

Convex Geometry II

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1 Symmetrisations

1.1 Definition [Steiner-Symmetrisation]. Let $K \in \mathcal{K}^n$, let $H \subset \mathbb{R}^n$ be a hyperplane with normal $\mathbf{a} \in S^{n-1}$ and let $K|H$ be the orthogonal projection of K onto H . For $\mathbf{x} \in K|H$, let $K \cap (\mathbf{x} + \text{lin}\{\mathbf{a}\}) = \text{conv}\{\mathbf{v}_x, \mathbf{w}_x\}$. The set

$$\text{st}_H(K) = \bigcup_{\mathbf{x} \in K|H} \left[\left(\mathbf{x} - \frac{1}{2}(\mathbf{v}_x + \mathbf{w}_x) \right) + \text{conv}\{\mathbf{v}_x, \mathbf{w}_x\} \right]$$

is called the Steiner-Symmetral of K with respect to H .

1.2 Definition [Inradius, Circumradius, Diameter]. For $K \in \mathcal{K}^n$,

- i) $r(K) = \max\{r \geq 0 : \mathbf{x} + rB_n \subseteq K, \mathbf{x} \in \mathbb{R}^n\}$ is the inradius of K ,
- ii) $R(K) = \min\{R \geq 0 : K \subseteq \mathbf{x} + RB_n, \mathbf{x} \in \mathbb{R}^n\}$ is the circumradius of K ,
- iii) $D(K) = \max\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{x}, \mathbf{y} \in K\} = \max\{h(K, \mathbf{u}) + h(K, -\mathbf{u}) : \mathbf{u} \in S^{n-1}\}$ is the diameter of K .
- iv) $w(K) = \min\{h(K, \mathbf{u}) + h(K, -\mathbf{u}) : \mathbf{u} \in S^{n-1}\}$ is the minimal width of K .

1.3 Proposition. Let $K, M \in \mathcal{K}^n$ and let H be a hyperplane.

- i) $\text{st}_H(K) \in \mathcal{K}^n$.
- ii) $\text{st}_H(\lambda K) = \lambda \text{st}_H(K)$ for $\lambda \geq 0$, up to translations.
- iii) $\text{st}_H(K) + \text{st}_H(M) \subseteq \text{st}_H(K + M)$, up to translations.
- iv) If $K \subseteq M$ then $\text{st}_H(K) \subseteq \text{st}_H(M)$.
- v) $\text{vol}(\text{st}_H(K)) = \text{vol}(K)$.
- vi) $F(\text{st}_H(K)) \leq F(K)$.
- vii) $D(\text{st}_H(K)) \leq D(K)$.
- viii) $r(\text{st}_H(K)) \geq r(K)$ and $R(\text{st}_H(K)) \leq R(K)$.
- ix) $\text{st}_H : \{K \in \mathcal{K}^n : \dim K = n\} \rightarrow \{K \in \mathcal{K}^n : \dim K = n\}$ is a continuous map. The sequence $K_i = \text{conv}\{(\frac{1}{i}, 0)^\top, (0, 1)^\top\} \subset \mathbb{R}^2$ shows that the assumption $\dim K = n$ is needed.

Proof. Without loss of generality we assume that $\mathbf{0} \in H$ and let $L = \text{lin}\{\mathbf{a}\}$ be the orthogonal complement of H . ii) and iv) are certainly true.

i) Obviously, $\text{st}_H(K)$ is a compact set (cf. Exercises). For the convexity of $\text{st}_H(K)$, let $\mathbf{x}, \mathbf{y} \in \text{st}_H(K)$ and consider the convex trapezoid $T = \text{conv}\{K \cap (\mathbf{x} + L), K \cap (\mathbf{y} + L)\} \subseteq K$. Clearly $\text{st}_H(T)$ is also a convex trapezoid with $\mathbf{x}, \mathbf{y} \in \text{st}_H(T)$, and so we have $\text{conv}\{\mathbf{x}, \mathbf{y}\} \subseteq \text{st}_H(T) \subseteq \text{st}_H(K)$.

iii) Let $\mathbf{x} \in \text{st}_H(K)$ and $\mathbf{y} \in \text{st}_H(M)$. Then we can write $\mathbf{x} = \bar{\mathbf{x}} + l_x$ and $\mathbf{y} = \bar{\mathbf{y}} + l_y$, with $\bar{\mathbf{x}} = \mathbf{x}|_H$, $\bar{\mathbf{y}} = \mathbf{y}|_H$ and $l_x, l_y \in L$. Then

$$\mathbf{x} + \mathbf{y} = \bar{\mathbf{x}} + \bar{\mathbf{y}} + (l_x + l_y),$$

and since $\mathbf{0} \in H$ we have $\bar{\mathbf{x}} + \bar{\mathbf{y}} \in (K + M)|_H$. It remains to show that $\|l_x + l_y\| \leq \frac{1}{2} \text{vol}_1((K + M) \cap (\mathbf{x} + \mathbf{y} + L))$, which follows from the observation

$$\begin{aligned} \|l_x + l_y\| &\leq \|l_x\| + \|l_y\| \leq \frac{1}{2} \text{vol}_1(K \cap (\mathbf{x} + L)) + \frac{1}{2} \text{vol}_1(M \cap (\mathbf{y} + L)) \\ &= \frac{1}{2} \text{vol}_1([K \cap (\mathbf{x} + L)] + [M \cap (\mathbf{y} + L)]) \\ &\leq \frac{1}{2} \text{vol}_1((K + M) \cap (\mathbf{x} + \mathbf{y} + L)). \end{aligned}$$

v) It is an immediate consequence of Fubini's theorem:

$$\text{vol}(K) = \int_{K|_H} \text{vol}_1(K \cap (\mathbf{x} + L)) \, d\mathbf{x} = \text{vol}(\text{st}_H(K)).$$

vi) By (ii) and (iii) we have $\text{st}_H(K) + \lambda B_n = \text{st}_H(K) + \lambda \text{st}_H(B_n) \subseteq \text{st}_H(K + \lambda B_n)$ for any $\lambda > 0$, and with Remark 5.31¹ and property (v) we get

$$\begin{aligned} \text{F}(\text{st}_H(K)) &= \lim_{\lambda \rightarrow 0} \frac{\text{vol}(\text{st}_H(K) + \lambda B_n) - \text{vol}(\text{st}_H(K))}{\lambda} \\ &\leq \lim_{\lambda \rightarrow 0} \frac{\text{vol}(\text{st}_H(K + \lambda B_n)) - \text{vol}(\text{st}_H(K))}{\lambda} \\ &= \lim_{\lambda \rightarrow 0} \frac{\text{vol}(K + \lambda B_n) - \text{vol}(K)}{\lambda} = \text{F}(K). \end{aligned}$$

vii) Let $\mathbf{x}, \mathbf{y} \in \text{st}_H(K)$ and let $T, \text{st}_H(T)$ be the convex trapezoids as in the proof of (i). Then one of the diagonals (or both) of T has length greater than or equal to $\|\mathbf{x} - \mathbf{y}\|$, which proves the assertion.

viii) Let $\mathbf{x} \in \mathbb{R}^n$ be such that $\mathbf{x} + r(K)B_n \subseteq K$. Then, using ii), iii) and iv) we get $\text{st}_H(K) \supseteq \text{st}_H(\mathbf{x} + r(K)B_n) \supseteq \text{st}_H(\mathbf{x}) + r(K) \text{st}_H(B_n) = \text{st}_H(\mathbf{x}) + r(K)B_n$. Hence, $r(\text{st}_H(K)) \geq r(K)$, and similarly, $R(\text{st}_H(K)) \leq R(K)$.

ix) Let $K_i \in \mathcal{K}^n$, $i \in \mathbb{N}$, be a sequence of full-dimensional convex bodies converging in the Hausdorff metric to $K \in \mathcal{K}^n$, with $\dim K = n$. Without loss of generality let $\mathbf{0} \in \text{int} K$. Since $K_i \rightarrow K$, for any $\varepsilon > 0$ we have $(1 - \varepsilon)K \subseteq K_i \subseteq (1 + \varepsilon)K$ (see Exercise) for sufficiently large i . Hence, by ii) and iv) we get $(1 - \varepsilon)\text{st}_H(K) \subseteq \text{st}_H(K_i) \subseteq (1 + \varepsilon)\text{st}_H(K)$ for sufficiently large i . Hence $\text{st}_H(K_i) \rightarrow \text{st}_H(K)$. \square

1.4 Remark. Regarding the monotonicity of the surface area (Proposition 1.3 vi)) it is also known that $\text{F}(\text{st}_H(K)) = \text{F}(K)$ if and only if K is symmetric to a hyperplane parallel to H .

¹Skript WS14

1.5 Theorem. *Let $K \in \mathcal{K}^n$ with $\dim K = n$. Let \mathcal{S}_K be the set of all convex bodies which are obtained by finitely many successive Steiner-symmetrisations of K with respect to hyperplanes containing $\mathbf{0}$. Then there exists a sequence $K_i \in \mathcal{S}_K$, $i \in \mathbb{N}$, such that*

$$\lim_{i \rightarrow \infty} K_i = \left(\frac{\text{vol}(K)}{\text{vol}(B_n)} \right)^{1/n} B_n.$$

Proof. For $M \in \mathcal{K}^n$ let $\rho(M) = \min\{R > 0 : M \subseteq RB_n\}$, let $\rho = \inf\{\rho(M) : M \in \mathcal{S}_K\}$ and let $K_i \in \mathcal{S}_K$ with $\rho(K_i) \rightarrow \rho$. Since $K_i \subset \text{rho}(K_i)B_n \subset \rho(K)B_n$ (cf. Proposition 1.3 iv)), we may assume by Blaschke's selection Theorem 4.9² that $K_i \rightarrow \overline{K} \in \mathcal{K}^n$, say. In the following we show

$$\overline{K} = \rho B_n,$$

which implies the assertion, since then $K_i \rightarrow \rho B_n$ and $\rho^n \text{vol}(B_n) = \text{vol}(\overline{K}) = \lim_{i \rightarrow \infty} \text{vol}(K_i) = \text{vol}(K)$. (cf.)

Suppose $\overline{K} \neq \rho B_n$. By the continuity of $\rho(\cdot)$ we have $\rho(\overline{K}) = \lim_{i \rightarrow \infty} \rho(K_i) = \rho$, and so we have $\overline{K} \subset \rho B_n$ (strictly). Hence we might choose $\mathbf{x} \in \rho B_n \setminus \overline{K}$ and a hyperplane $H(\mathbf{a}, \alpha)$ strictly separating \overline{K} and \mathbf{x} . Let $\langle \mathbf{a}, \mathbf{x} \rangle > \alpha$, and let $C = \{\mathbf{y} \in \rho S^{n-1} : \langle \mathbf{a}, \mathbf{y} \rangle \geq \alpha\}$. Let H_1, \dots, H_k be hyperplanes such that the successive reflections of C with respect to these planes, i.e., first we reflect C at H_1 then the image at H_2 and so on, cover ρS^{n-1} . Then for $\widehat{K} = \text{st}_{H_k} \text{st}_{H_{k-1}} \cdots \text{st}_{H_1}(\overline{K})$ we have $\widehat{K} \subset \text{int}(\rho B_n)$, i.e., $\rho(\widehat{K}) < \rho$.

On the other hand, by Proposition 1.3 ix) we know

$$\text{st}_{H_k} \cdots \text{st}_{H_1}(K_i) \rightarrow \widehat{K},$$

and since the bodies on the left side belong to \mathcal{S}_K and ρ is continuous we arrive at the contradiction $\rho(\widehat{K}) \geq \rho$. \square

1.6 Corollary. *Let $K_1, K_2 \in \mathcal{K}^n$ with $\dim K_1 = \dim K_2 = n$. Then there exist Steiner-symmetrisations st_{H_i} , $i \in \mathbb{N}$, such that*

$$\begin{aligned} \lim_{i \rightarrow \infty} \text{st}_{H_i} \cdots \text{st}_{H_1}(K_1) &= \left(\frac{\text{vol}(K_1)}{\text{vol}(B_n)} \right)^{1/n} B_n \\ \lim_{i \rightarrow \infty} \text{st}_{H_i} \cdots \text{st}_{H_1}(K_2) &= \left(\frac{\text{vol}(K_2)}{\text{vol}(B_n)} \right)^{1/n} B_n. \end{aligned}$$

Proof. With out loss of generality we assume $\text{vol}(K_1) = \text{vol}(K_2) = \text{vol}(B_n)$. Let $\varepsilon > 0$. By Theorem 1.5 there exist hyperplanes H_1, \dots, H_k containing $\mathbf{0}$ such that

$$(1 - \varepsilon)B_n \subseteq \text{st}_{H_k} \cdots \text{st}_{H_1}(K_1) \subseteq (1 + \varepsilon)B_n.$$

Applying again Theorem 1.5 with respect to $\text{st}_{H_k} \cdots \text{st}_{H_1}(K_2)$, we get hyperplanes H_{k+1}, \dots, H_l containing the origin such that

$$(1 - \varepsilon)B_n \subseteq \text{st}_{H_l} \cdots \text{st}_{H_{k+1}}(\text{st}_{H_k} \cdots \text{st}_{H_1}(K_2)) \subseteq (1 + \varepsilon)B_n.$$

²Skript WS14

Due to Proposition 1.3 ii) and iv) we also know that for this sequence of the symmetrisations with respect to K_1 still holds

$$(1 - \varepsilon)B_n \subseteq \text{st}_{H_1} \cdots \text{st}_{H_{k+1}}(\text{st}_{H_k} \cdots \text{st}_{H_1}(K_1)) \subseteq (1 + \varepsilon)B_n.$$

Thus applying the same argumentation to $\text{st}_{H_1} \cdots \text{st}_{H_1}(K_1)$ and $\text{st}_{H_1} \cdots \text{st}_{H_1}(K_2)$ with $\varepsilon/2, \varepsilon/3 \dots$ we get the required sequence of hyperplanes. \square

1.7 Corollary. *Let $K, M \in \mathcal{K}^n$, $\dim K = \dim M = n$. Then*

- i) isodiametric inequality: $\text{vol}(K) \leq \left(\frac{D(K)}{2}\right)^n \text{vol}(B_n)$
(and equality holds only for a ball).
- ii) isoperimetric inequality: $F(K)^n / \text{vol}(K)^{n-1} \geq F(B_n)^n / \text{vol}(B_n)^{n-1}$ and equality holds only for a ball.
- iii) Brunn-Minkowski inequality: $\text{vol}(K + M)^{1/n} \geq \text{vol}(K)^{1/n} + \text{vol}(M)^{1/n}$.

Proof. i) Let \bar{K} be the convex body which is obtained by applying successive Steiner-symmetrisations to K with respect to the coordinate hyperplanes. Then \bar{K} is symmetric with respect to the origin. Thus $R(\bar{K}) = D(\bar{K})/2$ and $\bar{K} \subseteq (D(\bar{K})/2)B_n$ and by Proposition 1.3 v), vii) we get

$$\text{vol}(K) = \text{vol}(\bar{K}) \leq \left(\frac{D(\bar{K})}{2}\right)^n \text{vol}(B_n) \leq \left(\frac{D(K)}{2}\right)^n \text{vol}(B_n).$$

ii) We may assume $\text{vol}(K) = \text{vol}(B_n)$. By Corollary 1.6 there exist hyperplanes H_i , $i \in \mathbb{N}$, containing the origin, such that

$$\lim_{i \rightarrow \infty} \text{st}_{H_i} \cdots \text{st}_{H_1}(K) = B_n.$$

Then by Proposition 1.3 vi) and by the continuity of the surface area F we get

$$F(B_n) = \lim_{i \rightarrow \infty} F(\text{st}_{H_i} \cdots \text{st}_{H_1}(K)) \leq F(K).$$

In the case of equality we must have $K = B_n$, since otherwise, by Remark 1.4 we can find a Steiner-symmetrisation strictly decreasing the surface area.

iii) By Corollary 1.6 there exist hyperplanes H_i , $i \in \mathbb{N}$, containing the origin, such that

$$\text{st}_{H_i} \cdots \text{st}_{H_1}(K) \rightarrow \left(\frac{\text{vol}(K)}{\text{vol}(B_n)}\right)^{1/n} B_n \text{ and } \text{st}_{H_i} \cdots \text{st}_{H_1}(M) \rightarrow \left(\frac{\text{vol}(M)}{\text{vol}(B_n)}\right)^{1/n} B_n.$$

By Proposition 1.3 iii), v) and the continuity of the volume (Lemma 5.12³) we

³Skript WS14

find

$$\begin{aligned}
\text{vol}(K + M) &= \lim_{i \rightarrow \infty} \text{vol}(\text{st}_{H_i} \cdots \text{st}_{H_1}(K + M)) \\
&\geq \lim_{i \rightarrow \infty} \text{vol}(\text{st}_{H_i} \cdots \text{st}_{H_1}(K) + \text{st}_{H_i} \cdots \text{st}_{H_1}(M)) \\
&= \text{vol} \left(\left(\frac{\text{vol}(K)}{\text{vol}(B_n)} \right)^{1/n} B_n + \left(\frac{\text{vol}(M)}{\text{vol}(B_n)} \right)^{1/n} B_n \right) \\
&= \text{vol} \left(\frac{\text{vol}(K)^{1/n} + \text{vol}(M)^{1/n}}{\text{vol}(B_n)^{1/n}} B_n \right) \\
&= \left(\text{vol}(K)^{1/n} + \text{vol}(M)^{1/n} \right)^n.
\end{aligned}$$

□

1.8 Lemma. Let $K \in \mathcal{K}_0^n$ with $\dim K = n$ and let H be a hyperplane containing $\mathbf{0}$. Then

$$\text{vol}(\text{st}_H(K)^*) \geq \text{vol}(K^*).$$

Proof. Without loss of generality let $H = \{\mathbf{x} \in \mathbb{R}^n : x_n = 0\}$. Then

$$\text{st}_H(K) = \left\{ \left(\mathbf{x}, \frac{1}{2}(a-b) \right)^\top \in \mathbb{R}^n : (\mathbf{x}, 0)^\top \in K|H, (\mathbf{x}, a)^\top, (\mathbf{x}, b)^\top \in K \right\}$$

and we also may write

$$\begin{aligned}
K^* &= \{(\mathbf{y}, t)^\top \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{y} \rangle + st \leq 1, (\mathbf{x}, 0)^\top \in K|H, (\mathbf{x}, s)^\top \in K\}, \\
\text{st}_H(K)^* &= \left\{ (\mathbf{y}, t)^\top \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{y} \rangle + \frac{1}{2}(a-b)t \leq 1, \right. \\
&\quad \left. (\mathbf{x}, 0)^\top \in K|H, (\mathbf{x}, a)^\top, (\mathbf{x}, b)^\top \in K \right\}.
\end{aligned}$$

For $A \subset \mathbb{R}^n$, let $A(t) = \{\mathbf{x} \in \mathbb{R}^{n-1} : (\mathbf{x}, t)^\top \in A\}$. Next we observe that

$$\frac{1}{2}(K^*(t) + K^*(-t)) \subseteq \text{st}_H(K)^*(t), \quad (1.8.1)$$

i.e., for $\mathbf{y}_1 \in K^*(t)$ and $\mathbf{y}_2 \in K^*(-t)$, we have to show $((\mathbf{y}_1 + \mathbf{y}_2)/2, t)^\top \in \text{st}_H(K)^*$. To this end let $(\mathbf{x}, a)^\top, (\mathbf{x}, b)^\top \in K$, then

$$\left\langle \mathbf{x}, \frac{1}{2}(\mathbf{y}_1 + \mathbf{y}_2) \right\rangle + \frac{1}{2}(a-b)t = \frac{1}{2} \langle \mathbf{x}, \mathbf{y}_1 \rangle + \frac{1}{2} \langle \mathbf{x}, \mathbf{y}_2 \rangle + \frac{1}{2}at + \frac{1}{2}b(-t) \leq 1,$$

and this shows (1.8.1). Moreover, since K is o -symmetric, $K^* \in \mathcal{K}_0^n$ and hence $K^*(t) = -K^*(-t)$, and so $\text{vol}_{n-1}(K^*(t)) = \text{vol}_{n-1}(K^*(-t))$. Thus, (1.8.1) and Brunn-Minkowski inequality imply for all $t \in \mathbb{R}$

$$\begin{aligned}
\text{vol}_{n-1}(\text{st}_H(K)^*(t))^{1/(n-1)} &\geq \text{vol}_{n-1} \left(\frac{1}{2}(K^*(t) + K^*(-t)) \right)^{1/(n-1)} \\
&\geq \frac{1}{2} \text{vol}_{n-1}(K^*(t))^{1/(n-1)} + \frac{1}{2} \text{vol}_{n-1}(K^*(-t))^{1/(n-1)} \\
&= \text{vol}_{n-1}(K^*(t))^{1/(n-1)}.
\end{aligned}$$

Integrating over t gives $\text{vol}(\text{st}_H(K)^*) \geq \text{vol}(K^*)$. □

1.9 Theorem [Blaschke-Santaló]. ^{4,5} Let $K \in \mathcal{K}_0^n$ with $\dim K = n$. Then

$$\operatorname{vol}(K) \operatorname{vol}(K^*) \leq \operatorname{vol}(B_n)^2,$$

(and equality holds if and only if K is an ellipsoid).

Proof. Let $\operatorname{vol}(K) = \operatorname{vol}(B_n)$. It is to show $\operatorname{vol}(K^*) \leq \operatorname{vol}(B_n)$. By Corollary 1.6 there exist hyperplanes H_i , $i \in \mathbb{N}$, containing the origin such that

$$K_i = \operatorname{st}_{H_i} \cdots \operatorname{st}_{H_1}(K) \rightarrow B_n.$$

By Lemma 1.8 we know

$$\begin{aligned} \operatorname{vol}(K^*) &\leq \operatorname{vol}(\operatorname{st}_{H_1}(K)^*) = \operatorname{vol}(K_1^*) \\ &\leq \operatorname{vol}(\operatorname{st}_{H_2}(K_1)^*) = \operatorname{vol}(K_2^*) \leq \dots \leq \operatorname{vol}(K_i^*) \end{aligned}$$

for every $i \in \mathbb{N}$. Since $K_i^* \rightarrow B_n$, as well, we get by the continuity of the volume

$$\operatorname{vol}(K^*) \leq \lim_{i \rightarrow \infty} \operatorname{vol}(K_i^*) = \operatorname{vol}(B_n).$$

□

1.10 Proposition. ^{6,7} Let $K \in \mathcal{K}_0^n$ with $\dim K = n$. Then

$$\operatorname{vol}(K) \operatorname{vol}(K^*) \geq \frac{4^n}{(n!)^2}.$$

Proof. Without loss of generality we may assume that $C_n^* = \operatorname{conv}\{\pm e_i : 1 \leq i \leq n\} \subseteq K$ has maximum volume amongst all AC_n^* with $A \in \operatorname{GL}(n, \mathbb{R})$. This implies that $K \subseteq C_n$, since otherwise there exists a $\mathbf{v} \in K$ with $v_j > 1$, say, and replacing $\pm e_j$ by $\pm \mathbf{v}$ yields a crosspolytope in K having larger volume than C_n^* . Hence, by polarity we also have $C_n^* \subseteq K^*$ and so

$$\operatorname{vol}(K) \operatorname{vol}(K^*) \geq \operatorname{vol}(C_n^*)^2 = \left(\frac{2^n}{n!}\right)^2.$$

□

1.11 Definition [Schwarz symmetral]. ⁸ Let $K \in \mathcal{K}^n$, let L be a $(n - k)$ -dimensional subspace, $1 \leq k \leq n - 1$, with orthogonal complement L^\perp . For any $\mathbf{y} \in K|L$ (orthogonal projection of K onto L) let $B_k(\mathbf{y}, r_y) \subset \mathbf{y} + L^\perp$ be the k -dimensional ball with center \mathbf{y} and radius r_y such that $\operatorname{vol}_k(B_k(\mathbf{y}, r_y)) = \operatorname{vol}_k(K \cap (\mathbf{y} + L^\perp))$. Then

$$S_L(K) = \bigcup_{\mathbf{y} \in K|L} B_k(\mathbf{y}, r_y)$$

is called the Schwarz symmetral of K with respect to L .

⁴Wilhelm Blaschke, 1885–1962.

⁵Luis Santaló, 1911–2001.

⁶The Mahler conjecture states $\operatorname{vol}(K) \operatorname{vol}(K^*) \geq \frac{4^n}{n!}$.

⁷Kurt Mahler, 1903 – 1988

⁸Hermann Schwarz, 1843 – 1921

1.12 Proposition. $S_L(K)$ is a convex body.

Proof. $S_L(K)$ is certainly bounded, and in order to show that it is also closed let $\mathbf{z}_i \in S_L(K)$ be a sequence converging to \mathbf{z} . Let \mathbf{y}_i and \mathbf{y} be the projections of \mathbf{z}_i and \mathbf{z} onto L , respectively. Then $\mathbf{y}_i \rightarrow \mathbf{y}$ and to due the continuity of $\text{vol}_k((\mathbf{y}_i + L^\perp) \cap K)$ on its support $K|L$ we get

$$\begin{aligned} \|\mathbf{z} - \mathbf{y}\| &= \lim_{m \rightarrow \infty} \|\mathbf{z}_i - \mathbf{y}_i\| \\ &\leq \limsup_{m \rightarrow \infty} \left(\frac{\text{vol}_k((\mathbf{y}_i + L^\perp) \cap K)}{\text{vol}_k(B_k)} \right)^{1/k} = \left(\frac{\text{vol}_k((\mathbf{y} + L^\perp) \cap K)}{\text{vol}_k(B_k)} \right)^{1/k}. \end{aligned}$$

Hence, $\mathbf{z} \in S_L(K)$. It remains to establish the convexity of $S_L(K)$. Let $\mathbf{z}_1, \mathbf{z}_2 \in S_L(K)$, and let $\mathbf{y}_i = \mathbf{z}_i|L$, $i = 1, 2$. Then for $\lambda \in [0, 1]$ we have $(\lambda\mathbf{z}_1 + (1 - \lambda)\mathbf{z}_2)|L = \lambda\mathbf{y}_1 + (1 - \lambda)\mathbf{y}_2$ and

$$\begin{aligned} \|\lambda\mathbf{z}_1 + (1 - \lambda)\mathbf{z}_2 - (\lambda\mathbf{y}_1 + (1 - \lambda)\mathbf{y}_2)\| &\leq \lambda\|\mathbf{z}_1 - \mathbf{y}_1\| + (1 - \lambda)\|\mathbf{z}_2 - \mathbf{y}_2\| \\ &\leq \lambda \left(\frac{\text{vol}_k((\mathbf{y}_1 + L^\perp) \cap K)}{\text{vol}_k(B_k)} \right)^{1/k} + (1 - \lambda) \left(\frac{\text{vol}_k((\mathbf{y}_2 + L^\perp) \cap K)}{\text{vol}_k(B_k)} \right)^{1/k} \\ &\leq \left(\frac{\text{vol}_k(\lambda[(\mathbf{y}_1 + L^\perp) \cap K] + (1 - \lambda)[(\mathbf{y}_2 + L^\perp) \cap K])}{\text{vol}_k(B_k)} \right)^{1/k}, \end{aligned}$$

where for the last inequality we use the Brunn-Minkowski theorem. Since $\lambda[(\mathbf{y}_1 + L) \cap K] + (1 - \lambda)[(\mathbf{y}_2 + L) \cap K] \subseteq (\lambda\mathbf{y}_1 + (1 - \lambda)\mathbf{y}_2 + L) \cap K$ we are done. \square

1.13 Definition [Central symmetrisation]. Let $K \in \mathcal{K}^n$. The central symmetrisation of K is defined as the body $\frac{1}{2}(K - K)$.

1.14 Proposition. Let $K \in \mathcal{K}^n$ and $\mathbf{u} \in S^{n-1}$. Then

$$h(K, \mathbf{u}) + h(K, -\mathbf{u}) = h\left(\frac{1}{2}(K - K), \mathbf{u}\right) + h\left(\frac{1}{2}(K - K), -\mathbf{u}\right).$$

In particular, $D(K) = D\left(\frac{1}{2}(K - K)\right)$ and $w(K) = w\left(\frac{1}{2}(K - K)\right)$.

1.15 Theorem [Rogers&Shephard].⁹ Let $K \in \mathcal{K}^n$ with $\dim K = n$. Then

$$\text{vol}(K) \leq \text{vol}\left(\frac{1}{2}(K - K)\right) \leq \frac{1}{2^n} \binom{2n}{n} \text{vol}(K).$$

The lower bound is attained if and only if K is centrally symmetric, (and the upper bound if and only if K is a simplex).

Proof. The lower bound is an immediate consequence of the Brunn-Minkowski inequality

$$\text{vol}\left(\frac{1}{2}(K - K)\right) \geq \left(\frac{1}{2}\text{vol}(K)^{1/n} + \frac{1}{2}\text{vol}(-K)^{1/n}\right)^n = \text{vol}(K).$$

⁹Geoffrey Shephard, 1927

Since K is full-dimensional, equality holds here if and only if K and $-K$ are homothetic (up to translations), i.e., if and only if $K = -K + \mathbf{t}$ for some $\mathbf{t} \in \mathbb{R}^n$, i.e., K is symmetric to $\mathbf{t}/2$.

For the upper bound let $\chi_K(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ be the characteristic function of K . Then

$$\begin{aligned}
 \text{vol}(K)^2 &= \int_{\mathbb{R}^n} \chi_K(\mathbf{y}) \text{vol}(K) \, d\mathbf{y} \\
 &= \int_{\mathbb{R}^n} \chi_K(\mathbf{y}) \left(\int_{\mathbb{R}^n} \chi_K(\mathbf{y} - \mathbf{x}) \, d\mathbf{x} \right) \, d\mathbf{y} \\
 &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \chi_K(\mathbf{y} - \mathbf{x}) \chi_K(\mathbf{y}) \, d\mathbf{y} \right) \, d\mathbf{x} \quad (1.15.1) \\
 &= \int_{\mathbb{R}^n} \text{vol}(K \cap (\mathbf{x} + K)) \, d\mathbf{x} \\
 &= \int_{K-K} \text{vol}(K \cap (\mathbf{x} + K)) \, d\mathbf{x}.
 \end{aligned}$$

Now, for any $\mathbf{x} \in K - K$ let $\rho = \rho(\mathbf{x}) \in [0, 1]$ such that $\mathbf{x} \in \rho \text{bd}(K - K)$, i.e., $\rho = |\mathbf{x}|_{K-K}$ is just the norm of \mathbf{x} induced by $K - K$. Let $\mathbf{x} = \rho(\mathbf{v} - \mathbf{w})$ with $\mathbf{v}, \mathbf{w} \in K$. Then

$$(1 - \rho)K + \rho\mathbf{v} \subseteq K \text{ and } (1 - \rho)K + \rho\mathbf{v} = (1 - \rho)K + \rho\mathbf{w} + \mathbf{x} \subseteq \mathbf{x} + K.$$

Hence $\text{vol}(K \cap (\mathbf{x} + K)) \geq \text{vol}((1 - \rho)K + \rho\mathbf{v}) = (1 - \rho)^n \text{vol}(K)$, and so with (1.15.1)

$$\text{vol}(K)^2 \geq \int_{K-K} (1 - \rho(\mathbf{x}))^n \text{vol}(K) \, d\mathbf{x},$$

which leads to

$$\begin{aligned}
 \text{vol}(K) &\geq \int_{K-K} (1 - \rho(\mathbf{x}))^n \, d\mathbf{x} \\
 &= \int_{K-K} \left(\int_{\rho(\mathbf{x})}^1 n(1-t)^{n-1} \, dt \right) \, d\mathbf{x} \\
 &= \int_0^1 \left(\int_{\{\mathbf{x} \in K-K : \rho(\mathbf{x}) \leq t\}} n(1-t)^{n-1} \, d\mathbf{x} \right) \, dt \\
 &= \int_0^1 n(1-t)^{n-1} \text{vol}(t(K - K)) \, dt \\
 &= \text{vol}(K - K) \int_0^1 n(1-t)^{n-1} t^n \, dt \\
 &= \frac{(n!)^2}{(2n)!} \text{vol}(K - K).
 \end{aligned}$$

□

2 Brunn-Minkowski revisited

2.1 Theorem [Brunn-Minkowski Inequality]. *Let $K, L \subset \mathcal{K}^n$, and $\lambda \in [0, 1]$. Then*

$$\text{vol}(\lambda K + (1 - \lambda)L)^{1/n} \geq \lambda \text{vol}(K)^{1/n} + (1 - \lambda)\text{vol}(L)^{1/n}, \quad (2.1.1)$$

and for $0 < \lambda < 1$ equality holds if and only if either K and L are homothetic or K and L lie in parallel hyperplanes.

Proof. For $\lambda \in [0, 1]$ let $K_\lambda = \lambda K + (1 - \lambda)L$. First we check that the theorem holds for convex bodies $K, L \in \mathcal{K}^n$ of which one is lower dimensional. With out loss of generality let $\dim L < n$.

If also $\dim K < n$ then $\text{vol}(K) = \text{vol}(L) = 0$ and (2.1.1) holds trivially. If equality holds for a $\lambda \in (0, 1)$ then $\text{vol}(K_\lambda) = 0$ and so K_λ lies in a hyperplane H , say. But this implies that K and L lie in hyperplanes parallel to H . On the other hand, if K and L lie in parallel hyperplanes then also K_λ is contained in a hyperplane and $\text{vol}(K_\lambda) = 0$ for $0 \leq \lambda \leq 1$.

