|B_n \subseteq \Pi(A\,K)$, and hence, with Proposition 3.24 ii) we get

$$\mathrm{vol}_{n-1}((A\,K)|\boldsymbol{u}^\perp) = h\big(\Pi(A\,K),\boldsymbol{u}\big) \geq |\det A|,$$

for all $\boldsymbol{u} \in S^{n-1}$.

By Theorem 6.11[36] there exist $\boldsymbol{u}_i \in S^{n-1} \cap \mathrm{bd}\,\big(|\det A|^{-1}\Pi(AK)\big)$ and $\lambda_i > 0$, $1 \leq i \leq m$, such that $\mathrm{I}_n = \sum_{i=1}^n \lambda_i\,(\boldsymbol{u}_i\boldsymbol{u}_i^\intercal)$ and so $\sum_{i=1}^m \lambda_i = n$. In particular,

$$\mathrm{vol}_{n-1}((AK)|\boldsymbol{u}_i^\perp) = h\big(\Pi(AK),\boldsymbol{u}_i\big) = |\det A|.$$

Together with Theorem 3.15 we get

$$\begin{aligned}\mathrm{vol}\,(AK)^{n-1} &\leq \prod_{i=1}^m \mathrm{vol}_{n-1}(AK|\boldsymbol{u}_i^\perp)^{\lambda_i} = \prod_{i=1}^m |\det A|^{\lambda_i} \\ &= |\det A|^{\sum_{i=1}^m \lambda_i} = |\det A|^n \leq \mathrm{vol}_{n-1}\big(AK|\boldsymbol{u}^\perp\big)^n,\end{aligned}$$

for all $\boldsymbol{u} \in S^{n-1}$. □

3.26 Theorem* [Vaaler, 1979]. *Let* L *be a* k*-dimensional linear subspace. Then* $\mathrm{vol}_k(C_n \cap L) \geq 2^k$.

3.27 Theorem*. *Let* $H = \{\boldsymbol{x} \in \mathbb{R}^n : \langle a, \boldsymbol{x}\rangle = b\}$, $\boldsymbol{a} \in S^{n-1}$, $b \in \mathbb{R}$, *and assume* $a_i \neq 0$, $1 \leq i \leq n$. *Then*

$$\mathrm{vol}\,(C_n \cap H) = \frac{1}{2\,(n-1)!}\left(\prod_{i=1}^n a_i\right)^{-1} \sum_{\boldsymbol{v} \in \mathrm{vert}\,C_n} (\langle \boldsymbol{a},\boldsymbol{v}\rangle + b)^{n-1}\mathrm{sgn}\,(\langle \boldsymbol{a},\boldsymbol{v}\rangle + b)\prod_{i=1}^n \boldsymbol{v}_i.$$

3.28 Theorem* [Chakerian& Filliman, 1986]. *Let* L *be a* k*-dimensional linear subspace. Then*

$$\mathrm{vol}_k(C_n|L) \leq 2^k \min\left\{\frac{\kappa_{k-1}^k}{\kappa_k^{k-1}}\binom{n}{k}^{\frac{k}{2}}, \sqrt{\binom{n}{k}}\right\}.$$

3.29 Theorem [McMullen, 1984]. *Let* L *be a* k*-dimensional linear subspace with orthogonal complement* $L^\perp$. *Then* $\mathrm{vol}_k(C_n|L) = \mathrm{vol}_{n-k}(C_n|L^\perp)$.

[36] Skript WS14

Proof. Exercise. The main tool is here the so-called Jacobi's determinat identity which for a regular matrix $A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$ and its inverse $A^{-1} = \begin{pmatrix} B' & C' \\ D' & E' \end{pmatrix}$ where B, B' are $(k \times k)$ matrices says that

$$\det B = \det E' \det A.$$

□

4 A few remarks on isotropic positions

4.1 Notation. *Let* $\mathcal{K}_c^n \subset \mathcal{K}^n$ *be the set of all convex bodies with* $\mathrm{vol}\,(K) = 1$ *and centroid at the origin, i.e.,* $\int_K \boldsymbol{x}\,\mathrm{d}\boldsymbol{x} = \boldsymbol{0}$.

4.2 Definition [Isotropic Position, isotropic constant]. *Let* $K \in \mathcal{K}^n$. *An* isotropic position *of* K *is defined as* $\overline{K} = A\,K + \boldsymbol{t}$, $A \in \mathrm{GL}(n, \mathbb{R})$, $\boldsymbol{t} \in \mathbb{R}^n$, *sucht that* $\overline{K} \in \mathcal{K}_c^n$ *and there exists a positive constant* L_K, *called the* isotropic constant, *such that for all* $\boldsymbol{u} \in S^{n-1}$

$$\int_{\overline{K}} \langle \boldsymbol{u}, \boldsymbol{x} \rangle^2 \,\mathrm{d}\boldsymbol{x} = \mathrm{L}_K^2.$$

4.3 Definition [Moment matrix]. *For* $K \in \mathcal{K}^n$ *the* $n \times n$*-matrix*

$$A_K = \int_K \boldsymbol{x}\boldsymbol{x}^\intercal \,\mathrm{d}\boldsymbol{x}$$

is called the moment matrix *or* matrix of inertia *of* K.

4.4 Remark [Binet ellipsoid]. *Observe that for* $\boldsymbol{u} \in \mathbb{R}^n$

$$\boldsymbol{u}^\intercal A_K \boldsymbol{u} = \int_K \boldsymbol{u}^\intercal \boldsymbol{x}\boldsymbol{x}^\intercal \boldsymbol{u}\,\mathrm{d}\boldsymbol{x} = \int_K \langle \boldsymbol{u}, \boldsymbol{x} \rangle^2 \,\mathrm{d}\boldsymbol{x}.$$

Hence A_K *is a positive definite symmetric matrix and it can be decomposed as* $A_K = (D\,U)^\intercal\,(D\,U)$ *where* $U \in \mathrm{O}(n, \mathbb{R})$ *is an orthogonal matrix and* D *is a diagonalmatrix. The ellipsoid* $\mathcal{E}_K = (DU)^{-1}\,B_n$ *has norm* $|\boldsymbol{u}|_{\mathcal{E}_K} = \sqrt{\boldsymbol{u}^\intercal A_K \boldsymbol{u}}$ *and it is called the* Binet (fundamental) Ellipsoid *of* K.

4.5 Proposition.

i) $K \in \mathcal{K}_c^n$ *is in isotropic position if and only if* $\mathcal{E}_K = (1/\mathrm{L}_K)\,B_n$ *which is equivalent to* $A_K = \mathrm{L}_K^2\,\mathrm{I}_n$.

ii) *If* $K \in \mathcal{K}^n$ *is in isotropic position, then*

$$\int_K \|\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x} = n\,\mathrm{L}_K^2.$$

iii) *Let* $K \in \mathcal{K}^n$ *and let* $T \in \mathrm{GL}(n, \mathbb{R})$. *Then*

$$A_{T\,K} = |\det T|\,T\,A_K\,T^\intercal.$$

iv) *If* $K \in \mathcal{K}_c^n$ *is in isotropic position then* $T\,K$ *is in isotropic position for all* $T \in \mathrm{O}(n, \mathbb{R})$, *and it is* $L_K = L_{T\,K}$.

Proof. K is isotropic if and only if $\boldsymbol{u}^\intercal(A_k - \mathrm{L}_K^2\,\mathrm{I}_n)\boldsymbol{u} = 0$ for all $\boldsymbol{u} \in S^{n-1}$. Since $A_k - \mathrm{L}_K^2\,\mathrm{I}_n$ is symmetric we conclude $A_K = \mathrm{L}_K^2\,\mathrm{I}_n$ which is equivalent to the fact that $\mathcal{E}_K$ is a ball of radius $1/\mathrm{L}_K$. This shwows i).

ii) is obvious since $\int_K x_i^2\,\mathrm{d}\boldsymbol{x} = \mathrm{L}_K^2$ for each $1 \le i \le n$, if K is in isotropic position. For iii) we note

$$\begin{aligned}\boldsymbol{u}^\intercal T\,A_K\,T^\intercal\boldsymbol{u} &= \int_K \langle T^\intercal\boldsymbol{u},\boldsymbol{x}\rangle^2\,\mathrm{d}\,\boldsymbol{x} = \int_K \langle\boldsymbol{u},T\,\boldsymbol{x}\rangle^2\,\mathrm{d}\,\boldsymbol{x}\\ &= \frac{1}{|\det T|}\int_{TK}\langle\boldsymbol{u},\boldsymbol{x}\rangle^2\,\mathrm{d}\,\boldsymbol{x} = \frac{1}{|\det T|}\boldsymbol{u}^\intercal A_{TK}\boldsymbol{u}\end{aligned}$$

for all $\boldsymbol{u} \in S^{n-1}$. Hence $|\det T|\,T\,A_K\,T^\intercal = A_{TK}$.

For iv) let $T \in \mathrm{O}(n,\mathbb{R})$ and K (in) isotropic (position); iii) and i) gives $A_{TK} = T A_k T^\intercal = T\,\mathrm{L}_K^2\mathrm{I}_n\,T^\intercal = \mathrm{L}_K^2\,\mathrm{I}_n$ and so by i) TK is isotropic as well with teh same isoctropic ocnstant as K. □

4.6 Remark. *Let* $\overline{C}_n = \frac{1}{2}C_n$, $\overline{B}_n = (1/\kappa_n)^{1/n}B_n$ *be the cube and ball of volume 1, respectively. Then*

$$\mathrm{L}_{\overline{C}_n} = \sqrt{\frac{1}{12}} \quad \textit{and} \quad \mathrm{L}_{\overline{B}_n} = \sqrt{\frac{1}{n+2}\frac{\Gamma(\frac{n}{2}+1)^{\frac{2}{n}}}{\pi}} \sim \sqrt{\frac{1}{2\pi\mathrm{e}}}.$$

Proof. First we note that for $K \in \mathcal{K}^n$ and $\lambda \in \mathbb{R}_{>0}$

$$\int_{\lambda K}\langle\boldsymbol{u},\boldsymbol{x}\rangle^2\,\mathrm{d}\boldsymbol{x} = \lambda^n\int_K\langle\boldsymbol{u},\lambda\,\boldsymbol{x}\rangle^2\,\mathrm{d}\boldsymbol{x} = \lambda^{n+2}\int_K\langle\boldsymbol{u},\boldsymbol{x}\rangle^2\,\mathrm{d}\boldsymbol{x}. \tag{4.6.1}$$

Form (3.21.1) we know already $\int_{C_n}\langle\boldsymbol{u},\boldsymbol{x}\rangle^2\,\mathrm{d}\boldsymbol{x} = \frac{2}{3}2^{n-1}$, and hence $\overline{C}_n$ is in isotropic position. With (4.6.1) we find $\mathrm{L}_{\overline{C}_n}^2 = 1/12$.

Obviously, the ball $\overline{B}_n$ is in isotropic position, since the value of $\int_{\overline{B}_n}\langle\boldsymbol{u},\boldsymbol{x}\rangle^2\,\mathrm{d}\boldsymbol{x}$ is independent of the chosen direction $\boldsymbol{u}$. For the trace of the moment matrix we compute via spherical coordinates

$$\int_{B_n}\|\boldsymbol{x}\|^2\,\mathrm{d}\boldsymbol{x} = n\kappa_n\int_0^1 r^{n+1}\,\mathrm{d}r = \frac{n}{n+2}\kappa_n.$$

Hence $\int_{B_n} x_i^2\,\mathrm{d}\boldsymbol{x} = \frac{\kappa_n}{n+2}$ and thus

$$L_{\overline{B}_n}^2 = \left(\frac{1}{\kappa_n}\right)^{\frac{n+2}{n}}\frac{\kappa_n}{n+2} = \frac{1}{n+2}\kappa_n^{-2/n} = \frac{1}{n+2}\frac{\Gamma(\frac{n}{2}+1)^{2/n}}{\pi}.$$

□

4.7 Lemma. *Let* $K \in \mathcal{K}^n$ *be centered. Then there exists a* $T \in \mathrm{GL}(n,\mathbb{R})$ *such that* $T\,K$ *is in isotropic position. Moreover,* T *is uniquely determined up to orthogonal transformations.*

Proof. We may assume that also $\operatorname{vol} K = 1$. Since A_K is a positive definite symmetric matrix there exists an $U \in \mathrm{O}(n,\mathbb{R})$ such that $U A_k U^\intercal = \operatorname{diag}(s_1,\dots,s_n)$ with $s_i > 0$. Let $\sigma = (s_1 \cdot \dots \cdot s_n)^{1/n}$ and let $V = \operatorname{diag}(\sqrt{\sigma/s_1},\dots,\sqrt{\sigma/s_n})$. Then $VU \in \mathrm{GL}(n,\mathbb{R})$, $\det(VU) = 1$ and

$$VU\,A_K\,(VU)^\intercal = \sigma \mathrm{I}_n$$

and with Proposition 4.5 iii) and i), we see that VUK is in isotropic position. Moreover, VU is uniquely determined up to orthogonal transformations, because $SA_KS^\intercal = \sigma\mathrm{I}_n = TA_KT^\intercal$ implies $S^{-1}S^{-\intercal} = T^{-1}T^{-\intercal}$ or $\mathrm{I}_n = (TS^{-1})(TS^{-1})^\intercal$. Thus $TS^{-1} \in \mathrm{O}(n,\mathbb{R})$ which means $T = U\,S$ for an $U \in \mathrm{O}(n,\mathbb{R})$. □

4.8 Notation. *Due to Lemma 4.7 and Proposition 4.5 iv) isotropic constant is an affine invariant functional, i.e.,* $L_{A\,K} = L_K$ *for any* $A \in \mathrm{GL}(n,\mathbb{R})$.

4.9 Proposition. *Let* $K \in \mathcal{K}^n_c$. *Then* $\det A_K = \mathrm{L}_K^{2n}$ *and (thus)* $\operatorname{vol}(\mathcal{E}_K) = \mathrm{L}_K^{-n}\,\kappa_n$. *Moreover, there exist* $\hat{\boldsymbol{u}}, \tilde{\boldsymbol{u}} \in S^{n-1}$ *such that*

$$\int_K \langle \hat{\boldsymbol{u}}, \boldsymbol{x}\rangle^2 \,\mathrm{d}\boldsymbol{x} \le \mathrm{L}_K^2 \le \int_K \langle \tilde{\boldsymbol{u}}, \boldsymbol{x}\rangle^2 \,\mathrm{d}\boldsymbol{x}.$$

Proof. According to Lemma 4.7 we can find a $T \in \mathrm{GL}(n,\mathbb{R})$, $|\det T| = 1$, such that $T\,K$ is in isotropic position, i.e., $\det(A_{TK}) = L_K^{2n}$. By Proposition 4.5 iii) it is $A_{TK} = |\det T| T A_K T^\intercal$, and hence $\det A_K = \mathrm{L}_K^{2n}$. Due to the definition of $\mathcal{E}_K$ we have $\operatorname{vol}(\mathcal{E}_K) = \kappa_n \det(A_K)^{-1/2}$. Hence, together with Proposition 3.8 iii) we have

$$\mathrm{L}_K^{-n} = \frac{\operatorname{vol}(\mathcal{E}_K)}{\kappa_n} = \int_{S^{n-1}} (|\boldsymbol{u}|_{\mathcal{E}_K})^{-n} \,\mathrm{d}\sigma(\boldsymbol{u}).$$

Thus we can find $\tilde{\boldsymbol{u}} \in S^{n-1}$ such that (cf. Remark 4.4)

$$\mathrm{L}_K^2 \le |\tilde{\boldsymbol{u}}|^2_{\mathcal{E}_K} = \tilde{\boldsymbol{u}}^\intercal A_K \tilde{\boldsymbol{u}} = \int_K \langle \tilde{\boldsymbol{u}}, \boldsymbol{x}\rangle^2 \,\mathrm{d}\boldsymbol{x}.$$

In the same way we can find a lower bound on L_K. □

4.10 Theorem. *Let* $K \in \mathcal{K}^n$. *Then* $\mathrm{L}_K \ge \mathrm{L}_{B_n}$, *and equality holds if and only if* K *is an ellipsoid.*

Proof. After an affine transformation we may assume that K is in isotropic position and let $\overline{B}_n$ be the ball of volume 1. Since K and $\overline{B}_n$ have the same volume we conclude $\operatorname{vol}(K \setminus \overline{B}_n) = \operatorname{vol}(\overline{B}_n \setminus K)$. Hence

$$\int_{K\setminus\overline{B}_n} \|\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x} \ge \left(\frac{1}{\kappa_n}\right)^{\frac{2}{n}} \operatorname{vol}(K\setminus\overline{B}_n) = \left(\frac{1}{\kappa_n}\right)^{\frac{2}{n}} \operatorname{vol}(\overline{B}_n\setminus K) \ge \int_{\overline{B}_n\setminus K} \|\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x}$$

and so

$$
\begin{aligned}
n\,\mathrm{L}_K^2 &= \int_K \|\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x} = \int_{K\cap \overline{B}_n} \|\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x} + \int_{K\setminus \overline{B}_n} \|\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x} \\
&\geq \int_{\overline{B}_n\cap K} \|\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x} + \int_{\overline{B}_n\setminus K} \|\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x} = \int_{\overline{B}_n} \|\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x} = n\,\mathrm{L}_{\overline{B}_n}^2,
\end{aligned}
$$

and here we have equality if and only if $K = \overline{B}_n$. □

4.11 Theorem. *Let $K \in \mathcal{K}_c^n$. K is in isotropic position if and only if*

$$
\int_K \|\boldsymbol{x}\|^2 \,\mathrm{d}\,\boldsymbol{x} \leq \int_{TK} \|\boldsymbol{x}\|^2 \,\mathrm{d}\,\boldsymbol{x}
$$

for all $T \in \mathrm{GL}(n,\mathbb{R})$, $|\det T| = 1$.

Proof. First we note that via eigenvalues and the arithmetic/geometric mean inequality we have for any matrix $T \in \mathrm{GL}(n,\mathbb{R})$, $|\det T| = 1$

$$
\mathrm{tr}(TT^\intercal) \geq n\,\det(T\,T^\intercal)^{1/n} = n.
$$

Let K be in isotropic position, then $A_K = \mathrm{L}_K^2\,\mathrm{I}_n$ and in view (cf. Proposition 4.5) we get for any $T \in \mathrm{GL}(n,\mathbb{R})$, $|\det T| = 1$

$$
\begin{aligned}
\int_K \|\boldsymbol{x}\|^2 \,\mathrm{d}\,\boldsymbol{x} &= n\,\mathrm{L}_K^2 \leq \mathrm{tr}(TT^\intercal)\,\mathrm{L}_K^2 = \mathrm{tr}(T\,A_K\,T^\intercal) \\
&= \mathrm{tr}(A_{TK}) = \int_{TK} \|\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

For the reverse direction let $B \in \mathbb{R}^{n\times n}$. Then

$$
\det(\mathrm{I}_n + \epsilon\,B) = 1 + \epsilon\,\mathrm{tr}(B) + \cdots + \epsilon^n \det B.
$$

For all sufficiently small $|\epsilon|$ we have $\det(\mathrm{I}_n + \epsilon\,B) \neq 0$, and due to our assumption we find

$$
\begin{aligned}
\int_K \|\boldsymbol{x}\|^2 \,\mathrm{d}\,\boldsymbol{x} &\leq \int_{\left(\frac{1}{\det(\mathrm{I}_n+\epsilon\,B)}\right)^{1/n}(\mathrm{I}_n+\epsilon\,B)\,K} \|\boldsymbol{x}\|^2 \,\mathrm{d}\,\boldsymbol{x} \\
&= |\det(\mathrm{I}_n + \epsilon\,B)|^{\frac{-2}{n}} \int_K \|\boldsymbol{x} + \epsilon B\boldsymbol{x}\|^2 \,\mathrm{d}\,\boldsymbol{x}.
\end{aligned}
$$

Or equivalently,

$$
\begin{aligned}
|\det(\mathrm{I}_n + \epsilon\,B)|^{\frac{2}{n}} \int_K \|\boldsymbol{x}\|^2 \,\mathrm{d}\,\boldsymbol{x} &\leq \int_K \|\boldsymbol{x} + \epsilon B\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x} \\
&= \int_K \|\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x} + 2\epsilon \int_K \langle \boldsymbol{x}, B\boldsymbol{x}\rangle \,\mathrm{d}\boldsymbol{x} + \epsilon^2 \int_K \|B\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x}.
\end{aligned}
\tag{4.11.1}
$$

Looking at the Taylor series of $|\det(\mathrm{I}_n + \epsilon\,B)|^{\frac{2}{n}}$ at $\epsilon = 0$ gives

$$
|\det(\mathrm{I}_n + \epsilon\,B)|^{\frac{2}{n}} = 1 + \epsilon\,\frac{2}{n}\mathrm{tr}B + O(\epsilon^2)
$$

for $\epsilon \to 0$, and hence (4.11.1) yields

$$\frac{\operatorname{tr} B}{n} \int_K \|\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x} \leq \int_K \langle \boldsymbol{x}, B\boldsymbol{x}\rangle \,\mathrm{d}\boldsymbol{x}.$$

Replacing B by $-B$ shows that we must equality in the inequality above and, especially, for $B = \boldsymbol{e}_i \boldsymbol{e}_j^\intercal$ we get

$$\frac{\delta_{i,j}}{n} \int_K \|\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x} = \int_K x_i\, x_j \,\mathrm{d}\boldsymbol{x} = (A_K)_{i,j}.$$

Hence, the moment matrix A_K is a multiple of I_n which means that K is in isotropic position. □

4.12 Corollary. *Let $K \in \mathcal{K}^n_c \cap \mathcal{K}^n_o$. Then $\mathrm{L}_K \leq c\sqrt{n}$ for an universal constant c.*

Proof. Let K be in isotropic position. Let $T \in \mathrm{GL}(n, \mathbb{R})$, $|\det T| = 1$, such that the volume maximal ellipsoid in $T\,K$ is a ball of radius α, say. Observe that the volume maximal ellipsoid is centered at the origin, since $K \in \mathcal{K}^n_o$. By Corollary 6.16[37] we have $\alpha\, B_n \subseteq T\,K \subset \sqrt{n}\alpha\, B_n$ and thus with Theorem 4.11 we get

$$n\,\mathrm{L}_K^2 \leq \int_{T\,K} \|\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x} \leq \alpha^2 n.$$

Hence $\mathrm{L}_K \leq \alpha$. By the inclusion $\alpha\, B_n \subseteq T\,K$ we have $\alpha^n \leq \mathrm{vol}\,(T\,K)/\kappa_n$ and so

$$\alpha \leq \kappa_n^{-1/n} \leq c\sqrt{n},$$

for a suitable constant c. Hence we have found the bound $\mathrm{L}_K \leq c\sqrt{n}$. □

4.13 Conjecture [Isotropic Constant Conjecture]. *There exists an absolute constant $C > 0$ such that $\mathrm{L}_K \leq C$ for any $K \in \mathcal{K}^n_o$.*[38]

4.14 Proposition. *Let $K \in \mathcal{K}^n_o$. Then (cf. Notation 4.8)*

$$\mathrm{L}_K^2 \cdot \mathrm{L}_{K^\star}^2 \leq c\,n \frac{1}{\mathrm{vol}\,(K)\mathrm{vol}\,(K^\star)} \int_K \int_{K^\star} \langle \boldsymbol{y}, \boldsymbol{x}\rangle^2 \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}.$$

In particular, $\mathrm{L}_K \cdot \mathrm{L}_{K^\star} \leq c\sqrt{n}$.[39]

Proof. Let

$$\Phi(K) = \frac{1}{\mathrm{vol}\,(K)\mathrm{vol}\,(K^\star)} \int_K \int_{K^\star} \langle \boldsymbol{y}, \boldsymbol{x}\rangle^2 \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}.$$

[37] Skript WS14

[38] Bourgain, 1991 proved an upper bound of order $n^{1/4}\log(n)$ for o-symmetric bodies, which also holds true for arbitrary bodies (Paouris, 2000). The current general best bound is of size $n^{1/4}$ due to Klartag, 2006

[39] Here c is always an absolute constant, which may vary from statement to statement.

Oberve, that $\Phi(K) \leq 1$. For every $T \in \mathrm{GL}(n,\mathbb{R})$ we have $(TK)^\star = T^{-\intercal}K^\star$ and thus $\Phi(K) = \Phi(TK)$. Hence for the proof we may assume that $K^\star$ is in isotropic position. Then

$$\begin{aligned}\left(\frac{1}{\mathrm{vol}\,(K)}\right)^{\frac{n+2}{n}} \int_K \int_{K^\star} \langle \boldsymbol{y}, \boldsymbol{x}\rangle^2 \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} &= \left(\frac{1}{\mathrm{vol}\,(K)}\right)^{\frac{n+2}{n}} \mathrm{L}_{K^\star}^2 \int_K \|\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x} \\ &= \mathrm{L}_{K^\star}^2 \int_{K/\mathrm{vol}\,(K)^{1/n}} \|\boldsymbol{x}\|^2 \,\mathrm{d}\boldsymbol{x} \\ &\geq n\mathrm{L}_{K^\star}^2\mathrm{L}_K^2,\end{aligned}$$

by Theorem 4.11. On account of $\mathrm{vol}\,(K^\star) = 1$ we may write

$$n\,\mathrm{L}_{K^\star}^2\mathrm{L}_K^2 \leq \left(\frac{1}{\mathrm{vol}\,(K)\mathrm{vol}\,(K^\star)}\right)^{\frac{2}{n}} \Phi(K).$$

By [40] we know $\mathrm{vol}\,(K)\mathrm{vol}\,(K^\star) \geq c^n/n!$ and thus $n\,\mathrm{L}_{K^\star}^2\mathrm{L}_K^2 \leq c\,n^2\Phi(K)$.

□

4.15 Theorem. *Let $K \in \mathcal{K}_c^n$ be in isotropic position. Then $K \subset (n+1)\,\mathrm{L}_K\, B_n$.* [41]

Proof. For $\boldsymbol{w} \in K$ we consider the local radial function $\rho_{\boldsymbol{w}} : S^{n-1} \to \mathbb{R}_{\geq 0}$ with $\rho_{\boldsymbol{w}}(\boldsymbol{v}) = \max\{\rho \geq 0 : \boldsymbol{w} + \rho\,\boldsymbol{v} \in K\}$. Let $\boldsymbol{u} \in S^{n-1}$. Then

$$\begin{aligned}\mathrm{L}_K^2 &= \int_K \langle \boldsymbol{u}, \boldsymbol{x}\rangle^2 \,\mathrm{d}\boldsymbol{x} \\ &= n\,\kappa_n \int_{S^{n-1}} \int_0^{\rho_{\boldsymbol{w}}(\boldsymbol{v})} t^{n-1} \langle \boldsymbol{u}, \boldsymbol{w} + t\,\boldsymbol{v}\rangle^2 \,\mathrm{d}t\,\mathrm{d}\sigma(\boldsymbol{v}) \\ &= n\,\kappa_n \int_{S^{n-1}} \int_0^{\rho_{\boldsymbol{w}}(\boldsymbol{v})} \Big(t^{n-1} \langle \boldsymbol{u}, \boldsymbol{w}\rangle^2 + 2\,t^n \langle \boldsymbol{u}, \boldsymbol{v}\rangle \langle \boldsymbol{u}, \boldsymbol{w}\rangle \\ &\qquad + t^{n+1} \langle \boldsymbol{u}, \boldsymbol{v}\rangle^2\Big) \,\mathrm{d}t\,\mathrm{d}\sigma(\boldsymbol{v}) \\ &= n\,\kappa_n \int_{S^{n-1}} \Big(\frac{\rho_{\boldsymbol{w}}(\boldsymbol{v})^n}{n} \langle \boldsymbol{u}, \boldsymbol{w}\rangle^2 + 2\frac{\rho_{\boldsymbol{w}}(\boldsymbol{v})^{n+1}}{n+1} \langle \boldsymbol{u}, \boldsymbol{v}\rangle \langle \boldsymbol{u}, \boldsymbol{w}\rangle \\ &\qquad + \frac{\rho_{\boldsymbol{w}}(\boldsymbol{v})^{n+2}}{n+2} \langle \boldsymbol{u}, \boldsymbol{v}\rangle^2\Big) \,\mathrm{d}\sigma(\boldsymbol{v}) \\ &= n\,\kappa_n \int_{S^{n-1}} \Big(\frac{1}{n(n+1)^2}\rho_{\boldsymbol{w}}(\boldsymbol{v})^n \langle \boldsymbol{u}, \boldsymbol{w}\rangle^2 + \frac{\rho_{\boldsymbol{w}}(\boldsymbol{v})^n}{n} \times \\ &\qquad \left[\sqrt{\frac{n}{n+2}}\rho_{\boldsymbol{w}}(\boldsymbol{v}) \langle \boldsymbol{u}, \boldsymbol{v}\rangle + \frac{\sqrt{n(n+2)}}{n+1} \langle \boldsymbol{u}, \boldsymbol{w}\rangle\right]^2\Big) \,\mathrm{d}\sigma(\boldsymbol{v}) \\ &\geq \frac{\langle \boldsymbol{u}, \boldsymbol{w}\rangle^2}{(n+1)^2}\,\kappa_n \int_{S^{n-1}} \rho_{\boldsymbol{w}}(\boldsymbol{v})^n \,\mathrm{d}\sigma(\boldsymbol{v}) = \frac{\langle \boldsymbol{u}, \boldsymbol{w}\rangle^2}{(n+1)^2},\end{aligned}$$

[40] Bourgain& Milman proved in 1987 that there exists an absolute constant c such that $\mathrm{vol}\,(K)\mathrm{vol}\,(K^\star) \geq c^n/n!$ for every $K \in \mathcal{K}_o^n$. See also Wikipedia Mahler volume.

[41] Kannan, Lovasz, Simonovits proved for an isotropic body $(n+2)/n\,\mathrm{L}_K\, B_n \subset K \subset \sqrt{n(n+2)}\,\mathrm{L}_K\, B_n$, which are best possible.

where the last identity comes from $\mathrm{vol}(K) = 1$ and Proposition 3.8 ii). Replacing $\boldsymbol{u}$ by $\boldsymbol{w}/\|\boldsymbol{w}\|$ shows that $\|\boldsymbol{w}\|^2 \leq (n+1)^2 \mathrm{L}_K^2$ for all $\boldsymbol{w} \in K$. □

4.16 Lemma. *Let $K \in \mathcal{K}^n$ be centered. Let $\boldsymbol{a} \in S^{n-1}$, and for $t \in \mathbb{R}$ let $f(t) = \mathrm{vol}_{n-1}(K \cap H(\boldsymbol{a}, t))$. Then*

$$\max\{f(t) : t \in \mathbb{R}\} \leq \left(\frac{n+1}{n}\right)^{n-1} f(0).$$

Observe that $\left(\frac{n+1}{n}\right)^{n-1} \to \mathrm{e}$ as n to infinity.

Proof. Let $H(\boldsymbol{a}, t) \cap K \neq \emptyset$ for $t \in [-a, b]$, say, with $a, b > 0$. Let $t^\star$ such that $f(t^\star) = \max\{f(t) : t \in [-a, b]\}$. With loss of generality we may assume $f(0) < f(t^\star)$ and $t^\star > 0$. For if $f(t^\star) = f(0)$ we are done and if $t^\star < 0$ we replace $\boldsymbol{a}$ by $-\boldsymbol{a}$. Since $\mathbf{0}$ is the centroid we know $0 = \int_{-a}^{b} t\, f(t) \mathrm{d}t$. Hence

$$\int_{-a}^{0} (-t)\, f(t)\, \mathrm{d}t = \int_{0}^{b} t\, f(t)\, \mathrm{d}t \geq \int_{0}^{t^\star} t\, f(t)\, \mathrm{d}t. \tag{4.16.1}$$

Now let $h : [-a, b] \to \mathbb{R}_{\geq 0}$ be the concave function $h(t) = f(t)^{\frac{1}{n-1}}$, and let $g : [-a, b] \to \mathbb{R}_{\geq 0}$ be the linear function

$$g(t) = h(0) + \frac{t}{t^\star}\left(h(t^\star) - h(0)\right).$$

Then $g(0) = h(0)$ and $g(t^\star) = h(t^\star)$ and due to the concavity of h we conclude

$$g(t) \geq h(t), \quad t \in [-a, 0] \text{ and } g(t) \leq h(t), \quad t \in [0, t^\star].$$

Let $g(-c) = 0$ for some $c \geq a$, and let $g(t) = \alpha(t + c)$ with $\alpha > 0$. Then with (4.16.1)

$$\begin{aligned}\int_{-c}^{0} (-t)\, g(t)^{n-1}\, \mathrm{d}t &\geq \int_{-a}^{0} (-t)\, g(t)^{n-1}\, \mathrm{d}t \geq \int_{-a}^{0} (-t)\, h(t)^{n-1}\, \mathrm{d}t \\ &\geq \int_{0}^{t^\star} t\, h(t)^{n-1}\, \mathrm{d}t \geq \int_{0}^{t^\star} t\, g(t)^{n-1}\, \mathrm{d}t.\end{aligned}$$

Thus

$$\begin{aligned}0 &\geq \int_{-c}^{t^\star} t\, g(t)^{n-1}\, \mathrm{d}t = \alpha^{n-1} \int_{-c}^{t^\star} t(t+c)^{n-1}\, \mathrm{d}t \\ &= \alpha^{n-1} \int_{0}^{t^\star + c} (t - c) t^{n-1}\, \mathrm{d}t = \alpha^{n-1} \left(\frac{1}{n+1} t^{n+1} - \frac{c}{n} t^n\right)\Big|_0^{t^\star + c} \\ &= \alpha^{n-1} (t^\star + c)^n \left(\frac{t^\star + c}{n+1} - \frac{c}{n}\right).\end{aligned}$$