Now suppose $\dim K = n$. Since $K_\lambda \supseteq \lambda K + (1 - \lambda)\mathbf{x}$ for any $\mathbf{x} \in L$, we have $\text{vol}(K_\lambda) \geq \text{vol}(\lambda K + (1 - \lambda)\mathbf{x}) = \lambda^n \text{vol}(K)$ with equality if and only if $L = \{\mathbf{x}\}$ and then K and L are homothetic.

Thus, in the following let $\dim K = \dim L = n$, and next we observe that it is sufficient to prove (2.1.1) in the particular situation when $\text{vol}(K) = \text{vol}(L) = 1$. The general case can be reduced to this setting via the normalisation $\bar{K} = \text{vol}(K)^{-1/n}K$, $\bar{L} = \text{vol}(L)^{-1/n}L$, and

$$\bar{\lambda} = \frac{\lambda \text{vol}(K)^{1/n}}{\lambda \text{vol}(K)^{1/n} + (1 - \lambda)\text{vol}(L)^{1/n}},$$

since

$$\begin{aligned} & \text{vol}(\bar{\lambda}\bar{K} + (1 - \bar{\lambda})\bar{L}) \\ &= \text{vol}\left(\frac{\lambda}{\lambda \text{vol}(K)^{1/n} + (1 - \lambda)\text{vol}(L)^{1/n}}K + \frac{1 - \lambda}{\lambda \text{vol}(K)^{1/n} + (1 - \lambda)\text{vol}(L)^{1/n}}L\right) \\ &= \frac{1}{(\lambda \text{vol}(K)^{1/n} + (1 - \lambda)\text{vol}(L)^{1/n})^n} \text{vol}(\lambda K + (1 - \lambda)L). \end{aligned}$$

So let $\text{vol}(K) = \text{vol}(L) = 1$ and we have to show $\text{vol}(K_\lambda) \geq 1$. This will be done by induction on the dimension. If $n = 1$ the result is certainly true, since K and L are intervals which are homothetic. Let $n \geq 2$. For an arbitrary but fixed $\mathbf{u} \in S^{n-1}$ and for $\mu \in \mathbb{R}$ we set

$$\begin{aligned} v_K(\mu) &= \text{vol}_{n-1}(K \cap H(\mathbf{u}, \mu)), & v_L(\mu) &= \text{vol}_{n-1}(L \cap H(\mathbf{u}, \mu)), \\ w_K(\mu) &= \text{vol}(K \cap H^-(\mathbf{u}, \mu)), & w_L(\mu) &= \text{vol}(L \cap H^-(\mathbf{u}, \mu)). \end{aligned}$$

Then

$$w_K(\mu) = \int_{-h(K, -\mathbf{u})}^{\mu} v_K(s) \, ds \quad \text{and} \quad w_L(\mu) = \int_{-h(L, -\mathbf{u})}^{\mu} v_L(s) \, ds.$$

The functions v_K, v_L are continuous on the intervals $(-h(K, -\mathbf{u}), h(K, \mathbf{u}))$ and $(-h(L, -\mathbf{u}), h(L, \mathbf{u}))$, respectively, which ensures that w_K, w_L are differentiable with $w'_K(\mu) = v_K(\mu)$ and $w'_L(\mu) = v_L(\mu)$. Moreover, if we denote by z_K, z_L the inverse function of w_K, w_L respectively, then

$$z'_K(\eta) = \frac{1}{v_K(z_K(\eta))}, \quad z'_L(\eta) = \frac{1}{v_L(z_L(\eta))}, \quad \text{for } 0 < \eta < 1.$$

Now in order to apply induction we compare we compare $(n-1)$ -dimensional sections of the bodies via the function $z_K(\eta)$ and set

$$\begin{aligned} \tilde{K}_\eta &= K \cap H(\mathbf{u}, z_K(\eta)), & \tilde{L}_\eta &= L \cap H(\mathbf{u}, z_L(\eta)), \\ z_\lambda(\eta) &= \lambda z_K(\eta) + (1-\lambda)z_L(\eta). \end{aligned}$$

Observe that $z_\lambda(0) = \lambda(-h(K, -\mathbf{u})) + (1-\lambda)(-h(L, -\mathbf{u})) = -h(K_\lambda, -\mathbf{u})$ and in the same way we find $z_\lambda(1) = h(K_\lambda, \mathbf{u})$. $\tilde{K}_\eta, \tilde{L}_\eta$ are $(n-1)$ -dimensional with $\text{vol}_{n-1}(\tilde{K}_\eta) = v_K(z_K(\eta))$ and $\text{vol}_{n-1}(\tilde{L}_\eta) = v_L(z_L(\eta))$. Since $\lambda \tilde{K}_\eta + (1-\lambda)\tilde{L}_\eta \subseteq K_\lambda \cap H(\mathbf{u}, z_\lambda(\eta))$ we get in view of our induction hypothesis

$$\begin{aligned} \text{vol}(K_\lambda) &= \int_{-h(K_\lambda, -\mathbf{u})}^{h(K_\lambda, \mathbf{u})} \text{vol}_{n-1}(K_\lambda \cap H(\mathbf{u}, \mu)) \, d\mu \\ &= \int_0^1 \text{vol}_{n-1}(K_\lambda \cap H(\mathbf{u}, z_\lambda(\eta))) z'_\lambda(\eta) \, d\eta \\ &\geq \int_0^1 \text{vol}_{n-1}(\lambda \tilde{K}_\eta + (1-\lambda)\tilde{L}_\eta) \left[\frac{\lambda}{v_K(z_K(\eta))} + \frac{1-\lambda}{v_L(z_L(\eta))} \right] \, d\eta \\ &\geq \int_0^1 \left(\lambda v_K(z_K(\eta))^{\frac{1}{n-1}} + (1-\lambda)v_L(z_L(\eta))^{\frac{1}{n-1}} \right)^{n-1} \left[\frac{\lambda}{v_K(z_K(\eta))} + \frac{1-\lambda}{v_L(z_L(\eta))} \right] \, d\eta \\ &\geq \int_0^1 1 \, d\eta = 1, \end{aligned} \tag{2.1.2}$$

where the second to last inequality follows by the (weighted) arithmetic/geometric inequality.

We certainly have equality if K and L are homothetic. Thus, suppose we have equality in (2.1.1). Again we may assume $\text{vol}(K) = \text{vol}(L) = 1$ and so $\text{vol}(K_\lambda) = 1$ for some $\lambda \in (0, 1)$. Then we have equality in (2.1.2) and by the equality case of the arithmetic/geometric inequality we conclude $v_K(z_K(\eta)) = v_L(z_L(\eta))$ for $\eta \in [0, 1]$ (remember v_K, v_L, z_K, z_L are continuous functions). Hence $z'_K(\eta) = z'_L(\eta)$ for $0 \leq \eta \leq 1$, i.e., $z_K(\eta) - z_L(\eta)$ is constant.

Let g_K, g_L be the centroids of K, L , respectively, and let $g_K = g_L = \mathbf{0}$. Then $\mathbf{0} = \int_K \mathbf{x} \, d\mathbf{x}$ and so

$$\begin{aligned} 0 &= \int_K \langle \mathbf{x}, \mathbf{u} \rangle \, d\mathbf{x} \\ &= \int_{-h(K, -\mathbf{u})}^{h(K, \mathbf{u})} t \, \text{vol}_{n-1}(K \cap H(\mathbf{u}, t)) \, dt \\ &= \int_0^1 \text{vol}_{n-1}(K \cap H(\mathbf{u}, z_K(\eta))) z_K(\eta) \frac{1}{v_K(z_K(\eta))} \, d\eta = \int_0^1 z_K(\eta) \, d\eta. \end{aligned}$$

Analogously we obtain the relation $0 = \int_0^1 z_L(\eta) d\eta$, and thus $z_K(\eta) = z_L(\eta)$ for $0 \leq \eta \leq 1$. Hence, $h(K, \mathbf{u}) = h(L, \mathbf{u})$. By the arbitrariness of $\mathbf{u} \in S^{n-1}$ we conclude $h(K, \mathbf{u}) = h(L, \mathbf{u})$ for all $\mathbf{u} \in S^{n-1}$ and thus $K = L$. \square

2.2 Lemma. *Let $K, L \in \mathcal{K}^n$ and suppose there exists a hyperplane H such that $K|H = L|H$. Then for $\lambda \in [0, 1]$*

$$\text{vol}(\lambda K + (1 - \lambda)L) \geq \lambda \text{vol}(K) + (1 - \lambda) \text{vol}(L).$$

Proof. Let $H = H(\mathbf{a}, 0)$ with $\|\mathbf{a}\| = 1$. For abbreviation we write $U = K|H = L|H$, and we set $K_\lambda = \lambda K + (1 - \lambda)L$. Then $K_\lambda|H = \lambda(K|H) + (1 - \lambda)(L|H) = U$ for all $\lambda \in [0, 1]$, and K_λ can be described as

$$K_\lambda = \{\mathbf{y} + t\mathbf{a} : \mathbf{y} \in U, f_\lambda(\mathbf{y}) \leq t \leq g_\lambda(\mathbf{y})\},$$

where the functions f_λ and g_λ satisfy $f_\lambda \leq g_\lambda$, f_λ is convex and g_λ is concave. For $\mathbf{y} \in U$ and $t_1, t_2 \in \mathbb{R}$ with $\mathbf{y} + t_1\mathbf{a} \in K$ and $\mathbf{y} + t_2\mathbf{a} \in L$ we have

$$\mathbf{y} + (\lambda t_1 + (1 - \lambda)t_2)\mathbf{a} = \lambda(\mathbf{y} + t_1\mathbf{a}) + (1 - \lambda)(\mathbf{y} + t_2\mathbf{a}) \in K_\lambda.$$

Hence, $f_\lambda(\mathbf{y}) \leq \lambda t_1 + (1 - \lambda)t_2 \leq g_\lambda(\mathbf{y})$. For $t_1 = f_1(\mathbf{y})$ and $t_2 = f_0(\mathbf{y})$ we obtain $f_\lambda(\mathbf{y}) \leq \lambda f_1(\mathbf{y}) + (1 - \lambda)f_0(\mathbf{y})$, and for $t_1 = g_1(\mathbf{y})$ and $t_2 = g_0(\mathbf{y})$ we get $g_\lambda(\mathbf{y}) \geq \lambda g_1(\mathbf{y}) + (1 - \lambda)g_0(\mathbf{y})$. Therefore

$$\begin{aligned} \text{vol}(K_\lambda) &= \int_U [g_\lambda(\mathbf{y}) - f_\lambda(\mathbf{y})] d\mathbf{y} \\ &\geq \int_U [\lambda g_1(\mathbf{y}) + (1 - \lambda)g_0(\mathbf{y}) - \lambda f_1(\mathbf{y}) - (1 - \lambda)f_0(\mathbf{y})] d\mathbf{y} \\ &= \lambda \int_U [g_1(\mathbf{y}) - f_1(\mathbf{y})] d\mathbf{y} + (1 - \lambda) \int_U [g_0(\mathbf{y}) - f_0(\mathbf{y})] d\mathbf{y} \\ &= \lambda \text{vol}(K) + (1 - \lambda) \text{vol}(L), \end{aligned}$$

which concludes the proof. \square

2.3 Theorem. *Let $K, L \in \mathcal{K}^n$ and suppose there exists a hyperplane H such that $\text{vol}_{n-1}(K|H) = \text{vol}_{n-1}(L|H)$. Then for $\lambda \in [0, 1]$*

$$\text{vol}(\lambda K + (1 - \lambda)L) \geq \lambda \text{vol}(K) + (1 - \lambda) \text{vol}(L).$$

Proof. Let $H = H(\mathbf{a}, 0)$, and let $K' = \text{st}_H(K)$ and $L' = \text{st}_H(L)$ be the Steiner-symmetrals of K and L with respect to H , respectively. By Proposition 1.3¹⁰ we have $\lambda K' + (1 - \lambda)L' \subseteq \text{st}_H(\lambda K + (1 - \lambda)L)$ and since the Steiner symmetrisation preserves the volume, it suffices to prove

$$\text{vol}(\lambda K' + (1 - \lambda)L') \geq \lambda \text{vol}(K') + (1 - \lambda) \text{vol}(L').$$

¹⁰Skript WS14

Observe that $K|H = K' \cap H$ and $L|H = L' \cap H$. According to Corollary 1.6¹¹ we can find hyperplanes $H_i = H(\mathbf{a}_i, 0)$, $\mathbf{a}_i \in H$, $i \in \mathbb{N}$, such that

$$\begin{aligned} \lim_{i \rightarrow \infty} \text{st}_{H_i} \cdots \text{st}_{H_1}(K' \cap H) &= \left(\frac{\text{vol}_{n-1}(K' \cap H)}{\kappa_{n-1}} \right)^{\frac{1}{n-1}} B_{n-1} \\ &= \left(\frac{\text{vol}_{n-1}(L' \cap H)}{\kappa_{n-1}} \right)^{\frac{1}{n-1}} B_{n-1} \\ &= \lim_{i \rightarrow \infty} \text{st}_{H_i} \cdots \text{st}_{H_1}(L' \cap H), \end{aligned} \quad (2.3.1)$$

where B_{n-1} is the ball of radius 1 centered at the origin in H with volume κ_{n-1} . Now let

$$K'' = \lim_{i \rightarrow \infty} \text{st}_{H_i} \cdots \text{st}_{H_1}(K') \text{ and } L'' = \lim_{i \rightarrow \infty} \text{st}_{H_i} \cdots \text{st}_{H_1}(L').$$

Since K' is symmetric to H and $\mathbf{a}_i \in H$, the symmetrals $\text{st}_{H_i} \cdots \text{st}_{H_1}(K')$ are also symmetric to H and so

$$\begin{aligned} [\text{st}_{H_i} \cdots \text{st}_{H_1}(K')]|H &= [\text{st}_{H_i} \cdots \text{st}_{H_1}(K')] \cap H \\ &= [\text{st}_{H_i} \cdots \text{st}_{H_1}(K' \cap H)] = \text{st}_{H_i} \cdots \text{st}_{H_1}(K'|H). \end{aligned}$$

Hence

$$K''|H = \lim_{i \rightarrow \infty} \text{st}_{H_i} \cdots \text{st}_{H_1}(K' \cap H) \text{ and } L''|H = \lim_{i \rightarrow \infty} \text{st}_{H_i} \cdots \text{st}_{H_1}(L' \cap H).$$

we get with Lemma 2.2 applied to K'', L''

$$\begin{aligned} \text{vol}(\lambda K' + (1-\lambda)L') &= \text{vol} \left(\lim_{i \rightarrow \infty} \text{st}_{H_i} \cdots \text{st}_{H_1}(\lambda K' + (1-\lambda)L') \right) \\ &\geq \text{vol}(\lambda K'' + (1-\lambda)L'') \geq \lambda \text{vol}(K'') + (1-\lambda) \text{vol}(L'') \\ &= \lambda \text{vol}(K') + (1-\lambda) \text{vol}(L'). \end{aligned}$$

□

2.4 Theorem [Grünbaum].¹² Let $K \in \mathcal{K}^n$ with $\dim K = n$ and center of gravity $\mathbf{c}(K)$. Let H^+ be a subspace containing $\mathbf{c}(K)$. Then

$$\frac{\text{vol}(K \cap H^+)}{\text{vol}(K)} \geq \left(\frac{n}{n+1} \right)^n.$$

(Observe that $(n/(n+1))^n \rightarrow 1/e$.)

Proof. Without loss of generality let $\mathbf{c}(K) = \mathbf{0}$ and $H^+ = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{e}_1, \mathbf{x} \rangle = x_1 \geq 0\}$. Let $H^- = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{e}_1, \mathbf{x} \rangle \leq 0\}$, and for $t \in \mathbb{R}$ let $H_t = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{e}_1, \mathbf{x} \rangle = t\}$ and $f(t) = \text{vol}_{n-1}(K \cap H_t)$.

¹¹Skript WS14

¹²Branko Grünbaum, 1929

Let $b = h(K, \mathbf{e}_1)$ and $a = h(K, -\mathbf{e}_1)$, where $h(K, \cdot)$ denotes the support function of K . Since $\mathbf{0} = \int_K \mathbf{x} \, d\mathbf{x}$ we have

$$0 = \int_K \langle \mathbf{x}, \mathbf{e}_1 \rangle \, d\mathbf{x} = \int_{-a}^b t f(t) \, dt. \quad (2.4.1)$$

First we apply Schwarz-Symmetrization to K with respect to the hyperplane H , i.e., we replace $K \cap H_t$ by an $(n-1)$ -dimensional ball of volume $f(t)$, i.e., of radius $(f(t)/\kappa_{n-1})^{1/(n-1)}$ and centred at the point $(t, \mathbf{0})^\top \in \mathbb{R}^n$. Let \bar{K} be the resulting Schwarz-Symmetrization. By construction, $\text{vol}(K \cap H_t) = \text{vol}(\bar{K} \cap H_t)$ and, in particular, $\text{vol}(K) = \text{vol}(\bar{K})$, $\text{vol}(K \cap H^+) = \text{vol}(\bar{K} \cap H^+)$, and we also have $\mathbf{c}(\bar{K}) = \mathbf{c}(K) = \mathbf{0}$. For the last statement we observe that \bar{K} is rotational symmetric with respect to the axis $\{\lambda \mathbf{e}_1 : \lambda \in \mathbb{R}\}$. Hence $\mathbf{c}(\bar{K})$ has to be on this axis, i.e., $\mathbf{c}(\bar{K}) = \gamma \mathbf{e}_1$. Thus, on account of (2.4.1), we get

$$\begin{aligned} \text{vol}(\bar{K}) \langle \gamma \mathbf{e}_1, \mathbf{e}_1 \rangle &= \int_{\bar{K}} \langle \mathbf{x}, \mathbf{e}_1 \rangle \, d\mathbf{x} \\ &= \int_{-a}^b t \, \text{vol}_{n-1}(\bar{K} \cap H_t) \, dt = \int_{-a}^b t f(t) \, dt = 0. \end{aligned}$$

Hence $\gamma = 0$, i.e., $\mathbf{c}(\bar{K}) = \mathbf{0}$ and in the following we may assume $K = \bar{K}$.

Let $K_0 = K \cap H_0$, $K^+ = K \cap H^+$, $K^- = K \cap H^-$, and let $P^+ = \text{conv}\{K_0, \beta \mathbf{e}_1\}$ be the pyramid with basis K_0 such that $\text{vol}(P^+) = \text{vol}(K^+)$, i.e., $\beta = n \text{vol}(K^+) / \text{vol}_{n-1}(K_0)$.

We now extend this pyramid in the other direction such that for $\alpha > 0$ and $P^- = \text{conv}\{K_0, -\alpha \mathbf{e}_1 + \frac{\alpha+\beta}{\beta} K_0\}$ holds $\text{vol}(P^-) = \text{vol}(K^-)$. Let $P = P^+ \cup P^-$, i.e.,

$$P = \text{conv} \left\{ -\alpha \mathbf{e}_1 + \frac{\alpha+\beta}{\beta} K_0, \beta \mathbf{e}_1 \right\}$$

is a circular pyramid of height $\alpha + \beta$ and $\text{vol}(P) = \text{vol}(K)$, and we have

$$\frac{\text{vol}(K^+)}{\text{vol}(K)} = \frac{\text{vol}(P^+)}{\text{vol}(P)} = \left(\frac{\beta}{\alpha + \beta} \right)^n.$$

Hence, it remains to show $\beta/(\alpha + \beta) \geq n/(n+1)$. The crucial observation here is that due to the concavity of $f(t)^{1/(n-1)}$ the centroid $\mathbf{c}(P)$ is on the non-negative x -axis, i.e., $\mathbf{c}(P) = (\gamma, \mathbf{0})^\top \in \mathbb{R}^n$ with $\gamma \geq 0$ (see the end of the proof). Then with

$$l(t) = \text{vol}_{n-1}(P \cap H_t) = \left(\frac{\beta - t}{\beta} \right)^{n-1} \text{vol}_{n-1}(K_0)$$

for $t \in [-\alpha, \beta]$ we may write

$$\begin{aligned} 0 \leq \gamma = \text{vol}(P) \langle \mathbf{c}(P), \mathbf{e}_1 \rangle &= \int_{-\alpha}^{\beta} t l(t) \, dt \\ &= \frac{\text{vol}_{n-1}(K_0)}{\beta^{n-1}} \int_{-\alpha}^{\beta} t (\beta - t)^{n-1} \, dt \\ &= \frac{\text{vol}_{n-1}(K_0)}{\beta^{n-1}} \left(-\frac{t}{n} (\beta - t)^n - \frac{(\beta - t)^{n+1}}{n(n+1)} \Big|_{-\alpha}^{\beta} \right) \\ &= \frac{\text{vol}_{n-1}(K_0)}{\beta^{n-1}} \frac{(\beta + \alpha)^n}{n} \left(-\alpha + \frac{\alpha + \beta}{n+1} \right). \end{aligned}$$

Hence $\alpha/(\alpha + \beta) \leq 1/(n + 1)$ which is equivalent to $\beta/(\alpha + \beta) \geq n/(n + 1)$.

Here we give a more formal argument why $\langle \mathbf{e}_1, \mathbf{c}(P) \rangle \geq 0$. Since

$$\text{vol}(P) \langle \mathbf{e}_1, \mathbf{c}(P) \rangle = \int_{-\alpha}^{\beta} t l(t) dt = - \int_{-\alpha}^0 (-t) l(t) dt + \int_0^{\beta} t l(t) dt,$$

it is to show

$$\int_0^{\beta} t l(t) dt \geq \int_{-\alpha}^0 (-t) l(t) dt.$$

In view of (2.4.1) this will follow from the two inequalities

$$\int_0^{\beta} t l(t) dt \geq \int_0^b t f(t) dt \text{ and } \int_{-\alpha}^0 (-t) f(t) dt \geq \int_{-\alpha}^0 (-t) l(t) dt. \quad (2.4.2)$$

We will just prove the first relation, the second is treated analogously. $P \cap H_t$ and $K \cap H_t$ are $(n - 1)$ -dimensional balls of radius $(l(t)/\kappa_{n-1})^{1/(n-1)}$ and $(f(t)/\kappa_{n-1})^{1/(n-1)}$, respectively. Moreover, $l(0) = f(0)$, $l(t)^{1/(n-1)}$ is a linear function and according to the Brunn-Minkowski theorem $f(t)^{1/(n-1)}$ is a concave function. Hence we have

$$\gamma = \min\{\langle \mathbf{e}_1, \mathbf{x} \rangle : \mathbf{x} \in P^+ \setminus K^+\} = \max\{\langle \mathbf{e}_1, \mathbf{x} \rangle : \mathbf{x} \in K^+ \setminus P^+\}.$$

We also have $P^+ = P^+ \setminus K^+ \cup (P^+ \cap K^+)$ and $K^+ = K^+ \setminus P^+ \cup (K^+ \cap P^+)$, and so $\text{vol}(P^+ \setminus K^+) = \text{vol}(K^+ \setminus P^+)$. Altogether we may write

$$\begin{aligned} \int_0^{\beta} t l(t) dt &= \int_{P^+} \langle \mathbf{e}_1, \mathbf{x} \rangle d\mathbf{x} \\ &= \int_{P^+ \setminus K^+} \langle \mathbf{e}_1, \mathbf{x} \rangle d\mathbf{x} + \int_{P^+ \cap K^+} \langle \mathbf{e}_1, \mathbf{x} \rangle d\mathbf{x} \\ &\geq \text{vol}(P^+ \setminus K^+) \gamma + \int_{P^+ \cap K^+} \langle \mathbf{e}_1, \mathbf{x} \rangle d\mathbf{x} \\ &= \text{vol}(K^+ \setminus P^+) \gamma + \int_{K^+ \cap P^+} \langle \mathbf{e}_1, \mathbf{x} \rangle d\mathbf{x} \\ &\geq \int_{K^+ \setminus P^+} \langle \mathbf{e}_1, \mathbf{x} \rangle d\mathbf{x} + \int_{K^+ \cap P^+} \langle \mathbf{e}_1, \mathbf{x} \rangle d\mathbf{x} \\ &= \int_{K^+} \langle \mathbf{e}_1, \mathbf{x} \rangle d\mathbf{x} = \int_0^b t f(t) dt. \end{aligned}$$

□

We recall from Proposition 5.18¹³ that the Brunn-Minkowski inequality is equivalent to its multiplicative version

$$\text{vol}(\lambda K + (1 - \lambda)L) \geq \text{vol}(K)^\lambda \text{vol}(L)^{1-\lambda}, \quad \text{for all } \lambda \in [0, 1]. \quad (2.4.3)$$

If we denote the characteristic functions of K and L by χ_K and χ_L , respectively, and if for a given $\lambda \in [0, 1]$ the characteristic function of $\lambda K + (1 - \lambda)L$ is denoted

¹³Skript WS14

by χ_λ , then we have $\chi_\lambda(\lambda \mathbf{x} + (1 - \lambda) \mathbf{y}) \geq \chi_K(\mathbf{x}) \chi_L(\mathbf{y}) = \chi_K(\mathbf{x})^\lambda \chi_L(\mathbf{y})^{1-\lambda}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and (2.4.3) becomes

$$\int_{\mathbb{R}^n} \chi_\lambda(\mathbf{x}) \, d\mathbf{x} \geq \left(\int_{\mathbb{R}^n} \chi_K(\mathbf{x}) \, d\mathbf{x} \right)^\lambda \left(\int_{\mathbb{R}^n} \chi_L(\mathbf{x}) \, d\mathbf{x} \right)^{1-\lambda}.$$

2.5 Theorem [Prékopa-Leindler Inequality]. ^{14 15} Let $\lambda \in (0, 1)$ and let $f, g, h : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ (Lebesgue-)measurable functions with

$$h(\lambda \mathbf{x} + (1 - \lambda) \mathbf{y}) \geq f(\mathbf{x})^\lambda g(\mathbf{y})^{1-\lambda}, \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \quad (2.5.1)$$

and let f, g be (Lebesgue-)integrable. Then

$$\int_{\mathbb{R}^n} h(\mathbf{x}) \, d\mathbf{x} \geq \left(\int_{\mathbb{R}^n} f(\mathbf{x}) \, d\mathbf{x} \right)^\lambda \left(\int_{\mathbb{R}^n} g(\mathbf{x}) \, d\mathbf{x} \right)^{1-\lambda}. \quad (2.5.2)$$

Proof. Without loss of generality let $f, g \neq 0$ and bounded; otherwise, we consider for an arbitrary integer m the function $\min\{f, m\}$ or $\min\{g, m\}$ instead of f or g and apply Beppo Levi's ¹⁶ monotone convergence theorem. Since

$$\frac{h(\lambda \mathbf{x} + (1 - \lambda) \mathbf{y})}{(\sup f)^\lambda (\sup g)^{1-\lambda}} \geq \left(\frac{f(\mathbf{x})}{\sup f} \right)^\lambda \left(\frac{g(\mathbf{y})}{\sup g} \right)^{1-\lambda}$$

let $\sup f = \sup g = 1$. Finally, since we can approximate absolute integrable functions by continuous functions arbitrary well and due to the Lebesgue dominated convergence theorem we may assume that f, g are continuous.

We show the result by induction on the dimension n . Let $n = 1$. For a measurable function $\psi : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ and a $t \geq 0$ we consider the super level set $[\psi \geq t] = \{x \in \mathbb{R} : \psi(x) \geq t\}$, and we write $|\psi \geq t|$ to denote the volume of the set $[\psi \geq t]$. Observe that the super level sets are measurable since ψ is measurable and $\int_{\mathbb{R}} \psi(x) \, dx = \int_0^\infty |\psi \geq t| \, dt$.

If $f(x) \geq t$ and $g(y) \geq t$, then by (2.5.1) we also have $h(\lambda x + (1 - \lambda)y) \geq t$ and so

$$\lambda[f \geq t] + (1 - \lambda)[g \geq t] \subseteq [h \geq t].$$

For $t \in [0, 1)$ the sets on the left-hand side are non-empty, bounded and closed, and thus, the (general) Brunn-Minkowski theorem for compact sets Theorem 2.1¹⁷ in the case $n = 1$ gives $|h \geq t| \geq \lambda|f \geq t| + (1 - \lambda)|g \geq t|$ and so we obtain

$$\begin{aligned} \int_{\mathbb{R}} h(x) \, dx &\geq \int_0^1 |h \geq t| \, dt \geq \lambda \int_0^1 |f \geq t| \, dt + (1 - \lambda) \int_0^1 |g \geq t| \, dt \\ &= \lambda \int_0^\infty f(x) \, dx + (1 - \lambda) \int_0^\infty g(x) \, dx \\ &\geq \left(\int_{\mathbb{R}} f(x) \, dx \right)^\lambda \left(\int_{\mathbb{R}} g(x) \, dx \right)^{1-\lambda}, \end{aligned}$$

¹⁴András Prékopa, 1929

¹⁵László Leindler, 1935

¹⁶Beppo Levi, 1875–1961

¹⁷Skript WS14

where the last inequality follows from the arithmetic-geometric mean inequality. This proves the case $n = 1$.

Now let $n > 1$. As usual we identify \mathbb{R}^n with $\mathbb{R}^{n-1} \times \mathbb{R}$ and for $\mathbf{z} \in \mathbb{R}^{n-1}$, $s \in \mathbb{R}$ we define $h_s(\mathbf{z}) = h(\mathbf{z}, s)$, $f_s(\mathbf{z}) = f(\mathbf{z}, s)$ and $g_s(\mathbf{z}) = g(\mathbf{z}, s)$ on \mathbb{R}^{n-1} . Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n-1}$, $\alpha, \beta \in \mathbb{R}$ and $\gamma = \lambda\alpha + (1 - \lambda)\beta$. Then

$$\begin{aligned} h_\gamma(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) &= h(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}, \lambda\alpha + (1 - \lambda)\beta) \\ &= h(\lambda(\mathbf{x}, \alpha) + (1 - \lambda)(\mathbf{y}, \beta)) \\ &\geq f_\alpha(\mathbf{x})^\lambda g_\beta(\mathbf{y})^{1-\lambda} = f_\alpha(\mathbf{x})^\lambda g_\beta(\mathbf{y})^{1-\lambda}. \end{aligned}$$

Thus, by our inductive argument we get

$$\int_{\mathbb{R}^{n-1}} h_\gamma(\mathbf{z}) \, d\mathbf{z} \geq \left(\int_{\mathbb{R}^{n-1}} f_\alpha(\mathbf{z}) \, d\mathbf{z} \right)^\lambda \left(\int_{\mathbb{R}^{n-1}} g_\beta(\mathbf{z}) \, d\mathbf{z} \right)^{1-\lambda}.$$

With

$$H(\gamma) = \int_{\mathbb{R}^{n-1}} h_\gamma(\mathbf{z}) \, d\mathbf{z}, \quad F(\alpha) = \int_{\mathbb{R}^{n-1}} f_\alpha(\mathbf{z}) \, d\mathbf{z}, \quad G(\beta) = \int_{\mathbb{R}^{n-1}} g_\beta(\mathbf{z}) \, d\mathbf{z},$$

this becomes $H(\lambda\alpha + (1 - \lambda)\beta) \geq F(\alpha)^\lambda G(\beta)^{1-\lambda}$. Hence we may apply the case $n = 1$ to these functions and get the desired result:

$$\begin{aligned} \int_{\mathbb{R}^n} h(\mathbf{x}) \, d\mathbf{x} &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} h_\gamma(\mathbf{z}) \, d\mathbf{z} \right) d\gamma \\ &= \int_{\mathbb{R}} H(\gamma) \, d\gamma \geq \left(\int_{\mathbb{R}} F(\alpha) \, d\alpha \right)^\lambda \left(\int_{\mathbb{R}} G(\beta) \, d\beta \right)^{1-\lambda} \\ &= \left(\int_{\mathbb{R}^n} f(\mathbf{x}) \, d\mathbf{x} \right)^\lambda \left(\int_{\mathbb{R}^n} g(\mathbf{x}) \, d\mathbf{x} \right)^{1-\lambda}. \end{aligned}$$

□

2.6 Remark. The Prékopa-Leindler inequality can be extended to m functions: Let $h : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, $f_i : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be measurable functions, $1 \leq i \leq m$, and let $\lambda_i > 0$ with $\sum_{i=1}^m \lambda_i = 1$ such that

$$h \left(\sum_{i=1}^m \lambda_i \mathbf{x}_i \right) \geq \prod_{i=1}^m f_i(\mathbf{x}_i)^{\lambda_i},$$

and let f_i be integrable, $1 \leq i \leq m$. Then

$$\int_{\mathbb{R}^n} h(\mathbf{x}) \, d\mathbf{x} \geq \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(\mathbf{x}) \, d\mathbf{x} \right)^{\lambda_i}.$$

2.7 Remark. Hölder's-inequality ¹⁸

$$\int_{\mathbb{R}^n} f(\mathbf{x}) g(\mathbf{x}) \, d\mathbf{x} \leq \left(\int_{\mathbb{R}^n} f(\mathbf{x})^p \, d\mathbf{x} \right)^{1/p} \left(\int_{\mathbb{R}^n} g(\mathbf{x})^q \, d\mathbf{x} \right)^{1/q}.$$

¹⁸Otto Hölder, 1859 - 1937

for integrable functions $f, g : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and real numbers $p, q > 1$ with $1/p + 1/q = 1$, may be regarded as a counterpart to the Prékopa-Leindler inequality in the special setting $p = 1/\lambda$, $q = 1/(1 - \lambda)$, $0 < \lambda < 1$, and with respect to the functions $f^{1/p}, g^{1/q}$:

$$\int_{\mathbb{R}^n} f(\mathbf{x})^\lambda g(\mathbf{x})^{1-\lambda} d\mathbf{x} \leq \left(\int_{\mathbb{R}^n} f(\mathbf{x}) d\mathbf{x} \right)^\lambda \left(\int_{\mathbb{R}^n} g(\mathbf{x}) d\mathbf{x} \right)^{1-\lambda}.$$

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3 Applications of the inequalities of Brascamp-Lieb and Barthe

3.1 Remark. Let $f_i : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be integrable functions, $1 \leq i \leq m$, and let $\lambda_i \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$. If we set

$$h(\mathbf{y}) = \sup \left\{ \prod_{i=1}^m f_i(\mathbf{y}_i)^{\lambda_i} : \mathbf{y} = \sum_{i=1}^m \lambda_i \mathbf{y}_i, \mathbf{y}_i \in \mathbb{R}^n \right\}$$

then $h(\sum_{i=1}^m \lambda_i \mathbf{x}_i) \geq \prod_{i=1}^m f_i(\mathbf{x}_i)^{\lambda_i}$, for all $\mathbf{x}_i \in \mathbb{R}^n$, $1 \leq i \leq m$. Hence, Prékopa-Leindler inequality (2.5.2) (cf. Remark 2.6) can also be equivalently written in the form

$$\int_{\mathbb{R}^n}^* \sup \left\{ \prod_{i=1}^m f_i(\mathbf{x}_i)^{\lambda_i} : \mathbf{y} = \sum_{i=1}^m \lambda_i \mathbf{x}_i, \mathbf{x}_i \in \mathbb{R}^n \right\} d\mathbf{y} \geq \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(\mathbf{x}) d\mathbf{x} \right)^{\lambda_i}.$$

Here $\int_{\mathbb{R}^n}^* g(\mathbf{y}) d\mathbf{y}$ denotes the outer integral, which for non-negative functions $g : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is defined as

$$\int_{\mathbb{R}^n}^* g(\mathbf{y}) d\mathbf{y} = \inf \left\{ \int_{\mathbb{R}^n} u(\mathbf{y}) d\mathbf{y} : u \geq g, u \text{ measurable} \right\}.$$

3.2 Theorem* [Brascamp-Lieb, Barthe].^{19 20 21} Let $c_i > 0$ and $n_i \in \mathbb{N}$, $1 \leq i \leq m$, satisfying $\sum_{i=1}^m c_i n_i = n$. Let $M_i \in \mathbb{R}^{n_i \times n}$, $1 \leq i \leq m$, with $\text{rg}(M_i) = n_i$, and let

$$\alpha = \inf \left\{ \frac{\det(\sum_{i=1}^m c_i M_i^\top A_i M_i)}{\prod_{i=1}^m (\det A_i)^{c_i}} : A_i \in \mathbb{R}^{n_i \times n_i} \text{ positive definite} \right\}.$$

Let $f_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_{\geq 0}$ be integrable functions. The BRASCAMP-LIEB inequality states

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(M_i \mathbf{x})^{c_i} \right) d\mathbf{x} \leq \frac{1}{\sqrt{\alpha}} \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(\mathbf{x}_i) d\mathbf{x}_i \right)^{c_i}, \quad (\text{BL-I})$$

and the BARTHE inequality states

$$\int_{\mathbb{R}^n}^* \sup \left\{ \prod_{i=1}^m f_i(\mathbf{x}_i)^{c_i} : \mathbf{y} = \sum_{i=1}^m c_i M_i^\top \mathbf{x}_i, \mathbf{x}_i \in \mathbb{R}^{n_i} \right\} d\mathbf{y} \geq \sqrt{\alpha} \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(\mathbf{x}_i) d\mathbf{x}_i \right)^{c_i}. \quad (\text{B-I})$$

¹⁹Brascamp-Lieb inequality

²⁰Elliott H. Lieb, 1932

²¹Franck Barthe, ??

3.3 Corollary [Hölder and Prékopa-Leindler inequalities]. Let $f_i : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be integrable functions and let $\lambda_i > 0$, $1 \leq i \leq m$, with $\sum_{i=1}^m \lambda_i = 1$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(\mathbf{x})^{\lambda_i} \right) d\mathbf{x} &\leq \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(\mathbf{x}) d\mathbf{x} \right)^{\lambda_i} \\ &\leq \int_{\mathbb{R}^n}^* \sup \left\{ \prod_{i=1}^m f_i(\mathbf{x}_i)^{\lambda_i} : \mathbf{y} = \sum_{i=1}^m \lambda_i \mathbf{x}_i, \mathbf{x}_i \in \mathbb{R}^n \right\} d\mathbf{y}. \end{aligned}$$

Proof. We start with Hölder's inequality for which we set $n_i = n$, $c_i = \lambda_i$ and $M_i = \mathbf{I}_n$, $1 \leq i \leq m$. The (BL-I) gives

$$\int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(\mathbf{x})^{\lambda_i} \right) d\mathbf{x} \leq \frac{1}{\sqrt{\alpha}} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(\mathbf{x}) d\mathbf{x} \right)^{\lambda_i},$$

and it remains to prove

$$1 \leq \alpha = \inf \left\{ \frac{\det \left(\sum_{i=1}^m \lambda_i A_i \right)}{\prod_{i=1}^m (\det A_i)^{\lambda_i}} : A_i \in \mathbb{R}^{n \times n} \text{ positive definite} \right\}.$$

Applying Minkowski's Theorem 6.4²² we get

$$\det \left(\sum_{i=1}^m \lambda_i A_i \right)^{1/n} \geq \sum_{i=1}^m (\det(\lambda_i A_i))^{1/n} = \sum_{i=1}^m \lambda_i (\det A_i)^{1/n} \geq \prod_{i=1}^m (\det A_i)^{\lambda_i/n},$$

where the last inequality follows from the arithmetic/geometric-mean inequality. Therefore $\alpha \geq 1$.

In fact, taking $A_i = \mathbf{I}_n$ for all $i = 1, \dots, m$ shows $\alpha = 1$, and Prékopa-Leindler's inequality follows immediately from (B-I). \square

3.4 Corollary [Young inequality].²³ Let $p_i > 0$, $1 \leq i \leq 3$, with $1/p_1 + 1/p_2 + 1/p_3 = 2$, and let $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $1 \leq i \leq 3$, be integrable functions. Then

$$\int_{\mathbb{R}^n} f_1(\mathbf{x}) \left(\int_{\mathbb{R}^n} f_2(\mathbf{x} - \mathbf{y}) f_3(\mathbf{y}) d\mathbf{y} \right) d\mathbf{x} \leq c^n \prod_{i=1}^3 \left(\int_{\mathbb{R}^n} f_i(\mathbf{x})^{p_i} d\mathbf{x} \right)^{1/p_i},$$

where $c = c_{p_1} c_{p_2} c_{p_3}$ and $c_p = p^{1/p} / q^{1/q}$, with q such that $1/q + 1/p = 1$.

Proof. We apply (BL-I) to the functions $f_i^{p_i}$, $c_i = 1/p_i$, $n_i = n$ and to the matrices $M_i \in \mathbb{R}^{n \times 2n}$, $1 \leq i \leq 3 = m$, given by $M_1(\mathbf{x}, \mathbf{y}) = \mathbf{x}$, $M_2(\mathbf{x}, \mathbf{y}) = \mathbf{x} - \mathbf{y}$ and $M_3(\mathbf{x}, \mathbf{y}) = \mathbf{y}$ for $(\mathbf{x}, \mathbf{y})^\top \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$. Then $\sum_{i=1}^m c_i n_i = \sum_{i=1}^3 n/p_i = 2n$ and

$$\begin{aligned} \int_{\mathbb{R}^{2n}} f_1(M_1(\mathbf{x}, \mathbf{y})) f_2(M_2(\mathbf{x}, \mathbf{y})) f_3(M_3(\mathbf{x}, \mathbf{y})) d(\mathbf{x}, \mathbf{y}) &\leq \\ \frac{1}{\sqrt{\alpha}} \prod_{i=1}^3 \left(\int_{\mathbb{R}^n} f_i(\mathbf{x})^{p_i} d\mathbf{x} \right)^{1/p_i}. & \end{aligned}$$

The computation of the constant α is quite involved and we omit it here. \square

²²Skript WS14

²³William Henry Young, 1863-1942

3.5 Lemma. Let $c_i > 0$ and let $\mathbf{v}_i \in S^{n-1}$, $1 \leq i \leq m$, $m \geq n$, such that $\sum_{i=1}^m c_i (\mathbf{v}_i \mathbf{v}_i^\top) = \mathbf{I}_n$. Then

$$\inf \left\{ \frac{\det(\sum_{i=1}^m c_i \alpha_i (\mathbf{v}_i \mathbf{v}_i^\top))}{\prod_{i=1}^m \alpha_i^{c_i}} : \alpha_i \in \mathbb{R}_{>0} \right\} = 1.$$

Proof. Let the infimum on the left hand side be denoted by α . Setting $\alpha_i = 1$, $1 \leq i \leq m$, shows $\alpha \leq 1$, and in the following we prove $\alpha \geq 1$. To this end, let $\mathbf{w}_i = \sqrt{c_i} \mathbf{v}_i$, $1 \leq i \leq m$. Then $\mathbf{I}_n = \sum_{i=1}^m c_i (\mathbf{v}_i \mathbf{v}_i^\top) = \sum_{i=1}^m (\mathbf{w}_i \mathbf{w}_i^\top) = W W^\top$, where $W = (\mathbf{w}_1, \dots, \mathbf{w}_m) \in \mathbb{R}^{n \times m}$. Hence, the Cauchy-Binet formula²⁴ gives

$$1 = \det(W W^\top) = \sum_{\#I=n} \beta_I, \quad (3.5.1)$$

$\beta_I = (\det(\mathbf{w}_i : i \in I))^2$ for each $I \subseteq \{1, \dots, m\}$ with $\#I = n$. In the same way we see that

$$\begin{aligned} \det \left(\sum_{i=1}^m \alpha_i (\mathbf{w}_i \mathbf{w}_i^\top) \right) &= \sum_{\#I=n} (\det(\sqrt{\alpha_i} \mathbf{w}_i : i \in I))^2 \\ &= \sum_{\#I=n} \left(\prod_{i \in I} \alpha_i \right) \beta_I = \sum_{\#I=n} \alpha_I \beta_I, \end{aligned}$$

with $\alpha_I = \prod_{i \in I} \alpha_i$. In view of (3.5.1) we may apply the arithmetic/geometric-mean inequality and get

$$\det \left(\sum_{i=1}^m \alpha_i (\mathbf{w}_i \mathbf{w}_i^\top) \right) \geq \prod_{\#I=n} \alpha_I^{\beta_I}.$$

The exponent of a fixed α_i in the right-hand side is given by

$$\begin{aligned} \sum_{i \in I, \#I=n} \beta_I &= \sum_{\#I=n} \beta_I - \sum_{i \notin I, \#I=n} \beta_I \\ &= 1 - \det \left(\sum_{j=1, j \neq i}^m \mathbf{w}_j \mathbf{w}_j^\top \right) = 1 - \det(\mathbf{I}_n - \mathbf{w}_i \mathbf{w}_i^\top) \\ &= 1 - (1 - \|\mathbf{w}_i\|^2) = c_i, \end{aligned}$$

where for the last step we observe that the eigenvalues of $\mathbf{I}_n - (\mathbf{w}_i \mathbf{w}_i^\top)$ are $1 - \|\mathbf{w}_i\|^2$ (with eigenvector \mathbf{w}_i) and $(n-1)$ -times 1 (with eigenvectors orthogonal to \mathbf{w}_i). Hence $\prod_{\#I=n} \alpha_I^{\beta_I} = \prod_{i=1}^m \alpha_i^{c_i}$ and so

$$\frac{\det(\sum_{i=1}^m c_i \alpha_i (\mathbf{v}_i \mathbf{v}_i^\top))}{\prod_{i=1}^m \alpha_i^{c_i}} = \frac{\det(\sum_{i=1}^m \alpha_i (\mathbf{w}_i \mathbf{w}_i^\top))}{\prod_{i=1}^m \alpha_i^{c_i}} \geq \frac{\prod_{\#I=n} \alpha_I^{\beta_I}}{\prod_{i=1}^m \alpha_i^{c_i}} = 1,$$

which shows that $\alpha \geq 1$, as required. \square

²⁴In its general form it says, that for $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times n}$ we have $\det AB = \sum_{I \in \binom{m}{n}} \det A^I \det B_I$, where A^I , B_I are the $n \times n$ submatrices of A and B with row and column indices in I , respectively. For $m < n$ it just gives/means $\det AB = 0$.

3.6 Theorem. Let $f_i : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ be measurable functions, and let $c_i > 0$ and $\mathbf{v}_i \in S^{n-1}$, $1 \leq i \leq m$, such that $\sum_{i=1}^m c_i (\mathbf{v}_i \mathbf{v}_i^\top) = \mathbf{I}_n$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(\langle \mathbf{v}_i, \mathbf{x} \rangle) \right)^{c_i} d\mathbf{x} &\leq \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t) dt \right)^{c_i} \\ &\leq \int_{\mathbb{R}^n}^* \sup \left\{ \prod_{i=1}^m f_i(t_i)^{c_i} : \mathbf{y} = \sum_{i=1}^m c_i t_i \mathbf{v}_i, t_i \in \mathbb{R} \right\} d\mathbf{y}. \end{aligned} \quad (3.6.1)$$

For $m = n$ we have equality in both inequalities and $\mathbf{v}_1, \dots, \mathbf{v}_n$ build an orthonormal basis.

Proof. Since $\mathbf{I}_n = \sum_{i=1}^m c_i (\mathbf{v}_i \mathbf{v}_i^\top)$ we may write $\mathbf{x} = \sum_{i=1}^m c_i \langle \mathbf{v}_i, \mathbf{x} \rangle \mathbf{v}_i$ for all $\mathbf{x} \in \mathbb{R}^n$ and thus, in particular, we have $\text{lin}\{\mathbf{v}_1, \dots, \mathbf{v}_m\} = \mathbb{R}^n$. First we discuss the case $m = n$. For any $k \in \{1, \dots, n\}$ we can write $\mathbf{v}_k = \sum_{i=1}^n c_i \langle \mathbf{v}_i, \mathbf{v}_k \rangle \mathbf{v}_i$ and thus we get

$$0 = \sum_{i \neq k} c_i \langle \mathbf{v}_i, \mathbf{v}_k \rangle \mathbf{v}_i + (c_k - 1) \mathbf{v}_k,$$

which implies that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ has to be an orthonormal basis and $c_i = 1$, $1 \leq i \leq n$. Hence we get with $V = (\mathbf{v}_1, \dots, \mathbf{v}_n)^\top$

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\prod_{i=1}^n f_i(\langle \mathbf{v}_i, \mathbf{x} \rangle) \right) d\mathbf{x} &= \int_{\mathbb{R}^n} \left(\prod_{i=1}^n f_i(\mathbf{e}_i^\top V \mathbf{x}) \right) d\mathbf{x} \\ &= \int_{\mathbb{R}^n} \left(\prod_{i=1}^n f_i(z_i) \right) |\det V| dz = \prod_{i=1}^n \left(\int_{\mathbb{R}} f_i(t) dt \right). \end{aligned}$$

For the upper bound we observe that for $\mathbf{y} = \sum_{i=1}^n t_i \mathbf{v}_i$ we have $t_i = \langle \mathbf{v}_i, \mathbf{y} \rangle$ and hence we get as before

$$\begin{aligned} \int_{\mathbb{R}^n}^* \sup \left\{ \prod_{i=1}^n f_i(t_i) : \mathbf{y} = \sum_{i=1}^n t_i \mathbf{v}_i, t_i \in \mathbb{R} \right\} d\mathbf{y} &= \int_{\mathbb{R}^n} \left(\prod_{i=1}^n f_i(\langle \mathbf{v}_i, \mathbf{y} \rangle) \right) d\mathbf{y} \\ &= \prod_{i=1}^n \left(\int_{\mathbb{R}} f_i(t) dt \right). \end{aligned}$$

The general validity of the inequalities is reduced to the (BL-I) and (B-I). Since

$$n = \text{tr } \mathbf{I}_n = \sum_{i=1}^m c_i \text{tr}(\mathbf{v}_i \mathbf{v}_i^\top) = \sum_{i=1}^m c_i \|\mathbf{v}_i\|^2 = \sum_{i=1}^m c_i,$$

we may apply (BL-I) and (B-I) with $n_i = 1$ and $M_i = \mathbf{v}_i^\top$, for $1 \leq i \leq m$, and get

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(\langle \mathbf{v}_i, \mathbf{x} \rangle) \right)^{c_i} d\mathbf{x} &\leq \frac{1}{\sqrt{\alpha}} \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t) dt \right)^{c_i} \quad \text{and} \\ \int_{\mathbb{R}^n}^* \sup \left\{ \prod_{i=1}^m f_i(t_i)^{c_i} : \mathbf{y} = \sum_{i=1}^m c_i t_i \mathbf{v}_i, t_i \in \mathbb{R} \right\} d\mathbf{y} &\geq \sqrt{\alpha} \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t) dt \right)^{c_i}, \end{aligned}$$

with

$$\alpha = \inf \left\{ \frac{\det(\sum_{i=1}^m c_i \alpha_i (\mathbf{v}_i \mathbf{v}_i^\top))}{\prod_{i=1}^m \alpha_i^{c_i}} : \alpha_i > 0 \right\}.$$

Lemma 3.5 shows that $\alpha \geq 1$. \square

3.7 Theorem*. Let $n \geq 2$, $\mathbf{v}_1, \dots, \mathbf{v}_m \in S^{n-1}$ be pairwise different and let $c_i > 0$ such that $\sum_{i=1}^m c_i \mathbf{v}_i \mathbf{v}_i^\top = \mathbf{I}_n$. Let $f_i : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ be measurable function which are neither identical 0 nor Gaussian, i.e., of the type $c e^{-\gamma x}$ for positive numbers c, γ . If we have equality in one of the inequalities (3.6.1), then $m = n$ and $\mathbf{v}_1, \dots, \mathbf{v}_n$ build an orthonormal basis.

3.8 Proposition.

i) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be an integrable function. Via spherical coordinates we get

$$\begin{aligned} \int_{\mathbb{R}^n} f(\mathbf{x}) \, d\mathbf{x} &= \int_0^\infty \int_{S^{n-1}} r^{n-1} f(r\mathbf{u}) \, d\mathbf{u} \, dr \\ &= n\kappa_n \int_0^\infty \int_{S^{n-1}} r^{n-1} f(r\mathbf{u}) \, d\sigma(\mathbf{u}) \, dr. \end{aligned}$$

Here $d\mathbf{u}$ denotes the rotational invariant area measure on the sphere of total mass $F(S^{n-1}) = n \text{vol}(B_n)$, and $d\sigma(\mathbf{u})$ its normalization to a probability measure, i.e., $\int_{S^{n-1}} 1 \, d\sigma(\mathbf{u}) = 1$.

ii) Let $K \in \mathcal{K}^n$ with $\mathbf{0} \in K$. For $\mathbf{u} \in S^{n-1}$ let $r_K(\mathbf{u}) = \max\{\rho \geq 0 : \rho \mathbf{u} \in K\}$ be its radial function. Then

$$\text{vol}(K) = \kappa_n \int_{S^{n-1}} r_K(\mathbf{u})^n \, d\sigma(\mathbf{u}).$$

iii) Let $K \in \mathcal{K}_0^n$. Then

$$\text{vol}(K) = \kappa_n \int_{S^{n-1}} |\mathbf{u}|_K^{-n} \, d\sigma(\mathbf{u}).$$

iv) ²⁵ Let $K \in \mathcal{K}_0^n$ and $1 \leq p < \infty$. Then

$$\text{vol}(K) = \frac{1}{\Gamma\left(\frac{n}{p} + 1\right)} \int_{\mathbb{R}^n} e^{-|\mathbf{x}|_K^p} \, d\mathbf{x}.$$

v)

$$\kappa_n = \text{vol}(B_n) = \frac{\pi^{n/2}}{\Gamma((n/2) + 1)} \approx \left(\frac{2\pi e}{n}\right)^{n/2} \frac{1}{\sqrt{\pi n}}.$$

²⁵It is $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$. In particular, $\Gamma(t+1) = t\Gamma(t)$, $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$.

Proof. i) is just a coordinate transformation. For ii) we apply i) to the characteristic function χ_K of K

$$\begin{aligned} \text{vol}(K) &= \int_{\mathbb{R}^n} \chi_K(\mathbf{x}) \, d\mathbf{x} = n\kappa_n \int_0^\infty \int_{S^{n-1}} \chi_K(r\mathbf{u}) r^{n-1} \, d\sigma(\mathbf{u}) \, dr \\ &= n\kappa_n \int_{S^{n-1}} \left(\int_0^{r_K(\mathbf{u})} r^{n-1} \, dr \right) d\sigma(\mathbf{u}) = \kappa_n \int_{S^{n-1}} r_K(\mathbf{u})^n \, d\sigma(\mathbf{u}). \end{aligned}$$

In iii) K is 0-symmetric, and so we have $r_K(\mathbf{u}) = 1/|\mathbf{u}|_K$ (it was an exercise). Hence, in this case iii) is just a reformulation of ii).

For iv) we observe that

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-|\mathbf{x}|_K^p} \, d\mathbf{x} &= n\kappa_n \int_0^\infty \int_{S^{n-1}} r^{n-1} e^{-|r\mathbf{u}|_K^p} \, d\sigma(\mathbf{u}) \, dr \\ &= n\kappa_n \int_{S^{n-1}} \int_0^\infty r^{n-1} e^{-|r\mathbf{u}|_K^p} \, dr \, d\sigma(\mathbf{u}) \\ &\stackrel{s=r|\mathbf{u}|_K}{=} n\kappa_n \left(\int_{S^{n-1}} |\mathbf{u}|_K^{-n} \, d\sigma(\mathbf{u}) \right) \left(\int_0^\infty e^{-s^p} s^{n-1} \, ds \right) \\ &\stackrel{\text{ii)}}{=} n \text{vol}(K) \int_0^\infty e^{-s^p} s^{n-1} \, ds \\ &\stackrel{t=s^p}{=} \frac{n}{p} \text{vol}(K) \int_0^\infty e^{-t} t^{n/p-1} \, dt \\ &= \frac{n}{p} \text{vol}(K) \Gamma\left(\frac{n}{p}\right) = \text{vol}(K) \Gamma\left(\frac{n}{p} + 1\right). \end{aligned}$$

For v) let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be the function $f(\mathbf{x}) = e^{-\|\mathbf{x}\|^2}$. Then by iv) we have

$$\int_{\mathbb{R}^n} f(\mathbf{x}) \, d\mathbf{x} = \kappa_n \Gamma\left(\frac{n}{2} + 1\right).$$

On the other hand we may write

$$\begin{aligned} \int_{\mathbb{R}^n} f(\mathbf{x}) \, d\mathbf{x} &= \int_{\mathbb{R}^n} e^{\sum_{i=1}^n x_i^2} \, d\mathbf{x} = \int_{\mathbb{R}^n} \left(\prod_{i=1}^n e^{-x_i^2} \right) \, d\mathbf{x} = \prod_{i=1}^n \int_{-\infty}^\infty e^{-x_i^2} \, dx_i \\ &= 2^n \prod_{i=1}^n \int_0^\infty e^{-x_i^2} \, dx_i = \prod_{i=1}^n \int_0^\infty e^{-t} t^{-1/2} \, dt \\ &= \Gamma\left(\frac{1}{2}\right)^n = (\sqrt{\pi})^n. \end{aligned}$$

Hence together with *Stirling's formula* $n! \approx \sqrt{2\pi n}(n/e)^n$ we get

$$\text{vol}(B_n) = \kappa_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} \approx \frac{\pi^{n/2}}{\sqrt{2\pi} \sqrt{\frac{n}{2}} \left(\frac{n}{2e}\right)^{n/2}} = \left(\frac{2\pi e}{n}\right)^{n/2} \frac{1}{\sqrt{\pi n}}.$$

□

3.9 Remark.

- i) The radius of an n -dimensional ball of volume 1 is about $\sqrt{\frac{n}{2\pi e}}$.
- ii) Let $C_n = [-1, 1]^n$ the cube of volume 2^n . From

$$2^n = \text{vol}(C_n) = \kappa_n \int_{S^{n-1}} r_{C_n}(\mathbf{u})^n d\sigma(\mathbf{u})$$

we get that the average value of the radial function of C_n is $2/\kappa_n^{1/n} \approx \sqrt{2n/(\pi e)}$. Observe, that $1 \leq r_{C_n}(\mathbf{u}) \leq \sqrt{n}$.

- iii) For the crosspolytope $C_n^* = \text{conv}\{\pm \mathbf{e}_i : 1 \leq i \leq n\}$ we find

$$\frac{2^n}{n!} = \text{vol}(C_n^*) = \kappa_n \int_{S^{n-1}} r_{C_n^*}(\mathbf{u})^n d\sigma(\mathbf{u})$$

and hence the average value of the radial function of the crosspolytope C_n^* is

$$\frac{2}{(\kappa_n n!)^{1/n}} \approx 2 \frac{\sqrt{n} e}{\sqrt{2\pi e} (\sqrt{2\pi n})^{1/n} n} \approx \sqrt{\frac{2e/\pi}{n}}.$$

Observe, that $\frac{1}{\sqrt{n}} \leq r_{C_n^*}(\mathbf{u}) \leq 1$.

3.10 Lemma. Let $c_i > 0$ and $\mathbf{v}_i \in S^{n-1}$, $1 \leq i \leq m$, such that $\sum_{i=1}^m c_i \mathbf{v}_i \mathbf{v}_i^\top = \mathbf{I}_n$. Moreover, let $\alpha_i > 0$, $1 \leq i \leq m$ and $1 \leq p < \infty$.

- i) Let $K \in \mathcal{K}_0^n$ be the 0-symmetric convex body with gauge function

$$|\mathbf{x}|_K = \left(\sum_{i=1}^m \alpha_i |\langle \mathbf{v}_i, \mathbf{x} \rangle|^p \right)^{1/p}.$$

Then

$$\text{vol}(K) \leq 2^n \frac{\Gamma\left(1 + \frac{1}{p}\right)^n}{\Gamma\left(1 + \frac{n}{p}\right)} \prod_{i=1}^m \left(\frac{c_i}{\alpha_i}\right)^{c_i/p}.$$

For the l_p -balls $B_n^p = \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \leq 1\}$ we have $\text{vol}(B_n^p) = 2^n \Gamma\left(1 + \frac{1}{p}\right)^n / \Gamma\left(1 + \frac{n}{p}\right)$.

- ii) Let Z be the zonotope $Z = \sum_{i=1}^m \text{conv}\{-\alpha_i \mathbf{v}_i, \alpha_i \mathbf{v}_i\}$.²⁶ Then

$$\text{vol}(Z) \geq 2^n \prod_{i=1}^m \left(\frac{\alpha_i}{c_i}\right)^{c_i}.$$

Equality holds if and only if $m = n$, $c_i = 1$, and $\mathbf{v}_1, \dots, \mathbf{v}_n$ form an orthonormal basis of \mathbb{R}^n , i.e., Z is an orthogonal box.

²⁶The volume of such a zonotope Z is $\text{vol}(Z) = 2^n \sum_{I \in \binom{[m]}{n}} |\det(\mathbf{v}_{j_i} : i \in I)|$.

Proof. By Proposition 3.8 iv) we have

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|_K^p} d\mathbf{x} = \text{vol}(K) \Gamma\left(\frac{n}{p} + 1\right). \quad (3.10.1)$$

On the other hand, due to the definition of the gauge function we may evaluate the integral as

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|_K^p} d\mathbf{x} = \int_{\mathbb{R}^n} \left(\prod_{i=1}^m e^{-\alpha_i |\langle \mathbf{v}_i, \mathbf{x} \rangle|^p} \right) d\mathbf{x} = \int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(\langle \mathbf{v}_i, \mathbf{x} \rangle)^{c_i} \right) d\mathbf{x},$$

with $f_i(t) = e^{-(\alpha_i/c_i)|t|^p}$. Now we apply the upper bound of Theorem 3.6 and get²⁷

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-|\mathbf{x}|_K^p} d\mathbf{x} &= \int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(\langle \mathbf{v}_i, \mathbf{x} \rangle)^{c_i} \right) d\mathbf{x} \leq \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t) dt \right)^{c_i} \\ &= \prod_{i=1}^m \left(2 \int_0^\infty e^{-\frac{\alpha_i}{c_i} t^p} dt \right)^{c_i} = \prod_{i=1}^m \left(2 \left(\frac{c_i}{\alpha_i} \right)^{1/p} \Gamma\left(\frac{1}{p} + 1\right) \right)^{c_i} \\ &= 2^n \Gamma\left(\frac{1}{p} + 1\right)^n \prod_{i=1}^m \left(\frac{c_i}{\alpha_i} \right)^{c_i/p}, \end{aligned}$$

since, by assumption, $\sum_{i=1}^m c_i = \sum_{i=1}^m c_i \|\mathbf{v}_i\|^2 = \sum_{i=1}^m c_i \text{tr}(\mathbf{v}_i \mathbf{v}_i^\top) = \text{tr} \mathbf{I}_n = n$. Together with (3.10.1) we get the result. Observe, that for $m = n$, $c_i = \alpha_i = 1$, $\mathbf{v}_i = \mathbf{e}_i$ we have equality (see Theorem 3.6).

ii) For $1 \leq i \leq m$ let $f_i(t) = \chi_{[-\alpha_i/c_i, \alpha_i/c_i]}(t)$ be the characteristic functions of the intervals $[-\alpha_i/c_i, \alpha_i/c_i]$, i.e., $f_i(t) = 1$ if and only if $c_i t \in [-\alpha_i, \alpha_i]$. Hence, if $\mathbf{z} \in \mathbb{R}^n$ satisfies

$$\sup \left\{ \prod_{i=1}^m f_i(t_i)^{c_i} : \mathbf{z} = \sum_{i=1}^m c_i t_i \mathbf{v}_i, t_i \in \mathbb{R} \right\} = 1,$$

then $\mathbf{z} \in Z = \sum_{i=1}^m \text{conv} \{-\alpha_i \mathbf{v}_i, \alpha_i \mathbf{v}_i\}$. With the upper bound in Theorem 3.6 we get

$$\begin{aligned} \text{vol}(Z) &= \int_{\mathbb{R}^n} \chi_Z(\mathbf{x}) d\mathbf{x} \geq \int_{\mathbb{R}^n} \sup \left\{ \prod_{i=1}^m f_i(t_i)^{c_i} : \mathbf{z} = \sum_{i=1}^m c_i t_i \mathbf{v}_i, t_i \in \mathbb{R} \right\} d\mathbf{z} \\ &\geq \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t) dt \right)^{c_i} = \prod_{i=1}^m \left(2 \frac{\alpha_i}{c_i} \right)^{c_i} = 2^n \prod_{i=1}^m \left(\frac{\alpha_i}{c_i} \right)^{c_i}, \end{aligned}$$

since, again, $\sum_{i=1}^m c_i = n$. The characterization of the equality case follows from Theorem 3.7. \square

²⁷Observe that $\int_0^\infty e^{-\gamma t^p} dt = \gamma^{-1/p} (1/p) \int_0^\infty e^{-s} s^{1/p-1} ds = \gamma^{-1/p} \Gamma(1/p + 1)$ for $\gamma > 0$.

3.11 Theorem. Let $K \in \mathcal{K}_0^n$ and let B_n be the maximum volume ellipsoid contained in K .²⁸ Then

$$\text{vol}(K) \leq \text{vol}(C_n) = 2^n.$$

Moreover equality holds if and only if K is a cube of edge length 2, i.e., $K = \{\mathbf{x} \in \mathbb{R}^n : |\langle \mathbf{v}_i, \mathbf{x} \rangle| \leq 1, 1 \leq i \leq n\}$, where $\mathbf{v}_1, \dots, \mathbf{v}_n$ form an orthonormal basis.