Hence $(t^\star + c)/c \leq (n+1)/n$, and so

$$\frac{f(t^\star)}{f(0)} = \frac{h(t^\star)^{n-1}}{h(0)^{n-1}} = \left(\frac{g(t^\star)}{g(0)}\right)^{n-1} = \left(\frac{t^\star + c}{c}\right)^{n-1} \leq \left(\frac{n+1}{n}\right)^{n-1}.$$

□

4.17 Theorem. *There exists absolute constants c_1, c_2 such that for every $K \in \mathcal{K}_c^n$ and for every $\boldsymbol{u} \in S^{n-1}$*

$$c_1 \frac{1}{\mathrm{vol}_{n-1}(K \cap H(\boldsymbol{u}, 0))} \leq \left(\int_K (\boldsymbol{u}^\intercal \boldsymbol{x})^2 \mathrm{d}\, \boldsymbol{x} \right)^{1/2} \leq c_2 \frac{1}{\mathrm{vol}_{n-1}(K \cap H(\boldsymbol{u}, 0))}.$$

Proof. Let $\boldsymbol{u} \in S^{n-1}$ and for $t \in \mathbb{R}$ let $f(t) = \mathrm{vol}_{n-1}(K \cap H(\boldsymbol{u}, t))$. Here $H(\boldsymbol{u}, t) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}, \boldsymbol{x} \rangle = t\}$. Moreover, let $t^\star \in \mathbb{R}$ such that $f(t^\star) = \max\{f(t) : t \in \mathbb{R}\}$ and let

$$A = \int_{-\infty}^0 f(t)\, \mathrm{d}t \text{ and } B = \int_0^\infty f(t)\, \mathrm{d}t.$$

We note that $1 = \mathrm{vol}\,(K) = A + B$ and

$$\int_K \langle \boldsymbol{u}, \boldsymbol{x} \rangle^2 \, \mathrm{d}\boldsymbol{x} = \int_{-\infty}^\infty t^2 f(t)\, \mathrm{d}t = \int_{-\infty}^0 t^2 f(t)\, \mathrm{d}t + \int_0^\infty t^2 f(t)\, \mathrm{d}t. \qquad (4.17.1)$$

First we prove the lower bound. To this end let $g(t) = f(t^\star)\chi_{[0,B/f(t^\star)]}$. Then we certainly have $f(t) \leq g(t)$ for all $t \in [0, B/f(t^\star)]$ and so

$$\int_0^s f(t)\, \mathrm{d}t \leq \int_0^s g(t)\, \mathrm{d}t$$

for all $s \in [0, B/f(t^\star)]$. Since we also have

$$\int_0^\infty g(t)\, \mathrm{d}t = B = \int_0^\infty f(t)\, \mathrm{d}t$$

we conclude

$$\int_s^\infty f(t)\, \mathrm{d}t \geq \int_s^\infty g(t)\, \mathrm{d}t$$

for all $s \geq 0$. Hence

$$\begin{aligned} \int_0^\infty t^2 f(t)\, \mathrm{d}t &= \int_0^\infty \int_0^t 2\, s\, f(t)\, \mathrm{d}s\, \mathrm{d}t = \int_0^\infty \left(\int_s^\infty 2\, s\, f(t)\, \mathrm{d}t \right) \mathrm{d}s \\ &\geq \int_0^\infty 2s \left(\int_s^\infty g(t)\, \mathrm{d}t \right) \mathrm{d}s = \int_0^\infty t^2 g(t)\, \mathrm{d}t \\ &= \int_0^{B/f(t^\star)} t^2\, f(t^\star)\, \mathrm{d}t = \frac{B^3}{3} \frac{1}{f(t^\star)^2}. \end{aligned}$$

In the same way it can be shown

$$\int_{-\infty}^0 t^2 f(t)\, \mathrm{d}t \geq \frac{A^3}{3} \frac{1}{f(t^\star)^2}.$$

Thus, in view of (4.17.1) we have found

$$\int_K \langle \boldsymbol{u}, \boldsymbol{x} \rangle^2 \, \mathrm{d}\boldsymbol{x} \geq \frac{A^3 + B^3}{3 f(t^\star)^2}.$$

Since $A + B = \mathrm{vol}\,(K) = 1$ we conclude $A^3 + B^3 \geq 1/8 + 1/8 = 1/4$, and due to Lemma 4.16 we also know $f(t^\star) \leq \mathrm{e} f(0)$. Therefore,

$$\int_K \langle \boldsymbol{u}, \boldsymbol{x} \rangle^2 \, \mathrm{d}\boldsymbol{x} \geq \frac{1}{12\,\mathrm{e}} \frac{1}{f(0)^2},$$

which shows the lower bound. For the upper bound we distinguish two cases. Firstly we assume that there exists an $\bar{t} > 0$ with the property that

$$f(\bar{t}) = f(0)/2. \tag{4.17.2}$$

By the Brunn-Minkowski inequality the function $f(t)$ is log-concave and so for every $\lambda \in [0, 1]$ we find

$$f(\lambda\,\bar{t}) = f((1-\lambda)\,0 + \lambda\,\bar{t}) \geq f(0)^{1-\lambda} f(\bar{t})^\lambda \geq f(\bar{t}).$$

Hence $f(t) \geq f(\bar{t})$ for all $t \in [0, \bar{t}]$ and so

$$1 \geq B = \int_0^{\bar{t}} f(t)\,\mathrm{d}t \geq \bar{t} f(\bar{t}) = \bar{t}\frac{f(0)}{2}.$$

This shows

$$\bar{t} \leq 2\frac{1}{f(0)}. \tag{4.17.3}$$

Now, for $t > \bar{t}$ we have

$$f(\bar{t}) = f\left((\bar{t}/t)t\right) = f((1-\bar{t}/t)0 + (\bar{t}/t)t) \geq f(0)^{(1-\bar{t}/t)} f(t)^{\bar{t}/t}$$

Hence

$$f(t) \leq \left(f(\bar{t})\, f(0)^{(\bar{t}/t-1)}\right)^{\frac{t}{\bar{t}}} = \left(\frac{1}{2}\right)^{\frac{t}{\bar{t}}} f(0). \tag{4.17.4}$$

Using these bounds as well as Lemma 4.16 leads to

$$\begin{aligned}
\int_0^\infty t^2 f(t)\,\mathrm{d}t &= \int_0^{\bar{t}} t^2 f(t)\,\mathrm{d}t + \int_{\bar{t}}^\infty t^2 f(t)\,\mathrm{d}t \\
&\leq f(t^\star) \int_0^{\bar{t}} t^2\,\mathrm{d}t \overset{(4.17.4)}{+} f(0) \int_{\bar{t}}^\infty t^2\, 2^{-t/\bar{t}}\,\mathrm{d}t \\
&\overset{\text{Lemma 4.16}}{\leq} f(0) \left(\mathrm{e} \int_0^{\bar{t}} t^2\,\mathrm{d}t + \int_{\bar{t}}^\infty t^2\, 2^{-t/\bar{t}}\,\mathrm{d}t\right) \\
&= f(0) \left(\mathrm{e}\frac{\bar{t}^3}{3} \overset{u=t/\bar{t}}{+} \bar{t}^3 \int_1^\infty u^2 2^{-u}\,\mathrm{d}u\right) \\
&= f(0)\,\bar{t}^3 \cdot c \overset{(4.17.3)}{\leq} 8\,c\frac{1}{f(0)^2}.
\end{aligned} \tag{4.17.5}$$

Now assume the assumption (4.17.2) does not hold. Then by the continuity of f in the interior of its support we know $f(t) > f(0)/2$ for all $t > 0$ for which

$f(t) \neq 0$. Let $\widehat{t} = \sup\{t > 0 : f(t) \neq 0\}$. Then $1 \geq B = \int_0^{\widehat{t}} f(t)\,\mathrm{d}t \geq \widehat{t} f(0)/2$ and so

$$\widehat{t} \leq 2\frac{1}{f(0)}$$

and so

$$\int_0^{\infty} t^2 f(t)\,\mathrm{d}t = \int_0^{\widehat{t}} t^2 f(t)\,\mathrm{d}t \overset{\text{Lemma 4.16}}{\leq} \mathrm{e}\frac{\widehat{t}^3}{3} f(0) \leq \frac{8\mathrm{e}}{3}\frac{1}{f(0)^2}.$$

Together with (4.17.5) we have shown that there exists a constant $\overline{c}$, say, such that

$$\int_0^{\infty} t^2\, f(t)\,\mathrm{d}t \leq \overline{c}\frac{1}{f(0)^2}.$$

Via the same argumentation we can bound the integral on the negative x-axis, i.e., $\int_{-\infty}^0 t^2\, f(t)\,\mathrm{d}t$ and we are done (cf.(4.17.1)). □

4.18 Corollary. *There exists absolute constants c_1, c_2 such that for every $K \in \mathcal{K}_c^n$ in isotropic position and for every $\boldsymbol{u} \in S^{n-1}$*

$$\frac{c_1}{\mathrm{L}_K} \leq \mathrm{vol}_{n-1}(K \cap H(\boldsymbol{u}, 0)) \leq \frac{c_2}{\mathrm{L}_K}.$$

Proof. Immediate consequence of Theorem 4.17. □

4.19 Conjecture [Slicing conjecture]. *There exists an absolute constant $c > 0$ with the property that for every $K \in \mathcal{K}_o^n$ with $\mathrm{vol}(K) = 1$ there exits an $\boldsymbol{v} \in S^{n-1}$ such that*

$$\mathrm{vol}_{n-1}(K \cap H(\boldsymbol{v}, 0)) \geq c.$$

4.20 Remark. *The Slicing conjecture 4.19 is equivalent to the Isotropic constant conjecture 4.13*

Proof. If the Slicing conjecture is true we get from the upper bound in Corollary 4.18 that the isotropic constant is bounded from above by an absolute constant. On the other hand, if the Isotropic constant conjecture holds true then we know by Proposition 4.9 that for every $K \in \mathcal{K}_o^n$ there exists a $\hat{\boldsymbol{u}} \in S^{n-1}$ that $\int_K \langle \hat{\boldsymbol{u}}, \boldsymbol{x}\rangle^2\,\mathrm{d}\boldsymbol{x} \leq L_K^2 \leq c^2$. Hence from the lower bound in Theorem 4.17 we get $\mathrm{vol}_{n-1}(K \cap H(\hat{\boldsymbol{u}}, 0)) \geq c_1/c$. □

4.21 Notation. [42] *For $K \in \mathcal{K}^n$ with $\mathrm{vol}(K) = 1$ let*

$$S(K) = \int_K \cdots \int_K \mathrm{vol}\,(\mathrm{conv}\,\{\boldsymbol{0}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\})^2\,\mathrm{d}\boldsymbol{x}_n \ldots \mathrm{d}\boldsymbol{x}_1.$$

[42] The Sylvester problem asks for the convex bodies $K \in \mathcal{K}^n$, $\mathrm{vol}(K) = 1$, for which $\int_K \cdots \int_K \mathrm{vol}\,(\mathrm{conv}\,\{\boldsymbol{0}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \boldsymbol{x}_{n+1}\})\,\mathrm{d}\boldsymbol{x}_{n+1} \ldots \mathrm{d}\boldsymbol{x}_1$ is minimized or maximized. It is known to be minimized for a ball and it is conjectured that it is maximized for a simplex.

4.22 Theorem. *Let $K \in \mathcal{K}^n$ with* $\mathrm{vol}\,(K) = 1$. *Then*

$$n!\,S(K) = \det(A_K) = \mathrm{L}_K^{2n}.$$

Proof. It is $\mathrm{vol}\,(\mathrm{conv}\,\{\mathbf{0}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}) = \frac{1}{n!}|\det(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)|$, and so

$$\begin{aligned}
n!^2\,S(K) &= \int_K \cdots \int_K |\det(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)|^2 \,\mathrm{d}\boldsymbol{x}_n \cdots \mathrm{d}\boldsymbol{x}_1 \\
&= \int_K \cdots \int_K \left(\sum_{\sigma \in S_n} \mathrm{sgn}\,\sigma \prod_{i=1}^n (\boldsymbol{x}_i)_{\sigma(i)} \right) \left(\sum_{\tau \in S_n} \mathrm{sgn}\,\tau \prod_{i=1}^n (\boldsymbol{x}_i)_{\tau(i)} \right) \mathrm{d}\boldsymbol{x}_n \cdots \mathrm{d}\boldsymbol{x}_1 \\
&= \int_K \cdots \int_K \left(\sum_{\sigma,\tau \in S_n} \mathrm{sgn}\,\sigma \mathrm{sgn}\,\tau \prod_{i=1}^n (\boldsymbol{x}_i)_{\sigma(i)} (\boldsymbol{x}_i)_{\tau(i)} \right) \mathrm{d}\boldsymbol{x}_n \cdots \mathrm{d}\boldsymbol{x}_1 \\
&= \int_K \cdots \int_K \left(\sum_{\sigma,\tau \in S_n} \mathrm{sgn}\,\tau \prod_{i=1}^n (\boldsymbol{x}_i)_{\sigma(i)} (\boldsymbol{x}_i)_{\tau(\sigma(i))} \right) \mathrm{d}\boldsymbol{x}_n \cdots \mathrm{d}\boldsymbol{x}_1 \\
&= \sum_{\sigma,\tau \in S_n} \mathrm{sgn}\,\tau \left(\prod_{i=1}^n \int_K (\boldsymbol{x}_i)_{\sigma(i)} (\boldsymbol{x}_i)_{\tau(\sigma(i))} \,\mathrm{d}\boldsymbol{x}_i \right) \\
&= \sum_{\sigma,\tau \in S_n} \mathrm{sgn}\,\tau \left(\prod_{i=1}^n \int_K y_{\sigma(i)} y_{\tau(\sigma(i))} \,\mathrm{d}\boldsymbol{y} \right) \\
&= \sum_{\sigma,\tau \in S_n} \mathrm{sgn}\,\tau \left(\prod_{i=1}^n \int_K y_i\, y_{\tau(i)} \,\mathrm{d}\boldsymbol{y} \right) \\
&= n! \sum_{\tau \in S_n} \mathrm{sgn}\,\tau \prod_{i=1}^n a_{i,\tau(i)} = n! \det A_K.
\end{aligned}$$

By Proposition 4.9 we also know $\det A_K = \mathrm{L}_K^{2n}$. □

4.23 Corollary. *Let $K \in \mathcal{K}^n$ with centroid* $\mathbf{0}$. *Then $\mathrm{L}_K \leq c\sqrt{n}$, where c is an absoulte constant (cf. Corollary 4.12).*

Proof. By Theorem 4.22 we have $\mathrm{L}_K^{2n} \leq n!$ and thus $\mathrm{L}_K \leq \sqrt[2n]{n!} \leq c\sqrt{n}$. □

4.24 Definition. *$K \in \mathcal{K}^n$ is called* unconditional *if it is symmetric to each coordinate hyperplane, i.e., $(\pm x_1, \ldots, \pm x_n)^\intercal \in K$ for all $\boldsymbol{x} \in K$.*

4.25 Proposition. *Let $K \in \mathcal{K}^n$ be unconditional.*

i) $\int_K x_i x_j \,\mathrm{d}\boldsymbol{x} = 0$ *for all* $1 \leq i \neq j \leq n$.

ii) *For each coordinate hyperplane we have $K \cap H = K|H$.*

iii) *If $\mathrm{vol}\,(K) = 1$ then there exists a diagonal matrix $T \in \mathrm{VP}(n, \mathbb{R})$ such TK is in isotropic position.*

4.26 Theorem. *There exists an absolute constant c such that $\mathrm{L}_K \leq c$ for all unconditional convex bodies $K \in \mathcal{K}^n$.*

Proof. According to Proposition 4.25 there exists a diagonalmatrix T such that $TK \in \mathcal{K}_c^n$ and TK is in isotropic position. Since T is a diagonalmatrix, TK is still an unconditional body.