Proof. Since B_n is the maximum volume ellipsoid contained in K , we know by Theorem 6.11³⁰ that there exist $\mathbf{v}_i \in S^{n-1} \cap \text{bd } K$ and $\lambda_i > 0, 1 \leq i \leq m$, with $\sum_{i=1}^m \lambda_i = n$ and $\mathbf{I}_n = \sum_{i=1}^m \lambda_i (\mathbf{v}_i \mathbf{v}_i^\top)$. Let $U = \{\mathbf{x} \in \mathbb{R}^n : |\langle \mathbf{v}_i, \mathbf{x} \rangle| \leq 1, 1 \leq i \leq m\}$. Clearly $K \subseteq U$ and so $\text{vol}(K) \leq \text{vol}(U)$. With $f_i = \chi_{[-1,1]}$, $1 \leq i \leq m$, we can write

$$U = \left\{ \mathbf{x} \in \mathbb{R}^n : \prod_{i=1}^m f_i(\langle \mathbf{v}_i, \mathbf{x} \rangle)^{\lambda_i} = 1 \right\}$$

and with the lower bound in Theorem 3.6 we get

$$\text{vol}(U) = \int_{\mathbb{R}^n} \left(\prod_{i=1}^m f_i(\langle \mathbf{v}_i, \mathbf{x} \rangle)^{\lambda_i} \right) d\mathbf{x} \leq \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t) dt \right)^{\lambda_i} = 2^{\sum_{i=1}^m \lambda_i} = 2^n.$$

The characterization of the equality case follows from Theorem 3.7. \square

3.12 Theorem. Let $K \in \mathcal{K}^n$ and let B_n be the maximum volume ellipsoid contained in K . Then

$$\text{vol}(K) \leq \text{vol}(T_n) = \frac{(\sqrt{n})^n (\sqrt{n+1})^{n+1}}{n!},$$

where T_n is a regular simplex with inradius 1. Moreover, equality holds if and only if K is a regular simplex with inradius 1, i.e., $K = \{x \in \mathbb{R}^n : \langle \mathbf{v}_i, \mathbf{x} \rangle \leq 1, 1 \leq i \leq n+1\}$ with $\mathbf{v}_i \in S^{n-1}$ and $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = -\frac{1}{n}$ for $i \neq j$.

Proof. Since B_n is the maximum volume ellipsoid contained in K , we know by Theorem 6.14³¹ that there exist $\mathbf{v}_i \in S^{n-1} \cap \text{bd } K$ and $\lambda_i > 0, 1 \leq i \leq m$ and $m \geq n+1$, with $\sum_{i=1}^m \lambda_i = n$, $\mathbf{I}_n = \sum_{i=1}^m \lambda_i (\mathbf{v}_i \mathbf{v}_i^\top)$ and $\sum_{i=1}^m \lambda_i \mathbf{v}_i = \mathbf{0}$. For $U = \{x \in \mathbb{R}^n : \langle \mathbf{v}_i, \mathbf{x} \rangle \leq 1, 1 \leq i \leq m\}$ it holds $K \subseteq U$ and so $\text{vol}(K) \leq \text{vol}(U)$. Now let

$$\mathbf{w}_i = \sqrt{\frac{n}{n+1}} \begin{pmatrix} -\mathbf{v}_i \\ \frac{1}{\sqrt{n}} \end{pmatrix} \in \mathbb{R}^{n+1} \quad \text{and} \quad c_i = \frac{n+1}{n} \lambda_i, \quad 1 \leq i \leq m.$$

Then $\mathbf{w}_i \in S^n$ for all $1 \leq i \leq m$, $\sum_{i=1}^m c_i = n+1$ and

$$\sum_{i=1}^m c_i (\mathbf{w}_i \mathbf{w}_i^\top) = \sum_{i=1}^m \lambda_i \begin{pmatrix} (\mathbf{v}_i \mathbf{v}_i^\top) & -\frac{1}{\sqrt{n}} \mathbf{v}_i \\ -\frac{1}{\sqrt{n}} \mathbf{v}_i^\top & \frac{1}{n} \end{pmatrix} = \mathbf{I}_{n+1}.$$

²⁸According to Theorem 6.11²⁹ B_n is the uniquely determined ellipsoid of maximum volume in K if and only if there exist $\mathbf{u}_i \in S^{n-1} \cap \text{bd } K$ and $\lambda_i > 0, 1 \leq i \leq m$, with $n \leq m \leq n(n+1)/2$ such that $\mathbf{I}_n = \sum_{i=1}^m \lambda_i (\mathbf{u}_i \mathbf{u}_i^\top)$.

³⁰Skript WS14

³¹Skript WS14

Now let

$$f_i : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \quad \text{given by} \quad f_i(t) = \begin{cases} e^{-t} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases}$$

for all $1 \leq i \leq m$. With $F(\mathbf{x}) = \prod_{i=1}^m f_i(\langle \mathbf{w}_i, \mathbf{x} \rangle)^{c_i}$ and Theorem 3.6 we obtain

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} F(\mathbf{x}) \, d\mathbf{x} &= \int_{\mathbb{R}^{n+1}} \left(\prod_{i=1}^m f_i(\langle \mathbf{w}_i, \mathbf{x} \rangle)^{c_i} \right) d\mathbf{x} \leq \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(t) \, dt \right)^{c_i} \\ &= \prod_{i=1}^m \left(\int_0^{\infty} e^{-t} \, dt \right)^{c_i} = 1. \end{aligned} \tag{3.12.1}$$

Next for $\mathbf{x} \in \mathbb{R}^{n+1}$ we write $\mathbf{x} = \begin{pmatrix} \mathbf{y} \\ r \end{pmatrix} \in \mathbb{R}^{n+1}$ with $\mathbf{y} \in \mathbb{R}^n$, $r \in \mathbb{R}$. Then

$$\langle \mathbf{w}_i, \mathbf{x} \rangle = \frac{r}{\sqrt{n+1}} - \sqrt{\frac{n}{n+1}} \langle \mathbf{v}_i, \mathbf{y} \rangle.$$

an so

$$F(\mathbf{x}) = e^{-\sum_{i=1}^m c_i \langle \mathbf{w}_i, \mathbf{x} \rangle} = e^{-\sum_{i=1}^m c_i \frac{r}{\sqrt{n+1}} + \sum_{i=1}^m c_i \sqrt{\frac{n}{n+1}} \langle \mathbf{v}_i, \mathbf{y} \rangle} = e^{-r\sqrt{n+1}}.$$

Due to definition of $F(\mathbf{x})$ we have $F(\mathbf{x}) \neq 0$, if and only if

$$\langle \mathbf{v}_i, \mathbf{y} \rangle \leq \frac{r}{\sqrt{n}}, \quad 1 \leq i \leq m. \tag{3.12.2}$$

Since $\sum_{i=1}^m \lambda_i \mathbf{v}_i = 0$, $\lambda_i > 0$, we know that for each $\mathbf{y} \in \mathbb{R}^n \setminus \{0\}$ there exists $j \in \{1, \dots, m\}$ such that $\langle \mathbf{v}_j, \mathbf{y} \rangle \geq 0$, and hence, if $r < 0$ (3.12.2) is never fulfilled. In the case $r \geq 0$, (3.12.2) is equivalent to $\mathbf{y} \in \frac{r}{\sqrt{n}}U$, and for such a $\mathbf{y} \in \frac{r}{\sqrt{n}}U$, $r \geq 0$, and $\mathbf{x} = \begin{pmatrix} \mathbf{y} \\ r \end{pmatrix}$, we have

Together with Fubini's theorem we get

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} F(\mathbf{x}) \, d\mathbf{x} &= \int_0^{\infty} \int_{\frac{r}{\sqrt{n}}U} F(\mathbf{y}, r) \, d\mathbf{y} \, dr = \int_0^{\infty} \int_{\frac{r}{\sqrt{n}}U} e^{-r\sqrt{n+1}} \, d\mathbf{y} \, dr \\ &= \int_0^{\infty} e^{-r\sqrt{n+1}} \left(\frac{r}{\sqrt{n}} \right)^n \text{vol}(U) \, dr \\ &= \text{vol}(U) \left(\frac{1}{\sqrt{n}} \right)^n \int_0^{\infty} e^{-t} \left(\frac{t}{\sqrt{n+1}} \right)^n \frac{1}{\sqrt{n+1}} \, dt \\ &= \frac{n! \text{vol}(U)}{(\sqrt{n})^n (\sqrt{n+1})^{n+1}}. \end{aligned}$$

Together with (3.12.1), the result follows. The characterization of the equality case follows from Theorem 3.7; observe, if we have equality we have $m = n + 1$ and the vectors \mathbf{w}_i build an orthonormal basis in \mathbb{R}^{n+1} . Hence $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = -\frac{1}{n}$ for $i \neq j$, which shows that U and thus K is a regular simplex. \square

3.13 Corollary. For a convex body $K \in \mathcal{K}^n$ the volume ratio $\text{vr}(K)$ is defined as

$$\text{vr}(K) = \inf_{E \subseteq K, E \text{ ellipsoid}} \left(\frac{\text{vol}(K)}{\text{vol}(E)} \right)^{\frac{1}{n}}.$$

- i) $\text{vr}(K) \leq \sqrt{n} \sqrt{n+1}^{1+1/n} / n!^{1/n}$ with equality if and only if K is a simplex.
- ii) For $K \in \mathcal{K}_o^n$ it is $\text{vr}(K) \leq 2$ with equality if and only if K is (an affine image of) the cube C_n .

Proof. Immediate consequence of the Theorems 3.11, 3.12. \square

3.14 Theorem [Reverse Isoperimetric Inequality]. Let $K \in \mathcal{K}^n$, $\dim K = n$. There exists a regular affine transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$\frac{F(TK)^n}{\text{vol}(TK)^{n-1}} \leq \frac{F(T_n)^n}{\text{vol}(T_n)^{n-1}} = n^n \frac{(\sqrt{n})^n (\sqrt{n+1})^{n+1}}{n!},$$

where T_n is a regular simplex. If K is o -symmetric then

$$\frac{F(TK)^n}{\text{vol}(TK)^{n-1}} \leq \frac{F(C_n)^n}{\text{vol}(C_n)^{n-1}} = (2n)^n,$$

where C_n is a cube. Both inequalities are best possible.

Proof. For the general case let the affine transformation T be chosen such that B_n is the maximum volume ellipsoid contained in TK . By the formula for the surface area given in Remark 5.31 iii)³², we obtain

$$\begin{aligned} F(TK) &= \lim_{\lambda \rightarrow 0} \frac{\text{vol}(TK + \lambda B_n) - \text{vol}(TK)}{\lambda} \leq \lim_{\lambda \rightarrow 0} \frac{\text{vol}(TK + \lambda TK) - \text{vol}(TK)}{\lambda} \\ &= \lim_{\lambda \rightarrow 0} \frac{(1 + \lambda)^n - 1}{\lambda} \text{vol}(TK) = n \text{vol}(TK) \\ &= n \text{vol}(TK)^{1/n} \text{vol}(TK)^{(n-1)/n} \leq n \text{vol}(T_n)^{1/n} \text{vol}(TK)^{(n-1)/n}, \end{aligned}$$

where the last inequality follows from Theorem 3.12 and T_n is here a regular simplex with inradius 1. Hence $F(T_n) = n \text{vol}(T_n)$ and we get

$$\frac{F(TK)^n}{\text{vol}(TK)^{n-1}} \leq n^n \text{vol}(T_n) = \frac{F(T_n)^n}{\text{vol}(T_n)^{n-1}}.$$

Now suppose that S is an n -simplex. Then we have $r(S) F(S) = n \text{vol}(S)$ and hence we have

$$\frac{F(TS)^n}{\text{vol}(TS)^{n-1}} = n^n \frac{\text{vol}(TS)}{r(TS)^n}$$

for any regular affine transformation T . Now for a given inradius the regular simplex has smallest volume and hence we conclude

$$\frac{F(TS)^n}{\text{vol}(TS)^{n-1}} = n^n \frac{\text{vol}(TS)}{r(TS)^n} \geq n^n \text{vol}(T_n) = \frac{F(T_n)^n}{\text{vol}(T_n)^{n-1}}.$$

Hence for a simplex the inequality can be improved.

The proof for symmetric convex bodies is analogous, but now we apply Theorem 3.11. \square

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3.15 Theorem. Let $K \in \mathcal{K}^n$, and let $c_i > 0$ and $\mathbf{v}_i \in S^{n-1}$, for $1 \leq i \leq m$, be such that $\sum_{i=1}^m c_i (\mathbf{v}_i \mathbf{v}_i^\top) = \mathbf{I}_n$. Then

$$\text{vol}(K)^{n-1} \leq \prod_{i=1}^m \text{vol}_{n-1}(K|\mathbf{v}_i^\perp)^{c_i}.$$

Here $K|\mathbf{v}_i^\perp$ denotes the orthogonal projection of K onto the orthogonal complement of \mathbf{v}_i , i.e., the hyperplane $H(\mathbf{v}_i, 0)$.

Moreover, equality holds if and only if $m = n$, $\mathbf{v}_1, \dots, \mathbf{v}_n$ are an orthonormal basis and K is an orthogonal box with respect to \mathbf{v}_i , i.e., $K = \{\mathbf{x} \in \mathbb{R}^n : \alpha_i \leq \langle \mathbf{v}_i, \mathbf{x} \rangle \leq \beta_i, 1 \leq i \leq n\}$ and $\alpha_i < \beta_i \in \mathbb{R}$.

Proof. Let $\alpha_i = c_i / \text{vol}_{n-1}(K|\mathbf{v}_i^\perp)$, $1 \leq i \leq m$, and let Z be the zonotope $Z = \sum_{i=1}^m \text{conv}\{-\alpha_i \mathbf{v}_i, \alpha_i \mathbf{v}_i\}$. Applying first Minkowski's inequality Theorem 5.32³³, then Corollary 5.27³⁴ and finally using the linearity of the mixed volumes Lemma 5.23 iv)³⁵, we get

$$\begin{aligned} \text{vol}(Z)^{1/n} \text{vol}(K)^{(n-1)/n} &\leq V(K, n-1; Z, 1) \\ &= \sum_{i=1}^m V(K, n-1; \text{conv}\{-\alpha_i \mathbf{v}_i, \alpha_i \mathbf{v}_i\}, 1) \\ &= \frac{2}{n} \sum_{i=1}^m \alpha_i \text{vol}_{n-1}(K|\mathbf{v}_i^\perp) = \frac{2}{n} \sum_{i=1}^m c_i = 2. \end{aligned}$$

Therefore, $\text{vol}(K)^{n-1} \leq 2^n / \text{vol}(Z)$. Finally, from Lemma 3.10 we get

$$\text{vol}(Z) \geq 2^n \prod_{i=1}^m \left(\frac{\alpha_i}{c_i} \right)^{c_i} = 2^n \prod_{i=1}^m \left(\frac{1}{\text{vol}_{n-1}(K|\mathbf{v}_i^\perp)} \right)^{c_i},$$

and the result follows.

Now, if we have equality then by the equality case of the Minkowski inequality, K and Z are homothetic and from Lemma 3.10 we conclude $m = n$, $c_i = 1$, and $\mathbf{v}_1, \dots, \mathbf{v}_n$ are an orthonormal basis. Hence K is an orthogonal box with edge directions \mathbf{v}_i . \square

3.16 Corollary [Loomis-Whitney inequality]. Let $K \in \mathcal{K}^n$. Then

$$\text{vol}(K)^{n-1} \leq \prod_{i=1}^n \text{vol}_{n-1}(K|\mathbf{e}_i^\perp),$$

and equality holds if and only if K is an orthogonal box with facets parallel to the coordinate axes.

Proof. Apply Theorem 3.15 with $m = n$, $\mathbf{v}_i = \mathbf{e}_i$ and $c_i = 1$, $1 \leq i \leq n$. \square

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³⁴Skript WS14

³⁵Skript WS14

3.17 Theorem. Let S be a k -cover of $[n] = \{1, \dots, n\}$, i.e., $S = \{S_1, S_2, \dots, S_l\}$ with $S_i \subset [n]$, $1 \leq i \leq l$, and each $j \in [n]$ is contained in exactly k sets S_i . Let $K \in \mathcal{K}^n$ and let $K|_{S_i}$ be the projection onto $\text{lin}\{\mathbf{e}_j : j \in S_i\}$. Then

$$\text{vol}(K)^k \geq \prod_{S_i \in S} \text{vol}_{|S_i|}(K|_{S_i}).$$

Proof. To simplify notation, in the following $\text{vol}(A)$ will always denote the volume measured with respect to $\text{aff} A$. We use induction on n . For $n = 1$ it is certainly true. So let $n \geq 2$ and let \square

3.18 Theorem [Meyer-Inequality]. Let $K \in \mathcal{K}^n$ and let $H_i = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{e}_i, \mathbf{x} \rangle = 0\}$, $1 \leq i \leq n$. Then

$$\text{vol}(K)^{n-1} \geq \frac{(n-1)!}{n^{n-1}} \prod_{i=1}^n \text{vol}_{n-1}(K \cap H_i),$$

and equality holds if and only if K is a generalized cross-polytope, i.e., $K = \text{conv}\{-\alpha_i \mathbf{e}_i, \beta_i \mathbf{e}_i : \alpha_i, \beta_i > 0, 1 \leq i \leq n\}$.

Proof. Applying Steiner-Symmetrization to K with respect to the hyperplane H_1 yields a convex body K_1 , say, with $\text{vol}(K_1) = \text{vol}(K)$, $\text{vol}_{n-1}(K \cap H_1) \leq \text{vol}_{n-1}(K_1 \cap H_1) = \text{vol}_{n-1}(K|_{\mathbf{e}_1^\perp})$ and $\text{vol}_{n-1}(K \cap H_i) = \text{vol}_{n-1}(K_1 \cap H_i)$ for all $i > 1$. Hence it suffices to prove the inequality for K_1 and repeating this argument to all coordinate hyperplanes H_i shows that it suffices to prove the inequality for a body L which is symmetric to all coordinate hyperplanes, i.e., for $\mathbf{x} \in L$ we also have $(\pm x_1, \dots, \pm x_n)^\top \in L$. Such a convex body is also called an unconditional convex body. In particular, we have $L \cap H_i = L|_{\mathbf{e}_i^\perp}$ for such a body.

Let $L^s = L \cap \mathbb{R}_{\geq 0}^n$ and $L_i^s = L^s \cap H_i$. By the symmetries we have $\text{vol}(L) = 2^n \text{vol}(L^s)$ and $\text{vol}(L \cap H_i) = 2^{n-1} \text{vol}_{n-1}(L_i^s)$. Hence it suffices to verify the inequality for L^s and its sections L_i^s , for which we rewrite the inequality as

$$\text{vol}(L^s) \leq \frac{n^{n-1}}{(n-1)!} \prod_{i=1}^n \frac{\text{vol}(L^s)}{\text{vol}(L_i^s)}. \quad (3.18.1)$$

For $\mathbf{y} \in L^s$ the intersection of the pyramid $\text{conv}\{L_i^s, \mathbf{y}\}$ with another pyramid $\text{conv}\{L_j^s, \mathbf{y}\}$, $j \neq i$, is contained in an $(n-1)$ -dimensional subspace. Thus we have

$$\text{vol}(L^s) \geq \sum_{i=1}^n \text{vol}(\text{conv}\{L_i^s, \mathbf{y}\}) = \frac{1}{n} \sum_{i=1}^n y_i \text{vol}_{n-1}(L_i^s)$$

for every $\mathbf{y} \in L^s$, and so

$$L^s \subseteq \left\{ \mathbf{y} \in \mathbb{R}_{\geq 0}^n : \sum_{i=1}^n y_i \text{vol}_{n-1}(L_i^s) \leq n \text{vol}(L^s) \right\}.$$

Now the set on the left hand side is an n -simplex with vertices $\mathbf{0}$, $\frac{n \text{vol}(L^s)}{\text{vol}_{n-1}(L_i^s)} \mathbf{e}_i$, $1 \leq i \leq n$, and of volume

$$\frac{1}{n!} \prod_{i=1}^n \frac{n \text{vol}(L^s)}{\text{vol}_{n-1}(L_i^s)}.$$

and (3.18.1) is proven.

Now if we equality for K then the proof above shows that we also have equality for all the bodies K_i created by successively Steiner-symmetrizations at the planes H_i . In particular, we have equality for the unconditional body $K_n = L$, i.e., we have equality in (3.18.1). Hence L^s is a simplex with vertices $\mathbf{0}$ and at the coordinate axis, and thus $L = \text{conv} \{\pm\gamma_i e_i : 1 \leq i \leq n\}$, with $\gamma_i \in \mathbb{R}_{>0}$, and $\text{vol}(L) = \text{vol}(K)$.

By the equality assumption we have $K_{n-1} \cap H_n = K_{n-1}|H_n = L \cap H_n$ and so we conclude $\text{conv} \{L \cap H_n, -\alpha_n e_n, \beta_n e_n\} \subseteq K_{n-1}$ with $\alpha_n + \beta_n = 2\gamma_n$. Comparing the volumes of the two sets shows that we have indeed $K_{n-1} = \text{conv} \{L \cap H_n, -\alpha_n e_n, \beta_n e_n\}$ and $\alpha_n, \beta_n > 0$. Hence $-\alpha_n e_n, \beta_n e_n \in K$. Repeating backwards this argument shows that K has to be a generalized crosspolytope. \square

3.19 Theorem. *Let L be a k -dimensional linear subspace of \mathbb{R}^n . Then*

$$\text{vol}_k(C_n \cap L) \leq 2^k \left(\frac{n}{k}\right)^{k/2}.$$

If k is a divisor of n then the inequality is best possible.

Proof. Here we want to apply Theorem 3.6 in the k -dimensional space L . To this end let P be the orthogonal projection of \mathbb{R}^n onto L . Let $\mathbf{u}_i = Pe_i$ and $c_i = \|\mathbf{u}_i\|^2$, $1 \leq i \leq n$. If $c_j = 0$ then $L \subseteq e_j^\perp$ and $C_n \cap L = (C_n \cap e_j^\perp) \cap L$. Hence the problem is reduced to a cube of dimension one less and the result follows inductively.

So we assume $c_i > 0$ for $1 \leq i \leq n$. For $\mathbf{x} \in \mathbb{R}^n$ we have

$$P\mathbf{x} = P \left(\sum_{i=1}^n \langle \mathbf{x}, e_i \rangle e_i \right) = \sum_{i=1}^n \langle \mathbf{x}, e_i \rangle Pe_i = \sum_{i=1}^n \langle \mathbf{x}, e_i \rangle \mathbf{u}_i,$$

and since for $\mathbf{x} \in L$ we also have $\langle \mathbf{x}, e_i \rangle = \langle \mathbf{x}, \mathbf{u}_i \rangle$ we get

$$\mathbf{x} = P\mathbf{x} = \sum_{i=1}^n \langle \mathbf{x}, \mathbf{u}_i \rangle \mathbf{u}_i = \sum_{i=1}^n (\mathbf{u}_i \mathbf{u}_i^T) \mathbf{x} = \sum_{i=1}^n c_i (\mathbf{v}_i \mathbf{v}_i^T) \mathbf{x},$$

with $\mathbf{v}_i = \mathbf{u}_i / \sqrt{c_i} \in L \cap S^{n-1}$. Hence, in L the unit vectors \mathbf{v}_i give a decomposition of the identity and since $\text{tr}P = k$ we get $\sum_{i=1}^n c_i = k$. Moreover we have

$$\begin{aligned} C_n \cap L &= \left\{ \mathbf{x} \in L : |\langle \mathbf{x}, e_i \rangle| \leq 1, 1 \leq i \leq n \right\} \\ &= \left\{ \mathbf{x} \in L : |\langle \mathbf{x}, \mathbf{v}_i \rangle| \leq \frac{1}{\sqrt{c_i}}, 1 \leq i \leq n \right\}, \end{aligned}$$

and with $f_i = \chi_{[-1/\sqrt{c_i}, 1/\sqrt{c_i}]}$ we get by Theorem 3.6 (applied in L)

$$\begin{aligned} \text{vol}_k(C_n \cap L) &= \int_L \left(\prod_{i=1}^n f_i(\langle \mathbf{v}_i, \mathbf{x} \rangle)^{c_i} \right) d\mathbf{x} \leq \prod_{i=1}^n \left(\int_{\mathbb{R}} f_i(t) dt \right)^{c_i} \\ &= \prod_{i=1}^n \left(\frac{2}{\sqrt{c_i}} \right)^{c_i} = 2^k \left(\prod_{i=1}^n c_i \right)^{-1/2}. \end{aligned}$$

The continuous function $\prod_{i=1}^n x_i^{x_i}$ attains a minimum on the compact set $\{\mathbf{x} \in \mathbb{R}_{\geq 0}^n : \sum_{i=1}^n x_i = k\}$. Inductively it can be shown that such a minimum satisfies $x_1 = \cdots = x_n$. Therefore,

exercise

$$\prod_{i=1}^n c_i^{c_i} \geq \prod_{i=1}^n \left(\frac{\sum_{i=1}^n c_i}{n} \right)^{\sum_{i=1}^n c_i/n} = \left(\frac{k}{n} \right)^k.$$

Thus $\text{vol}_k(C_n \cap L) \leq 2^k (n/k)^{k/2}$.

Now suppose that k is a divisor of n , and let $m = n/k$. For each $1 \leq i \leq k$, let $\mathbf{u}_i = \sum_{j=1}^m \mathbf{e}_{(i-1)m+j}$. These vectors are pairwise orthogonal and with $L = \text{lin}\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ we get that

$$Q = \left\{ \sum_{i=1}^k \lambda_i \mathbf{u}_i : \lambda_i \in [-1, 1] \right\} \subset C_n \cap L.$$

Since Q is a k -dimensional cube with edge-length $\sqrt{m} = \sqrt{n/k}$ we have equality in this case. \square

3.20 Theorem*. *Let L be a k -dimensional linear subspace of \mathbb{R}^n . Then*

$$\text{vol}_k(C_n \cap L) \leq 2^k 2^{(n-k)/2}.$$

If $k \geq n/2$ the inequality is best possible.

3.21 Theorem [Vaaler]. *Let L be an $(n-1)$ -dimensional linear subspace of \mathbb{R}^n . Then*

$$\text{vol}_{n-1}(C_n \cap L) \geq 2^{n-1},$$

and equality holds if and only if L is a coordinate hyperplane.

Proof. For an arbitrary but fixed $\mathbf{u} \in S^{n-1}$ let $H(t) = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{u} \rangle = t\}$, $t \in \mathbb{R}$, and let $f(t) = \text{vol}_{n-1}(C_n \cap H(t))$. Since C_n is a 0-symmetric convex body, we get by the Brunn-Minkowski theorem that $f(t) \leq f(0)$ for all $t \in \mathbb{R}$.

$$F(t) = \int_0^t f(s) ds,$$

we know $F(t) \leq t f(0)$ for $t \geq 0$ where for $t > 0$ equality holds if and only if $f(t) = f(0)$ for all $t > 0$ with $f(t) \neq 0$; also observe that $F(t)' = f(t)$. Therefore we find

$$\begin{aligned} \int_{C_n} \langle \mathbf{x}, \mathbf{u} \rangle^2 d\mathbf{x} &= \int_{\mathbb{R}} t^2 f(t) dt = \frac{2}{f(0)^2} \int_0^\infty (t f(0))^2 f(t) dt \\ &\geq \frac{2}{f(0)^2} \int_0^\infty F(t)^2 f(t) dt = \frac{2}{3 f(0)^2} \int_0^\infty [F(t)^3]' dt \\ &= \frac{2}{3 f(0)^2} [F(\infty)^3 - F(0)^3] = \frac{2}{3 f(0)^2} \left(\frac{\text{vol}(C_n)}{2} \right)^3 \\ &= \frac{2}{3 f(0)^2} (2^{n-1})^3, \end{aligned}$$

with equality if and only if $F(t) = t f(0)$ for all $t \in \mathbb{R}$ with $f(t) > 0$, i.e., if and only if $f(t) = f(0)$ for all $t \in \mathbb{R}$ with $f(t) > 0$. By the continuity of $f(t)$ on its support we conclude that this is equivalent to $\mathbf{u} \in \{\pm \mathbf{e}_i, 1 \leq i \leq n\}$.

On the other hand, we may evaluate the left hand side integral by

$$\begin{aligned}
 \int_{C_n} \langle \mathbf{x}, \mathbf{u} \rangle^2 d\mathbf{x} &= \int_{C_n} \left(\sum_{i=1}^n u_i x_i \right)^2 d\mathbf{x} \\
 &= \int_{C_n} \left(\sum_{i=1}^n u_i^2 x_i^2 + 2 \sum_{1 \leq i < j \leq n} u_i u_j x_i x_j \right) d\mathbf{x} \\
 &= \sum_{i=1}^n \int_{C_n} u_i^2 x_i^2 d\mathbf{x} + 2 \int_{C_n} \left(\sum_{1 \leq i < j \leq n} u_i x_i u_j x_j \right) d\mathbf{x} \quad (3.21.1) \\
 &= \sum_{i=1}^n \left(\int_{-1}^1 \cdots \int_{-1}^1 u_i^2 x_i^2 \right) dx_1 \dots dx_n \\
 &= \frac{2}{3} 2^{n-1} \sum_{i=1}^n u_i^2 = \frac{2}{3} 2^{n-1}.
 \end{aligned}$$

Comparing the two integrals gives $f(0) \geq 2^{n-1}$ with equality if and only if $\mathbf{u} \in \{\pm \mathbf{e}_i, 1 \leq i \leq n\}$. \square

3.22 Remark [Busemann-Petty problem]. *The Busemann-Petty problem was the question whether for two 0-symmetric convex bodies $K, L \in \mathcal{K}_0^n$, the inequalities*

$$\text{vol}_{n-1}(K \cap H(\mathbf{u}, 0)) \geq \text{vol}_{n-1}(L \cap H(\mathbf{u}, 0)), \quad \text{for all } \mathbf{u} \in S^{n-1},$$

imply $\text{vol}(K) \geq \text{vol}(L)$? Taking $K = C_n$ and L a ball of volume 2^n , Theorem 3.20 gives

$$\text{vol}(K \cap H(\mathbf{u}, 0)) \leq 2^{n-1} \sqrt{2} < \text{vol}(L \cap H(\mathbf{u}, 0)) = 2^{n-1} \frac{\Gamma(\frac{n}{2} + 1)^{\frac{n-1}{n}}}{\Gamma(\frac{n-1}{2} + 1)} \rightarrow 2^{n-1} \sqrt{e}.$$

And so the answer is No for $n \geq 10$. In the meantime the problem has been completely solved: the answer is affirmative for $n \leq 4$ and negative for $n \geq 5$.

3.23 Definition. *Let $K \in \mathcal{K}^n$. The set*

$$\Pi(K) = \{x \in \mathbb{R}^n : |\langle \mathbf{u}, \mathbf{x} \rangle| \leq \text{vol}_{n-1}(K|\mathbf{u}^\perp), \text{ for all } \mathbf{u} \in S^{n-1}\}$$

is called the projection body of K .

3.24 Proposition. *Let $K \in \mathcal{K}^n$. Then*

- i) $h(\Pi(K), \mathbf{u}) = \text{vol}_{n-1}(K|\mathbf{u}^\perp)$.
- ii) $\Pi(K)$ is *o*-symmetric.

iii) $\Pi(AK) = |\det A| A^{-\top} \Pi(K)$ for $A \in \text{GL}(n, \mathbb{R})$, and $\Pi(\mathbf{t} + K) = \Pi(K)$ for $\mathbf{t} \in \mathbb{R}^n$.

3.25 Theorem. *Let $K \in \mathcal{K}^n$. There exists a regular affine transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that*

$$\text{vol}_{n-1}((TK)|\mathbf{u}^\perp)^n \geq \text{vol}(TK)^{n-1}, \text{ for all } \mathbf{u} \in S^{n-1}.$$

Proof. According to Proposition 3.24 iii), we may find a linear transformation $A \in \text{GL}(n, \mathbb{R})$ such that B_n is the maximum volume ellipsoid contained in $A^{-\top} \Pi(K) = |\det A|^{-1} \Pi(AK)$. In particular, $|\det A| B_n \subseteq \Pi(AK)$, and hence, with Proposition 3.24 ii) we get

$$\text{vol}_{n-1}((AK)|\mathbf{u}^\perp) = h(\Pi(AK), \mathbf{u}) \geq |\det A|,$$

for all $\mathbf{u} \in S^{n-1}$.

By Theorem 6.11³⁶ there exist $\mathbf{u}_i \in S^{n-1} \cap \text{bd}(|\det A|^{-1} \Pi(AK))$ and $\lambda_i > 0$, $1 \leq i \leq m$, such that $\mathbf{I}_n = \sum_{i=1}^m \lambda_i (\mathbf{u}_i \mathbf{u}_i^\top)$ and so $\sum_{i=1}^m \lambda_i = n$. In particular,

$$\text{vol}_{n-1}((AK)|\mathbf{u}_i^\perp) = h(\Pi(AK), \mathbf{u}_i) = |\det A|.$$

Together with Theorem 3.15 we get

$$\begin{aligned} \text{vol}(AK)^{n-1} &\leq \prod_{i=1}^m \text{vol}_{n-1}(AK|\mathbf{u}_i^\perp)^{\lambda_i} = \prod_{i=1}^m |\det A|^{\lambda_i} \\ &= |\det A|^{\sum_{i=1}^m \lambda_i} = |\det A|^n \leq \text{vol}_{n-1}(AK|\mathbf{u}^\perp)^n, \end{aligned}$$

for all $\mathbf{u} \in S^{n-1}$. □

3.26 Theorem* [Vaaler, 1979]. *Let L be a k -dimensional linear subspace. Then $\text{vol}_k(C_n \cap L) \geq 2^k$.*

3.27 Theorem*. *Let $H = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{a}, \mathbf{x} \rangle = b\}$, $\mathbf{a} \in S^{n-1}$, $b \in \mathbb{R}$, and assume $a_i \neq 0$, $1 \leq i \leq n$. Then*

$$\text{vol}(C_n \cap H) = \frac{1}{2(n-1)!} \left(\prod_{i=1}^n a_i \right)^{-1} \sum_{\mathbf{v} \in \text{vert } C_n} (\langle \mathbf{a}, \mathbf{v} \rangle + b)^{n-1} \text{sgn}(\langle \mathbf{a}, \mathbf{v} \rangle + b) \prod_{i=1}^n \mathbf{v}_i.$$

3.28 Theorem* [Chakerian & Filliman, 1986]. *Let L be a k -dimensional linear subspace. Then*

$$\text{vol}_k(C_n|L) \leq 2^k \min \left\{ \frac{\kappa_{k-1}^k}{\kappa_k^{k-1}} \left(\frac{n}{k} \right)^{\frac{k}{2}}, \sqrt{\binom{n}{k}} \right\}.$$

3.29 Theorem [McMullen, 1984]. *Let L be a k -dimensional linear subspace with orthogonal complement L^\perp . Then $\text{vol}_k(C_n|L) = \text{vol}_{n-k}(C_n|L^\perp)$.*

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Proof. Exercise. The main tool is here the so-called Jacobi's determinat identity which for a regular matrix $A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$ and its inverse $A^{-1} = \begin{pmatrix} B' & C' \\ D' & E' \end{pmatrix}$ where B, B' are $(k \times k)$ matrices says that

$$\det B = \det E' \det A.$$

□

Preliminary Version – Draft 2016

4 A few remarks on isotropic positions

4.1 Notation. Let $\mathcal{K}_c^n \subset \mathcal{K}^n$ be the set of all convex bodies with $\text{vol}(K) = 1$ and centroid at the origin, i.e., $\int_K \mathbf{x} \, d\mathbf{x} = \mathbf{0}$.

4.2 Definition [Isotropic Position, isotropic constant]. Let $K \in \mathcal{K}^n$. An isotropic position of K is defined as $\bar{K} = AK + \mathbf{t}$, $A \in \text{GL}(n, \mathbb{R})$, $\mathbf{t} \in \mathbb{R}^n$, such that $\bar{K} \in \mathcal{K}_c^n$ and there exists a positive constant L_K , called the isotropic constant, such that for all $\mathbf{u} \in S^{n-1}$

$$\int_{\bar{K}} \langle \mathbf{u}, \mathbf{x} \rangle^2 \, d\mathbf{x} = L_K^2.$$

4.3 Definition [Moment matrix]. For $K \in \mathcal{K}^n$ the $n \times n$ -matrix

$$A_K = \int_K \mathbf{x}\mathbf{x}^\top \, d\mathbf{x}$$

is called the moment matrix or matrix of inertia of K .

4.4 Remark [Binet ellipsoid]. Observe that for $\mathbf{u} \in \mathbb{R}^n$

$$\mathbf{u}^\top A_K \mathbf{u} = \int_K \mathbf{u}^\top \mathbf{x}\mathbf{x}^\top \mathbf{u} \, d\mathbf{x} = \int_K \langle \mathbf{u}, \mathbf{x} \rangle^2 \, d\mathbf{x}.$$

Hence A_K is a positive definite symmetric matrix and it can be decomposed as $A_K = (DU)^\top (DU)$ where $U \in \text{O}(n, \mathbb{R})$ is an orthogonal matrix and D is a diagonal matrix. The ellipsoid $\mathcal{E}_K = (DU)^{-1} B_n$ has norm $|\mathbf{u}|_{\mathcal{E}_K} = \sqrt{\mathbf{u}^\top A_K \mathbf{u}}$ and it is called the Binet (fundamental) Ellipsoid of K .

4.5 Proposition.

- i) $K \in \mathcal{K}_c^n$ is in isotropic position if and only if $\mathcal{E}_K = (1/L_K) B_n$ which is equivalent to $A_K = L_K^2 I_n$.
- ii) If $K \in \mathcal{K}^n$ is in isotropic position, then

$$\int_K \|\mathbf{x}\|^2 \, d\mathbf{x} = n L_K^2.$$

- iii) Let $K \in \mathcal{K}^n$ and let $T \in \text{GL}(n, \mathbb{R})$. Then

$$A_{TK} = |\det T| T A_K T^\top.$$

- iv) If $K \in \mathcal{K}_c^n$ is in isotropic position then TK is in isotropic position for all $T \in \text{O}(n, \mathbb{R})$, and it is $L_K = L_{TK}$.

Proof. K is isotropic if and only if $\mathbf{u}^\top (A_k - L_K^2 \mathbf{I}_n) \mathbf{u} = 0$ for all $\mathbf{u} \in S^{n-1}$. Since $A_k - L_K^2 \mathbf{I}_n$ is symmetric we conclude $A_K = L_K^2 \mathbf{I}_n$ which is equivalent to the fact that \mathcal{E}_K is a ball of radius $1/L_K$. This shows i).

ii) is obvious since $\int_K x_i^2 d\mathbf{x} = L_K^2$ for each $1 \leq i \leq n$, if K is in isotropic position. For iii) we note

$$\begin{aligned} \mathbf{u}^\top T A_K T^\top \mathbf{u} &= \int_K \langle T^\top \mathbf{u}, \mathbf{x} \rangle^2 d\mathbf{x} = \int_K \langle \mathbf{u}, T \mathbf{x} \rangle^2 d\mathbf{x} \\ &= \frac{1}{|\det T|} \int_{TK} \langle \mathbf{u}, \mathbf{x} \rangle^2 d\mathbf{x} = \frac{1}{|\det T|} \mathbf{u}^\top A_{TK} \mathbf{u} \end{aligned}$$

for all $\mathbf{u} \in S^{n-1}$. Hence $|\det T| T A_K T^\top = A_{TK}$.

For iv) let $T \in O(n, \mathbb{R})$ and K (in) isotropic (position); iii) and i) gives $A_{TK} = T A_K T^\top = T L_K^2 \mathbf{I}_n T^\top = L_K^2 \mathbf{I}_n$ and so by i) TK is isotropic as well with the same isotropic constant as K . \square

4.6 Remark. Let $\bar{C}_n = \frac{1}{2}C_n$, $\bar{B}_n = (1/\kappa_n)^{1/n} B_n$ be the cube and ball of volume 1, respectively. Then

$$L_{\bar{C}_n} = \sqrt{\frac{1}{12}} \quad \text{and} \quad L_{\bar{B}_n} = \sqrt{\frac{1}{n+2} \frac{\Gamma(\frac{n}{2} + 1)^{\frac{2}{n}}}{\pi}} \sim \sqrt{\frac{1}{2\pi e}}.$$

Proof. First we note that for $K \in \mathcal{K}^n$ and $\lambda \in \mathbb{R}_{>0}$

$$\int_{\lambda K} \langle \mathbf{u}, \mathbf{x} \rangle^2 d\mathbf{x} = \lambda^n \int_K \langle \mathbf{u}, \lambda \mathbf{x} \rangle^2 d\mathbf{x} = \lambda^{n+2} \int_K \langle \mathbf{u}, \mathbf{x} \rangle^2 d\mathbf{x}. \quad (4.6.1)$$

From (3.21.1) we know already $\int_{C_n} \langle \mathbf{u}, \mathbf{x} \rangle^2 d\mathbf{x} = \frac{2}{3}2^{n-1}$, and hence \bar{C}_n is in isotropic position. With (4.6.1) we find $L_{\bar{C}_n}^2 = 1/12$.

Obviously, the ball \bar{B}_n is in isotropic position, since the value of $\int_{\bar{B}_n} \langle \mathbf{u}, \mathbf{x} \rangle^2 d\mathbf{x}$ is independent of the chosen direction \mathbf{u} . For the trace of the moment matrix we compute via spherical coordinates

$$\int_{B_n} \|\mathbf{x}\|^2 d\mathbf{x} = n\kappa_n \int_0^1 r^{n+1} dr = \frac{n}{n+2} \kappa_n.$$

Hence $\int_{B_n} x_i^2 d\mathbf{x} = \frac{\kappa_n}{n+2}$ and thus

$$L_{\bar{B}_n}^2 = \left(\frac{1}{\kappa_n} \right)^{\frac{n+2}{n}} \frac{\kappa_n}{n+2} = \frac{1}{n+2} \kappa_n^{-2/n} = \frac{1}{n+2} \frac{\Gamma(\frac{n}{2} + 1)^{2/n}}{\pi}.$$

\square

4.7 Lemma. Let $K \in \mathcal{K}^n$ be centered. Then there exists a $T \in GL(n, \mathbb{R})$ such that TK is in isotropic position. Moreover, T is uniquely determined up to orthogonal transformations.

Proof. We may assume that also $\text{vol } K = 1$. Since A_K is a positive definite symmetric matrix there exists an $U \in O(n, \mathbb{R})$ such that $UA_kU^\top = \text{diag}(s_1, \dots, s_n)$ with $s_i > 0$. Let $\sigma = (s_1 \cdot \dots \cdot s_n)^{1/n}$ and let $V = \text{diag}(\sqrt{\sigma/s_1}, \dots, \sqrt{\sigma/s_n})$. Then $VU \in \text{GL}(n, \mathbb{R})$, $\det(VU) = 1$ and

$$VU A_K (VU)^\top = \sigma I_n$$

and with Proposition 4.5 iii) and i), we see that VUK is in isotropic position. Moreover, VU is uniquely determined up to orthogonal transformations, because $SA_K S^\top = \sigma I_n = TA_K T^\top$ implies $S^{-1}S^{-\top} = T^{-1}T^{-\top}$ or $I_n = (TS^{-1})(TS^{-1})^\top$. Thus $TS^{-1} \in O(n, \mathbb{R})$ which means $T = US$ for an $U \in O(n, \mathbb{R})$. \square

4.8 Notation. Due to Lemma 4.7 and Proposition 4.5 iv) isotropic constant is an affine invariant functional, i.e., $L_{AK} = L_K$ for any $A \in \text{GL}(n, \mathbb{R})$.

4.9 Proposition. Let $K \in \mathcal{K}_c^n$. Then $\det A_K = L_K^{2n}$ and (thus) $\text{vol}(\mathcal{E}_K) = L_K^{-n} \kappa_n$. Moreover, there exist $\hat{\mathbf{u}}, \tilde{\mathbf{u}} \in S^{n-1}$ such that

$$\int_K \langle \hat{\mathbf{u}}, \mathbf{x} \rangle^2 d\mathbf{x} \leq L_K^2 \leq \int_K \langle \tilde{\mathbf{u}}, \mathbf{x} \rangle^2 d\mathbf{x}.$$

Proof. According to Lemma 4.7 we can find a $T \in \text{GL}(n, \mathbb{R})$, $|\det T| = 1$, such that TK is in isotropic position, i.e., $\det(A_{TK}) = L_K^{2n}$. By Proposition 4.5 iii) it is $A_{TK} = |\det T|TA_K T^\top$, and hence $\det A_K = L_K^{2n}$. Due to the definition of \mathcal{E}_K we have $\text{vol}(\mathcal{E}_K) = \kappa_n \det(A_K)^{-1/2}$. Hence, together with Proposition 3.8 iii) we have

$$L_K^{-n} = \frac{\text{vol}(\mathcal{E}_K)}{\kappa_n} = \int_{S^{n-1}} (|\mathbf{u}|_{\mathcal{E}_K})^{-n} d\sigma(\mathbf{u}).$$

Thus we can find $\tilde{\mathbf{u}} \in S^{n-1}$ such that (cf. Remark 4.4)

$$L_K^2 \leq |\tilde{\mathbf{u}}|_{\mathcal{E}_K}^2 = \tilde{\mathbf{u}}^\top A_K \tilde{\mathbf{u}} = \int_K \langle \tilde{\mathbf{u}}, \mathbf{x} \rangle^2 d\mathbf{x}.$$

In the same way we can find a lower bound on L_K . \square

4.10 Theorem. Let $K \in \mathcal{K}^n$. Then $L_K \geq L_{B_n}$, and equality holds if and only if K is an ellipsoid.

Proof. After an affine transformation we may assume that K is in isotropic position and let \bar{B}_n be the ball of volume 1. Since K and \bar{B}_n have the same volume we conclude $\text{vol}(K \setminus \bar{B}_n) = \text{vol}(\bar{B}_n \setminus K)$. Hence

$$\int_{K \setminus \bar{B}_n} \|\mathbf{x}\|^2 d\mathbf{x} \geq \left(\frac{1}{\kappa_n}\right)^{\frac{2}{n}} \text{vol}(K \setminus \bar{B}_n) = \left(\frac{1}{\kappa_n}\right)^{\frac{2}{n}} \text{vol}(\bar{B}_n \setminus K) \geq \int_{\bar{B}_n \setminus K} \|\mathbf{x}\|^2 d\mathbf{x}$$

and so

$$\begin{aligned} n L_K^2 &= \int_K \|\mathbf{x}\|^2 d\mathbf{x} = \int_{K \cap \bar{B}_n} \|\mathbf{x}\|^2 d\mathbf{x} + \int_{K \setminus \bar{B}_n} \|\mathbf{x}\|^2 d\mathbf{x} \\ &\geq \int_{\bar{B}_n \cap K} \|\mathbf{x}\|^2 d\mathbf{x} + \int_{\bar{B}_n \setminus K} \|\mathbf{x}\|^2 d\mathbf{x} = \int_{\bar{B}_n} \|\mathbf{x}\|^2 d\mathbf{x} = n L_{\bar{B}_n}^2, \end{aligned}$$

and here we have equality if and only if $K = \bar{B}_n$. \square

4.11 Theorem. *Let $K \in \mathcal{K}_c^n$. K is in isotropic position if and only if*

$$\int_K \|\mathbf{x}\|^2 d\mathbf{x} \leq \int_{TK} \|\mathbf{x}\|^2 d\mathbf{x}$$

for all $T \in \text{GL}(n, \mathbb{R})$, $|\det T| = 1$.

Proof. First we note that via eigenvalues and the arithmetic/geometric mean inequality we have for any matrix $T \in \text{GL}(n, \mathbb{R})$, $|\det T| = 1$

$$\text{tr}(TT^\top) \geq n \det(TT^\top)^{1/n} = n.$$

Let K be in isotropic position, then $A_K = L_K^2 \mathbf{I}_n$ and in view (cf. Proposition 4.5) we get for any $T \in \text{GL}(n, \mathbb{R})$, $|\det T| = 1$

$$\begin{aligned} \int_K \|\mathbf{x}\|^2 d\mathbf{x} &= n L_K^2 \leq \text{tr}(TT^\top) L_K^2 = \text{tr}(T A_K T^\top) \\ &= \text{tr}(A_{TK}) = \int_{TK} \|\mathbf{x}\|^2 d\mathbf{x}. \end{aligned}$$

For the reverse direction let $B \in \mathbb{R}^{n \times n}$. Then

$$\det(\mathbf{I}_n + \epsilon B) = 1 + \epsilon \text{tr}(B) + \dots + \epsilon^n \det B.$$

For all sufficiently small $|\epsilon|$ we have $\det(\mathbf{I}_n + \epsilon B) \neq 0$, and due to our assumption we find

$$\begin{aligned} \int_K \|\mathbf{x}\|^2 d\mathbf{x} &\leq \int_{\left(\frac{1}{\det(\mathbf{I}_n + \epsilon B)}\right)^{1/n} (\mathbf{I}_n + \epsilon B) K} \|\mathbf{x}\|^2 d\mathbf{x} \\ &= |\det(\mathbf{I}_n + \epsilon B)|^{-\frac{2}{n}} \int_K \|\mathbf{x} + \epsilon B\mathbf{x}\|^2 d\mathbf{x}. \end{aligned}$$

Or equivalently,

$$\begin{aligned} |\det(\mathbf{I}_n + \epsilon B)|^{\frac{2}{n}} \int_K \|\mathbf{x}\|^2 d\mathbf{x} &\leq \int_K \|\mathbf{x} + \epsilon B\mathbf{x}\|^2 d\mathbf{x} \\ &= \int_K \|\mathbf{x}\|^2 d\mathbf{x} + 2\epsilon \int_K \langle \mathbf{x}, B\mathbf{x} \rangle d\mathbf{x} + \epsilon^2 \int_K \|B\mathbf{x}\|^2 d\mathbf{x}. \end{aligned} \tag{4.11.1}$$

Looking at the Taylor series of $|\det(\mathbf{I}_n + \epsilon B)|^{\frac{2}{n}}$ at $\epsilon = 0$ gives

$$|\det(\mathbf{I}_n + \epsilon B)|^{\frac{2}{n}} = 1 + \epsilon \frac{2}{n} \text{tr} B + O(\epsilon^2)$$

for $\epsilon \rightarrow 0$, and hence (4.11.1) yields

$$\frac{\operatorname{tr} B}{n} \int_K \|\mathbf{x}\|^2 \, d\mathbf{x} \leq \int_K \langle \mathbf{x}, B\mathbf{x} \rangle \, d\mathbf{x}.$$

Replacing B by $-B$ shows that we must equality in the inequality above and, especially, for $B = \mathbf{e}_i \mathbf{e}_j^\top$ we get

$$\frac{\delta_{i,j}}{n} \int_K \|\mathbf{x}\|^2 \, d\mathbf{x} = \int_K x_i x_j \, d\mathbf{x} = (A_K)_{i,j}.$$

Hence, the moment matrix A_K is a multiple of I_n which means that K is in isotropic position. \square

4.12 Corollary. *Let $K \in \mathcal{K}_c^n \cap \mathcal{K}_o^n$. Then $L_K \leq c\sqrt{n}$ for an universal constant c .*

Proof. Let K be in isotropic position. Let $T \in \operatorname{GL}(n, \mathbb{R})$, $|\det T| = 1$, such that the volume maximal ellipsoid in TK is a ball of radius α , say. Observe that the volume maximal ellipsoid is centered at the origin, since $K \in \mathcal{K}_o^n$. By Corollary 6.16³⁷ we have $\alpha B_n \subseteq TK \subseteq \sqrt{n}\alpha B_n$ and thus with Theorem 4.11 we get

$$n L_K^2 \leq \int_{TK} \|\mathbf{x}\|^2 \, d\mathbf{x} \leq \alpha^2 n.$$

Hence $L_K \leq \alpha$. By the inclusion $\alpha B_n \subseteq TK$ we have $\alpha^n \leq \operatorname{vol}(TK)/\kappa_n$ and so

$$\alpha \leq \kappa_n^{-1/n} \leq c\sqrt{n},$$

for a suitable constant c . Hence we have found the bound $L_K \leq c\sqrt{n}$. \square

4.13 Conjecture [Isotropic Constant Conjecture]. *There exists an absolute constant $C > 0$ such that $L_K \leq C$ for any $K \in \mathcal{K}_o^n$.*³⁸

4.14 Proposition. *Let $K \in \mathcal{K}_o^n$. Then (cf. Notation 4.8)*

$$L_K^2 \cdot L_{K^*}^2 \leq cn \frac{1}{\operatorname{vol}(K)\operatorname{vol}(K^*)} \int_K \int_{K^*} \langle \mathbf{y}, \mathbf{x} \rangle^2 \, d\mathbf{y} \, d\mathbf{x}.$$

In particular, $L_K \cdot L_{K^} \leq c\sqrt{n}$.*³⁹

Proof. Let

$$\Phi(K) = \frac{1}{\operatorname{vol}(K)\operatorname{vol}(K^*)} \int_K \int_{K^*} \langle \mathbf{y}, \mathbf{x} \rangle^2 \, d\mathbf{y} \, d\mathbf{x}.$$

³⁷Skript WS14

³⁸Bourgain, 1991 proved an upper bound of order $n^{1/4} \log(n)$ for o -symmetric bodies, which also holds true for arbitrary bodies (Paouris, 2000). The current general best bound is of size $n^{1/4}$ due to Klartag, 2006

³⁹Here c is always an absolute constant, which may vary from statement to statement.

Observe, that $\Phi(K) \leq 1$. For every $T \in \text{GL}(n, \mathbb{R})$ we have $(TK)^* = T^{-\top}K^*$ and thus $\Phi(K) = \Phi(TK)$. Hence for the proof we may assume that K^* is in isotropic position. Then

$$\begin{aligned} \left(\frac{1}{\text{vol}(K)}\right)^{\frac{n+2}{n}} \int_K \int_{K^*} \langle \mathbf{y}, \mathbf{x} \rangle^2 d\mathbf{y} d\mathbf{x} &= \left(\frac{1}{\text{vol}(K)}\right)^{\frac{n+2}{n}} L_{K^*}^2 \int_K \|\mathbf{x}\|^2 d\mathbf{x} \\ &= L_{K^*}^2 \int_{K/\text{vol}(K)^{1/n}} \|\mathbf{x}\|^2 d\mathbf{x} \\ &\geq n L_{K^*}^2 L_K^2, \end{aligned}$$

by Theorem 4.11. On account of $\text{vol}(K^*) = 1$ we may write

$$n L_{K^*}^2 L_K^2 \leq \left(\frac{1}{\text{vol}(K)\text{vol}(K^*)}\right)^{\frac{2}{n}} \Phi(K).$$

By ⁴⁰ we know $\text{vol}(K)\text{vol}(K^*) \geq c^n/n!$ and thus $n L_{K^*}^2 L_K^2 \leq c n^2 \Phi(K)$. □

4.15 Theorem. *Let $K \in \mathcal{K}_c^n$ be in isotropic position. Then $K \subset (n+1)L_K B_n$.*

Proof. For $\mathbf{w} \in K$ we consider the local radial function $\rho_{\mathbf{w}} : S^{n-1} \rightarrow \mathbb{R}_{\geq 0}$ with $\rho_{\mathbf{w}}(\mathbf{v}) = \max\{\rho \geq 0 : \mathbf{w} + \rho \mathbf{v} \in K\}$. Let $\mathbf{u} \in S^{n-1}$. Then

$$\begin{aligned} L_K^2 &= \int_K \langle \mathbf{u}, \mathbf{x} \rangle^2 d\mathbf{x} \\ &= n \kappa_n \int_{S^{n-1}} \int_0^{\rho_{\mathbf{w}}(\mathbf{v})} t^{n-1} \langle \mathbf{u}, \mathbf{w} + t \mathbf{v} \rangle^2 dt d\sigma(\mathbf{v}) \\ &= n \kappa_n \int_{S^{n-1}} \int_0^{\rho_{\mathbf{w}}(\mathbf{v})} \left(t^{n-1} \langle \mathbf{u}, \mathbf{w} \rangle^2 + 2 t^n \langle \mathbf{u}, \mathbf{v} \rangle \langle \mathbf{u}, \mathbf{w} \rangle \right. \\ &\quad \left. + t^{n+1} \langle \mathbf{u}, \mathbf{v} \rangle^2 \right) dt d\sigma(\mathbf{v}) \\ &= n \kappa_n \int_{S^{n-1}} \left(\frac{\rho_{\mathbf{w}}(\mathbf{v})^n}{n} \langle \mathbf{u}, \mathbf{w} \rangle^2 + 2 \frac{\rho_{\mathbf{w}}(\mathbf{v})^{n+1}}{n+1} \langle \mathbf{u}, \mathbf{v} \rangle \langle \mathbf{u}, \mathbf{w} \rangle \right. \\ &\quad \left. + \frac{\rho_{\mathbf{w}}(\mathbf{v})^{n+2}}{n+2} \langle \mathbf{u}, \mathbf{v} \rangle^2 \right) d\sigma(\mathbf{v}) \\ &= n \kappa_n \int_{S^{n-1}} \left(\frac{1}{n(n+1)^2} \rho_{\mathbf{w}}(\mathbf{v})^n \langle \mathbf{u}, \mathbf{w} \rangle^2 + \frac{\rho_{\mathbf{w}}(\mathbf{v})^n}{n} \times \right. \\ &\quad \left. \left[\sqrt{\frac{n}{n+2}} \rho_{\mathbf{w}}(\mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle + \frac{\sqrt{n(n+2)}}{n+1} \langle \mathbf{u}, \mathbf{w} \rangle \right]^2 \right) d\sigma(\mathbf{v}) \\ &\geq \frac{\langle \mathbf{u}, \mathbf{w} \rangle^2}{(n+1)^2} \kappa_n \int_{S^{n-1}} \rho_{\mathbf{w}}(\mathbf{v})^n d\sigma(\mathbf{v}) = \frac{\langle \mathbf{u}, \mathbf{w} \rangle^2}{(n+1)^2}, \end{aligned}$$

⁴⁰Bourgain& Milman proved in 1987 that there exists an absolute constant c such that $\text{vol}(K)\text{vol}(K^*) \geq c^n/n!$ for every $K \in \mathcal{K}_c^n$. See also [Wikipedia Mahler volume](#).

⁴¹Kannan, Lovasz, Simonovits proved for an isotropic body $(n+2)/n L_K B_n \subset K \subset \sqrt{n(n+2)} L_K B_n$, which are best possible.

where the last identity comes from $\text{vol}(K) = 1$ and Proposition 3.8 ii). Replacing \mathbf{u} by $\mathbf{w}/\|\mathbf{w}\|$ shows that $\|\mathbf{w}\|^2 \leq (n+1)^2 L_K^2$ for all $\mathbf{w} \in K$. \square

4.16 Lemma. *Let $K \in \mathcal{K}^n$ be centered. Let $\mathbf{a} \in S^{n-1}$, and for $t \in \mathbb{R}$ let $f(t) = \text{vol}_{n-1}(K \cap H(\mathbf{a}, t))$. Then*

$$\max\{f(t) : t \in \mathbb{R}\} \leq \left(\frac{n+1}{n}\right)^{n-1} f(0).$$

Observe that $\left(\frac{n+1}{n}\right)^{n-1} \rightarrow e$ as n to infinity.

Proof. Let $H(\mathbf{a}, t) \cap K \neq \emptyset$ for $t \in [-a, b]$, say, with $a, b > 0$. Let t^* such that $f(t^*) = \max\{f(t) : t \in [-a, b]\}$. With loss of generality we may assume $f(0) < f(t^*)$ and $t^* > 0$. For if $f(t^*) = f(0)$ we are done and if $t^* < 0$ we replace \mathbf{a} by $-\mathbf{a}$. Since $\mathbf{0}$ is the centroid we know $0 = \int_{-a}^b t f(t) dt$. Hence

$$\int_{-a}^0 (-t) f(t) dt = \int_0^b t f(t) dt \geq \int_0^{t^*} t f(t) dt. \quad (4.16.1)$$

Now let $h : [-a, b] \rightarrow \mathbb{R}_{\geq 0}$ be the concave function $h(t) = f(t)^{\frac{1}{n-1}}$, and let $g : [-a, b] \rightarrow \mathbb{R}_{\geq 0}$ be the linear function

$$g(t) = h(0) + \frac{t}{t^*} (h(t^*) - h(0)).$$

Then $g(0) = h(0)$ and $g(t^*) = h(t^*)$ and due to the concavity of h we conclude

$$g(t) \geq h(t), \quad t \in [-a, 0] \text{ and } g(t) \leq h(t), \quad t \in [0, t^*].$$

Let $g(-c) = 0$ for some $c \geq a$, and let $g(t) = \alpha(t+c)$ with $\alpha > 0$. Then with (4.16.1)

$$\begin{aligned} \int_{-c}^0 (-t) g(t)^{n-1} dt &\geq \int_{-a}^0 (-t) g(t)^{n-1} dt \geq \int_{-a}^0 (-t) h(t)^{n-1} dt \\ &\geq \int_0^{t^*} t h(t)^{n-1} dt \geq \int_0^{t^*} t g(t)^{n-1} dt. \end{aligned}$$

Thus

$$\begin{aligned} 0 &\geq \int_{-c}^{t^*} t g(t)^{n-1} dt = \alpha^{n-1} \int_{-c}^{t^*} t(t+c)^{n-1} dt \\ &= \alpha^{n-1} \int_0^{t^*+c} (t-c)t^{n-1} dt = \alpha^{n-1} \left(\frac{1}{n+1} t^{n+1} - \frac{c}{n} t^n \right) \Big|_0^{t^*+c} \\ &= \alpha^{n-1} (t^*+c)^n \left(\frac{t^*+c}{n+1} - \frac{c}{n} \right). \end{aligned}$$

Hence $(t^*+c)/c \leq (n+1)/n$, and so

$$\frac{f(t^*)}{f(0)} = \frac{h(t^*)^{n-1}}{h(0)^{n-1}} = \left(\frac{g(t^*)}{g(0)} \right)^{n-1} = \left(\frac{t^*+c}{c} \right)^{n-1} \leq \left(\frac{n+1}{n} \right)^{n-1}.$$

\square

4.17 Theorem. *There exists absolute constants c_1, c_2 such that for every $K \in \mathcal{K}_c^n$ and for every $\mathbf{u} \in S^{n-1}$*

$$c_1 \frac{1}{\text{vol}_{n-1}(K \cap H(\mathbf{u}, 0))} \leq \left(\int_K (\mathbf{u}^\top \mathbf{x})^2 d\mathbf{x} \right)^{1/2} \leq c_2 \frac{1}{\text{vol}_{n-1}(K \cap H(\mathbf{u}, 0))}.$$

Proof. Let $\mathbf{u} \in S^{n-1}$ and for $t \in \mathbb{R}$ let $f(t) = \text{vol}_{n-1}(K \cap H(\mathbf{u}, t))$. Here $H(\mathbf{u}, t) = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{u}, \mathbf{x} \rangle = t\}$. Moreover, let $t^* \in \mathbb{R}$ such that $f(t^*) = \max\{f(t) : t \in \mathbb{R}\}$ and let

$$A = \int_{-\infty}^0 f(t) dt \text{ and } B = \int_0^{\infty} f(t) dt.$$

We note that $1 = \text{vol}(K) = A + B$ and

$$\int_K \langle \mathbf{u}, \mathbf{x} \rangle^2 d\mathbf{x} = \int_{-\infty}^{\infty} t^2 f(t) dt = \int_{-\infty}^0 t^2 f(t) dt + \int_0^{\infty} t^2 f(t) dt. \quad (4.17.1)$$

First we prove the lower bound. To this end let $g(t) = f(t^*)\chi_{[0, B/f(t^*)]}$. Then we certainly have $f(t) \leq g(t)$ for all $t \in [0, B/f(t^*)]$ and so

$$\int_0^s f(t) dt \leq \int_0^s g(t) dt$$

for all $s \in [0, B/f(t^*)]$. Since we also have

$$\int_0^{\infty} g(t) dt = B = \int_0^{\infty} f(t) dt$$

we conclude

$$\int_s^{\infty} f(t) dt \geq \int_s^{\infty} g(t) dt$$

for all $s \geq 0$. Hence

$$\begin{aligned} \int_0^{\infty} t^2 f(t) dt &= \int_0^{\infty} \int_0^t 2s f(t) ds dt = \int_0^{\infty} \left(\int_s^{\infty} 2s f(t) dt \right) ds \\ &\geq \int_0^{\infty} 2s \left(\int_s^{\infty} g(t) dt \right) ds = \int_0^{\infty} t^2 g(t) dt \\ &= \int_0^{B/f(t^*)} t^2 f(t^*) dt = \frac{B^3}{3} \frac{1}{f(t^*)^2}. \end{aligned}$$

In the same way it can be shown

$$\int_{-\infty}^0 t^2 f(t) dt \geq \frac{A^3}{3} \frac{1}{f(t^*)^2}.$$

Thus, in view of (4.17.1) we have found

$$\int_K \langle \mathbf{u}, \mathbf{x} \rangle^2 d\mathbf{x} \geq \frac{A^3 + B^3}{3f(t^*)^2}.$$

Since $A + B = \text{vol}(K) = 1$ we conclude $A^3 + B^3 \geq 1/8 + 1/8 = 1/4$, and due to Lemma 4.16 we also know $f(t^*) \leq ef(0)$. Therefore,

$$\int_K \langle \mathbf{u}, \mathbf{x} \rangle^2 d\mathbf{x} \geq \frac{1}{12e} \frac{1}{f(0)^2},$$

which shows the lower bound. For the upper bound we distinguish two cases. Firstly we assume that there exists an $\bar{t} > 0$ with the property that

$$f(\bar{t}) = f(0)/2. \quad (4.17.2)$$

By the Brunn-Minkowski inequality the function $f(t)$ is log-concave and so for every $\lambda \in [0, 1]$ we find

$$f(\lambda \bar{t}) = f((1 - \lambda)0 + \lambda \bar{t}) \geq f(0)^{1-\lambda} f(\bar{t})^\lambda \geq f(\bar{t}).$$

Hence $f(t) \geq f(\bar{t})$ for all $t \in [0, \bar{t}]$ and so

$$1 \geq B = \int_0^{\bar{t}} f(t) dt \geq \bar{t} f(\bar{t}) = \bar{t} \frac{f(0)}{2}.$$

This shows

$$\bar{t} \leq 2 \frac{1}{f(0)}. \quad (4.17.3)$$

Now, for $t > \bar{t}$ we have

$$f(\bar{t}) = f((\bar{t}/t)t) = f((1 - \bar{t}/t)0 + (\bar{t}/t)t) \geq f(0)^{(1-\bar{t}/t)} f(t)^{\bar{t}/t}$$

Hence

$$f(t) \leq \left(f(\bar{t}) f(0)^{(\bar{t}/t-1)} \right)^{\frac{t}{\bar{t}}} = \left(\frac{1}{2} \right)^{\frac{t}{\bar{t}}} f(0). \quad (4.17.4)$$

Using these bounds as well as Lemma 4.16 leads to

$$\begin{aligned} \int_0^\infty t^2 f(t) dt &= \int_0^{\bar{t}} t^2 f(t) dt + \int_{\bar{t}}^\infty t^2 f(t) dt \\ &\leq f(t^*) \int_0^{\bar{t}} t^2 dt + f(0) \int_{\bar{t}}^\infty t^2 2^{-t/\bar{t}} dt \\ &\stackrel{\text{Lemma 4.16}}{\leq} f(0) \left(e \int_0^{\bar{t}} t^2 dt + \int_{\bar{t}}^\infty t^2 2^{-t/\bar{t}} dt \right) \\ &= f(0) \left(e \frac{\bar{t}^3}{3} + \bar{t}^3 \int_1^\infty u^2 2^{-u} du \right) \\ &= f(0) \bar{t}^3 \cdot c \stackrel{(4.17.3)}{\leq} 8c \frac{1}{f(0)^2}. \end{aligned} \quad (4.17.5)$$

Now assume the assumption (4.17.2) does not hold. Then by the continuity of f in the interior of its support we know $f(t) > f(0)/2$ for all $t > 0$ for which

$f(t) \neq 0$. Let $\hat{t} = \sup\{t > 0 : f(t) \neq 0\}$. Then $1 \geq B = \int_0^{\hat{t}} f(t) dt \geq \hat{t}f(0)/2$ and so

$$\hat{t} \leq 2 \frac{1}{f(0)}$$

and so

$$\int_0^{\infty} t^2 f(t) dt = \int_0^{\hat{t}} t^2 f(t) dt \stackrel{\text{Lemma 4.16}}{\leq} e \frac{\hat{t}^3}{3} f(0) \leq \frac{8e}{3} \frac{1}{f(0)^2}.$$

Together with (4.17.5) we have shown that there exists a constant \bar{c} , say, such that

$$\int_0^{\infty} t^2 f(t) dt \leq \bar{c} \frac{1}{f(0)^2}.$$

Via the same argumentation we can bound the integral on the negative x -axis, i.e., $\int_{-\infty}^0 t^2 f(t) dt$ and we are done (cf.(4.17.1)). \square

4.18 Corollary. *There exists absolute constants c_1, c_2 such that for every $K \in \mathcal{K}_c^n$ in isotropic position and for every $\mathbf{u} \in S^{n-1}$*

$$\frac{c_1}{L_K} \leq \text{vol}_{n-1}(K \cap H(\mathbf{u}, 0)) \leq \frac{c_2}{L_K}.$$

Proof. Immediate consequence of Theorem 4.17. \square

4.19 Conjecture [Slicing conjecture]. *There exists an absolute constant $c > 0$ with the property that for every $K \in \mathcal{K}_o^n$ with $\text{vol}(K) = 1$ there exists an $\mathbf{v} \in S^{n-1}$ such that*

$$\text{vol}_{n-1}(K \cap H(\mathbf{v}, 0)) \geq c.$$

4.20 Remark. *The Slicing conjecture 4.19 is equivalent to the Isotropic constant conjecture 4.13*

Proof. If the Slicing conjecture is true we get from the upper bound in Corollary 4.18 that the isotropic constant is bounded from above by an absolute constant. On the other hand, if the Isotropic constant conjecture holds true then we know by Proposition 4.9 that for every $K \in \mathcal{K}_o^n$ there exists a $\hat{\mathbf{u}} \in S^{n-1}$ that $\int_K \langle \hat{\mathbf{u}}, \mathbf{x} \rangle^2 d\mathbf{x} \leq L_K^2 \leq c^2$. Hence from the lower bound in Theorem 4.17 we get $\text{vol}_{n-1}(K \cap H(\hat{\mathbf{u}}, 0)) \geq c_1/c$. \square

4.21 Notation. ⁴² For $K \in \mathcal{K}^n$ with $\text{vol}(K) = 1$ let

$$S(K) = \int_K \dots \int_K \text{vol}(\text{conv}\{\mathbf{0}, \mathbf{x}_1, \dots, \mathbf{x}_n\})^2 d\mathbf{x}_n \dots d\mathbf{x}_1.$$

⁴²The Sylvester problem asks for the convex bodies $K \in \mathcal{K}^n$, $\text{vol}(K) = 1$, for which $\int_K \dots \int_K \text{vol}(\text{conv}\{\mathbf{0}, \mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{x}_{n+1}\}) d\mathbf{x}_{n+1} \dots d\mathbf{x}_1$ is minimized or maximized. It is known to be minimized for a ball and it is conjectured that it is maximized for a simplex.