For $1 \leq i \leq n$ let H_i be the coordinate hyperplanes. By Corollary 4.18 there exists an absolute constant c_2 such that for $1 \leq i \leq n$

$$\mathrm{L}_k \leq c_2 \frac{1}{\mathrm{vol}_{n-1}(TK \cap H_i)} = c_2 \frac{1}{\mathrm{vol}_{n-1}(TK|H_i)}, \tag{4.26.1}$$

where for the last identity we have used Proposition 4.25 ii). By Corollary 3.16 we know that

$$1 = \mathrm{vol}\,(TK)^{n-1} \leq \prod_{i=1}^{n} \mathrm{vol}_{n-1}(TK|H_i),$$

and thus there exists and $i^\star \in \{1, \ldots, n\}$ with $\mathrm{vol}_{n-1}(TK|H_{i^\star}) \geq 1$. Hence, for that $i^\star$, (4.26.1) gives the desired bound. □

5 Almost spherical

5.1 Definition [Banach-Mazur distance]. *Let $K, L \in \mathcal{K}^n$ be n-dimensional convex bodies. Then*

$$\mathrm{d_{BM}}(K,L) = \inf\bigl\{\lambda > 0 : \exists\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n,\, T \in \mathrm{GL}(n,\mathbb{R}) \text{ with } \boldsymbol{y} + K \subseteq T(\boldsymbol{x} + L) \subseteq \lambda(\boldsymbol{y} + K)\bigr\}$$

is called the Banach-Mazur *distance of K and L .*

5.2 Remark.

i) *If $K, L \in \mathcal{K}^n_o$ then we can assume $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{0}$, and $\mathrm{d_{BM}}(K,L)$ may be interpreted as a distance between norms defined by K and L.*

ii) *Let $K, L, M \in \mathcal{K}^n$. Then $\mathrm{d_{BM}}(K,L) = \mathrm{d_{BM}}(L,K)$, $\mathrm{d_{BM}}(K,L) = \mathrm{d_{BM}}(\boldsymbol{x} + TK, \boldsymbol{y} + \tilde{T}L)$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $T, \tilde{T} \in \mathrm{GL}(n,\mathbb{R})$.*

iii) *$\mathrm{d_{BM}}$ verifies the multiplicative triangular inequality, i.e.,*

$$\mathrm{d_{BM}}(K,M) \leq \mathrm{d_{BM}}(K,L)\, \mathrm{d_{BM}}(L,M).$$

5.3 Proposition.

i) *$\mathrm{d_{BM}}(K,B_n) \leq \sqrt{n}$ for all $K \in \mathcal{K}^n_o$, and $\mathrm{d_{BM}}(C_n,B_n) = \mathrm{d_{BM}}(C_n^\star,B_n) = \sqrt{n}$.*

ii) *$\mathrm{d_{BM}}(K,B_n) \leq n$ for all $K \in \mathcal{K}^n$, and $\mathrm{d_{BM}}(S_n,B_n) = n$ for an n-simplex S_n.*

iii) *$\mathrm{d_{BM}}(K,L) \leq n$ for all $K, L \in \mathcal{K}^n_o$, and $\mathrm{d_{BM}}(K,L) \leq n^2$ for all $K, L \in \mathcal{K}^n$.*

Proof. This is Proposition 6.21[43] .[44] □

5.4 Theorem*. *Let $K \in \mathcal{K}^n$ with $\mathrm{d_{BM}}(K,B_n) = n$. Then K is a simplex.*

5.5 Definition [Spherical cap]. *Let $\boldsymbol{v} \in S^{n-1}$. For $\varepsilon \in [-1,1]$ the set*

$$C(\varepsilon,\boldsymbol{v}) = \bigl\{\boldsymbol{u} \in S^{n-1} : \langle \boldsymbol{v},\boldsymbol{u}\rangle \geq \varepsilon\bigr\} = \bigl\{\boldsymbol{u} \in S^{n-1} : \|\boldsymbol{u} - \boldsymbol{v}\| \leq \sqrt{2 - 2\,\varepsilon}\bigr\}$$

is called an ε-cap or a spherical cap of radius $r = \sqrt{2 - 2\varepsilon}$, $r \in [0, \sqrt{2}]$.

5.6 Notation. *In the following we will denote for a measurable set $A \subset S^{n-1}$ by*

$$\mu(A) = \int_{S^{n-1}} \chi_A(\boldsymbol{u})\, \mathrm{d}\mu(\boldsymbol{u})$$

its normalized surface are measure (cf. Proposition 3.8 i)).

[43] Skript WS14

[44] It was shown by Gluskin (1981) that there exists $K, L \in \mathcal{K}^n_o$ such that $\mathrm{d_{BM}}(K,L) \geq c\,n$, where c is an absolute constant. In the general case, Rudelson (2001) proved an upper bound of order $n^{4/3} \log^9(n)$. See also Banach-Mazur compactum.

5.7 Lemma. *Let $\boldsymbol{v} \in S^{n-1}$ and $\varepsilon \in (0,1)$. Then*

$$\frac{1}{2}\left(\frac{\sqrt{2-2\varepsilon}}{2}\right)^{n-1} \leq \mu\big(C(\varepsilon, \boldsymbol{v})\big) \leq \mathrm{e}^{-n\varepsilon^2/2}.$$

Proof. We set $\overline{C}(\varepsilon, \boldsymbol{v}) = \operatorname{conv}\big\{\mathbf{0}, C(\varepsilon, \boldsymbol{v})\big\}$, and since in the following $\boldsymbol{v}$ will be a fixed unit vector we will just write $C(\varepsilon)$ and $\overline{C}(\varepsilon)$, respectively. We start with the upper bound. Since (cf. Proposition 3.8 ii))

$$\operatorname{vol}\big(\overline{C}(\varepsilon)\big) = \kappa_n\, \mu\big(C(\varepsilon)\big),$$

it suffices to bound $\operatorname{vol}\big(\overline{C}(\varepsilon)\big)$. The diameter of $\overline{C}(\varepsilon)$, denoted by $\mathrm{D}(\overline{C}(\varepsilon))$ is given either by the diameter of the $(n-1)$-dimensional ball determined by the cap $C(\varepsilon)$, i.e., by $2\sqrt{1-\varepsilon^2}$ or by $\|\boldsymbol{v}\| = 1$. Hence

$$\mathrm{D}(\overline{C}(\varepsilon)) = \begin{cases} 2\sqrt{1-\varepsilon^2}, & \varepsilon \leq \sqrt{3}/2, \\ 1, & \text{otherwise.} \end{cases}$$

In the first case we get by the isodiametric inequality (see Proposition 1.7)

$$\mu\big(C(\varepsilon)\big) = \frac{\operatorname{vol}\big(\overline{C}(\varepsilon)\big)}{\kappa_n} \leq \frac{1}{2^n}\mathrm{D}\big(\overline{C}(\varepsilon)\big)^n = (1-\varepsilon^2)^{n/2} \leq \mathrm{e}^{-n\varepsilon^2/2},$$

where for the last inequality we used $\mathrm{e}^{-t} \geq 1-t$. In the same way we also obtain in the second case

$$\mu\big(C(\varepsilon)\big) \leq \frac{1}{2^n}\mathrm{D}\big(\overline{C}(\varepsilon)\big)^n = \frac{1}{2^n} \leq \mathrm{e}^{-n\varepsilon^2/2},$$

since $\mathrm{e}^{\varepsilon^2/2} \leq 2$ when $\varepsilon \leq 1$.

For the proof of the lower bound let $H = \big\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{x}, \boldsymbol{v}\rangle = \varepsilon\big\}$ and $B' = B_n \cap H$. Let $K = \operatorname{conv}\{B', \mathbf{0}, \boldsymbol{v}\}$. Then $K \subset \overline{C}(\varepsilon)$ and so

$$\mu\big(C(\varepsilon, v)\big) = \frac{\operatorname{vol}\big(\overline{C}(\varepsilon, v)\big)}{\kappa_n} \geq \frac{\operatorname{vol}(K)}{\kappa_n} = \frac{1}{n}\frac{\kappa_{n-1}}{\kappa_n}\big(\sqrt{1-\varepsilon^2}\big)^{n-1},$$

and it remains to verify

$$\frac{1}{n}\frac{\kappa_{n-1}}{\kappa_n}\big(\sqrt{1-\varepsilon^2}\big)^{n-1} \geq \frac{\big(\sqrt{2}\big)^{n-1}}{2^n}\big(\sqrt{1-\varepsilon}\big)^{n-1}.$$

For $\varepsilon < 1$ the inequality is equivalent to

$$\frac{n}{2^{(n+1)/2}}\frac{\kappa_n}{\kappa_{n-1}} \leq \sqrt{1+\varepsilon}^{\,n-1},$$

and we may assume that $\varepsilon = 0$. Now

$$\frac{n}{2^{(n+1)/2}}\frac{\kappa_n}{\kappa_{n-1}} = \frac{n}{2^{(n+1)/2}}\sqrt{\pi}\frac{\Gamma(n/2+1/2)}{\Gamma(n/2+1)} \leq \frac{n}{2^{(n+1)/2}}\sqrt{\pi} \leq 1,$$

for $n \geq 6$, and the remaining dimensions can easily be checked by a computer. □

5.8 Theorem. *Let $P \in \mathcal{P}^n$ be an o-symmetric n-polytope with $\mathrm{d_{BM}}(P, B_n) \leq \delta$. Then $f_{n-1}(P) \geq \mathrm{e}^{n/(2\delta^2)}$, i.e., P has at least $\mathrm{e}^{n/(2\delta^2)}$ facets.*

On the other hand, there is a polytope $P \in \mathcal{P}^n$ with 4^n facets such that $B_n \subset P \subset 2B_n$, and thus $\mathrm{d_{BM}}(P, B_n) \leq 2$.

Proof. Let $P = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{v}_i, \boldsymbol{x} \rangle \leq 1, 1 \leq i \leq k\}$ be a symmetric polytope with k facets and $\mathrm{d_{BM}}(P, B_n) = \delta$. We may assume that

$$B_n \subset P \subset \delta B_n.$$

Since $B_n \subset P$ we have $\boldsymbol{v}_i / \|\boldsymbol{v}_i\| \in P$ and thus $1 \geq \langle \boldsymbol{v}_i, \boldsymbol{v}_i / \|\boldsymbol{v}_i\| \rangle = \|\boldsymbol{v}_i\|$, $1 \leq i \leq k$. On the other hand since $P \subset \delta B_n$, we know that for $\boldsymbol{x} \in \delta S^{n-1}$ there exists an index $j \in \{1, \ldots, k\}$ with $\langle \boldsymbol{v}_j, \boldsymbol{x} \rangle \geq 1$. Therefore,

$$\left\langle \frac{\boldsymbol{v}_j}{\|\boldsymbol{v}_j\|}, \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \right\rangle \geq \frac{1}{\|\boldsymbol{v}_j\|} \frac{1}{\delta} \geq \frac{1}{\delta},$$

i.e., $\boldsymbol{x} / \|\boldsymbol{x}\| \in C\big(1/\delta, \boldsymbol{v}_j / \|\boldsymbol{v}_j\|\big)$. Hence

$$S^{n-1} = \bigcup_{i=1}^{k} C\big(1/\delta, \boldsymbol{v}_i / \|\boldsymbol{v}_i\|\big).$$

By Lemma 5.7 we get $\mu\big(C\big(1/\delta, \boldsymbol{v}_i / \|\boldsymbol{v}_i\|\big)\big) \leq \mathrm{e}^{-n/(2\delta^2)}$, and so $k \geq \mathrm{e}^{n/(2\delta^2)}$.

In order to prove the second assertion, we "construct" $k = 4^n$ points $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in S^{n-1}$ such that

$$S^{n-1} = \bigcup_{i=1}^{k} C(1/2, \boldsymbol{v}_i). \tag{5.8.1}$$

Then $P = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{v}_i, \boldsymbol{x} \rangle \leq 1, 1 \leq i \leq k\}$ is the required polytope: we certainly have $B_n \subset P$ and due to (5.26.2) we know for $\boldsymbol{x} \in \mathbb{R}^n$ there exist a $\boldsymbol{v}_j$ such that

$$\langle \boldsymbol{v}_j, \boldsymbol{x} / \|\boldsymbol{x}\| \rangle \geq 1/2, \quad \text{i.e.,} \quad \|\boldsymbol{x}\| \leq 2 \langle \boldsymbol{v}_j, \boldsymbol{x} \rangle.$$

Hence, if $\boldsymbol{x} \in P$ then $\|\boldsymbol{x}\| \leq 2$ and so $P \subset 2B_n$.

A set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \subset S^{n-1}$ with property (5.26.2) is usually called a 1-net, since 1/2-caps have radius 1. Now let $M \subset S^{n-1}$ be a set of maximal cardinality with the property $\|\boldsymbol{v} - \boldsymbol{w}\| \geq 1$ for any $\boldsymbol{v} \neq \boldsymbol{w} \in M$. Then M is a 1-net, and since the spherical caps of radius 1/2 centered at the points of M do not overlap, we get by Lemma 5.7 that

$$1 \geq \#M \cdot \frac{1}{2} \left(\frac{1}{4}\right)^{n-1}.$$

□

5.9 Corollary. *For the 4^n-dimensional cube C_{4^n} there exists an n-dimensional linear subspace L such that*

$$\mathrm{d_{BM}}\big(C_{4^n} \cap L, B_{4^n} \cap L\big) \leq 2.$$

Each linear subspace L fulfilling this bound has dimension at most $c\,n$, where c is an absolute constant. [45]

Proof. Let L be a k-dimensional linear subspace fulfilling the bound. Then $P = C_{4^n} \cap L$ is an o-symmetric k-dimensional polytope P with at most $2(4^n)$ facets. Since $\mathrm{d_{BM}}(P, B_{4^n} \cap L) \leq 2$, Theorem 5.8 leads to $\mathrm{e}^{k/8} \leq 2(4^n)$, and so there exists a constant c such that $k \leq c\,n$.

On the other hand, let $P = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{v}_i, \boldsymbol{x}\rangle \leq 1,\, 1 \leq i \leq 4^n\} \in \mathcal{P}^n$ be the polytope as in Theorem 5.8 with 4^n facets such that $B_n \subset P \subset 2B_n$. Let $\overline{P} = \{\boldsymbol{x} \in \mathbb{R}^n : \left|\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\right| \leq 1,\, 1 \leq i \leq 4^n\}$. Then $\overline{P}$ is o-symmetric and it still holds $B_n \subset \overline{P} \subset 2B_n$. Let $L \subset \mathbb{R}^{4^n}$ be the n-dimensional linear subspace given by $L = \{V\,\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n\}$ where $V \in \mathbb{R}^{4^n \times n}$ with rows $\boldsymbol{v}_1^\intercal, \ldots, \boldsymbol{v}_{4^n}^\intercal$. Then

$$\begin{aligned} L \cap C_{4^n} &= \{V\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n\} \cap \left\{\boldsymbol{x} \in \mathbb{R}^{4^n} : |x_i| \leq 1,\, 1 \leq i \leq 4^n\right\} \\ &= \left\{V\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n,\, \left|\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle\right| \leq 1,\, 1 \leq i \leq 4^n\right\} \\ &= V\overline{P}. \end{aligned}$$

and by the choice of P we know $\mathrm{d_{BM}}(C_{4^n} \cap L, B_{4^n} \cap L) \leq 2$. □

5.10 Proposition. *For every $(2n-1)$-dimensional ellipsoid $E \in \mathcal{K}_o^n$ there exists an n-dimensional linear subspace L such that $E \cap L$ is an Euclidean ball.*

Proof. After a suitable rotation we may assume

$$E = \left\{\boldsymbol{x} \in \mathbb{R}^{2n-1} : \sum_{i=1}^{2n-1} \frac{x_i^2}{\alpha_i^2} \leq 1\right\},$$

with $0 < \alpha_1 \leq \cdots \leq \alpha_{2n-1}$, and let

$$L = \left\{\boldsymbol{x} \in \mathbb{R}^{2n-1} : x_i\sqrt{\frac{1}{\alpha_i^2} - \frac{1}{\alpha_n^2}} = x_{2n-i}\sqrt{\frac{1}{\alpha_n^2} - \frac{1}{\alpha_{2n-i}^2}},\, i = 1, \ldots, n-1\right\}.$$

Then $\dim L \geq 2n-1-(n-1) = n$. For $\boldsymbol{x} \in L$ we have $x_i^2/\alpha_i^2 + x_{2n-i}^2/\alpha_{2n-i}^2 = (x_i^2 + x_{2n-i}^2)/\alpha_n^2$, $1 \leq i \leq n-1$, and hence, for $\boldsymbol{x} \in E \cap L$ we find

$$1 \geq \sum_{i=1}^{2n-1} \frac{x_i^2}{\alpha_i^2} = \sum_{i=1}^{n-1}\left(\frac{x_i^2}{\alpha_i^2} + \frac{x_{2n-i}^2}{\alpha_{2n-i}^2}\right) + \frac{x_n^2}{\alpha_n^2} = \frac{1}{\alpha_n^2}\sum_{i=1}^{2n-1} x_i^2.$$

Therefore, $\|\boldsymbol{x}\| \leq \alpha_n$ and so $E \cap L \subseteq \alpha_n(B_{2n-1} \cap L)$. The same argument shows that $\alpha_n(B_{2n-1} \cap L) \subseteq E \cap L$, and thus $E \cap L$ is an n-dimensional ball of radius α_n. □

[45] This property is also expressed by saying, that the 4^n-dimensional cube contains an almost 2-spherical section of dimension n.

5.11 Remark.

i) *Let* $\mathrm{O}(n)$ *be the* $n \times n$*-real orthogonal matrices. As a (locally) compact group it has a unique probability (Haar) measure* $\theta(\cdot)$*. Due to its uniqueness it is related to* $\mu(\cdot)$ *by*

$$\theta(A) = \mu(U\,\boldsymbol{v} : U \in A),$$

for $A \subseteq \mathrm{O}(n)$ *and a fixed, but arbitrary* $\boldsymbol{v} \in S^{n-1}$*.*

ii) *For* $1 \leq k \leq n$ *let* $\mathrm{Gr}(k,n)$ *be the* Grassmannian manifold *consisting of all* k*-dimensional linear spaces of* $\mathbb{R}^n$*. Via the Hausdorff distance of unit balls contained in the subspaces* $\mathrm{Gr}(k,n)$ *it becomes a metric space. With respect to the action of* $O(n)$ *there exists a unique probability (Haar) measure on* $\nu_{k,n}$ *of* $\mathrm{Gr}(n,k)$*:*

$$\nu_{k,n}(A) = \theta(\{U \in \mathrm{O}(n) : U\,F \in A\}),$$

for a fixed $F \in \mathrm{Gr}(k,n)$*. In particular, for* $f : S^{n-1} \to \mathbb{R}$ *we have*

$$\int_{S^{n-1}} f(\boldsymbol{u})\,\mathrm{d}\mu(\boldsymbol{u}) = \int_{\mathrm{Gr}(k,n)} \left(\int_{S^{n-1}\cap F} f(\boldsymbol{v})\,\mathrm{d}\mu_F(\boldsymbol{v}) \right) \mathrm{d}\nu(k,n)(F).$$

Here μ_F *is the unique probability (Haar) measure on* $S^{n-1} \cap \mathrm{lin}\,F$*.*

5.12 Theorem. *Let* $K \in \mathcal{K}_o^n$ *with* $B_n \subseteq K$ *and* $\big(\mathrm{vol}\,(K)/\mathrm{vol}\,(B_n)\big)^{1/n} = r$*. Then there exists an orthogonal transformation* $U \in \mathrm{O}(n)$ *such that for all* $\boldsymbol{u} \in S^{n-1}$

$$\frac{|\boldsymbol{u}|_{UK} + |\boldsymbol{u}|_K}{2} \geq \frac{1}{8r^2}.$$

In particular, $K \cap UK \subset (8r^2)B_n$*.*

Proof. Recall that $K \subseteq L$ if and only if $|\boldsymbol{x}|_L \leq |\boldsymbol{x}|_K$ for all $\boldsymbol{x} \in S^{n-1}$, and $|\boldsymbol{x}|_{\mu L} = |\boldsymbol{x}/\mu|_L = (1/\mu)\,|\boldsymbol{x}|_L$ for $\mu > 0$. Next we observe

$$\begin{aligned} |\boldsymbol{x}|_{UK\cap K} &= \min\{\lambda \geq 0 : \boldsymbol{x} \in \lambda\,(UK \cap K)\} \\ &= \max\big\{|\boldsymbol{x}|_{UK}\,, |\boldsymbol{x}|_K\big\} \\ &\geq \frac{|\boldsymbol{x}|_{UK} + |\boldsymbol{x}|_K}{2} = \frac{\left|U^{-1}\boldsymbol{x}\right|_K + |\boldsymbol{x}|_K}{2}. \end{aligned}$$

Hence $(|\boldsymbol{u}|_{UK} + |\boldsymbol{u}|_K)/2 \geq (8r^2)^{-1}\,\|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in S^{n-1}$ implies the inclusion $UK \cap K \subset 8r^2 B_n$.

For a fixed but arbitary $U \in \mathrm{O}(n)$ let $N_U(\boldsymbol{x}) = \big(|U\,\boldsymbol{x}|_K + |\boldsymbol{x}|_K\big)/2$ for $\boldsymbol{x} \in \mathbb{R}^n$. It is easy to see that $N_U(\cdot)$ is a norm in $\mathbb{R}^n$ and since $B_n \subseteq K$ we have

$$N_U(\boldsymbol{x}) = \frac{|U\boldsymbol{x}|_K + |\boldsymbol{x}|_K}{2} \leq \frac{\|U\boldsymbol{x}\| + \|\boldsymbol{x}\|}{2} = \|\boldsymbol{x}\|\,.$$

Next we are going to prove that there exists an $\overline{U} \in \mathrm{O}(n)$ with

$$\int_{S^{n-1}} \frac{1}{N_{\overline{U}}(\boldsymbol{u})^{2n}}\,\mathrm{d}\mu(\boldsymbol{u}) \leq r^{2n}. \tag{5.12.1}$$

By the geometric-arithmetic mean ineuality we have $N_{\overline{U}}(\boldsymbol{u})^2 \geq |U\boldsymbol{u}|_K\, |\boldsymbol{u}|_K$ and so it suffices to prove

$$\int_{S^{n-1}} \frac{1}{|U\boldsymbol{u}|_K^n\, |\boldsymbol{u}|_K^n}\, \mathrm{d}\mu(\boldsymbol{u}) \leq r^{2n}.$$

In view of Remark 5.11 i) we may write

$$\begin{aligned}\int_{O(n)} \left(\int_{S^{n-1}} \frac{1}{|U\boldsymbol{u}|_K^n\, |\boldsymbol{u}|_K^n}\, \mathrm{d}\mu(u)\right) \mathrm{d}\theta(U) &= \int_{S^{n-1}} \left(\int_{O(n)} \frac{1}{|U\boldsymbol{u}|_K^n}\, \mathrm{d}\theta(U)\right) \frac{1}{|\boldsymbol{u}|_K^n}\, \mathrm{d}\mu(\boldsymbol{u}) \\ &= \int_{S^{n-1}} \left(\int_{S^{n-1}} \frac{1}{|\boldsymbol{v}|_K^n}\, \mathrm{d}\mu(\boldsymbol{v})\right) \frac{1}{|\boldsymbol{u}|_K^n}\, \mathrm{d}\mu(\boldsymbol{u}) \\ &= \left(\int_{S^{n-1}} \frac{1}{|\boldsymbol{u}|_K^n}\, \mathrm{d}\mu(\boldsymbol{u})\right)^2 \leq r^{2n},\end{aligned}$$

where for the last inequality we have used $\mathrm{vol}\,(K) = \kappa_n \int_{S^{n-1}} |\boldsymbol{u}|_K^{-n}\, \mathrm{d}\mu(\boldsymbol{u})$ (cf. Proposition 3.8 iii)) and the assumption. Therefore, there exists a matrix $\overline{U}|inO(n)$ with

$$\int_{S^{n-1}} \frac{1}{|\overline{U}\boldsymbol{u}|_K^n\, |\boldsymbol{u}|_K^n}\, \mathrm{d}\mu(\boldsymbol{u}) \leq r^{2n},$$

which shows (5.12.1).

Next we use (5.12.1) in order to lower bound $\nu = N_{\overline{U}}(\boldsymbol{v})$ for an arbitray point $\boldsymbol{v} \in S^{n-1}$. Observe, that $\nu = N_{\overline{U}}(\boldsymbol{v}) \leq \|\boldsymbol{v}\| = 1$, and for $\boldsymbol{w} \in S^{n-1}$ with $\|\boldsymbol{w} - \boldsymbol{v}\| \leq \nu$ we have $N_{\overline{U}}(\boldsymbol{w}) \leq N_{\overline{U}}(\boldsymbol{v}) + N_{\overline{U}}(\boldsymbol{w} - \boldsymbol{v}) \leq \nu + \|\boldsymbol{w} - \boldsymbol{v}\| \leq 2\,\nu$. Hence

$$N_{\overline{U}}(\boldsymbol{w}) \leq 2\nu$$

on a sperical cap A at $\boldsymbol{v}$ of radius ν. Together with (5.12.1) and Lemma 5.7 we get

$$\begin{aligned} r^{2n} &\geq \int_{S^{n-1}} \frac{1}{N_{\overline{U}}(\boldsymbol{u})^{2n}}\, \mathrm{d}\mu(\boldsymbol{u}) \geq \int_A \frac{1}{N_{\overline{U}}(\boldsymbol{u})^{2n}}\, \mathrm{d}\mu(\boldsymbol{u}) \\ &\geq \mu(A)(2\nu)^{-2n} \geq (\nu/2)^n(2\nu)^{-2n} = \nu^{-n}8^{-n}.\end{aligned}$$

Hence $\nu = N_{\overline{U}}(\boldsymbol{v}) \geq 1/(r^2 8)$. □

5.13 Corollary. *For the $2n$-dimensional cross-polytope $C_{2n}^\star$ there exists an n-dimensional linear subspace L such that*

$$\frac{1}{\sqrt{2n}}(B_{2n} \cap L) \subseteq (C_{2n}^\star \cap L) \subset 32 \frac{1}{\sqrt{2n}}(B_{2n} \cap L).$$

Proof. For abbreviation we write $|\cdot|_1$ instead of $|\cdot|_{C_n^\star}$, and we recall that $|\boldsymbol{x}|_1 = \sum_{i=1}^n |x_i|$. It is $B_n \subset \sqrt{n}\, C_n^\star$ and $\mathrm{vol}\,(\sqrt{n}\, C_n^\star)/\mathrm{vol}\,(B_n) \leq 2^n$. By Theorem 5.12 there exists an orthogonal matrix $U \in \mathrm{O}(n)$ such that for all $\boldsymbol{x} \in \mathbb{R}^n$

$$|U\boldsymbol{x}|_1 + |\boldsymbol{x}|_1 \geq \frac{\sqrt{n}}{16}\, \|\boldsymbol{x}\|\,.$$

Let

$$T:\mathbb{R}^n \to \mathbb{R}^{2n} \quad \text{given by} \quad T\boldsymbol{x} = \begin{pmatrix} U \\ \mathrm{I}_n \end{pmatrix} \boldsymbol{x}.$$

First we observe that $\|T\boldsymbol{x}\| = \sqrt{\|U\boldsymbol{x}\|^2 + \|\boldsymbol{x}\|^2} = \sqrt{2}\,\|\boldsymbol{x}\|$ and so we may write

$$|T\boldsymbol{x}|_1 = |U\boldsymbol{x}|_1 + |\boldsymbol{x}|_1 \geq \frac{\sqrt{n}}{16}\,\|\boldsymbol{x}\| = \frac{\sqrt{n}}{16}\frac{1}{\sqrt{2}}\,\|T\boldsymbol{x}\| = \frac{\sqrt{2n}}{32}\,\|T\boldsymbol{x}\|\,.$$

This shows

$$T\mathbb{R}^n \cap C_{2n}^\star \subset \frac{32}{\sqrt{2n}}(T\mathbb{R}^n \cap B_{2n}).$$

Finally, since $|T\boldsymbol{x}|_1 \leq \sqrt{2n}\,\|T\boldsymbol{x}\|$ we also get

$$\frac{1}{\sqrt{2n}}(T\mathbb{R}^n \cap B_{2n}) \subset (T\mathbb{R}^n \cap C_{2n}^\star),$$

and with $L = T\mathbb{R}^n$ the corollary is proved. □

5.14 Theorem. *Let $K \in \mathcal{K}_o^n$ with $B_n \subseteq K$ and $\big(\mathrm{vol}\,(K)/\mathrm{vol}\,(B_n)\big)^{1/n} = r$. For every $1 \leq k \leq n$, a random subspace $F \in \mathrm{Gr}(n,k)$ satisfies with probability greater than $1 - \mathrm{e}^{-n}$*

$$B_n \cap \mathrm{lin}\,F \subseteq K \cap F \subseteq (6\mathrm{e}\,r)^{\frac{n}{n-k}}(B_n \cap \mathrm{lin}\,F),$$

where c is an absolute constant.

Proof. Since $B_n \subseteq K$ we have $|\boldsymbol{x}|_K \leq \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{R}^n$. By Remark 5.11 ii) we observe that

$$r^n = \int_{S^{n-1}} |\boldsymbol{u}|_K^{-n}\,\mathrm{d}\mu(\boldsymbol{u}) = \int_{\mathrm{Gr}(k,n)} \left(\int_{S^{n-1}\cap F} |\boldsymbol{u}|_K^{-n}\,\mathrm{d}\mu_F(\boldsymbol{v}) \right) \mathrm{d}\nu(k,n)(F). \tag{5.14.1}$$

Let

$$A = \left\{ F \in \mathrm{Gr}(k,n) : \int_{S_F} |\boldsymbol{u}|_K^{-n}\,\mathrm{d}\mu_F(\boldsymbol{v}) \leq \mathrm{e}^n\,r^n \right\},$$

where $S_F = S^{n-1} \cap \mathrm{lin}\,F$. By (5.14.1), Markov's inequality leads to

$$\nu_{k,n}(A) = 1 - \mathbb{P}\left(\int_{S_F} |\boldsymbol{u}|_K^{-n}\,\mathrm{d}\mu_F(\boldsymbol{v}) \geq \mathrm{e}^n\,r^n \right) \geq 1 - \frac{r^n}{\mathrm{e}^n\,r^n} = 1 - \mathrm{e}^{-n},$$

and it suffices to show that $F \in A$ satisfies

$$B_n \cap \mathrm{lin}\,F \subseteq K \cap F \subseteq (c\,r)^{\frac{n}{n-k}}(B_n \cap \mathrm{lin}\,F).$$

So let $F \in A$. By the definiton of the set A, Markov's inequality strikes again and gives for $\alpha \in (0,1)$

$$\mu_F(\{\boldsymbol{v} \in S_F : |\boldsymbol{v}|_K \geq \alpha\}) = \mu_F(\{\boldsymbol{v} \in S_F : |\boldsymbol{v}|_K^{-n} \leq \alpha^{-n}\}) \geq 1 - (\mathrm{e}\,r\,\alpha)^n. \tag{5.14.2}$$

Now let $\boldsymbol{x} \in S_F$. According to Lemma 5.7 the measure of a spherical cap of S_F of radius $\alpha/2$ is at least $(\alpha/6)^k$. Let us choose an α such that

$$(\mathrm{e}\, r\, \alpha)^n < (\alpha/6)^k, \tag{5.14.3}$$

then in view of (5.14.2) we find

$$B_n(\boldsymbol{x}, \alpha/2) \cap \{\boldsymbol{v} \in S_F : |\boldsymbol{v}|_K \geq \alpha\} \neq \emptyset.$$

Hence there exists an $\boldsymbol{y} \in S_F$ such that $\|\boldsymbol{x} - \boldsymbol{y}\| \leq \alpha/2$ and $|\boldsymbol{y}|_K \geq \alpha$. Thus

$$|\boldsymbol{x}|_K \geq |\boldsymbol{y}|_K - |\boldsymbol{x} - \boldsymbol{y}|_K \geq \alpha - \|\boldsymbol{x} - \boldsymbol{y}\| \geq \alpha/2.$$

Since $\boldsymbol{x}$ was arbitrary we conclude

$$B_n \cap \operatorname{lin} F \subseteq K \subseteq \frac{\alpha}{2}(B_n \cap \operatorname{lin} F)$$

and computing a proper α via (5.14.3) gives the desired result. □

5.15 Lemma. *Let $S \subset B_n$ be an n-dimensional simplex, and let $\boldsymbol{x} \in S$. For $k \in \{1, \ldots, n\}$ there exists a $(k-1)$-dimensional face F of S such that*

$$\boldsymbol{x} \in F + \left(\sum_{i=k}^{n} i^{-2}\right)^{1/2} \big(B_n \cap (\operatorname{aff} F)^{\perp}\big).$$

Proof. Let $\rho(n,k) = (\sum_{i=k}^{n} i^{-2})^{1/2}$. For the given $\boldsymbol{x} \in S$ let $\rho \geq 0$ be the maximum value such that $\boldsymbol{x} + \rho B_n \subset S$. Then $\rho \leq \mathrm{r}(S) \leq (1/n)\,\mathrm{R}(S) \leq 1/n$ (cf. exercise). $\boldsymbol{x} + \rho B_n$ touches a facet $\overline{F}$, say, of S in $\overline{\boldsymbol{x}}$. Then $\boldsymbol{x} - \overline{\boldsymbol{x}} \in (\operatorname{aff} \overline{F})^{\perp}$ and $\|\boldsymbol{x} - \overline{\boldsymbol{x}}\| = \rho \leq 1/n$.[46]

We prove the lemma by induction on $n-k$. For $k = n$ we are done since $\rho(n,n) = 1/n \geq \rho$ and by discussion above we have $\boldsymbol{x} \in \overline{F} + \rho\big(B_n \cap (\operatorname{aff} \overline{F})^{\perp}\big)$.

So we may assume that $k \leq n-1$. By induction hypothesis applied to the $(n-1)$-simplex $\overline{F}$ and the point $\overline{\boldsymbol{x}} \in \overline{F}$, there exists a $(k-1)$-face F of $\overline{F}$ such that

$$\overline{\boldsymbol{x}} \in \Big(F + \rho(n-1,k)\big(B_n \cap (\operatorname{aff} F)^{\perp}\big)\Big) \cap \operatorname{aff} \overline{F}.$$

So let $\boldsymbol{y} \in F$ with $\|\overline{\boldsymbol{x}} - \boldsymbol{y}\| \leq \rho(n-1,k)$ and $\overline{\boldsymbol{x}} - \boldsymbol{y} \in (\operatorname{aff} F)^{\perp} \cap \operatorname{aff} \overline{F}$. Then

$$\begin{aligned}\|\boldsymbol{x} - \boldsymbol{y}\|^2 &= \|\boldsymbol{x} - \overline{\boldsymbol{x}} + \overline{\boldsymbol{x}} - \boldsymbol{y}\|^2 = \|\boldsymbol{x} - \overline{\boldsymbol{x}}\|^2 + \|\overline{\boldsymbol{x}} - \boldsymbol{y}\|^2 \\ &\leq \frac{1}{n^2} + \rho(n-1,k)^2 = \rho(n,k)^2.\end{aligned}$$

Finally, since also $\boldsymbol{x} - \overline{\boldsymbol{x}} \in (\operatorname{aff} \overline{F})^{\perp} \subset \operatorname{aff} F^{\perp}$ we have $\boldsymbol{x} - \boldsymbol{y} \in (\operatorname{aff} F)^{\perp}$. □

5.16 Theorem. *Let $P \subset B_n$ be an n-dimensional polytope with m vertices. Then*

$$\frac{\operatorname{vol}(P)}{\operatorname{vol}(B_n)} \leq \left(c\,\frac{\log\left(\frac{m}{n}+1\right)}{n}\right)^{n/2},$$

where c is an absolute constant.