4.22 Theorem. *Let $K \in \mathcal{K}^n$ with $\text{vol}(K) = 1$. Then*

$$n! S(K) = \det(A_K) = L_K^{2n}.$$

Proof. It is $\text{vol}(\text{conv}\{\mathbf{0}, \mathbf{x}_1, \dots, \mathbf{x}_n\}) = \frac{1}{n!} |\det(\mathbf{x}_1, \dots, \mathbf{x}_n)|$, and so

$$\begin{aligned} n!^2 S(K) &= \int_K \cdots \int_K |\det(\mathbf{x}_1, \dots, \mathbf{x}_n)|^2 d\mathbf{x}_n \cdots d\mathbf{x}_1 \\ &= \int_K \cdots \int_K \left(\sum_{\sigma \in S_n} \text{sgn } \sigma \prod_{i=1}^n (\mathbf{x}_i)_{\sigma(i)} \right) \left(\sum_{\tau \in S_n} \text{sgn } \tau \prod_{i=1}^n (\mathbf{x}_i)_{\tau(i)} \right) d\mathbf{x}_n \cdots d\mathbf{x}_1 \\ &= \int_K \cdots \int_K \left(\sum_{\sigma, \tau \in S_n} \text{sgn } \sigma \text{sgn } \tau \prod_{i=1}^n (\mathbf{x}_i)_{\sigma(i)} (\mathbf{x}_i)_{\tau(i)} \right) d\mathbf{x}_n \cdots d\mathbf{x}_1 \\ &= \int_K \cdots \int_K \left(\sum_{\sigma, \tau \in S_n} \text{sgn } \tau \prod_{i=1}^n (\mathbf{x}_i)_{\sigma(i)} (\mathbf{x}_i)_{\tau(\sigma(i))} \right) d\mathbf{x}_n \cdots d\mathbf{x}_1 \\ &= \sum_{\sigma, \tau \in S_n} \text{sgn } \tau \left(\prod_{i=1}^n \int_K (\mathbf{x}_i)_{\sigma(i)} (\mathbf{x}_i)_{\tau(\sigma(i))} d\mathbf{x}_i \right) \\ &= \sum_{\sigma, \tau \in S_n} \text{sgn } \tau \left(\prod_{i=1}^n \int_K y_{\sigma(i)} y_{\tau(\sigma(i))} d\mathbf{y} \right) \\ &= \sum_{\sigma, \tau \in S_n} \text{sgn } \tau \left(\prod_{i=1}^n \int_K y_i y_{\tau(i)} d\mathbf{y} \right) \\ &= n! \sum_{\tau \in S_n} \text{sgn } \tau \prod_{i=1}^n a_{i, \tau(i)} = n! \det A_K. \end{aligned}$$

By Proposition 4.9 we also know $\det A_K = L_K^{2n}$. \square

4.23 Corollary. *Let $K \in \mathcal{K}^n$ with centroid $\mathbf{0}$. Then $L_K \leq c\sqrt{n}$, where c is an absolute constant (cf. Corollary 4.12).*

Proof. By Theorem 4.22 we have $L_K^{2n} \leq n!$ and thus $L_K \leq \sqrt[n]{n!} \leq c\sqrt{n}$. \square

4.24 Definition. $K \in \mathcal{K}^n$ is called unconditional if it is symmetric to each coordinate hyperplane, i.e., $(\pm x_1, \dots, \pm x_n)^\top \in K$ for all $\mathbf{x} \in K$.

4.25 Proposition. *Let $K \in \mathcal{K}^n$ be unconditional.*

- i) $\int_K x_i x_j d\mathbf{x} = 0$ for all $1 \leq i \neq j \leq n$.
- ii) For each coordinate hyperplane we have $K \cap H = K|H$.
- iii) If $\text{vol}(K) = 1$ then there exists a diagonal matrix $T \in \text{VP}(n, \mathbb{R})$ such TK is in isotropic position.

4.26 Theorem. *There exists an absolute constant c such that $L_K \leq c$ for all unconditional convex bodies $K \in \mathcal{K}^n$.*

Proof. According to Proposition 4.25 there exists a diagonal matrix T such that $TK \in \mathcal{K}_c^n$ and TK is in isotropic position. Since T is a diagonal matrix, TK is still an unconditional body.

For $1 \leq i \leq n$ let H_i be the coordinate hyperplanes. By Corollary 4.18 there exists an absolute constant c_2 such that for $1 \leq i \leq n$

$$L_k \leq c_2 \frac{1}{\text{vol}_{n-1}(TK \cap H_i)} = c_2 \frac{1}{\text{vol}_{n-1}(TK|H_i)}, \quad (4.26.1)$$

where for the last identity we have used Proposition 4.25 ii). By Corollary 3.16 we know that

$$1 = \text{vol}(TK)^{n-1} \leq \prod_{i=1}^n \text{vol}_{n-1}(TK|H_i),$$

and thus there exists and $i^* \in \{1, \dots, n\}$ with $\text{vol}_{n-1}(TK|H_{i^*}) \geq 1$. Hence, for that i^* , (4.26.1) gives the desired bound. \square

5 Almost spherical

5.1 Definition [Banach-Mazur distance]. Let $K, L \in \mathcal{K}^n$ be n -dimensional convex bodies. Then

$$d_{\text{BM}}(K, L) = \inf \left\{ \lambda > 0 : \exists \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, T \in \text{GL}(n, \mathbb{R}) \text{ with} \right. \\ \left. \mathbf{y} + K \subseteq T(\mathbf{x} + L) \subseteq \lambda(\mathbf{y} + K) \right\}$$

is called the Banach-Mazur distance of K and L .

5.2 Remark.

- i) If $K, L \in \mathcal{K}_o^n$ then we can assume $\mathbf{x} = \mathbf{y} = \mathbf{0}$, and $d_{\text{BM}}(K, L)$ may be interpreted as a distance between norms defined by K and L .
- ii) Let $K, L, M \in \mathcal{K}^n$. Then $d_{\text{BM}}(K, L) = d_{\text{BM}}(L, K)$, $d_{\text{BM}}(K, L) = d_{\text{BM}}(\mathbf{x} + TK, \mathbf{y} + \tilde{T}L)$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $T, \tilde{T} \in \text{GL}(n, \mathbb{R})$.
- iii) d_{BM} verifies the multiplicative triangular inequality, i.e.,

$$d_{\text{BM}}(K, M) \leq d_{\text{BM}}(K, L) d_{\text{BM}}(L, M).$$

5.3 Proposition.

- i) $d_{\text{BM}}(K, B_n) \leq \sqrt{n}$ for all $K \in \mathcal{K}_o^n$, and $d_{\text{BM}}(C_n, B_n) = d_{\text{BM}}(C_n^*, B_n) = \sqrt{n}$.
- ii) $d_{\text{BM}}(K, B_n) \leq n$ for all $K \in \mathcal{K}^n$, and $d_{\text{BM}}(S_n, B_n) = n$ for an n -simplex S_n .
- iii) $d_{\text{BM}}(K, L) \leq n$ for all $K, L \in \mathcal{K}_o^n$, and $d_{\text{BM}}(K, L) \leq n^2$ for all $K, L \in \mathcal{K}^n$.

Proof. This is Proposition 6.21^{43, 44}. □

5.4 Theorem*. Let $K \in \mathcal{K}^n$ with $d_{\text{BM}}(K, B_n) = n$. Then K is a simplex.

5.5 Definition [Spherical cap]. Let $\mathbf{v} \in S^{n-1}$. For $\varepsilon \in [-1, 1]$ the set

$$C(\varepsilon, \mathbf{v}) = \{ \mathbf{u} \in S^{n-1} : \langle \mathbf{v}, \mathbf{u} \rangle \geq \varepsilon \} = \{ \mathbf{u} \in S^{n-1} : \|\mathbf{u} - \mathbf{v}\| \leq \sqrt{2 - 2\varepsilon} \}$$

is called an ε -cap or a spherical cap of radius $r = \sqrt{2 - 2\varepsilon}$, $r \in [0, \sqrt{2}]$.

5.6 Notation. In the following we will denote for a measurable set $A \subset S^{n-1}$ by

$$\mu(A) = \int_{S^{n-1}} \chi_A(\mathbf{u}) \, d\mu(\mathbf{u})$$

its normalized surface measure (cf. Proposition 3.8 i)).

⁴³Skript WS14

⁴⁴It was shown by Gluskin (1981) that there exists $K, L \in \mathcal{K}_o^n$ such that $d_{\text{BM}}(K, L) \geq cn$, where c is an absolute constant. In the general case, Rudelson (2001) proved an upper bound of order $n^{4/3} \log^9(n)$. See also [Banach-Mazur compactum](#).

5.7 Lemma. *Let $\mathbf{v} \in S^{n-1}$ and $\varepsilon \in (0, 1)$. Then*

$$\frac{1}{2} \left(\frac{\sqrt{2-2\varepsilon}}{2} \right)^{n-1} \leq \mu(C(\varepsilon, \mathbf{v})) \leq e^{-n\varepsilon^2/2}.$$

Proof. We set $\overline{C}(\varepsilon, \mathbf{v}) = \text{conv}\{\mathbf{0}, C(\varepsilon, \mathbf{v})\}$, and since in the following \mathbf{v} will be a fixed unit vector we will just write $C(\varepsilon)$ and $\overline{C}(\varepsilon)$, respectively. We start with the upper bound. Since (cf. Proposition 3.8 ii))

$$\text{vol}(\overline{C}(\varepsilon)) = \kappa_n \mu(C(\varepsilon)),$$

it suffices to bound $\text{vol}(\overline{C}(\varepsilon))$. The diameter of $\overline{C}(\varepsilon)$, denoted by $D(\overline{C}(\varepsilon))$ is given either by the diameter of the $(n-1)$ -dimensional ball determined by the cap $C(\varepsilon)$, i.e., by $2\sqrt{1-\varepsilon^2}$ or by $\|\mathbf{v}\| = 1$. Hence

$$D(\overline{C}(\varepsilon)) = \begin{cases} 2\sqrt{1-\varepsilon^2}, & \varepsilon \leq \sqrt{3}/2, \\ 1, & \text{otherwise.} \end{cases}$$

In the first case we get by the isodiametric inequality (see Proposition 1.7)

$$\mu(C(\varepsilon)) = \frac{\text{vol}(\overline{C}(\varepsilon))}{\kappa_n} \leq \frac{1}{2^n} D(\overline{C}(\varepsilon))^n = (1-\varepsilon^2)^{n/2} \leq e^{-n\varepsilon^2/2},$$

where for the last inequality we used $e^{-t} \geq 1-t$. In the same way we also obtain in the second case

$$\mu(C(\varepsilon)) \leq \frac{1}{2^n} D(\overline{C}(\varepsilon))^n = \frac{1}{2^n} \leq e^{-n\varepsilon^2/2},$$

since $e^{\varepsilon^2/2} \leq 2$ when $\varepsilon \leq 1$.

For the proof of the lower bound let $H = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{v} \rangle = \varepsilon\}$ and $B' = B_n \cap H$. Let $K = \text{conv}\{B', \mathbf{0}, \mathbf{v}\}$. Then $K \subset \overline{C}(\varepsilon)$ and so

$$\mu(C(\varepsilon, \mathbf{v})) = \frac{\text{vol}(\overline{C}(\varepsilon, \mathbf{v}))}{\kappa_n} \geq \frac{\text{vol}(K)}{\kappa_n} = \frac{1}{n} \frac{\kappa_{n-1}}{\kappa_n} (\sqrt{1-\varepsilon^2})^{n-1},$$

and it remains to verify

$$\frac{1}{n} \frac{\kappa_{n-1}}{\kappa_n} (\sqrt{1-\varepsilon^2})^{n-1} \geq \frac{(\sqrt{2})^{n-1}}{2^n} (\sqrt{1-\varepsilon})^{n-1}.$$

For $\varepsilon < 1$ the inequality is equivalent to

$$\frac{n}{2^{(n+1)/2}} \frac{\kappa_n}{\kappa_{n-1}} \leq \sqrt{1+\varepsilon}^{n-1},$$

and we may assume that $\varepsilon = 0$. Now

$$\frac{n}{2^{(n+1)/2}} \frac{\kappa_n}{\kappa_{n-1}} = \frac{n}{2^{(n+1)/2}} \sqrt{\pi} \frac{\Gamma(n/2 + 1/2)}{\Gamma(n/2 + 1)} \leq \frac{n}{2^{(n+1)/2}} \sqrt{\pi} \leq 1,$$

for $n \geq 6$, and the remaining dimensions can easily be checked by a computer. \square

5.8 Theorem. Let $P \in \mathcal{P}^n$ be an o -symmetric n -polytope with $d_{\text{BM}}(P, B_n) \leq \delta$. Then $f_{n-1}(P) \geq e^{n/(2\delta^2)}$, i.e., P has at least $e^{n/(2\delta^2)}$ facets.

On the other hand, there is a polytope $P \in \mathcal{P}^n$ with 4^n facets such that $B_n \subset P \subset 2B_n$, and thus $d_{\text{BM}}(P, B_n) \leq 2$.

Proof. Let $P = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{v}_i, \mathbf{x} \rangle \leq 1, 1 \leq i \leq k\}$ be a symmetric polytope with k facets and $d_{\text{BM}}(P, B_n) = \delta$. We may assume that

$$B_n \subset P \subset \delta B_n.$$

Since $B_n \subset P$ we have $\mathbf{v}_i / \|\mathbf{v}_i\| \in P$ and thus $1 \geq \langle \mathbf{v}_i, \mathbf{v}_i / \|\mathbf{v}_i\| \rangle = \|\mathbf{v}_i\|$, $1 \leq i \leq k$. On the other hand since $P \subset \delta B_n$, we know that for $\mathbf{x} \in \delta S^{n-1}$ there exists an index $j \in \{1, \dots, k\}$ with $\langle \mathbf{v}_j, \mathbf{x} \rangle \geq 1$. Therefore,

$$\left\langle \frac{\mathbf{v}_j}{\|\mathbf{v}_j\|}, \frac{\mathbf{x}}{\|\mathbf{x}\|} \right\rangle \geq \frac{1}{\|\mathbf{v}_j\|} \frac{1}{\delta} \geq \frac{1}{\delta},$$

i.e., $\mathbf{x} / \|\mathbf{x}\| \in C(1/\delta, \mathbf{v}_j / \|\mathbf{v}_j\|)$. Hence

$$S^{n-1} = \bigcup_{i=1}^k C(1/\delta, \mathbf{v}_i / \|\mathbf{v}_i\|).$$

By Lemma 5.7 we get $\mu(C(1/\delta, \mathbf{v}_i / \|\mathbf{v}_i\|)) \leq e^{-n/(2\delta^2)}$, and so $k \geq e^{n/(2\delta^2)}$.

In order to prove the second assertion, we “construct” $k = 4^n$ points $\mathbf{v}_1, \dots, \mathbf{v}_k \in S^{n-1}$ such that

$$S^{n-1} = \bigcup_{i=1}^k C(1/2, \mathbf{v}_i). \quad (5.8.1)$$

Then $P = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{v}_i, \mathbf{x} \rangle \leq 1, 1 \leq i \leq k\}$ is the required polytope: we certainly have $B_n \subset P$ and due to (5.26.2) we know for $\mathbf{x} \in \mathbb{R}^n$ there exist a \mathbf{v}_j such that

$$\langle \mathbf{v}_j, \mathbf{x} / \|\mathbf{x}\| \rangle \geq 1/2, \quad \text{i.e.,} \quad \|\mathbf{x}\| \leq 2 \langle \mathbf{v}_j, \mathbf{x} \rangle.$$

Hence, if $\mathbf{x} \in P$ then $\|\mathbf{x}\| \leq 2$ and so $P \subset 2B_n$.

A set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subset S^{n-1}$ with property (5.26.2) is usually called a 1-net, since $1/2$ -caps have radius 1. Now let $M \subset S^{n-1}$ be a set of maximal cardinality with the property $\|\mathbf{v} - \mathbf{w}\| \geq 1$ for any $\mathbf{v} \neq \mathbf{w} \in M$. Then M is a 1-net, and since the spherical caps of radius $1/2$ centered at the points of M do not overlap, we get by Lemma 5.7 that

$$1 \geq \#M \cdot \frac{1}{2} \left(\frac{1}{4}\right)^{n-1}.$$

□

5.9 Corollary. For the 4^n -dimensional cube C_{4^n} there exists an n -dimensional linear subspace L such that

$$d_{\text{BM}}(C_{4^n} \cap L, B_{4^n} \cap L) \leq 2.$$

Each linear subspace L fulfilling this bound has dimension at most cn , where c is an absolute constant.⁴⁵

Proof. Let L be a k -dimensional linear subspace fulfilling the bound. Then $P = C_{4^n} \cap L$ is an o -symmetric k -dimensional polytope P with at most $2(4^n)$ facets. Since $d_{\text{BM}}(P, B_{4^n} \cap L) \leq 2$, Theorem 5.8 leads to $e^{k/8} \leq 2(4^n)$, and so there exists a constant c such that $k \leq cn$.

On the other hand, let $P = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{v}_i, \mathbf{x} \rangle \leq 1, 1 \leq i \leq 4^n\} \in \mathcal{P}^n$ be the polytope as in Theorem 5.8 with 4^n facets such that $B_n \subset P \subset 2B_n$. Let $\bar{P} = \{\mathbf{x} \in \mathbb{R}^n : |\langle \mathbf{v}_i, \mathbf{x} \rangle| \leq 1, 1 \leq i \leq 4^n\}$. Then \bar{P} is o -symmetric and it still holds $B_n \subset \bar{P} \subset 2B_n$. Let $L \subset \mathbb{R}^{4^n}$ be the n -dimensional linear subspace given by $L = \{V\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$ where $V \in \mathbb{R}^{4^n \times n}$ with rows $\mathbf{v}_1^\top, \dots, \mathbf{v}_{4^n}^\top$. Then

$$\begin{aligned} L \cap C_{4^n} &= \{V\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\} \cap \{\mathbf{x} \in \mathbb{R}^{4^n} : |x_i| \leq 1, 1 \leq i \leq 4^n\} \\ &= \{V\mathbf{x} : \mathbf{x} \in \mathbb{R}^n, |\langle \mathbf{v}_i, \mathbf{x} \rangle| \leq 1, 1 \leq i \leq 4^n\} \\ &= V\bar{P}. \end{aligned}$$

and by the choice of P we know $d_{\text{BM}}(C_{4^n} \cap L, B_{4^n} \cap L) \leq 2$. \square

5.10 Proposition. For every $(2n - 1)$ -dimensional ellipsoid $E \in \mathcal{K}_o^n$ there exists an n -dimensional linear subspace L such that $E \cap L$ is an Euclidean ball.

Proof. After a suitable rotation we may assume

$$E = \left\{ \mathbf{x} \in \mathbb{R}^{2n-1} : \sum_{i=1}^{2n-1} \frac{x_i^2}{\alpha_i^2} \leq 1 \right\},$$

with $0 < \alpha_1 \leq \dots \leq \alpha_{2n-1}$, and let

$$L = \left\{ \mathbf{x} \in \mathbb{R}^{2n-1} : x_i \sqrt{\frac{1}{\alpha_i^2} - \frac{1}{\alpha_n^2}} = x_{2n-i} \sqrt{\frac{1}{\alpha_n^2} - \frac{1}{\alpha_{2n-i}^2}}, i = 1, \dots, n-1 \right\}.$$

Then $\dim L \geq 2n - 1 - (n - 1) = n$. For $\mathbf{x} \in L$ we have $x_i^2/\alpha_i^2 + x_{2n-i}^2/\alpha_{2n-i}^2 = (x_i^2 + x_{2n-i}^2)/\alpha_n^2$, $1 \leq i \leq n - 1$, and hence, for $\mathbf{x} \in E \cap L$ we find

$$1 \geq \sum_{i=1}^{2n-1} \frac{x_i^2}{\alpha_i^2} = \sum_{i=1}^{n-1} \left(\frac{x_i^2}{\alpha_i^2} + \frac{x_{2n-i}^2}{\alpha_{2n-i}^2} \right) + \frac{x_n^2}{\alpha_n^2} = \frac{1}{\alpha_n^2} \sum_{i=1}^{2n-1} x_i^2.$$

Therefore, $\|\mathbf{x}\| \leq \alpha_n$ and so $E \cap L \subseteq \alpha_n(B_{2n-1} \cap L)$. The same argument shows that $\alpha_n(B_{2n-1} \cap L) \subseteq E \cap L$, and thus $E \cap L$ is an n -dimensional ball of radius α_n . \square

⁴⁵This property is also expressed by saying, that the 4^n -dimensional cube contains an almost 2-spherical section of dimension n .

5.11 Remark.

- i) Let $O(n)$ be the $n \times n$ -real orthogonal matrices. As a (locally) compact group it has a unique probability (Haar) measure $\theta(\cdot)$. Due to its uniqueness it is related to $\mu(\cdot)$ by

$$\theta(A) = \mu(U \mathbf{v} : U \in A),$$

for $A \subseteq O(n)$ and a fixed, but arbitrary $\mathbf{v} \in S^{n-1}$.

- ii) For $1 \leq k \leq n$ let $\text{Gr}(k, n)$ be the Grassmannian manifold consisting of all k -dimensional linear spaces of \mathbb{R}^n . Via the Hausdorff distance of unit balls contained in the subspaces $\text{Gr}(k, n)$ it becomes a metric space. With respect to the action of $O(n)$ there exists a unique probability (Haar) measure on $\nu_{k,n}$ of $\text{Gr}(n, k)$:

$$\nu_{k,n}(A) = \theta(\{U \in O(n) : U F \in A\}),$$

for a fixed $F \in \text{Gr}(k, n)$. In particular, for $f : S^{n-1} \rightarrow \mathbb{R}$ we have

$$\int_{S^{n-1}} f(\mathbf{u}) d\mu(\mathbf{u}) = \int_{\text{Gr}(k,n)} \left(\int_{S^{n-1} \cap F} f(\mathbf{v}) d\mu_F(\mathbf{v}) \right) d\nu(k, n)(F).$$

Here μ_F is the unique probability (Haar) measure on $S^{n-1} \cap \text{lin } F$.

5.12 Theorem. Let $K \in \mathcal{K}_o^n$ with $B_n \subseteq K$ and $(\text{vol}(K)/\text{vol}(B_n))^{1/n} = r$. Then there exists an orthogonal transformation $U \in O(n)$ such that for all $\mathbf{u} \in S^{n-1}$

$$\frac{|\mathbf{u}|_{UK} + |\mathbf{u}|_K}{2} \geq \frac{1}{8r^2}.$$

In particular, $K \cap UK \subset (8r^2)B_n$.

Proof. Recall that $K \subseteq L$ if and only if $|\mathbf{x}|_L \leq |\mathbf{x}|_K$ for all $\mathbf{x} \in S^{n-1}$, and $|\mathbf{x}|_{\mu L} = |\mathbf{x}/\mu|_L = (1/\mu)|\mathbf{x}|_L$ for $\mu > 0$. Next we observe

$$\begin{aligned} |\mathbf{x}|_{UK \cap K} &= \min\{\lambda \geq 0 : \mathbf{x} \in \lambda(UK \cap K)\} \\ &= \max\{|\mathbf{x}|_{UK}, |\mathbf{x}|_K\} \\ &\geq \frac{|\mathbf{x}|_{UK} + |\mathbf{x}|_K}{2} = \frac{|U^{-1}\mathbf{x}|_K + |\mathbf{x}|_K}{2}. \end{aligned}$$

Hence $(|\mathbf{u}|_{UK} + |\mathbf{u}|_K)/2 \geq (8r^2)^{-1} \|\mathbf{u}\|$ for all $\mathbf{u} \in S^{n-1}$ implies the inclusion $UK \cap K \subset 8r^2 B_n$.

For a fixed but arbitrary $U \in O(n)$ let $N_U(\mathbf{x}) = (|U\mathbf{x}|_K + |\mathbf{x}|_K)/2$ for $\mathbf{x} \in \mathbb{R}^n$. It is easy to see that $N_U(\cdot)$ is a norm in \mathbb{R}^n and since $B_n \subseteq K$ we have

$$N_U(\mathbf{x}) = \frac{|U\mathbf{x}|_K + |\mathbf{x}|_K}{2} \leq \frac{\|U\mathbf{x}\| + \|\mathbf{x}\|}{2} = \|\mathbf{x}\|.$$

Next we are going to prove that there exists an $\bar{U} \in O(n)$ with

$$\int_{S^{n-1}} \frac{1}{N_{\bar{U}}(\mathbf{u})^{2n}} d\mu(\mathbf{u}) \leq r^{2n}. \quad (5.12.1)$$

By the geometric-arithmetic mean inequality we have $N_{\bar{U}}(\mathbf{u})^2 \geq |U\mathbf{u}|_K |\mathbf{u}|_K$ and so it suffices to prove

$$\int_{S^{n-1}} \frac{1}{|U\mathbf{u}|_K^n |\mathbf{u}|_K^n} d\mu(\mathbf{u}) \leq r^{2n}.$$

In view of Remark 5.11 i) we may write

$$\begin{aligned} \int_{O(n)} \left(\int_{S^{n-1}} \frac{1}{|U\mathbf{u}|_K^n |\mathbf{u}|_K^n} d\mu(\mathbf{u}) \right) d\theta(U) &= \int_{S^{n-1}} \left(\int_{O(n)} \frac{1}{|U\mathbf{u}|_K^n} d\theta(U) \right) \frac{1}{|\mathbf{u}|_K^n} d\mu(\mathbf{u}) \\ &= \int_{S^{n-1}} \left(\int_{S^{n-1}} \frac{1}{|\mathbf{v}|_K^n} d\mu(\mathbf{v}) \right) \frac{1}{|\mathbf{u}|_K^n} d\mu(\mathbf{u}) \\ &= \left(\int_{S^{n-1}} \frac{1}{|\mathbf{u}|_K^n} d\mu(\mathbf{u}) \right)^2 \leq r^{2n}, \end{aligned}$$

where for the last inequality we have used $\text{vol}(K) = \kappa_n \int_{S^{n-1}} |\mathbf{u}|_K^{-n} d\mu(\mathbf{u})$ (cf. Proposition 3.8 iii)) and the assumption. Therefore, there exists a matrix \bar{U} in $O(n)$ with

$$\int_{S^{n-1}} \frac{1}{|\bar{U}\mathbf{u}|_K^n |\mathbf{u}|_K^n} d\mu(\mathbf{u}) \leq r^{2n},$$

which shows (5.12.1).

Next we use (5.12.1) in order to lower bound $\nu = N_{\bar{U}}(\mathbf{v})$ for an arbitrary point $\mathbf{v} \in S^{n-1}$. Observe, that $\nu = N_{\bar{U}}(\mathbf{v}) \leq \|\mathbf{v}\| = 1$, and for $\mathbf{w} \in S^{n-1}$ with $\|\mathbf{w} - \mathbf{v}\| \leq \nu$ we have $N_{\bar{U}}(\mathbf{w}) \leq N_{\bar{U}}(\mathbf{v}) + N_{\bar{U}}(\mathbf{w} - \mathbf{v}) \leq \nu + \|\mathbf{w} - \mathbf{v}\| \leq 2\nu$. Hence

$$N_{\bar{U}}(\mathbf{w}) \leq 2\nu$$

on a spherical cap A at \mathbf{v} of radius ν . Together with (5.12.1) and Lemma 5.7 we get

$$\begin{aligned} r^{2n} &\geq \int_{S^{n-1}} \frac{1}{N_{\bar{U}}(\mathbf{u})^{2n}} d\mu(\mathbf{u}) \geq \int_A \frac{1}{N_{\bar{U}}(\mathbf{u})^{2n}} d\mu(\mathbf{u}) \\ &\geq \mu(A)(2\nu)^{-2n} \geq (\nu/2)^n (2\nu)^{-2n} = \nu^{-n} 8^{-n}. \end{aligned}$$

Hence $\nu = N_{\bar{U}}(\mathbf{v}) \geq 1/(r^2 8)$. \square

5.13 Corollary. *For the $2n$ -dimensional cross-polytope C_{2n}^* there exists an n -dimensional linear subspace L such that*

$$\frac{1}{\sqrt{2n}}(B_{2n} \cap L) \subseteq (C_{2n}^* \cap L) \subseteq 32 \frac{1}{\sqrt{2n}}(B_{2n} \cap L).$$

Proof. For abbreviation we write $|\cdot|_1$ instead of $|\cdot|_{C_n^*}$, and we recall that $|\mathbf{x}|_1 = \sum_{i=1}^n |x_i|$. It is $B_n \subset \sqrt{n} C_n^*$ and $\text{vol}(\sqrt{n} C_n^*)/\text{vol}(B_n) \leq 2^n$. By Theorem 5.12 there exists an orthogonal matrix $U \in O(n)$ such that for all $\mathbf{x} \in \mathbb{R}^n$

$$|U\mathbf{x}|_1 + |\mathbf{x}|_1 \geq \frac{\sqrt{n}}{16} \|\mathbf{x}\|.$$

Let

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^{2n} \quad \text{given by} \quad T\mathbf{x} = \begin{pmatrix} U \\ I_n \end{pmatrix} \mathbf{x}.$$

First we observe that $\|T\mathbf{x}\| = \sqrt{\|U\mathbf{x}\|^2 + \|\mathbf{x}\|^2} = \sqrt{2} \|\mathbf{x}\|$ and so we may write

$$|T\mathbf{x}|_1 = |U\mathbf{x}|_1 + |\mathbf{x}|_1 \geq \frac{\sqrt{n}}{16} \|\mathbf{x}\| = \frac{\sqrt{n}}{16} \frac{1}{\sqrt{2}} \|T\mathbf{x}\| = \frac{\sqrt{2n}}{32} \|T\mathbf{x}\|.$$

This shows

$$T\mathbb{R}^n \cap C_{2n}^* \subset \frac{32}{\sqrt{2n}} (T\mathbb{R}^n \cap B_{2n}).$$

Finally, since $|T\mathbf{x}|_1 \leq \sqrt{2n} \|T\mathbf{x}\|$ we also get

$$\frac{1}{\sqrt{2n}} (T\mathbb{R}^n \cap B_{2n}) \subset (T\mathbb{R}^n \cap C_{2n}^*),$$

and with $L = T\mathbb{R}^n$ the corollary is proved. \square

5.14 Theorem. Let $K \in \mathcal{K}_o^n$ with $B_n \subseteq K$ and $(\text{vol}(K)/\text{vol}(B_n))^{1/n} = r$. For every $1 \leq k \leq n$, a random subspace $F \in \text{Gr}(n, k)$ satisfies with probability greater than $1 - e^{-n}$

$$B_n \cap \text{lin } F \subseteq K \cap F \subseteq (6e r)^{\frac{n}{n-k}} (B_n \cap \text{lin } F),$$

where c is an absolute constant.