[46] $(\operatorname{aff} \overline{F})^{\perp} = \{\boldsymbol{w} \in \mathbb{R}^n : \langle \boldsymbol{w}, \boldsymbol{x} - \boldsymbol{y} \rangle = 0 \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \operatorname{aff} F\}$

Proof. Let $P = \operatorname{conv} V$ with $V = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$. By Carathéodory Theorem 2.7[47] and Lemma 5.15 we get that for every $k \in \{1, \dots, n\}$,

$$P = \operatorname{conv} V$$
$$\subset \bigcup_{\substack{J \subset \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\} \\ \# J = k}} \Big(\operatorname{conv} \{\boldsymbol{v}_j : j \in J\} + \big(\sum_{i=k}^{n} i^{-2}\big)^{1/2} \Big[B_n \cap \big(\operatorname{aff} \operatorname{conv} \{\boldsymbol{v}_j : j \in J\} \big)^{\perp} \Big] \Big).$$

Now we observe $\operatorname{vol}_{k-1}\big(\operatorname{conv} \{\boldsymbol{v}_j : j \in J\}\big) \leq \operatorname{vol}_{k-1}(S_{k-1})$, where S_{k-1} is the regular $(k-1)$-simplex with $S_{k-1} \subset B_{k-1}$ and circumradius 1 (exercise) which has volume

$$\operatorname{vol}_{k-1}(S_{k-1}) = \left(\frac{k}{k-1} \right)^{(k-1)/2} \frac{\sqrt{k}}{(k-1)!}.$$

Together with the estimate

$$\rho(n,k)^2 = \sum_{i=k}^{n} \frac{1}{i^2} \leq \sum_{i=k}^{n} \frac{1}{i(i-1)} = \sum_{i=k}^{n} \left(\frac{1}{i-1} - \frac{1}{i} \right) = \frac{1}{k-1} - \frac{1}{n} \leq \frac{1}{k-1},$$

we get

$$\begin{aligned} \frac{\operatorname{vol}(P)}{\operatorname{vol}(B_n)} &\leq \binom{m}{k} \left(\frac{k}{k-1} \right)^{(k-1)/2} \frac{\sqrt{k}}{(k-1)!} \rho(n,k)^{n-k+1} \frac{\kappa_{n-k+1}}{\kappa_n} \\ &\leq \binom{m}{k} \left(\frac{k}{k-1} \right)^{(k-1)/2} \frac{\sqrt{k}}{(k-1)!} \left(\frac{1}{k-1} \right)^{(n-k+1)/2} \frac{\kappa_{n-k+1}}{\kappa_n}. \end{aligned} \tag{5.16.1}$$

Let $k = \big\lfloor n / \ln\big((m/n) + 1\big) \big\rfloor$ and, in the following, we are going to bound the logarithms of all terms in (5.16.1). Since $k \ln k = k \ln n + O(n)$ and $\ln(k!) = k \ln k - k + O(\ln k) = k \ln n + O(n)$ we have

$$\begin{aligned} \ln \binom{m}{k} &\leq k \log m - \ln(k!) = k \left(\ln \frac{m}{n} + \ln n \right) - \ln(k!) \\ &= k \ln \frac{m}{n} + k \ln n - k \ln n + O(n) = k \ln \frac{m}{n} + O(n) = n + O(n) \leq c\, n. \end{aligned}$$

For the next terms we find

$$\begin{aligned} \ln \left(\left(\frac{k}{k-1} \right)^{(k-1)/2} \frac{\sqrt{k}}{(k-1)!} \right) &= \frac{k-1}{2} \ln \frac{k}{k-1} + \frac{1}{2} \ln k - \ln\big((k-1)!\big) \\ &= -k \ln n + O(n), \\ \ln \left(\frac{1}{k-1} \right)^{(n-k+1)/2} &\leq -(n-k) \ln \sqrt{k-1} \\ &= -n \ln \sqrt{k-1} + k \ln \sqrt{k-1} \\ &= -\frac{n}{2} \ln k + \frac{k}{2} \ln k + O(n) \\ &= \frac{n}{2} \left(-\ln n + \ln \ln \left(\frac{m}{n} + 1 \right) \right) + \frac{k}{2} \ln n + O(n), \end{aligned}$$

[47] Skript WS14

$$\ln \frac{\kappa_{n-k+1}}{\kappa_n} \leq \ln\left(\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n-k+1}{2}+1\right)}\right) + cn \leq \ln n^{k/2} + cn = \frac{k}{2}\ln n + cn.$$

Hence by (5.16.1) we get

$$\ln\left(\frac{\operatorname{vol} P}{\operatorname{vol} B_n}\right) \leq -k\ln n + \frac{n}{2}\left(-\ln n + \ln\ln\left(\frac{m}{n}+1\right)\right) + \frac{k}{2}\ln n + \frac{k}{2}\ln n + cn,$$

as required. □

5.17 Theorem [Measure concentration for the sphere]. *Let $A \subseteq S^{n-1}$ with $\mu(A) > 0$, $t \in [0,2]$, and let $A_t = \left\{\boldsymbol{u} \in S^{n-1} : \exists\, \boldsymbol{a} \in A \text{ with } \|\boldsymbol{u}-\boldsymbol{a}\| \leq t\right\}$ be the* spherical parallel set *of A at distance t. Then*

$$\mu(A_t) \geq 1 - \frac{1}{\mu(A)}\,\mathrm{e}^{-nt^2/4}.$$

Proof. Let $B = S^{n-1}\backslash A_t = \left\{\boldsymbol{u} \in S^{n-1} : \|\boldsymbol{u}-\boldsymbol{a}\| \geq t \text{ for all } \boldsymbol{a} \in A\right\}$ be the complement of A_t. It is to show

$$\mu(B)\,\mu(A) \leq \mathrm{e}^{-nt^2/4}. \tag{5.17.1}$$

To this end we complete both sets with respect to B_n, i.e., we consider $\widetilde{A} = \left\{\alpha\boldsymbol{x} : \alpha \in [0,1],\ \boldsymbol{x} \in A\right\}$ and analogously we define $\widetilde{B}$. Then $\mu(A) = \operatorname{vol}(\widetilde{A})/\kappa_n$ and $\mu(B) = \operatorname{vol}(\widetilde{B})/\kappa_n$. Next we show

$$\frac{1}{2}(\widetilde{A}+\widetilde{B}) \subseteq \left(1-\frac{t^2}{8}\right)B_n. \tag{5.17.2}$$

Let $\widetilde{\boldsymbol{a}} = \alpha\boldsymbol{a} \in \widetilde{A}$ and $\widetilde{\boldsymbol{b}} = \beta\boldsymbol{b} \in \widetilde{B}$, with $\boldsymbol{a} \in A$, $\boldsymbol{b} \in B$ and $\alpha, \beta \in [0,1]$. Then

$$\begin{aligned}\left\|\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})\right\|^2 &= \frac{1}{4}\left(2\,\|\boldsymbol{a}\|^2 + 2\,\|\boldsymbol{b}\|^2 - \|\boldsymbol{a}-\boldsymbol{b}\|^2\right)\\ &\leq \frac{1}{4}(4-t^2) = 1-\frac{t^2}{4} \leq \left(1-\frac{t^2}{8}\right)^2.\end{aligned}$$

Let $\alpha \leq \beta$ and $\gamma = \alpha/\beta \leq 1$. Then

$$\begin{aligned}\left\|\frac{1}{2}(\widetilde{\boldsymbol{a}}+\widetilde{\boldsymbol{b}})\right\| &= \left\|\frac{1}{2}(\alpha\boldsymbol{a}+\beta\boldsymbol{b})\right\| = \left\|\frac{\beta}{2}(\gamma\boldsymbol{a}+\boldsymbol{b})\right\| = \beta\left\|\gamma\frac{\boldsymbol{a}+\boldsymbol{b}}{2} + (1-\gamma)\frac{\boldsymbol{b}}{2}\right\|\\ &\leq \beta\left(\gamma\left\|\frac{\boldsymbol{a}+\boldsymbol{b}}{2}\right\| + (1-\gamma)\left\|\frac{\boldsymbol{b}}{2}\right\|\right) \leq \gamma\left(1-\frac{t^2}{8}\right) + (1-\gamma)\frac{1}{2} \leq 1-\frac{t^2}{8}.\end{aligned}$$

This shows (5.17.2) and so

$$\left(1-\frac{t^2}{8}\right)^n \kappa_n \geq \operatorname{vol}\left(\frac{1}{2}(\widetilde{A}+\widetilde{B})\right) \geq \sqrt{\operatorname{vol}(\widetilde{A})\operatorname{vol}(\widetilde{B})},$$

according to the Brunn-Minkowski inequality. Finally, we conclude

$$\mu(B)\mu(A) = \frac{\operatorname{vol}(\widetilde{B})}{\kappa_n}\frac{\operatorname{vol}(\widetilde{A})}{\kappa_n} \leq \left(1-\frac{t^2}{8}\right)^{2n} \leq \mathrm{e}^{-2nt^2/8} = \mathrm{e}^{-nt^2/4},$$

i.e., (5.17.1) is shown. □

5.18 Corollary. *Let $\boldsymbol{v} \in S^{n-1}$, $t \in [0,1]$ and let $U = \{\boldsymbol{u} \in S^{n-1} : |\langle \boldsymbol{v}, \boldsymbol{u} \rangle| \leq t\}$. Then $\mu(U) \geq 1 - 4\mathrm{e}^{-nt^2/4}$ and, in particular, for $t \geq 4/\sqrt{n}$ it holds*

$$\mu(U) \geq 0.9.$$

Proof. Exercise. □

5.19 Definition [Median]. *Let $f : S^{n-1} \to \mathbb{R}$. Then*

$$\operatorname{med}(f) = \sup \left\{ t \in \mathbb{R} : \mu_{n-1}(f \leq t) \leq \frac{1}{2} \right\}$$

is called the median *of f.*

5.20 Proposition. *Let $f : S^{n-1} \to \mathbb{R}$. Then $\mu_{n-1}(f < \operatorname{med}(f)) \leq 1/2$, and $\mu_{n-1}(f > \operatorname{med}(f)) \leq 1/2$; or equivalently,*

$$\mu_{n-1}(f \geq \operatorname{med}(f)) \geq \frac{1}{2} \text{ and } \mu_{n-1}(f \leq \operatorname{med}(f)) \geq \frac{1}{2}.$$

Proof. By the σ-additivity of μ_{n-1} we have

$$\begin{aligned} \mu_{n-1}(f < \operatorname{med}(f)) &= \mu_{n-1}\left(f \leq \operatorname{med}(f) - 1\right) \\ &\quad + \sum_{k=2}^{\infty} \mu_{n-1}\left(\operatorname{med}(f) - \frac{1}{k-1} < f \leq \operatorname{med}(f) - \frac{1}{k} \right) \\ &= \sup_{k \geq 1} \mu_{n-1}\left(f \leq \operatorname{med} f - \frac{1}{k} \right) \leq \frac{1}{2}. \end{aligned}$$

□

5.21 Lemma [Levy's Lemma]. [48] *Let $f : S^{n-1} \to \mathbb{R}$ be a 1-Lipschitz function and let $t > 0$. Then*

$$\mu_{n-1}(|f - \operatorname{med}(f)| > t) \leq 4\,\mathrm{e}^{-n\,t^2/4}.$$

Proof. First we consider $U^+ = \{\boldsymbol{u} \in S^{n-1} : f(\boldsymbol{u}) > \operatorname{med}(f) + t\}$. Let $A = \{\boldsymbol{a} \in S^{n-1} : f(\boldsymbol{a}) \leq \operatorname{med}(f)\}$ and $A_t = \{\boldsymbol{v} \in S^{n-1} : \exists\, \boldsymbol{a} \in A \text{ with } \|\boldsymbol{v} - \boldsymbol{a}\| \leq t\}$. Then $U^+ \subseteq S^{n-1} \setminus A_t$, since for $\boldsymbol{v} \in A_t$ there exists an $\boldsymbol{a} \in A$ with $\|\boldsymbol{v} - \boldsymbol{a}\| \leq t$ and so by the 1-Lipschitz property

$$f(\boldsymbol{v}) - \operatorname{med}(f) \leq f(\boldsymbol{v}) - f(\boldsymbol{a}) \leq \|\boldsymbol{v} - \boldsymbol{a}\| \leq t.$$

Hence $\mu_{n-1}(U^+) \leq 1 - \mu_{n-1}(A_t)$ and with Theorem 5.17 and Proposition 5.20 we obtain

$$\mu_{n-1}(U^+) \leq \frac{1}{\mu_{n-1}(A)} \mathrm{e}^{-n\,t^2/4} \leq 2\,\mathrm{e}^{-n\,t^2/4}.$$

The set $U^- = \{\boldsymbol{u} \in S^{n-1} : f(\boldsymbol{a}) < \operatorname{med}(f) - t\}$ can be treated analogously. □

[48] Paul Pierre Lévy, 1886–1971

5.22 Corollary. *Let $f : S^{n-1} \to \mathbb{R}$ be a 1-Lipschitz function, $t > 0$, and let $\mathbb{E}(f) = \int_{S^{n-1}} f(\boldsymbol{u})\,\mathrm{d}\mu(\boldsymbol{u})$. Then*

$$\mu_{n-1}(|f - \mathbb{E}(f)| > t) \leq \left(\mathrm{e}^{8\,\pi}\right)\,\mathrm{e}^{-n\,t^2/8}.$$

Proof. First we note that

$$\begin{aligned}
|\mathrm{med}(f) - \mathbb{E}(f)| &= |\mathbb{E}(\mathrm{med}(f) - f)| \leq \int_{S^{n-1}} |\mathrm{med}(f) - f(\boldsymbol{u})|\,\mathrm{d}\mu(\boldsymbol{u}) \\
&= \mathbb{E}(|\mathrm{med}(f) - f|) = \int_0^\infty \mu(|f - \mathrm{med}(f)| > t)\,\mathrm{d}t \qquad (5.22.1)\\
&\leq \int_0^\infty 4\,\mathrm{e}^{-n\,t^2/4}\,\mathrm{d}t = 4\,\sqrt{\pi/n}.
\end{aligned}$$

by Lemma 5.21. Now for $t \leq 8\,\sqrt{\pi/n}$ we have $\mathrm{e}^{8\,\pi}\,\mathrm{e}^{-n\,t^2/8} \geq 1$ and the statement is certainly true. So let $t > 8\,\sqrt{\pi/n}$. Then we may write

$$\begin{aligned}
\mu_{n-1}(|f - \mathbb{E}(f)| > t) &\leq \mu_{n-1}(|f - \mathrm{med}(f)| > t - |\mathrm{med}(f) - \mathbb{E}(f)|) \\
&\leq \mu_{n-1}(|f - \mathrm{med}(f)| > t - 4\,\sqrt{\pi/n}) \leq \mu_{n-1}(|f - \mathrm{med}(f)| > t/2) \\
&\leq 4\,\mathrm{e}^{-n\,t^2/8}.
\end{aligned}$$

□

5.23 Definition [δ-**net**]. *For $\delta > 0$ a subset $N_\delta \subseteq S^{n-1}$ is called* δ-net *if for all $\boldsymbol{u} \in S^{n-1}$ there exists a $\boldsymbol{v} \in N_\delta$ such that $\|\boldsymbol{u} - \boldsymbol{v}\| \leq \delta$, i.e., $S^{n-1} = \cup_{\boldsymbol{v}\in N} C((2-\delta^2)/2, \boldsymbol{v})$.*

5.24 Proposition. *There exists always a δ-net on S^{n-1} consisting of at most $(4/\delta)^n$ points.*

Proof. Exercise. □

5.25 Lemma. *Let $|\cdot|$ be a norm on $\mathbb{R}^n$, $M, \gamma > 0$, and let $N_\delta \subset S^{n-1}$ be a δ-net such that for all $\boldsymbol{v} \in N_\delta$*

$$M\,(1-\gamma) \leq |\boldsymbol{v}| \leq M\,(1+\gamma).$$

Then for all $\boldsymbol{u} \in S^{n-1}$

$$M\,\frac{(1-\gamma-2\,\delta)}{1-\delta} \leq |\boldsymbol{u}| \leq M\,\frac{1+\gamma}{1-\delta}.$$

Proof. By adopting the norm we may asume $M = 1$. For the upper bound let $\overline{\boldsymbol{u}} \in S^{n-1}$ with $|\overline{\boldsymbol{u}}| = c = \max\{|\boldsymbol{u}| : \boldsymbol{u} \in S^{n-1}\}$ and next we show

$$c \leq \frac{1+\gamma}{1-\delta}. \qquad (5.25.1)$$

To this end let $\boldsymbol{v} \in N_\delta$ with $\|\boldsymbol{v} - \overline{\boldsymbol{u}}\| \le \delta$. Then $|\boldsymbol{v}| \le 1 + \gamma$ by assumption and so we get

$$c = |\overline{\boldsymbol{u}}| \le |\boldsymbol{v}| + |\boldsymbol{v} - \overline{\boldsymbol{u}}| \le (1+\gamma) + c\,\|\boldsymbol{v} - \overline{\boldsymbol{u}}\| \le 1 + \gamma + c\,\delta.$$

Thus we get (5.25.1) and for the lower bound let $\boldsymbol{u} \in S^{n-1}$ and let $\boldsymbol{v} \in N_\delta$ with $\|\boldsymbol{u} - \boldsymbol{v}\| \le \delta$. By assumption and using (5.25.1) we conclude

$$1 - \gamma \le |\boldsymbol{v}| \le |\boldsymbol{u}| + |\boldsymbol{v} - \boldsymbol{u}| \le |\boldsymbol{u}| + \frac{1+\gamma}{1-\delta}\,\|\boldsymbol{v} - \boldsymbol{u}\| \le |\boldsymbol{u}| + \frac{1+\gamma}{1-\delta}\delta,$$

which gives the desired lower bound on $|\boldsymbol{u}|$. □

5.26 Lemma. *Let $|\cdot|$ be a norm $\mathbb{R}^n$ with $|\cdot| \le \|\cdot\|$, let $M = \int_{S^{n-1}} |\boldsymbol{u}|\,\mathrm{d}\mu(\boldsymbol{u})$ and let $\gamma > 0$, $0 \le \delta \le 2$. For*

$$k \le \frac{1}{8}\,\frac{\gamma^2}{\ln(\mathrm{e}^{34}/\delta)} n M^2,$$

there exists a k-dimensional linear subspace H and a δ-net $N_{\delta,H}$ in $H \cap S^{n-1}$ such that for $\boldsymbol{v} \in N_{\delta,H}$

$$M\,(1-\gamma) \le |\boldsymbol{v}| \le M\,(1+\gamma).$$

Proof. The function $|\cdot| : S^{n-1} \to \mathbb{R}$ is a 1-Lipschitz function since

$$\big||\boldsymbol{u}| - |\boldsymbol{v}|\big| \le |\boldsymbol{u} - \boldsymbol{v}| \le \|\boldsymbol{u} - \boldsymbol{v}\|\,.$$

In view of Corollary 5.22 we have

$$\mu_{n-1}(\big||\boldsymbol{u}| - M\big| \ge M\gamma) \le \left(\mathrm{e}^{32}\right) \mathrm{e}^{-n\,M^2\gamma^2/8},$$

i.e.,

$$M(1-\gamma) \le |\boldsymbol{u}| \le M(1+\gamma) \tag{5.26.1}$$

for all $\boldsymbol{u} \in S^{n-1}$ except for a set of measure at most $\mathrm{e}^{-n\,M^2\gamma^2/8+32}$. Let $\overline{H}$ be an arbitrary m-dimensional subspace and let $N_{\delta,\overline{H}}$ be a δ-net in $\overline{H} \cap S^{n-1}$ with $|N_{\delta,\overline{H}}| \le (4/\delta)^m$ (cf. Proposition 5.24). In view of Remark 5.11 and (5.26.1) we know that for each $\boldsymbol{v} \in N_{\delta,\overline{H}}$ there exists a set $A_{\boldsymbol{v}} \subset O(n)$ of measure at most $\mathrm{e}^{-n\,M^2\gamma^2/8+32}$ with

$$M(1-\gamma) \le |U\boldsymbol{v}| \le M(1+\gamma) \text{ for all } U \in O(n) \setminus A_{\boldsymbol{v}}.$$

Hence, as long as

$$|N_{\delta,\overline{H}}|\,\mathrm{e}^{-n\,M^2\gamma^2/8+32} < 1 \tag{5.26.2}$$

there exists an $\overline{U} \in O(n)$

$$M(1-\gamma) \le |\overline{U}\boldsymbol{v}| \le M(1+\gamma) \text{ for all } \boldsymbol{v} \in N_{\delta,\overline{H}},$$

or equivalently

$$M(1-\gamma) \le |\boldsymbol{v}| \le M(1+\gamma) \text{ for all } \boldsymbol{v} \in \overline{U}\,N_{\delta,\overline{H}}.$$

Since $\overline{U}\,N_{\delta,\overline{H}}$ is a δ-net in $\overline{U}\overline{H} \cap S^{n-1}$ it remains to bound m so that (5.26.2) holds. Since $|N_{\delta,\overline{H}}| \le (4/\delta)^m$ we get the desired bound. □

5.27 Theorem. *Let $K \in \mathcal{K}_o^n$, $rB_n \subseteq K$, $M = \int_{S^{n-1}} |\boldsymbol{u}|_K \, \mathrm{d}\mu(\boldsymbol{u})$, $1 \geq \epsilon > 0$ and*

$$k \leq \frac{1}{400} \frac{\epsilon^2}{\ln(\mathrm{e}^{36}\epsilon^{-1})} \, n \, (r\, M)^2.$$

Then there exists a k-dimensional linear subspace $H \in \mathrm{Gr}(k,n)$ with $\mathrm{d}_{BM}(K \cap H, B_n \cap H) \leq 1 + \epsilon$.

Proof. Since $rM = \int_{S^{n-1}} |\boldsymbol{u}|_{(1/r)K} \, \mathrm{d}\mu(\boldsymbol{u})$ we may assume with out loss of generality $r = 1$, i.e,, $|\boldsymbol{u}|_K \leq \|\boldsymbol{u}\|$. For $\gamma, \delta > 0$, $\delta \leq 2$, and for

$$k \leq \frac{1}{8} \frac{\gamma^2}{\ln(\mathrm{e}^{34}/\delta)} n M^2, \tag{5.27.1}$$

there exists by Lemma 5.26 and Lemma 5.25 a k-dimensional linear subspace H such that

$$M \frac{1 - \gamma - 2\delta}{1 - \delta} \leq |\boldsymbol{u}|_K \leq M \frac{1 + \gamma}{1 - \delta}, \text{ for all } \boldsymbol{u} \in S^{n-1} \cap H.$$

Hence,

$$\mathrm{d}_{\mathrm{BM}}(K \cap H, B_n \cap H) \leq \frac{1 + \gamma}{1 - \gamma - 2\delta}.$$

For $\gamma = \delta = \epsilon/7$ the right hand side is less than $1 + \epsilon$ and the right hand side of (5.27.1) is lower bounded by the bound on k given in the statement of the theorem. □

5.28 Notation. *Let $\gamma_n(\cdot)$ be the standard Gaussian measure with density $\sqrt{2\pi}^{-n} \mathrm{e}^{-\|\boldsymbol{x}\|^2/2}$, i.e.,*

$$\gamma_n(A) = \int_{\mathbb{R}^n} \chi_A(\boldsymbol{x}) \mathrm{d}\gamma_n(\boldsymbol{x}) = \frac{1}{\sqrt{2\pi}^n} \int_A \mathrm{e}^{-\|\boldsymbol{x}\|^2/2} \mathrm{d}\boldsymbol{x}.$$

5.29 Proposition. *Let $K \in \mathcal{K}_o^n$. Then*

$$\frac{1}{\sqrt{n-1}} \int_{\mathbb{R}^n} |\boldsymbol{x}|_K \, \mathrm{d}\gamma_n(\boldsymbol{x}) > \int_{S^{n-1}} |\boldsymbol{u}|_K \, \mathrm{d}\mu(\boldsymbol{u}) > \frac{1}{\sqrt{n}} \int_{\mathbb{R}^n} |\boldsymbol{x}|_K \, \mathrm{d}\gamma_n(\boldsymbol{x}).$$

Proof. In view of Proposition 3.8 we may write

$$\begin{aligned}
\frac{1}{n\kappa_n} \int_{\mathbb{R}^n} |\boldsymbol{x}|_K \, \mathrm{e}^{-\|x\|^2/2} \, \mathrm{d}\boldsymbol{x} &= \int_0^\infty \int_{S^{n-1}} r^{n-1} \, |r\boldsymbol{u}|_K \, \mathrm{e}^{-\|r\boldsymbol{u}\|^2/2} \, \mathrm{d}\mu(\boldsymbol{u}) \, \mathrm{d}r \\
&= \int_{S^{n-1}} |\boldsymbol{u}|_K \, \mathrm{d}\mu(\boldsymbol{u}) \int_0^\infty r^n \mathrm{e}^{-\|r\|^2/2} \, \mathrm{d}r \\
&= \int_{S^{n-1}} |\boldsymbol{u}|_K \, \mathrm{d}\mu(\boldsymbol{u}) \sqrt{2}^{n-1} \int_0^\infty t^{n/2 - 1/2} \mathrm{e}^{-t} \, \mathrm{d}t \\
&= \sqrt{2}^{n-1} \Gamma((n+1)/2) \int_{S^{n-1}} |\boldsymbol{u}|_K \, \mathrm{d}\mu(\boldsymbol{u}).
\end{aligned}$$

Hence,

$$\begin{aligned}\int_{S^{n-1}} |\boldsymbol{u}|_K \,\mathrm{d}\mu(\boldsymbol{u}) &= \frac{\Gamma(n/2+1)}{\Gamma((n+1)/2)}\frac{1}{n}\sqrt{2}\int_{\mathbb{R}^n} |\boldsymbol{x}|_K \left(\frac{1}{\sqrt{2\pi}}\right)^n \mathrm{e}^{-\|\boldsymbol{x}\|^2/2}\,\mathrm{d}\boldsymbol{x}\\ &= \frac{\Gamma(n/2)}{\Gamma((n+1)/2)}\frac{1}{\sqrt{2}}\int_{\mathbb{R}^n} |\boldsymbol{x}|_K \,\mathrm{d}\gamma_n(\boldsymbol{x})\\ &= \frac{1}{\sqrt{2\pi}}\frac{\kappa_{n-1}}{\kappa_{n-2}}\int_{\mathbb{R}^n} |\boldsymbol{x}|_K \,\mathrm{d}\gamma_n(\boldsymbol{x}),\end{aligned}$$

and it remains to show

$$\sqrt{\frac{2\pi}{n}} > \frac{\kappa_n}{\kappa_{n-1}} > \sqrt{\frac{2\pi}{n+1}},$$

which is left as an exercise. □

5.30 Proposition. *For $K \in \mathcal{K}_o^n$ let $M(K) = \int_{S^{n-1}} |\boldsymbol{u}|_K \,\mathrm{d}\mu(\boldsymbol{u})$.*

i) $M(K) = \frac{1}{2}\int_{S^{n-1}}(h(K^\star,\boldsymbol{u}) + h(K^\star,-\boldsymbol{u}))\,\mathrm{d}\mu(\boldsymbol{u})$. *i.e., one half of the mean witdh of $K^\star$.*

ii) $M(K)\,M(K^\star) \geq 1$.

iii) *For $1 \leq p < \infty$*

$$M(B_n^p) \geq \sqrt{\frac{2}{\pi}}\, n^{1/p-1/2}.$$

In particular, $M(\sqrt{n}\,C_n^\star) \geq \sqrt{2/\pi}$.

iv) *There exists an absolute constant c such that for n large*

$$M(C_n) \geq c\sqrt{\frac{\ln n}{n}}.$$

Proof. i) and ii) are left as exercises. For iii) let $R > 0$ such that $\gamma_n(R\,C_n) = \gamma_n(|\boldsymbol{x}|_{C_n} \leq R) = 1/2$ and so also $\gamma_n(|\boldsymbol{x}|_{C_n} \geq R) = 1/2$. Hence we know

$$\begin{aligned}\int_{S^{n-1}} |\boldsymbol{u}|_{C_n} \,\mathrm{d}\mu(\boldsymbol{u}) &> \frac{1}{\sqrt{n}}\int_{\mathbb{R}^n} |\boldsymbol{x}|_{C_n}\,\mathrm{d}\gamma_n(\boldsymbol{x})\\ &\geq \frac{1}{\sqrt{n}}\int_{\mathbb{R}^n, |\boldsymbol{x}|_{C_n}\geq R} |\boldsymbol{x}|_{C_n}\,\mathrm{d}\gamma_n(\boldsymbol{x}) \geq \frac{1}{\sqrt{n}}\frac{R}{2}.\end{aligned} \tag{5.30.1}$$

In order to lower bound R we note that

$$\frac{1}{2} = \gamma_n(|\boldsymbol{x}|_{C_n} \leq R) = \frac{1}{\sqrt{2\pi}^n}\int_{R\,C_n} \mathrm{e}^{-\|\boldsymbol{x}\|^2/2}\mathrm{d}\boldsymbol{x} = \left(\frac{1}{\sqrt{2\pi}}\int_{-R}^{R} \mathrm{e}^{-t^2/2}\mathrm{d}t\right)^n.$$

Hence,

$$\frac{1}{\sqrt{2\pi}}\int_{-R}^{R} \mathrm{e}^{-t^2/2}\mathrm{d}t = \mathrm{e}^{-\ln(2)/n} \geq 1 - \frac{\ln(2)}{n}.\text{[49]}$$

[49] $\mathrm{e}^{-t} \geq 1 - t$

For the left hand side we also have

$$\frac{1}{\sqrt{2\pi}}\int_{-R}^{R} \mathrm{e}^{-t^2/2}\mathrm{d}t = 1-\frac{2}{\sqrt{2\pi}}\int_{R}^{\infty} \mathrm{e}^{-t^2/2}\mathrm{d}t \le 1-c_1\,\mathrm{e}^{-(R+1)^2/2},$$

for a suitable constant c_1. Combining the last two inequalities leads to $R \ge c\sqrt{\ln n}$ and we are done by (5.30.1). $\square$

5.31 Lemma [Dvoretzky-Rogers]. *Let $K \in \mathcal{K}_o^n$ and let B_n be the volume maximal ellipsoid in K. Then there exists an orthonormal basis $\boldsymbol{v}_1,\dots,\boldsymbol{v}_n \in S^{n-1}$ such that for $1 \le i \le n$*

$$|\boldsymbol{v}_i|_K \ge \frac{1}{\mathrm{e}}\left(1-\frac{i-1}{n}\right).$$

Proof. We choose recursively the points $\boldsymbol{v}_i$ in the following way: Let $\boldsymbol{v}_1 \in S^{n-1} \subset K$ be a point of maximal norm $|\cdot|_K$. For $i > 1$ let $\boldsymbol{v}_i$ be a point of maximal norm $|\cdot|_K$ in $S^{n-1} \cap \operatorname{lin}\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_{i-1}\}^{\perp}$. Then $B_n = \{\sum_{i=1}^n \alpha_i \boldsymbol{v}_i : \sum_{i=1}^n (\alpha_i)^2 \le 1\}$, and by the choice of $\boldsymbol{v}_j$ we have for all $\boldsymbol{x} \in \operatorname{lin}\{\boldsymbol{v}_j,\dots,\boldsymbol{v}_n\} \cap S^{n-1}$

$$|\boldsymbol{x}|_K \le |\boldsymbol{v}_j|_K\,. \tag{5.31.1}$$

For $1 \le j \le n$ and some positive $\beta,\gamma > 0$ we consider the ellipsoid

$$\mathcal{E}_j = \left\{\sum_{i=1}^{n} \alpha_i\,\boldsymbol{v}_i : \sum_{i=1}^{j-1}\frac{\alpha_i^2}{\gamma^2} + \sum_{i=j}^{n}\frac{\alpha_i^2}{\beta^2} \le 1\right\}$$

of volume $\operatorname{vol}(\mathcal{E}_j) = \gamma^{j-1}\beta^{n-(j-1)}\kappa_n$. Let $\sum_{i=1}^n \alpha_i\,\boldsymbol{v}_i \in \mathcal{E}_j$. Then we certainly have $\sum_{i=1}^{j-1}\alpha_i\boldsymbol{v}_i \in \gamma B_n$ and so, since $B_n \subset K$, $\left|\sum_{i=1}^{j-1}\alpha_i\boldsymbol{v}_i\right|_K \le \gamma$.