Proof. Since $B_n \subseteq K$ we have $|\mathbf{x}|_K \leq \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$. By Remark 5.11 ii) we observe that

$$r^n = \int_{S^{n-1}} |\mathbf{u}|_K^{-n} d\mu(\mathbf{u}) = \int_{\text{Gr}(k, n)} \left(\int_{S^{n-1} \cap F} |\mathbf{u}|_K^{-n} d\mu_F(\mathbf{v}) \right) d\nu(k, n)(F). \quad (5.14.1)$$

Let

$$A = \left\{ F \in \text{Gr}(k, n) : \int_{S_F} |\mathbf{u}|_K^{-n} d\mu_F(\mathbf{v}) \leq e^n r^n \right\},$$

where $S_F = S^{n-1} \cap \text{lin } F$. By (5.14.1), Markov's inequality leads to

$$\nu_{k, n}(A) = 1 - \mathbb{P} \left(\int_{S_F} |\mathbf{u}|_K^{-n} d\mu_F(\mathbf{v}) \geq e^n r^n \right) \geq 1 - \frac{r^n}{e^n r^n} = 1 - e^{-n},$$

and it suffices to show that $F \in A$ satisfies

$$B_n \cap \text{lin } F \subseteq K \cap F \subseteq (c r)^{\frac{n}{n-k}} (B_n \cap \text{lin } F).$$

So let $F \in A$. By the definition of the set A , Markov's inequality strikes again and gives for $\alpha \in (0, 1)$

$$\mu_F(\{\mathbf{v} \in S_F : |\mathbf{v}|_K \geq \alpha\}) = \mu_F(\{\mathbf{v} \in S_F : |\mathbf{v}|_K^{-n} \leq \alpha^{-n}\}) \geq 1 - (e r \alpha)^n. \quad (5.14.2)$$

Now let $\mathbf{x} \in S_F$. According to Lemma 5.7 the measure of a spherical cap of S_F of radius $\alpha/2$ is at least $(\alpha/6)^k$. Let us choose an α such that

$$(er\alpha)^n < (\alpha/6)^k, \quad (5.14.3)$$

then in view of (5.14.2) we find

$$B_n(\mathbf{x}, \alpha/2) \cap \{\mathbf{v} \in S_F : |\mathbf{v}|_K \geq \alpha\} \neq \emptyset.$$

Hence there exists an $\mathbf{y} \in S_F$ such that $\|\mathbf{x} - \mathbf{y}\| \leq \alpha/2$ and $|\mathbf{y}|_K \geq \alpha$. Thus

$$|\mathbf{x}|_K \geq |\mathbf{y}|_K - \|\mathbf{x} - \mathbf{y}\| \geq \alpha - \alpha/2.$$

Since \mathbf{x} was arbitrary we conclude

$$B_n \cap \text{lin } F \subseteq K \subseteq \frac{\alpha}{2}(B_n \cap \text{lin } F)$$

and computing a proper α via (5.14.3) gives the desired result. \square

5.15 Lemma. *Let $S \subset B_n$ be an n -dimensional simplex, and let $\mathbf{x} \in S$. For $k \in \{1, \dots, n\}$ there exists a $(k-1)$ -dimensional face F of S such that*

$$\mathbf{x} \in F + \left(\sum_{i=k}^n i^{-2} \right)^{1/2} (B_n \cap (\text{aff } F)^\perp).$$

Proof. Let $\rho(n, k) = (\sum_{i=k}^n i^{-2})^{1/2}$. For the given $\mathbf{x} \in S$ let $\rho \geq 0$ be the maximum value such that $\mathbf{x} + \rho B_n \subset S$. Then $\rho \leq r(S) \leq (1/n)R(S) \leq 1/n$ (cf. exercise). $\mathbf{x} + \rho B_n$ touches a facet \bar{F} , say, of S in $\bar{\mathbf{x}}$. Then $\mathbf{x} - \bar{\mathbf{x}} \in (\text{aff } \bar{F})^\perp$ and $\|\mathbf{x} - \bar{\mathbf{x}}\| = \rho \leq 1/n$.⁴⁶

We prove the lemma by induction on $n - k$. For $k = n$ we are done since $\rho(n, n) = 1/n \geq \rho$ and by discussion above we have $\mathbf{x} \in \bar{F} + \rho(B_n \cap (\text{aff } \bar{F})^\perp)$.

So we may assume that $k \leq n - 1$. By induction hypothesis applied to the $(n-1)$ -simplex \bar{F} and the point $\bar{\mathbf{x}} \in \bar{F}$, there exists a $(k-1)$ -face F of \bar{F} such that

$$\bar{\mathbf{x}} \in \left(F + \rho(n-1, k)(B_n \cap (\text{aff } F)^\perp) \right) \cap \text{aff } \bar{F}.$$

So let $\mathbf{y} \in F$ with $\|\bar{\mathbf{x}} - \mathbf{y}\| \leq \rho(n-1, k)$ and $\bar{\mathbf{x}} - \mathbf{y} \in (\text{aff } F)^\perp \cap \text{aff } \bar{F}$. Then

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\|^2 &= \|\mathbf{x} - \bar{\mathbf{x}} + \bar{\mathbf{x}} - \mathbf{y}\|^2 = \|\mathbf{x} - \bar{\mathbf{x}}\|^2 + \|\bar{\mathbf{x}} - \mathbf{y}\|^2 \\ &\leq \frac{1}{n^2} + \rho(n-1, k)^2 = \rho(n, k)^2. \end{aligned}$$

Finally, since also $\mathbf{x} - \bar{\mathbf{x}} \in (\text{aff } \bar{F})^\perp \subset \text{aff } F^\perp$ we have $\mathbf{x} - \mathbf{y} \in (\text{aff } F)^\perp$. \square

5.16 Theorem. *Let $P \subset B_n$ be an n -dimensional polytope with m vertices. Then*

$$\frac{\text{vol}(P)}{\text{vol}(B_n)} \leq \left(c \frac{\log(\frac{m}{n} + 1)}{n} \right)^{n/2},$$

where c is an absolute constant.

⁴⁶ $(\text{aff } \bar{F})^\perp = \{\mathbf{w} \in \mathbb{R}^n : \langle \mathbf{w}, \mathbf{x} - \mathbf{y} \rangle = 0 \text{ for all } \mathbf{x}, \mathbf{y} \in \text{aff } F\}$

Proof. Let $P = \text{conv } V$ with $V = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$. By Carathéodory Theorem 2.7⁴⁷ and Lemma 5.15 we get that for every $k \in \{1, \dots, n\}$,

$$P = \text{conv } V \subset \bigcup_{\substack{J \subset \{\mathbf{v}_1, \dots, \mathbf{v}_m\} \\ \#J=k}} \left(\text{conv } \{\mathbf{v}_j : j \in J\} + \left(\sum_{i=k}^n i^{-2} \right)^{1/2} \left[B_n \cap (\text{aff conv } \{\mathbf{v}_j : j \in J\})^\perp \right] \right).$$

Now we observe $\text{vol}_{k-1}(\text{conv } \{\mathbf{v}_j : j \in J\}) \leq \text{vol}_{k-1}(S_{k-1})$, where S_{k-1} is the regular $(k-1)$ -simplex with $S_{k-1} \subset B_{k-1}$ and circumradius 1 (exercise) which has volume

$$\text{vol}_{k-1}(S_{k-1}) = \left(\frac{k}{k-1} \right)^{(k-1)/2} \frac{\sqrt{k}}{(k-1)!}.$$

Together with the estimate

$$\rho(n, k)^2 = \sum_{i=k}^n \frac{1}{i^2} \leq \sum_{i=k}^n \frac{1}{i(i-1)} = \sum_{i=k}^n \left(\frac{1}{i-1} - \frac{1}{i} \right) = \frac{1}{k-1} - \frac{1}{n} \leq \frac{1}{k-1},$$

we get

$$\begin{aligned} \frac{\text{vol}(P)}{\text{vol}(B_n)} &\leq \binom{m}{k} \left(\frac{k}{k-1} \right)^{(k-1)/2} \frac{\sqrt{k}}{(k-1)!} \rho(n, k)^{n-k+1} \frac{\kappa_{n-k+1}}{\kappa_n} \\ &\leq \binom{m}{k} \left(\frac{k}{k-1} \right)^{(k-1)/2} \frac{\sqrt{k}}{(k-1)!} \left(\frac{1}{k-1} \right)^{(n-k+1)/2} \frac{\kappa_{n-k+1}}{\kappa_n}. \end{aligned} \quad (5.16.1)$$

Let $k = \lfloor n / \ln((m/n) + 1) \rfloor$ and, in the following, we are going to bound the logarithms of all terms in (5.16.1). Since $k \ln k = k \ln n + O(n)$ and $\ln(k!) = k \ln k - k + O(\ln k) = k \ln n + O(n)$ we have

$$\begin{aligned} \ln \binom{m}{k} &\leq k \log m - \ln(k!) = k \left(\ln \frac{m}{n} + \ln n \right) - \ln(k!) \\ &= k \ln \frac{m}{n} + k \ln n - k \ln n + O(n) = k \ln \frac{m}{n} + O(n) = n + O(n) \leq cn. \end{aligned}$$

For the next terms we find

$$\begin{aligned} \ln \left(\left(\frac{k}{k-1} \right)^{(k-1)/2} \frac{\sqrt{k}}{(k-1)!} \right) &= \frac{k-1}{2} \ln \frac{k}{k-1} + \frac{1}{2} \ln k - \ln((k-1)!) \\ &= -k \ln n + O(n), \\ \ln \left(\frac{1}{k-1} \right)^{(n-k+1)/2} &\leq -(n-k) \ln \sqrt{k-1} \\ &= -n \ln \sqrt{k-1} + k \ln \sqrt{k-1} \\ &= -\frac{n}{2} \ln k + \frac{k}{2} \ln k + O(n) \\ &= \frac{n}{2} \left(-\ln n + \ln \ln \left(\frac{m}{n} + 1 \right) \right) + \frac{k}{2} \ln n + O(n), \end{aligned}$$

⁴⁷Skript WS14

$$\ln \frac{\kappa_{n-k+1}}{\kappa_n} \leq \ln \left(\frac{\Gamma(\frac{n}{2} + 1)}{\Gamma(\frac{n-k+1}{2} + 1)} \right) + cn \leq \ln n^{k/2} + cn = \frac{k}{2} \ln n + cn.$$

Hence by (5.16.1) we get

$$\ln \left(\frac{\text{vol } P}{\text{vol } B_n} \right) \leq -k \ln n + \frac{n}{2} \left(-\ln n + \ln \ln \left(\frac{m}{n} + 1 \right) \right) + \frac{k}{2} \ln n + \frac{k}{2} \ln n + cn,$$

as required. \square

5.17 Theorem [Measure concentration for the sphere]. *Let $A \subseteq S^{n-1}$ with $\mu(A) > 0$, $t \in [0, 2]$, and let $A_t = \{\mathbf{u} \in S^{n-1} : \exists \mathbf{a} \in A \text{ with } \|\mathbf{u} - \mathbf{a}\| \leq t\}$ be the spherical parallel set of A at distance t . Then*

$$\mu(A_t) \geq 1 - \frac{1}{\mu(A)} e^{-nt^2/4}.$$

Proof. Let $B = S^{n-1} \setminus A_t = \{\mathbf{u} \in S^{n-1} : \|\mathbf{u} - \mathbf{a}\| \geq t \text{ for all } \mathbf{a} \in A\}$ be the complement of A_t . It is to show

$$\mu(B)\mu(A) \leq e^{-nt^2/4}. \quad (5.17.1)$$

To this end we complete both sets with respect to B_n , i.e., we consider $\tilde{A} = \{\alpha \mathbf{x} : \alpha \in [0, 1], \mathbf{x} \in A\}$ and analogously we define \tilde{B} . Then $\mu(A) = \text{vol}(\tilde{A})/\kappa_n$ and $\mu(B) = \text{vol}(\tilde{B})/\kappa_n$. Next we show

$$\frac{1}{2}(\tilde{A} + \tilde{B}) \subseteq \left(1 - \frac{t^2}{8}\right) B_n. \quad (5.17.2)$$

Let $\tilde{\mathbf{a}} = \alpha \mathbf{a} \in \tilde{A}$ and $\tilde{\mathbf{b}} = \beta \mathbf{b} \in \tilde{B}$, with $\mathbf{a} \in A$, $\mathbf{b} \in B$ and $\alpha, \beta \in [0, 1]$. Then

$$\begin{aligned} \left\| \frac{1}{2}(\tilde{\mathbf{a}} + \tilde{\mathbf{b}}) \right\|^2 &= \frac{1}{4}(2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2) \\ &\leq \frac{1}{4}(4 - t^2) = 1 - \frac{t^2}{4} \leq \left(1 - \frac{t^2}{8}\right)^2. \end{aligned}$$

Let $\alpha \leq \beta$ and $\gamma = \alpha/\beta \leq 1$. Then

$$\begin{aligned} \left\| \frac{1}{2}(\tilde{\mathbf{a}} + \tilde{\mathbf{b}}) \right\| &= \left\| \frac{1}{2}(\alpha \mathbf{a} + \beta \mathbf{b}) \right\| = \left\| \frac{\beta}{2}(\gamma \mathbf{a} + \mathbf{b}) \right\| = \beta \left\| \gamma \frac{\mathbf{a} + \mathbf{b}}{2} + (1 - \gamma) \frac{\mathbf{b}}{2} \right\| \\ &\leq \beta \left(\gamma \left\| \frac{\mathbf{a} + \mathbf{b}}{2} \right\| + (1 - \gamma) \left\| \frac{\mathbf{b}}{2} \right\| \right) \leq \gamma \left(1 - \frac{t^2}{8}\right) + (1 - \gamma) \frac{1}{2} \leq 1 - \frac{t^2}{8}. \end{aligned}$$

This shows (5.17.2) and so

$$\left(1 - \frac{t^2}{8}\right)^n \kappa_n \geq \text{vol} \left(\frac{1}{2}(\tilde{A} + \tilde{B}) \right) \geq \sqrt{\text{vol}(\tilde{A})\text{vol}(\tilde{B})},$$

according to the Brunn-Minkowski inequality. Finally, we conclude

$$\mu(B)\mu(A) = \frac{\text{vol}(\tilde{B})}{\kappa_n} \frac{\text{vol}(\tilde{A})}{\kappa_n} \leq \left(1 - \frac{t^2}{8}\right)^{2n} \leq e^{-2nt^2/8} = e^{-nt^2/4},$$

i.e., (5.17.1) is shown. \square

5.18 Corollary. Let $\mathbf{v} \in S^{n-1}$, $t \in [0, 1]$ and let $U = \{\mathbf{u} \in S^{n-1} : |\langle \mathbf{v}, \mathbf{u} \rangle| \leq t\}$. Then $\mu(U) \geq 1 - 4e^{-nt^2/4}$ and, in particular, for $t \geq 4/\sqrt{n}$ it holds

$$\mu(U) \geq 0.9.$$

Proof. Exercise. □

5.19 Definition [Median]. Let $f : S^{n-1} \rightarrow \mathbb{R}$. Then

$$\text{med}(f) = \sup \left\{ t \in \mathbb{R} : \mu_{n-1}(f \leq t) \leq \frac{1}{2} \right\}$$

is called the median of f .

5.20 Proposition. Let $f : S^{n-1} \rightarrow \mathbb{R}$. Then $\mu_{n-1}(f < \text{med}(f)) \leq 1/2$, and $\mu_{n-1}(f > \text{med}(f)) \leq 1/2$; or equivalently,

$$\mu_{n-1}(f \geq \text{med}(f)) \geq \frac{1}{2} \text{ and } \mu_{n-1}(f \leq \text{med}(f)) \geq \frac{1}{2}.$$

Proof. By the σ -additivity of μ_{n-1} we have

$$\begin{aligned} \mu_{n-1}(f < \text{med}(f)) &= \mu_{n-1}(f \leq \text{med}(f) - 1) \\ &\quad + \sum_{k=2}^{\infty} \mu_{n-1} \left(\text{med}(f) - \frac{1}{k-1} < f \leq \text{med}(f) - \frac{1}{k} \right) \\ &= \sup_{k \geq 1} \mu_{n-1} \left(f \leq \text{med}(f) - \frac{1}{k} \right) \leq \frac{1}{2}. \end{aligned}$$

□

5.21 Lemma [Levy's Lemma]. ⁴⁸ Let $f : S^{n-1} \rightarrow \mathbb{R}$ be a 1-Lipschitz function and let $t > 0$. Then

$$\mu_{n-1}(|f - \text{med}(f)| > t) \leq 4e^{-nt^2/4}.$$

Proof. First we consider $U^+ = \{\mathbf{u} \in S^{n-1} : f(\mathbf{u}) > \text{med}(f) + t\}$. Let $A = \{\mathbf{a} \in S^{n-1} : f(\mathbf{a}) \leq \text{med}(f)\}$ and $A_t = \{\mathbf{v} \in S^{n-1} : \exists \mathbf{a} \in A \text{ with } \|\mathbf{v} - \mathbf{a}\| \leq t\}$. Then $U^+ \subseteq S^{n-1} \setminus A_t$, since for $\mathbf{v} \in A_t$ there exists an $\mathbf{a} \in A$ with $\|\mathbf{v} - \mathbf{a}\| \leq t$ and so by the 1-Lipschitz property

$$f(\mathbf{v}) - \text{med}(f) \leq f(\mathbf{v}) - f(\mathbf{a}) \leq \|\mathbf{v} - \mathbf{a}\| \leq t.$$

Hence $\mu_{n-1}(U^+) \leq 1 - \mu_{n-1}(A_t)$ and with Theorem 5.17 and Proposition 5.20 we obtain

$$\mu_{n-1}(U^+) \leq \frac{1}{\mu_{n-1}(A)} e^{-nt^2/4} \leq 2e^{-nt^2/4}.$$

The set $U^- = \{\mathbf{u} \in S^{n-1} : f(\mathbf{u}) < \text{med}(f) - t\}$ can be treated analogously. □

⁴⁸Paul Pierre Lévy, 1886–1971

5.22 Corollary. Let $f : S^{n-1} \rightarrow \mathbb{R}$ be a 1-Lipschitz function, $t > 0$, and let $\mathbb{E}(f) = \int_{S^{n-1}} f(\mathbf{u}) d\mu(\mathbf{u})$. Then

$$\mu_{n-1}(|f - \mathbb{E}(f)| > t) \leq (e^{8\pi}) e^{-nt^2/8}.$$

Proof. First we note that

$$\begin{aligned} |\text{med}(f) - \mathbb{E}(f)| &= |\mathbb{E}(\text{med}(f) - f)| \leq \int_{S^{n-1}} |\text{med}(f) - f(\mathbf{u})| d\mu(\mathbf{u}) \\ &= \mathbb{E}(|\text{med}(f) - f|) = \int_0^\infty \mu(|f - \text{med}(f)| > t) dt \quad (5.22.1) \\ &\leq \int_0^\infty 4e^{-nt^2/4} dt = 4\sqrt{\pi/n}. \end{aligned}$$

by Lemma 5.21. Now for $t \leq 8\sqrt{\pi/n}$ we have $e^{8\pi} e^{-nt^2/8} \geq 1$ and the statement is certainly true. So let $t > 8\sqrt{\pi/n}$. Then we may write

$$\begin{aligned} \mu_{n-1}(|f - \mathbb{E}(f)| > t) &\leq \mu_{n-1}(|f - \text{med}(f)| > t - |\text{med}(f) - \mathbb{E}(f)|) \\ &\leq \mu_{n-1}(|f - \text{med}(f)| > t - 4\sqrt{\pi/n}) \leq \mu_{n-1}(|f - \text{med}(f)| > t/2) \\ &\leq 4e^{-nt^2/8}. \end{aligned}$$

□

5.23 Definition [δ -net]. For $\delta > 0$ a subset $N_\delta \subseteq S^{n-1}$ is called δ -net if for all $\mathbf{u} \in S^{n-1}$ there exists a $\mathbf{v} \in N_\delta$ such that $\|\mathbf{u} - \mathbf{v}\| \leq \delta$, i.e., $S^{n-1} = \cup_{\mathbf{v} \in N} C((2 - \delta^2)/2, \mathbf{v})$.

5.24 Proposition. There exists always a δ -net on S^{n-1} consisting of at most $(4/\delta)^n$ points.

Proof. Exercise. □

5.25 Lemma. Let $|\cdot|$ be a norm on \mathbb{R}^n , $M, \gamma > 0$, and let $N_\delta \subset S^{n-1}$ be a δ -net such that for all $\mathbf{v} \in N_\delta$

$$M(1 - \gamma) \leq |\mathbf{v}| \leq M(1 + \gamma).$$

Then for all $\mathbf{u} \in S^{n-1}$

$$M \frac{(1 - \gamma - 2\delta)}{1 - \delta} \leq |\mathbf{u}| \leq M \frac{1 + \gamma}{1 - \delta}.$$

Proof. By adopting the norm we may assume $M = 1$. For the upper bound let $\bar{\mathbf{u}} \in S^{n-1}$ with $|\bar{\mathbf{u}}| = c = \max\{|\mathbf{u}| : \mathbf{u} \in S^{n-1}\}$ and next we show

$$c \leq \frac{1 + \gamma}{1 - \delta}. \quad (5.25.1)$$

To this end let $\mathbf{v} \in N_\delta$ with $\|\mathbf{v} - \bar{\mathbf{u}}\| \leq \delta$. Then $|\mathbf{v}| \leq 1 + \gamma$ by assumption and so we get

$$c = |\bar{\mathbf{u}}| \leq |\mathbf{v}| + \|\mathbf{v} - \bar{\mathbf{u}}\| \leq (1 + \gamma) + c \|\mathbf{v} - \bar{\mathbf{u}}\| \leq 1 + \gamma + c\delta.$$

Thus we get (5.25.1) and for the lower bound let $\mathbf{u} \in S^{n-1}$ and let $\mathbf{v} \in N_\delta$ with $\|\mathbf{u} - \mathbf{v}\| \leq \delta$. By assumption and using (5.25.1) we conclude

$$1 - \gamma \leq |\mathbf{v}| \leq |\mathbf{u}| + \|\mathbf{v} - \mathbf{u}\| \leq |\mathbf{u}| + \frac{1 + \gamma}{1 - \delta} \|\mathbf{v} - \mathbf{u}\| \leq |\mathbf{u}| + \frac{1 + \gamma}{1 - \delta} \delta,$$

which gives the desired lower bound on $|\mathbf{u}|$. \square

5.26 Lemma. *Let $|\cdot|$ be a norm \mathbb{R}^n with $|\cdot| \leq \|\cdot\|$, let $M = \int_{S^{n-1}} |\mathbf{u}| d\mu(\mathbf{u})$ and let $\gamma > 0$, $0 \leq \delta \leq 2$. For*

$$k \leq \frac{1}{8} \frac{\gamma^2}{\ln(e^{34}/\delta)} nM^2,$$

there exists a k -dimensional linear subspace H and a δ -net $N_{\delta,H}$ in $H \cap S^{n-1}$ such that for $\mathbf{v} \in N_{\delta,H}$

$$M(1 - \gamma) \leq |\mathbf{v}| \leq M(1 + \gamma).$$

Proof. The function $|\cdot| : S^{n-1} \rightarrow \mathbb{R}$ is a 1-Lipschitz function since

$$\left| |\mathbf{u}| - |\mathbf{v}| \right| \leq \|\mathbf{u} - \mathbf{v}\| \leq \|\mathbf{u} - \mathbf{v}\|.$$

In view of Corollary 5.22 we have

$$\mu_{n-1}(|\mathbf{u}| - M \geq M\gamma) \leq (e^{32}) e^{-nM^2\gamma^2/8},$$

i.e.,

$$M(1 - \gamma) \leq |\mathbf{u}| \leq M(1 + \gamma) \tag{5.26.1}$$

for all $\mathbf{u} \in S^{n-1}$ except for a set of measure at most $e^{-nM^2\gamma^2/8+32}$. Let \bar{H} be an arbitrary m -dimensional subspace and let $N_{\delta,\bar{H}}$ be a δ -net in $\bar{H} \cap S^{n-1}$ with $|N_{\delta,\bar{H}}| \leq (4/\delta)^m$ (cf. Proposition 5.24). In view of Remark 5.11 and (5.26.1) we know that for each $\mathbf{v} \in N_{\delta,\bar{H}}$ there exists a set $A_{\mathbf{v}} \subset O(n)$ of measure at most $e^{-nM^2\gamma^2/8+32}$ with

$$M(1 - \gamma) \leq |U\mathbf{v}| \leq M(1 + \gamma) \text{ for all } U \in O(n) \setminus A_{\mathbf{v}}.$$

Hence, as long as

$$|N_{\delta,\bar{H}}| e^{-nM^2\gamma^2/8+32} < 1 \tag{5.26.2}$$

there exists an $\bar{U} \in O(n)$

$$M(1 - \gamma) \leq |\bar{U}\mathbf{v}| \leq M(1 + \gamma) \text{ for all } \mathbf{v} \in N_{\delta,\bar{H}},$$

or equivalently

$$M(1 - \gamma) \leq |\mathbf{v}| \leq M(1 + \gamma) \text{ for all } \mathbf{v} \in \bar{U} N_{\delta,\bar{H}}.$$

Since $\bar{U} N_{\delta,\bar{H}}$ is a δ -net in $\bar{U}\bar{H} \cap S^{n-1}$ it remains to bound m so that (5.26.2) holds. Since $|N_{\delta,\bar{H}}| \leq (4/\delta)^m$ we get the desired bound. \square

5.27 Theorem. Let $K \in \mathcal{K}_o^n$, $rB_n \subseteq K$, $M = \int_{S^{n-1}} |\mathbf{u}|_K d\mu(\mathbf{u})$, $1 \geq \epsilon > 0$ and

$$k \leq \frac{1}{400} \frac{\epsilon^2}{\ln(e^{36}\epsilon^{-1})} n (rM)^2.$$

Then there exists a k -dimensional linear subspace $H \in \text{Gr}(k, n)$ with $d_{BM}(K \cap H, B_n \cap H) \leq 1 + \epsilon$.

Proof. Since $rM = \int_{S^{n-1}} |\mathbf{u}|_{(1/r)K} d\mu(\mathbf{u})$ we may assume with out loss of generality $r = 1$, i.e., $|\mathbf{u}|_K \leq \|\mathbf{u}\|$. For $\gamma, \delta > 0$, $\delta \leq 2$, and for

$$k \leq \frac{1}{8} \frac{\gamma^2}{\ln(e^{34}/\delta)} nM^2, \quad (5.27.1)$$

there exists by Lemma 5.26 and Lemma 5.25 a k -dimensional linear subspace H such that

$$M \frac{1 - \gamma - 2\delta}{1 - \delta} \leq |\mathbf{u}|_K \leq M \frac{1 + \gamma}{1 - \delta}, \text{ for all } \mathbf{u} \in S^{n-1} \cap H.$$

Hence,

$$d_{BM}(K \cap H, B_n \cap H) \leq \frac{1 + \gamma}{1 - \gamma - 2\delta}.$$

For $\gamma = \delta = \epsilon/7$ the right hand side is less than $1 + \epsilon$ and the right hand side of (5.27.1) is lower bounded by the bound on k given in the statement of the theorem. \square

5.28 Notation. Let $\gamma_n(\cdot)$ be the standard Gaussian measure with density $\sqrt{2\pi}^{-n} e^{-\|\mathbf{x}\|^2/2}$, i.e.,

$$\gamma_n(A) = \int_{\mathbb{R}^n} \chi_A(\mathbf{x}) d\gamma_n(\mathbf{x}) = \frac{1}{\sqrt{2\pi}^n} \int_A e^{-\|\mathbf{x}\|^2/2} d\mathbf{x}.$$

5.29 Proposition. Let $K \in \mathcal{K}_o^n$. Then

$$\frac{1}{\sqrt{n-1}} \int_{\mathbb{R}^n} |\mathbf{x}|_K d\gamma_n(\mathbf{x}) > \int_{S^{n-1}} |\mathbf{u}|_K d\mu(\mathbf{u}) > \frac{1}{\sqrt{n}} \int_{\mathbb{R}^n} |\mathbf{x}|_K d\gamma_n(\mathbf{x}).$$

Proof. In view of Proposition 3.8 we may write

$$\begin{aligned} \frac{1}{n\kappa_n} \int_{\mathbb{R}^n} |\mathbf{x}|_K e^{-\|\mathbf{x}\|^2/2} d\mathbf{x} &= \int_0^\infty \int_{S^{n-1}} r^{n-1} |r\mathbf{u}|_K e^{-\|r\mathbf{u}\|^2/2} d\mu(\mathbf{u}) dr \\ &= \int_{S^{n-1}} |\mathbf{u}|_K d\mu(\mathbf{u}) \int_0^\infty r^n e^{-\|r\|^2/2} dr \\ &= \int_{S^{n-1}} |\mathbf{u}|_K d\mu(\mathbf{u}) \sqrt{2}^{n-1} \int_0^\infty t^{n/2-1/2} e^{-t} dt \\ &= \sqrt{2}^{n-1} \Gamma((n+1)/2) \int_{S^{n-1}} |\mathbf{u}|_K d\mu(\mathbf{u}). \end{aligned}$$

Hence,

$$\begin{aligned} \int_{S^{n-1}} |\mathbf{u}|_K \, d\mu(\mathbf{u}) &= \frac{\Gamma(n/2 + 1)}{\Gamma((n+1)/2)} \frac{1}{n} \sqrt{2} \int_{\mathbb{R}^n} |\mathbf{x}|_K \left(\frac{1}{\sqrt{2\pi}} \right)^n e^{-\|\mathbf{x}\|^2/2} \, d\mathbf{x} \\ &= \frac{\Gamma(n/2)}{\Gamma((n+1)/2)} \frac{1}{\sqrt{2}} \int_{\mathbb{R}^n} |\mathbf{x}|_K \, d\gamma_n(\mathbf{x}) \\ &= \frac{1}{\sqrt{2\pi}} \frac{\kappa_{n-1}}{\kappa_{n-2}} \int_{\mathbb{R}^n} |\mathbf{x}|_K \, d\gamma_n(\mathbf{x}), \end{aligned}$$

and it remains to show

$$\sqrt{\frac{2\pi}{n}} > \frac{\kappa_n}{\kappa_{n-1}} > \sqrt{\frac{2\pi}{n+1}},$$

which is left as an exercise. \square

5.30 Proposition. For $K \in \mathcal{K}_o^n$ let $M(K) = \int_{S^{n-1}} |\mathbf{u}|_K \, d\mu(\mathbf{u})$.

- i) $M(K) = \frac{1}{2} \int_{S^{n-1}} (h(K^*, \mathbf{u}) + h(K^*, -\mathbf{u})) \, d\mu(\mathbf{u})$. i.e., one half of the mean width of K^* .
- ii) $M(K) M(K^*) \geq 1$.
- iii) For $1 \leq p < \infty$

$$M(B_n^p) \geq \sqrt{\frac{2}{\pi}} n^{1/p-1/2}.$$

In particular, $M(\sqrt{n} C_n^*) \geq \sqrt{2/\pi}$.