We also have $\left\|\sum_{i=j}^n \alpha_i\boldsymbol{v}_i\right\| \le \beta$ and since $\left(\sum_{i=j}^n \alpha_i\boldsymbol{v}_i\right)/\left\|\sum_{i=j}^n \alpha_i\boldsymbol{v}_i\right\| \in \operatorname{lin}\{\boldsymbol{v}_j,\dots,\boldsymbol{v}_n\} \cap S^{n-1}$ we conclude by (5.31.1) $\left|\sum_{i=j}^n \bar{\alpha}_i\boldsymbol{v}_i\right|_K \le \beta\,|\boldsymbol{v}_j|_K$. Hence

$$\left|\sum_{i=1}^{n}\bar{\alpha}_i\,\boldsymbol{v}_i\right|_K \le \gamma+\beta\,|\boldsymbol{v}_j|_K\,.$$

Thus, if $\gamma+\beta\,|\boldsymbol{v}_j|_K \le 1$ we have $\mathcal{E}_j \subseteq K$ and by the volume maximality of B_n we get for $j = 1,\dots,n$

$$\gamma^{j-1}\beta^{n-(j-1)} \le 1 \text{ for all } \gamma,\beta > 0 \text{ with } \gamma+\beta\,|\boldsymbol{v}_j|_K \le 1.$$

Hence, setting $\beta = (1-\gamma)/\,|\boldsymbol{v}_j|_K$ yields

$$|\boldsymbol{v}_j|_K^{n-j+1} \ge (1-\gamma)^{n-j+1}\gamma^{j-1},$$

which for $\gamma = \frac{j-1}{n}$ leads to

$$|\boldsymbol{v}_j|_K \ge \left(1-\frac{j-1}{n}\right)\left(\frac{j-1}{n}\right)^{\frac{j-1}{n-j+1}} \ge \left(1-\frac{j-1}{n}\right)\frac{1}{\mathrm{e}}.$$

For the last inequality observe that $(i/n)^{i/(n-i)} \ge \mathrm{e}^{-1}$ is equivalent to $\ln(n)-\ln(i) \le (n-i)\,(1/i)$ which is certainly true as the integral $\int_i^n (1/x)\,\mathrm{d}x$ shows. $\square$

5.32 Theorem [Dvoretzky]. *There exists an absolute constant c such that for all $K \in \mathcal{K}_o^n$ and $\epsilon > 0$ and for*

$$k \leq c \frac{\epsilon^2}{\ln(1+\epsilon^{-1})} \ln(n)$$

there exists a k-dimensional subspace H with $\mathrm{d}_{BM}(K \cap H, B_n \cap H) \leq 1 + \epsilon$.

Proof. We may assume that B_n is the volume maximal ellipsoid in K. In view of Theorem 5.27 it is to show that

$$M = \int_{S^{n-1}} |\boldsymbol{u}|_K \,\mathrm{d}\mu(\boldsymbol{u}) \geq c\sqrt{\frac{\ln n}{n}}.$$

Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be an orthonormal basis according to Lemma 5.31. Then $S^{n-1} = \{\sum_{i=1}^n \alpha_i \boldsymbol{v}_i : \sum_{i=1}^n \alpha_i^2 = 1\}$ and for $i \leq n/2$, say, we, in particular, have

$$|\boldsymbol{v}_i|_K \geq \bar{c},$$

for a certain constant $\bar{c}$. Hence, we may write

$$\begin{aligned}
\int_{S^{n-1}} |\boldsymbol{u}|_K \,\mathrm{d}\mu(\boldsymbol{u}) &= \int_{S^{n-1}} \left|\sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i\right|_K \mathrm{d}\mu(\alpha) \\
&= \int_{S^{n-1}} \frac{1}{2}\left(\left|\sum_{i=1}^{n-1} \alpha_i \boldsymbol{v}_i + \alpha_n \boldsymbol{v}_n\right|_K + \left|\sum_{i=1}^{n-1} \alpha_i \boldsymbol{v}_i - \alpha_n \boldsymbol{v}_n\right|_K\right) \mathrm{d}\mu(\alpha) \\
&\geq \int_{S^{n-1}} \max\left\{\left|\sum_{i=1}^{n-1} \alpha_i \boldsymbol{v}_i\right|_K, |\alpha_n \boldsymbol{v}_n|_K\right\} \mathrm{d}\mu(\alpha),
\end{aligned}$$

where we have used the triangle inequality $2\,|a|_K \leq |a+b+a-b|_K \leq |a+b|_K + |a-b|_K$. Repeating in this way we obtain

$$\begin{aligned}
\int_{S^{n-1}} |\boldsymbol{u}|_K \,\mathrm{d}\mu(\boldsymbol{u}) &\geq \int_{S^{n-1}} \max\{|\alpha_1 \boldsymbol{v}_1|_K, \cdots, |\alpha_n \boldsymbol{v}_n|_K\} \,\mathrm{d}\mu(\alpha) \\
&\geq \bar{c} \int_{S^{n-1}} \max\left\{|\alpha_1|, \cdots, |\alpha_{n/2}|\right\} \mathrm{d}\mu(\alpha) \\
&\geq \frac{\bar{c}}{\sqrt{n}} \int_{\mathbb{R}^n} \max\left\{|x_1|, \cdots, |x_{n/2}|\right\} \mathrm{d}\gamma_n(\boldsymbol{x}) \\
&= \frac{\bar{c}}{\sqrt{n}} \int_{\mathbb{R}^{n/2}} \max\left\{|x_1|, \cdots, |x_{n/2}|\right\} \mathrm{d}\gamma_{n/2}(\boldsymbol{x}) \\
&= \frac{\bar{c}}{\sqrt{n}} \int_{\mathbb{R}^{n/2}} |\boldsymbol{x}|_{C_{n/2}} \,\mathrm{d}\gamma_{n/2}(\boldsymbol{x}) \\
&\geq c \int_{S^{n/2-1}} |\boldsymbol{u}|_{C_{n/2}} \,\mathrm{d}\mu(\boldsymbol{u}) \geq c\sqrt{\frac{\ln n}{n}},
\end{aligned}$$

where we have applied Propositions 5.29 and 5.30. □

6 A few remarks on covering numbers

6.1 Definition [covering-, separation number]. *Let $K, L \in \mathcal{K}^n$.*

i)

$$\mathrm{N}\,(K, L) = \min\{|S| : S \subset \mathbb{R}^n \text{ with } K \subseteq S + L\},$$
$$\overline{\mathrm{N}}\,(K, L) = \min\{|S| : S \subset K \text{ with } K \subseteq S + L\}$$

are called covering numbers *of K by L.*

ii) *If $L = -L$,*

$$\mathrm{M}\,(K, L) = \max\{|S| : S \subset K \text{ with } |\boldsymbol{x}_i - \boldsymbol{x}_j|_L > 1 \text{ for all } \boldsymbol{x}_i \neq \boldsymbol{x}_j \in S\}$$

is called the separation number *of K by L.*

6.2 Proposition. *Show that*

i) *for $K, L, M \in \mathcal{K}^n$, $K \subseteq L$: $\mathrm{N}\,(K, M) \leq \mathrm{N}\,(L, M)$, $\mathrm{N}\,(M, L) \leq \mathrm{N}\,(M, K)$, and $\overline{\mathrm{N}}\,(M, L) \leq \overline{\mathrm{N}}\,(M, K)$*

ii) *for $K, L \in \mathcal{K}^n$: $\overline{\mathrm{N}}\,(K, L - L) \leq \mathrm{N}\,(K, L) \leq \overline{\mathrm{N}}\,(K, L)$.*

iii) *for $K \in \mathcal{K}^n$, $\lambda > 0$: $\mathrm{N}\,(K, \lambda\, B_n) = \overline{\mathrm{N}}\,(K, \lambda\, B_n)$.*

iv) *for $K, L, M \in \mathcal{K}^n$: $\mathrm{N}\,(K, L) \leq \mathrm{N}\,(K, M)\,\mathrm{N}\,(M, L)$.*

Proof. Exercise. □

6.3 Proposition. *Let $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{K}^n$ ellipsoids centered at the origin. Show that*

$$\mathrm{N}\,(\mathcal{E}_1, \mathcal{E}_2) = \mathrm{N}\,(\mathcal{E}_2{}^\star, \mathcal{E}_1{}^\star).$$

6.4 Proposition. *Let $K, L \in \mathcal{K}^n$.*

i)

$$\overline{\mathrm{N}}\,(K, (K - K) \cap L) = \overline{\mathrm{N}}\,(K, L).$$

ii)

$$\mathrm{N}\,(K, (K - K) \cap (L - L)) \leq \mathrm{N}\,(K, L).$$

iii) *for $r > 0$*

$$\mathrm{N}\,(K, (K - K) \cap rB_n) \leq \mathrm{N}\,(K, rB_n).$$

Proof. i) is left as an exercise. For ii) let $K \subseteq \cup_{i=1}^N (\boldsymbol{x}_i + L)$ be a minimal covering of K by L. By the minimality there exists points $\boldsymbol{y}_i \in (\boldsymbol{x}_i + L) \cap K$ for all $1 \leq i \leq N$.

iii) □

6.5 Proposition. *Show that*

$$\mathrm{M}\,(K, 2\,L) \leq \mathrm{N}\,(K, L) \leq \overline{\mathrm{N}}\,(K, L) \leq \mathrm{M}\,(K, L)$$

Proof. Exercise. □

6.6 Lemma. *Let $K, L \in \mathcal{K}^n$, $\dim K, \dim L = n$. Show that*

$$\mathrm{vol}\,(K)/\mathrm{vol}\,(L) \le \mathrm{N}\,(K, L).$$

If $L = -L$ then

$$\mathrm{N}\,(K, L) \le 2^n \mathrm{vol}\,(K + L/2)/\mathrm{vol}\,(L).$$

Proof. Exercise. □

6.7 Corollary. *Let $L, K \in \mathcal{K}^n$, $L \subseteq K$, $L = -L$. Then for $\epsilon > 0$*

$$\mathrm{N}\,(K, \epsilon L) \le \left(\frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(L)}\right)\left(1 + \frac{2}{\epsilon}\right)^n.$$

Proof. Immediate consequence by Lemma 6.6, since

$$\mathrm{N}\,(K, \epsilon\, L) \le \frac{2^n \mathrm{vol}\,\left(K + \frac{\epsilon}{2} L\right)}{\mathrm{vol}\,(\epsilon\, L)} \le 2^n \frac{\left(1 + \frac{\epsilon}{2}\right)^n}{\epsilon^n} \frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(L)}.$$

□

6.8 Corollary. *Let $K \in \mathcal{K}^n_o$. Then for $1 > \epsilon > 0$*

$$\left(\frac{1}{\epsilon}\right)^n \le \mathrm{N}\,(K, \epsilon K) \le \left(1 + \frac{2}{\epsilon}\right)^n.$$

Proof. Immediate consequence by Corollary 6.7 and the lower bound in Lemma 6.6. □

6.9 Lemma. *Let μ be a probability measure on $\mathbb{R}^n$, $\psi : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ a log-concave function with finite, positive integral Then*

$$\int_{\mathbb{R}^n} \psi(\boldsymbol{x})\,\mathrm{d}\mu(\boldsymbol{x}) \le \psi\left(\int_{\mathbb{R}^n} \boldsymbol{x} \frac{\psi(\boldsymbol{x})}{\int \psi\,\mathrm{d}\mu}\,\mathrm{d}\mu(\boldsymbol{x})\right).$$

Proof. to be written.... □

6.10 Theorem. *Let $K, L \in \mathcal{K}^n$ be centered convex bodies, i.e., $\mathbf{0} = \int_K \boldsymbol{x}\,\mathrm{d}\boldsymbol{x} = \int_L \boldsymbol{x}\,\mathrm{d}\boldsymbol{x}$. Then*

$$\mathrm{vol}\,(K)\mathrm{vol}\,(L) \le \mathrm{vol}\,(K - L)\mathrm{vol}\,(K \cap L).$$

In particular, if $L = -K$ then

$$\mathrm{vol}\,(K \cap -K) \ge 2^{-n}\mathrm{vol}\,(K).$$

6.11 Theorem. *Let $K, L \in \mathcal{K}^n$. Then*

$$4^{-n}\frac{\mathrm{vol}\,(K-L)}{\mathrm{vol}\,(L)} \le \mathrm{N}\,(K,L) \le 4^n\frac{\mathrm{vol}\,(K-L)}{\mathrm{vol}\,(L)}.$$

6.12 Corollary. *Let $K \in \mathcal{K}^n$. Then*

$$\mathrm{N}\,(K-K,K) \le 32^n.$$

Proof. Without loss of generality let $\mathbf{0} \in \mathrm{int}\,K$. In view of the upper bound of Theorem 6.11 we have

$$\mathrm{N}\,(K-K,K) \le 4^n\frac{\mathrm{vol}\,(K-2K)}{\mathrm{vol}\,(K)} \le 8^n\frac{\mathrm{vol}\,(K-K)}{\mathrm{vol}\,(K)} \le 32^n,$$

by the Theorem 1.15 of Rogers&Shephard. □

6.13 Lemma. *Let $K, L \in \mathcal{K}^n$, $K = -K$, $L \subseteq r\,K$, $r \ge 1$. Then*

$$\mathrm{N}\,(\mathrm{conv}\,(K \cup L), (1+1/n)\,K) \le 2rn\,\mathrm{N}\,(L,K).$$

6.14 Corollary. *Let $K, L \in \mathcal{K}^n$, $K = -K$, $L \subseteq r\,K$, $r \ge 1$. Then*

$$\mathrm{vol}\,(\mathrm{conv}\,(K \cup L)) \le 2\mathrm{e}\,rn\,\mathrm{N}\,(L,K)\mathrm{vol}\,(K).$$

6.15 Theorem [Pajor-Tomczak]. *Let $K \in \mathcal{K}^n_o$ and $t > 0$. Then*

$$\log \overline{\mathrm{N}}\,(B_n, t\,K) \le c\,n\left(\frac{M(K)}{t}\right)^2,$$

where c is an absolute constant.

6.16 Lemma. *Let $K \in \mathcal{K}^n_o$. Then*

$$\mathrm{N}\,(K, tB_n) \le \mathrm{N}\,(K, 2tB_n)\mathrm{N}\,(B_n, \frac{t}{8}K^\star).$$

6.17 Theorem. *Let $K \in \mathcal{K}^n_o$. Then*

$$\sup_{t>0} t\,(\log \mathrm{N}\,(K, t\,B_n))^{1/2} \le 10\,\sup_{t>0} t\,(\log \mathrm{N}\,(B_n, t\,K^\star))^{1/2}$$

6.18 Theorem. *Let $K \in \mathcal{K}^n_o$ and $t > 0$. Then*

$$\log \mathrm{N}\,(K, t\,B_n) \le c\,n\left(\frac{M(K^\star)}{t}\right)^2$$

6.19 Proposition. *Let $K \in \mathcal{K}^n$ and $t > 0$. Then*

$$t^2 \log \mathrm{N}\,(K, t\,B_n) \le c\,nM(K^\star)^2,$$

where c is an absolute constant.

6.20 Definition [M-position]. *Let $K \in \mathcal{K}_o^n$ and $r = (\mathrm{vol}\,(K)/\mathrm{vol}\,(B_n))^{1/n}$. K is called in M-position with constant $c > 0$ if*

$$\frac{1}{c}\mathrm{vol}\,(r\,B_n + L)^{\frac{1}{n}} \le \mathrm{vol}\,(K + L)^{\frac{1}{n}} \le c\mathrm{vol}\,(rB_n + L)^{\frac{1}{n}}$$
$$\frac{1}{c}\mathrm{vol}\,(r\,B_n{}^{\star} + L)^{\frac{1}{n}} \le \mathrm{vol}\,(K^{\star} + L)^{\frac{1}{n}} \le c\mathrm{vol}\,(rB_n{}^{\star} + L)^{\frac{1}{n}}$$

6.21 Theorem* [Milman]. *There exists an absolute $c > 0$ such that for all $K \in \mathcal{K}_o^n$ there exists an $A \in \mathrm{SL}(n, \mathbb{R})$ such that AK is in M-position.*

6.22 Theorem [Milman's reverse Brunn-Minkowski inequality]. *Let $K, L \in \mathcal{K}_o^n$. There exists $A_K, A_L \in \mathrm{SL}(n, \mathbb{R})$ such that for $t > 0$*

$$\mathrm{vol}\,(A_K\,K + t\,A_l\,L)^{\frac{1}{n}} \le C\left(\mathrm{vol}\,(K)^{\frac{1}{n}} + t\,\mathrm{vol}\,(L)^{\frac{1}{n}}\right).$$

Here C is an absolute constant independent of n and K, L.

6.23 Lemma. *Let $K \in \mathcal{K}_o^n$, $\mathrm{vol}\,(K) = \mathrm{vol}\,(B_n)$ and let*

$$\alpha(K) = \max\left\{\mathrm{N}\,(K, B_n), \mathrm{N}\,(B_n, K), \mathrm{N}\,(K^{\star}, B_n), \mathrm{N}\,(B_n, K^{\star})\right\}.$$

i) *If K is in M-position with constant c then $\alpha(K) \le \mathrm{e}^{\ln(3c)n}$.*

ii) *If $\alpha(K) \le \mathrm{e}^{\beta\,n}$ then K is in M-position with constant $\overline{c}\,\beta$, where $\overline{c}$ is an absolute constant.*

6.24 Theorem* [Bourgain-Milman]. *There exists an absolute $c > 0$ such that for $K \in \mathcal{K}_o^n$*

$$\mathrm{vol}\,(K)\mathrm{vol}\,(K^{\star}) \ge c^n\,\mathrm{vol}\,(B_n)^2.$$

6.25 Theorem. *There exists an absolute $c > 0$ such that for $K, L \in \mathcal{K}_o^n$*

$$c^{-n}\mathrm{N}\,(L^{\star}, K^{\star}) \le \mathrm{N}\,(K, L) \le c^n\mathrm{N}\,(L^{\star}, K^{\star}).$$

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