- iv) There exists an absolute constant c such that for n large

$$M(C_n) \geq c \sqrt{\frac{\ln n}{n}}.$$

Proof. i) and ii) are left as exercises. For iii) let $R > 0$ such that $\gamma_n(R C_n) = \gamma_n(|\mathbf{x}|_{C_n} \leq R) = 1/2$ and so also $\gamma_n(|\mathbf{x}|_{C_n} \geq R) = 1/2$. Hence we know

$$\begin{aligned} \int_{S^{n-1}} |\mathbf{u}|_{C_n} \, d\mu(\mathbf{u}) &> \frac{1}{\sqrt{n}} \int_{\mathbb{R}^n} |\mathbf{x}|_{C_n} \, d\gamma_n(\mathbf{x}) \\ &\geq \frac{1}{\sqrt{n}} \int_{\mathbb{R}^n, |\mathbf{x}|_{C_n} \geq R} |\mathbf{x}|_{C_n} \, d\gamma_n(\mathbf{x}) \geq \frac{1}{\sqrt{n}} \frac{R}{2}. \end{aligned} \quad (5.30.1)$$

In order to lower bound R we note that

$$\frac{1}{2} = \gamma_n(|\mathbf{x}|_{C_n} \leq R) = \frac{1}{\sqrt{2\pi}^n} \int_{R C_n} e^{-\|\mathbf{x}\|^2/2} \, d\mathbf{x} = \left(\frac{1}{\sqrt{2\pi}} \int_{-R}^R e^{-t^2/2} \, dt \right)^n.$$

Hence,

$$\frac{1}{\sqrt{2\pi}} \int_{-R}^R e^{-t^2/2} \, dt = e^{-\ln(2)/n} \geq 1 - \frac{\ln(2)}{n}.$$

⁴⁹ $e^{-t} \geq 1 - t$

For the left hand side we also have

$$\frac{1}{\sqrt{2\pi}} \int_{-R}^R e^{-t^2/2} dt = 1 - \frac{2}{\sqrt{2\pi}} \int_R^\infty e^{-t^2/2} dt \leq 1 - c_1 e^{-(R+1)^2/2},$$

for a suitable constant c_1 . Combining the last two inequalities leads to $R \geq c\sqrt{\ln n}$ and we are done by (5.30.1). \square

5.31 Lemma [Dvoretzky-Rogers]. *Let $K \in \mathcal{K}_o^n$ and let B_n be the volume maximal ellipsoid in K . Then there exists an orthonormal basis $\mathbf{v}_1, \dots, \mathbf{v}_n \in S^{n-1}$ such that for $1 \leq i \leq n$*

$$|\mathbf{v}_i|_K \geq \frac{1}{e} \left(1 - \frac{i-1}{n}\right).$$

Proof. We choose recursively the points \mathbf{v}_i in the following way: Let $\mathbf{v}_1 \in S^{n-1} \subset K$ be a point of maximal norm $|\cdot|_K$. For $i > 1$ let \mathbf{v}_i be a point of maximal norm $|\cdot|_K$ in $S^{n-1} \cap \text{lin}\{\mathbf{v}_1, \dots, \mathbf{v}_{i-1}\}^\perp$. Then $B_n = \{\sum_{i=1}^n \alpha_i \mathbf{v}_i : \sum_{i=1}^n (\alpha_i)^2 \leq 1\}$, and by the choice of \mathbf{v}_j we have for all $\mathbf{x} \in \text{lin}\{\mathbf{v}_j, \dots, \mathbf{v}_n\} \cap S^{n-1}$

$$|\mathbf{x}|_K \leq |\mathbf{v}_j|_K. \quad (5.31.1)$$

For $1 \leq j \leq n$ and some positive $\beta, \gamma > 0$ we consider the ellipsoid

$$\mathcal{E}_j = \left\{ \sum_{i=1}^n \alpha_i \mathbf{v}_i : \sum_{i=1}^{j-1} \frac{\alpha_i^2}{\gamma^2} + \sum_{i=j}^n \frac{\alpha_i^2}{\beta^2} \leq 1 \right\}$$

of volume $\text{vol}(\mathcal{E}_j) = \gamma^{j-1} \beta^{n-(j-1)} \kappa_n$. Let $\sum_{i=1}^n \alpha_i \mathbf{v}_i \in \mathcal{E}_j$. Then we certainly have $\sum_{i=1}^{j-1} \alpha_i \mathbf{v}_i \in \gamma B_n$ and so, since $B_n \subset K$, $\left| \sum_{i=1}^{j-1} \alpha_i \mathbf{v}_i \right|_K \leq \gamma$.

We also have $\left\| \sum_{i=j}^n \alpha_i \mathbf{v}_i \right\| \leq \beta$ and since $(\sum_{i=j}^n \alpha_i \mathbf{v}_i) / \left\| \sum_{i=j}^n \alpha_i \mathbf{v}_i \right\| \in \text{lin}\{\mathbf{v}_j, \dots, \mathbf{v}_n\} \cap S^{n-1}$ we conclude by (5.31.1) $\left| \sum_{i=j}^n \bar{\alpha}_i \mathbf{v}_i \right|_K \leq \beta |\mathbf{v}_j|_K$. Hence

$$\left| \sum_{i=1}^n \bar{\alpha}_i \mathbf{v}_i \right|_K \leq \gamma + \beta |\mathbf{v}_j|_K.$$

Thus, if $\gamma + \beta |\mathbf{v}_j|_K \leq 1$ we have $\mathcal{E}_j \subseteq K$ and by the volume maximality of B_n we get for $j = 1, \dots, n$

$$\gamma^{j-1} \beta^{n-(j-1)} \leq 1 \text{ for all } \gamma, \beta > 0 \text{ with } \gamma + \beta |\mathbf{v}_j|_K \leq 1.$$

Hence, setting $\beta = (1 - \gamma) / |\mathbf{v}_j|_K$ yields

$$|\mathbf{v}_j|_K^{n-j+1} \geq (1 - \gamma)^{n-j+1} \gamma^{j-1},$$

which for $\gamma = \frac{j-1}{n}$ leads to

$$|\mathbf{v}_j|_K \geq \left(1 - \frac{j-1}{n}\right) \left(\frac{j-1}{n}\right)^{\frac{j-1}{n-j+1}} \geq \left(1 - \frac{j-1}{n}\right) \frac{1}{e}.$$

For the last inequality observe that $(i/n)^{i/(n-i)} \geq e^{-1}$ is equivalent to $\ln(n) - \ln(i) \leq (n-i)(1/i)$ which is certainly true as the integral $\int_i^n (1/x) dx$ shows. \square

5.32 Theorem [Dvoretzky]. *There exists an absolute constant c such that for all $K \in \mathcal{K}_o^n$ and $\epsilon > 0$ and for*

$$k \leq c \frac{\epsilon^2}{\ln(1 + \epsilon^{-1})} \ln(n)$$

there exists a k -dimensional subspace H with $d_{BM}(K \cap H, B_n \cap H) \leq 1 + \epsilon$.

Proof. We may assume that B_n is the volume maximal ellipsoid in K . In view of Theorem 5.27 it is to show that

$$M = \int_{S^{n-1}} |\mathbf{u}|_K d\mu(\mathbf{u}) \geq c \sqrt{\frac{\ln n}{n}}.$$

Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be an orthonormal basis according to Lemma 5.31. Then $S^{n-1} = \{\sum_{i=1}^n \alpha_i \mathbf{v}_i : \sum_{i=1}^n \alpha_i^2 = 1\}$ and for $i \leq n/2$, say, we, in particular, have

$$|\mathbf{v}_i|_K \geq \bar{c},$$

for a certain constant \bar{c} . Hence, we may write

$$\begin{aligned} \int_{S^{n-1}} |\mathbf{u}|_K d\mu(\mathbf{u}) &= \int_{S^{n-1}} \left| \sum_{i=1}^n \alpha_i \mathbf{v}_i \right|_K d\mu(\alpha) \\ &= \int_{S^{n-1}} \frac{1}{2} \left(\left| \sum_{i=1}^{n-1} \alpha_i \mathbf{v}_i + \alpha_n \mathbf{v}_n \right|_K + \left| \sum_{i=1}^{n-1} \alpha_i \mathbf{v}_i - \alpha_n \mathbf{v}_n \right|_K \right) d\mu(\alpha) \\ &\geq \int_{S^{n-1}} \max \left\{ \left| \sum_{i=1}^{n-1} \alpha_i \mathbf{v}_i \right|_K, |\alpha_n \mathbf{v}_n|_K \right\} d\mu(\alpha), \end{aligned}$$

where we have used the triangle inequality $2|a|_K \leq |a+b|_K + |a-b|_K \leq |a+b|_K + |a-b|_K$. Repeating in this way we obtain

$$\begin{aligned} \int_{S^{n-1}} |\mathbf{u}|_K d\mu(\mathbf{u}) &\geq \int_{S^{n-1}} \max \{ |\alpha_1 \mathbf{v}_1|_K, \dots, |\alpha_n \mathbf{v}_n|_K \} d\mu(\alpha) \\ &\geq \bar{c} \int_{S^{n-1}} \max \{ |\alpha_1|, \dots, |\alpha_{n/2}| \} d\mu(\alpha) \\ &\geq \frac{\bar{c}}{\sqrt{n}} \int_{\mathbb{R}^n} \max \{ |x_1|, \dots, |x_{n/2}| \} d\gamma_n(\mathbf{x}) \\ &= \frac{\bar{c}}{\sqrt{n}} \int_{\mathbb{R}^{n/2}} \max \{ |x_1|, \dots, |x_{n/2}| \} d\gamma_{n/2}(\mathbf{x}) \\ &= \frac{\bar{c}}{\sqrt{n}} \int_{\mathbb{R}^{n/2}} |\mathbf{x}|_{C_{n/2}} d\gamma_{n/2}(\mathbf{x}) \\ &\geq c \int_{S^{n/2-1}} |\mathbf{u}|_{C_{n/2}} d\mu(\mathbf{u}) \geq c \sqrt{\frac{\ln n}{n}}, \end{aligned}$$

where we have applied Propositions 5.29 and 5.30. \square

Preliminary Version – Draft 2016

6 A few remarks on covering numbers

6.1 Definition [covering-, separation number]. Let $K, L \in \mathcal{K}^n$.

i)

$$\begin{aligned} N(K, L) &= \min\{|S| : S \subset \mathbb{R}^n \text{ with } K \subseteq S + L\}, \\ \bar{N}(K, L) &= \min\{|S| : S \subset K \text{ with } K \subseteq S + L\} \end{aligned}$$

are called covering numbers of K by L .

ii) If $L = -L$,

$$M(K, L) = \max\{|S| : S \subset K \text{ with } |\mathbf{x}_i - \mathbf{x}_j|_L > 1 \text{ for all } \mathbf{x}_i \neq \mathbf{x}_j \in S\}$$

is called the separation number of K by L .

6.2 Proposition. Show that

i) for $K, L, M \in \mathcal{K}^n$, $K \subseteq L$: $N(K, M) \leq N(L, M)$, $N(M, L) \leq N(M, K)$,
and $\bar{N}(M, L) \leq \bar{N}(M, K)$

ii) for $K, L \in \mathcal{K}^n$: $\bar{N}(K, L - L) \leq N(K, L) \leq \bar{N}(K, L)$.

iii) for $K \in \mathcal{K}^n$, $\lambda > 0$: $N(K, \lambda B_n) = \bar{N}(K, \lambda B_n)$.

iv) for $K, L, M \in \mathcal{K}^n$: $N(K, L) \leq N(K, M)N(M, L)$.

Proof. Exercise. □

6.3 Proposition. Let $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{K}^n$ ellipsoids centered at the origin. Show that

$$N(\mathcal{E}_1, \mathcal{E}_2) = N(\mathcal{E}_2^*, \mathcal{E}_1^*).$$

6.4 Proposition. Let $K, L \in \mathcal{K}^n$.

i)

$$\bar{N}(K, (K - K) \cap L) = \bar{N}(K, L).$$

ii)

$$N(K, (K - K) \cap (L - L)) \leq N(K, L).$$

iii) for $r > 0$

$$N(K, (K - K) \cap rB_n) \leq N(K, rB_n).$$

Proof. i) is left as an exercise. For ii) let $K \subseteq \cup_{i=1}^N (\mathbf{x}_i + L)$ be a minimal covering of K by L . By the minimality there exists points $\mathbf{y}_i \in (\mathbf{x}_i + L) \cap K$ for all $1 \leq i \leq N$.

iii) □

6.5 Proposition. Show that

$$M(K, 2L) \leq N(K, L) \leq \bar{N}(K, L) \leq M(K, L)$$

Proof. Exercise. □

6.6 Lemma. Let $K, L \in \mathcal{K}^n$, $\dim K, \dim L = n$. Show that

$$\text{vol}(K)/\text{vol}(L) \leq N(K, L).$$

If $L = -L$ then

$$N(K, L) \leq 2^n \text{vol}(K + L/2)/\text{vol}(L).$$

Proof. Exercise. □

6.7 Corollary. Let $L, K \in \mathcal{K}^n$, $L \subseteq K$, $L = -L$. Then for $\epsilon > 0$

$$N(K, \epsilon L) \leq \left(\frac{\text{vol}(K)}{\text{vol}(L)} \right) \left(1 + \frac{2}{\epsilon} \right)^n.$$

Proof. Immediate consequence by Lemma 6.6, since

$$N(K, \epsilon L) \leq \frac{2^n \text{vol}(K + \frac{\epsilon}{2}L)}{\text{vol}(\epsilon L)} \leq 2^n \frac{(1 + \frac{\epsilon}{2})^n \text{vol}(K)}{\epsilon^n \text{vol}(L)}.$$

□

6.8 Corollary. Let $K \in \mathcal{K}_o^n$. Then for $1 > \epsilon > 0$

$$\left(\frac{1}{\epsilon} \right)^n \leq N(K, \epsilon K) \leq \left(1 + \frac{2}{\epsilon} \right)^n.$$

Proof. Immediate consequence by Corollary 6.7 and the lower bound in Lemma 6.6. □

6.9 Lemma. Let μ be a probability measure on \mathbb{R}^n , $\psi : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ a log-concave function with finite, positive integral Then

$$\int_{\mathbb{R}^n} \psi(\mathbf{x}) \, d\mu(\mathbf{x}) \leq \psi \left(\int_{\mathbb{R}^n} \mathbf{x} \frac{\psi(\mathbf{x})}{\int \psi \, d\mu} \, d\mu(\mathbf{x}) \right).$$

Proof. to be written... □

6.10 Theorem. Let $K, L \in \mathcal{K}^n$ be centered convex bodies, i.e., $\mathbf{0} = \int_K \mathbf{x} \, d\mathbf{x} = \int_L \mathbf{x} \, d\mathbf{x}$. Then

$$\text{vol}(K)\text{vol}(L) \leq \text{vol}(K - L)\text{vol}(K \cap L).$$

In particular, if $L = -K$ then

$$\text{vol}(K \cap -K) \geq 2^{-n} \text{vol}(K).$$

6.11 Theorem. *Let $K, L \in \mathcal{K}^n$. Then*

$$4^{-n} \frac{\text{vol}(K - L)}{\text{vol}(L)} \leq N(K, L) \leq 4^n \frac{\text{vol}(K - L)}{\text{vol}(L)}.$$

6.12 Corollary. *Let $K \in \mathcal{K}^n$. Then*

$$N(K - K, K) \leq 32^n.$$

Proof. Without loss of generality let $\mathbf{0} \in \text{int } K$. In view of the upper bound of Theorem 6.11 we have

$$N(K - K, K) \leq 4^n \frac{\text{vol}(K - 2K)}{\text{vol}(K)} \leq 8^n \frac{\text{vol}(K - K)}{\text{vol}(K)} \leq 32^n,$$

by the Theorem 1.15 of Rogers&Shephard. □

6.13 Lemma. *Let $K, L \in \mathcal{K}^n$, $K = -K$, $L \subseteq rK$, $r \geq 1$. Then*

$$N(\text{conv}(K \cup L), (1 + 1/n)K) \leq 2rn N(L, K).$$

6.14 Corollary. *Let $K, L \in \mathcal{K}^n$, $K = -K$, $L \subseteq rK$, $r \geq 1$. Then*

$$\text{vol}(\text{conv}(K \cup L)) \leq 2ern N(L, K) \text{vol}(K).$$

6.15 Theorem [Pajor-Tomczak]. *Let $K \in \mathcal{K}_o^n$ and $t > 0$. Then*

$$\log \bar{N}(B_n, tK) \leq cn \left(\frac{M(K)}{t} \right)^2,$$

where c is an absolute constant.

6.16 Lemma. *Let $K \in \mathcal{K}_o^n$. Then*

$$N(K, tB_n) \leq N(K, 2tB_n) N(B_n, \frac{t}{8}K^*).$$

6.17 Theorem. *Let $K \in \mathcal{K}_o^n$. Then*

$$\sup_{t>0} t (\log N(K, tB_n))^{1/2} \leq 10 \sup_{t>0} t (\log N(B_n, tK^*))^{1/2}$$

6.18 Theorem. *Let $K \in \mathcal{K}_o^n$ and $t > 0$. Then*

$$\log N(K, tB_n) \leq cn \left(\frac{M(K^*)}{t} \right)^2$$

6.19 Proposition. *Let $K \in \mathcal{K}^n$ and $t > 0$. Then*

$$t^2 \log N(K, tB_n) \leq cnM(K^*)^2,$$

where c is an absolute constant.

6.20 Definition [M-position]. Let $K \in \mathcal{K}_o^n$ and $r = (\text{vol}(K)/\text{vol}(B_n))^{1/n}$. K is called in M -position with constant $c > 0$ if

$$\begin{aligned} \frac{1}{c} \text{vol}(r B_n + L)^{\frac{1}{n}} &\leq \text{vol}(K + L)^{\frac{1}{n}} \leq c \text{vol}(r B_n + L)^{\frac{1}{n}}, \\ \frac{1}{c} \text{vol}((r B_n)^* + L)^{\frac{1}{n}} &\leq \text{vol}(K^* + L)^{\frac{1}{n}} \leq c \text{vol}((r B_n)^* + L)^{\frac{1}{n}}. \end{aligned}$$

6.21 Theorem* [Milman]. There exists an absolute $c > 0$ such that for all $K \in \mathcal{K}_o^n$ there exists an $A \in \text{SL}(n, \mathbb{R})$ such that AK is in M -position.

6.22 Theorem [Milman's reverse Brunn-Minkowski inequality]. Let $K, L \in \mathcal{K}_o^n$. There exists $A_K, A_L \in \text{SL}(n, \mathbb{R})$ such that for $t > 0$

$$\text{vol}(A_K K + t A_L L)^{\frac{1}{n}} \leq C \left(\text{vol}(K)^{\frac{1}{n}} + t \text{vol}(L)^{\frac{1}{n}} \right).$$

Here C is an absolute constant independent of n and K, L .

6.23 Lemma. Let $K \in \mathcal{K}_o^n$, $\text{vol}(K) = \text{vol}(B_n)$ and let

$$\alpha(K) = \max \{N(K, B_n), N(B_n, K), N(K^*, B_n), N(B_n, K^*)\}.$$

- i) If K is in M -position with constant c then $\alpha(K) \leq e^{\ln(3c)n}$.
- ii) If $\alpha(K) \leq e^{\beta n}$ then K is in M -position with constant $\bar{c}\beta$, where \bar{c} is an absolute constant.

6.24 Theorem* [Bourgain-Milman]. There exists an absolute $c > 0$ such that for $K \in \mathcal{K}_o^n$

$$\text{vol}(K)\text{vol}(K^*) \geq c^n \text{vol}(B_n)^2.$$

6.25 Theorem. There exists an absolute $c > 0$ such that for $K, L \in \mathcal{K}_o^n$

$$c^{-n} N(L^*, K^*) \leq N(K, L) \leq c^n N(L^*, K^*).$$

7 Some (discrete) aspects of the L_p -Minkowski problem

7.1 Definition [Facet data]. Let $P \subset \mathbb{R}^n$, $\dim P = n$, be a polytope with m facets F_1, \dots, F_m . Let $\mathbf{u}_i \in S^{n-1}$ be the outer unit normal vector of the facet F_i , and let $\phi_i = \text{vol}_{n-1}(F_i)$. The data $\{(\mathbf{u}_i, \phi_i) \in S^{n-1} \times \mathbb{R}_{>0}^n : 1 \leq i \leq m\}$ are called the facet data of P .

7.2 Theorem [Minkowski, 1903]. $\{(\mathbf{u}_i, \phi_i) \in S^{n-1} \times \mathbb{R}_{>0} : 1 \leq i \leq m\}$ are the facet data of an n -dimensional polytope if and only if $\mathbf{u}_i \neq \mathbf{u}_j$, $i \neq j$, $\text{lin}\{\mathbf{u}_1, \dots, \mathbf{u}_m\} = \mathbb{R}^n$ and

$$\sum_{i=1}^m \phi_i \mathbf{u}_i = \mathbf{0}.$$

Moreover, P is by its facet data uniquely determined (up to translations).

Proof. For the necessity we observe that the volume is a translation-invariant functional, and so for $\mathbf{t} \in \mathbb{R}^n$

$$\begin{aligned} \text{vol}(P) &= \text{vol}(\mathbf{t} + P) = \frac{1}{n} \sum_{i=1}^m h(\mathbf{t} + P, \mathbf{u}_i) \phi_i \\ &= \frac{1}{n} \sum_{i=1}^m (h(P, \mathbf{u}_i) + \langle \mathbf{t}, \mathbf{u}_i \rangle) \phi_i = \frac{1}{n} \sum_{i=1}^m h(P, \mathbf{u}_i) \phi_i + \frac{1}{n} \sum_{i=1}^m \langle \mathbf{t}, \mathbf{u}_i \rangle \phi_i \\ &= \text{vol}(P) + \left\langle \mathbf{t}, \sum_{i=1}^m \phi_i \mathbf{u}_i \right\rangle, \end{aligned}$$

where $h(P, \cdot)$ is the support function of P . Since this holds true for any $\mathbf{t} \in \mathbb{R}^n$ we must have $\sum_{i=1}^m \phi_i \mathbf{u}_i = \mathbf{0}$. Since P is bounded we have $\text{lin}\{\mathbf{u}_1, \dots, \mathbf{u}_m\} = \mathbb{R}^n$, and of course, the outer normals are pairwise different.

In order to prove the sufficiency we consider for $\mathbf{b} \in \mathbb{R}^m$

$$P(\mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{u}_i, \mathbf{x} \rangle \leq b_i, 1 \leq i \leq m\}.$$

Observe, due to our assumption we actually know that $\text{pos}\{\mathbf{u}_1, \dots, \mathbf{u}_m\} = \mathbb{R}^n$, and hence $P(\mathbf{b})$ is a bounded set, i.e., a (possible empty) polytope. The outer unit normals of the facets of $P(\mathbf{b})$ are a subset of $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$, and so let for $1 \leq i \leq m$

$$\beta_i(\mathbf{b}) = \text{vol}_{n-1}\{\text{facet of } P(\mathbf{b}) \text{ with outer normal } \mathbf{u}_i\},$$

where we set $\beta_i(\mathbf{b}) = 0$ if $P(\mathbf{b})$ has no facet with outer normal \mathbf{u}_i . Observe that we may uniquely identify the facets of $P(\mathbf{b})$ via the vectors \mathbf{u}_i since they are pairwise different. So we have

$$\text{vol}(P(\mathbf{b})) = \frac{1}{n} \sum_{i=1}^m b_i \beta_i(\mathbf{b}).$$

Next we consider the linear function $\Psi : \mathbb{R}^m \rightarrow \mathbb{R}$ with

$$\Psi(\mathbf{b}) = \frac{1}{n} \sum_{i=1}^m b_i \phi_i.$$

Let $S = \{\mathbf{b} \in \mathbb{R}^m : \mathbf{b} \geq \mathbf{0} \text{ and } \text{vol}(P(\mathbf{b})) = 1\}$ and we set

$$\Psi^* = \inf\{\Psi(\mathbf{b}) : \mathbf{b} \in S\}.$$

Let $\mathbf{b}_i \in S$, $i \in \mathbb{N}$, with $\Psi^* = \lim_{i \rightarrow \infty} \Psi(\mathbf{b}_i)$. By the non-negativity of \mathbf{b}_i and the values ϕ , the sequence $\mathbf{b}_i \in S$, $i \in \mathbb{N}$, is bounded. Hence we may assume that it converges to $\mathbf{b}^* \in \mathbb{R}_{\geq 0}^m$. Since $\text{vol}(\cdot)$ is a continuous we also have $\text{vol}(P(\mathbf{b}^*)) = 1$, i.e., $\mathbf{b}^* \in S$, and so $\Psi(\mathbf{b}^*) = \Psi^* \leq \Psi(\mathbf{b})$ for all $\mathbf{b} \in S$. In fact, we have

$$\Psi(\mathbf{b}^*) \leq \Psi(\mathbf{b}) \text{ for all } \mathbf{b} \in \mathbb{R}^m \text{ with } \text{vol}(P(\mathbf{b})) = 1. \quad (7.2.1)$$

To this end we observe, that for any $\mathbf{b} \in \mathbb{R}^m$ there exists a $\mathbf{t} \in \mathbb{R}^n$ such that $b_i + \langle \mathbf{t}, \mathbf{u}_i \rangle \geq 0$, $1 \leq i \leq m$, because we just translate $P(\mathbf{b})$ such that $\mathbf{t} + P(\mathbf{b})$ contains the origin. Assuming $\text{vol}(P(\mathbf{b})) = 1$ we also have $1 = \text{vol}(\mathbf{t} + P(\mathbf{b})) = \text{vol}(P(\tilde{\mathbf{b}}))$ with $\tilde{b}_i = b_i + \langle \mathbf{t}, \mathbf{u}_i \rangle$, $1 \leq i \leq m$. Then $\tilde{\mathbf{b}} \in S$ and in view of our assumption we find

$$\begin{aligned} \Psi(\mathbf{b}) &= \frac{1}{n} \sum_{i=1}^m b_i \phi_i = \frac{1}{n} \sum_{i=1}^m \tilde{b}_i \phi_i - \frac{1}{n} \sum_{i=1}^m \langle \mathbf{t}, \phi_i \mathbf{u}_i \rangle \\ &= \frac{1}{n} \sum_{i=1}^m \tilde{b}_i \phi_i = \Psi(\tilde{\mathbf{b}}) \geq \Psi(\mathbf{b}^*). \end{aligned}$$

Let $\boldsymbol{\beta}^* = (\beta_1(\mathbf{b}^*), \dots, \beta_m(\mathbf{b}^*))^\top$ be the vector with facet areas of $P(\mathbf{b}^*)$, and let $\boldsymbol{\phi} = (\phi_1, \dots, \phi_m)^\top$ be the vector with the desired facet areas. Next we show that these vectors are proportional and hence, $P(\mathbf{b}^*)$ is a homothetic copy of the polytope we are looking for. To this we will prove that

$$\left\{ \mathbf{b} \in \mathbb{R}^m : \frac{1}{n} \langle \boldsymbol{\phi}, \mathbf{b} \rangle = \Psi(\mathbf{b}^*) \right\} = \left\{ \mathbf{b} \in \mathbb{R}^m : \frac{1}{n} \langle \boldsymbol{\beta}^*, \mathbf{b} \rangle = 1 \right\}.$$

In order to prove it, we first observe that \mathbf{b}^* is contained in both sets. Let U be an open neighbourhood of \mathbf{b}^* in the left hand set. It suffices to prove

$$U \subset \left\{ \mathbf{b} \in \mathbb{R}^m : \frac{1}{n} \langle \boldsymbol{\beta}^*, \mathbf{b} \rangle \leq 1 \right\}. \quad (7.2.2)$$

Next we choose the neighbourhood U of \mathbf{b}^* such that for all $\mathbf{b} \in U$

$$\beta_i(\mathbf{b}) \neq 0 \text{ if } \beta_i(\mathbf{b}^*) \neq 0. \quad (7.2.3)$$

Let $\mathbf{c} \in U$. Then $\Psi(\mathbf{c}) = \Psi(\mathbf{b}^*)$, and if $\text{vol}(P(\mathbf{c})) = \mu > 1$ then we get $\text{vol}(P(\mu^{-1/n} \mathbf{c})) = 1$ and $\Psi(\mu^{-1/n} \mathbf{c}) < \Psi(\mathbf{b}^*)$ violating the minimality of \mathbf{b}^* . Hence we have $\text{vol}(P(\mathbf{c})) \leq 1$ and for $\lambda \in [0, 1]$ let

$$P_\lambda = P(\lambda \mathbf{b}^* + (1 - \lambda) \mathbf{c}).$$

By the linearity of Ψ we have $\Psi(P_\lambda) = \Psi(\mathbf{b}^*)$ and as above we conclude $\text{vol}(P_\lambda) \leq 1$ for $0 \leq \lambda \leq 1$. On the other hand we have $P(\lambda \mathbf{b}^*) + P((1 - \lambda)\mathbf{c}) \subseteq P_\lambda$ and so we get

$$\begin{aligned} 1 &\geq \text{vol}(P_\lambda) \\ &\geq \text{vol}(P(\lambda \mathbf{b}^*) + P((1 - \lambda)\mathbf{c})) = \text{vol}(\lambda P(\mathbf{b}^*) + (1 - \lambda)P(\mathbf{c})) \\ &= \sum_{i=0}^n \binom{n}{i} V_{n-i}(P(\mathbf{b}^*), P(\mathbf{c})) \lambda^i (1 - \lambda)^{n-i}, \end{aligned}$$

where on the right hand side we have the so called Steiner polynomial whose coefficients are the mixed volumes (cf. Theorem 5.20⁵⁰). Since the Steiner polynomial above is 1 at $\lambda = 1$, the left hand side derivative of the polynomial at $\lambda = 1$ has to be non-negative which gives

$$nV_0(P(\mathbf{b}^*), P(\mathbf{c})) - nV_1(P(\mathbf{b}^*), P(\mathbf{c})) \geq 0.$$

Since $V_0(P(\mathbf{b}^*), P(\mathbf{c})) = \text{vol}(P(\mathbf{b}^*)) = 1$ we get (cf. Theorem 5.26⁵¹)

$$\begin{aligned} 1 &\geq V_1(P(\mathbf{b}^*), P(\mathbf{c})) = \underbrace{V(P(\mathbf{b}^*), \dots, P(\mathbf{b}^*), P(\mathbf{c}))}_{n-1} \\ &= \frac{1}{n} \sum_{i=1}^m h(P(\mathbf{c}), \mathbf{u}_i) \beta_i(\mathbf{b}^*) = \frac{1}{n} \langle \mathbf{c}, \boldsymbol{\beta}^* \rangle, \end{aligned}$$

where for the last equation we have used (7.2.3). Hence we have shown (7.2.2).

So we know $\boldsymbol{\beta}^* = \gamma \boldsymbol{\phi}$ for some $\gamma > 0$. From $n = \langle \boldsymbol{\beta}^*, \mathbf{b}^* \rangle = \gamma \langle \boldsymbol{\phi}, \mathbf{b}^* \rangle = \gamma n \Psi(\mathbf{b}^*)$ we conclude $\boldsymbol{\beta}^* = \frac{1}{\Psi(\mathbf{b}^*)} \boldsymbol{\phi}$ and $P(\Psi(\mathbf{b}^*)^{\frac{1}{n-1}} \mathbf{b}^*)$ is a polytope whose facet data are the given data $\{(\mathbf{u}_i, \phi_i) : 1 \leq i \leq m\}$.

Now suppose that there are two polytopes P, Q having the same facet data $\{(\mathbf{u}_i, \phi_i) : 1 \leq i \leq m\}$. Then we have

$$V(Q, \dots, Q, P) = \frac{1}{n} \sum_{i=1}^m h(P, \mathbf{u}_i) \phi_i = \text{vol}(P),$$

which gives with Minkowski's first inequality (cf. Theorem 5.32 i)⁵²)

$$\text{vol}(P)^n = V(Q, \dots, Q, P)^n \geq \text{vol}(Q)^{n-1} \text{vol}(P). \quad (7.2.4)$$

Hence $\text{vol}(P) \geq \text{vol}(Q)$ and interchanging the role of P and Q gives $\text{vol}(P) = \text{vol}(Q)$. Thus we have equality in (7.2.4) and by Theorem 5.32 i)⁵³ the polytopes P and Q must be homothetic, which implies P and Q are translates of each other. \square

7.3 Definition [Essential boundary]. Let $K \in \mathcal{K}^n$ with $\dim K = n$. The essential boundary of K , denoted by $\text{bd}'K$, is the set of all boundary points $\mathbf{x} \in \text{bd}K$ having an unique outer normal vector. Those boundary points are also called regular, and non-regular boundary points are also called singular.

⁵⁰Skript WS14

⁵¹Skript WS14

⁵²Skript WS14

⁵³Skript WS14

7.4 Proposition. $\mathcal{H}_{n-1}(\text{bd } K \setminus \text{bd}' K) = 0$, i.e., the set of singular boundary points has $(n-1)$ -dimensional Hausdorff measure zero.

Proof. Without loss of generality let $K \subset \text{int } B_n$ and for $\mathbf{x} \in \text{bd } K$ let $N_K(\mathbf{x}) = \text{pos}\{\mathbf{u} \in S^{n-1} : \mathbf{x} \in H(\mathbf{u}, h(K, \mathbf{u}))\}$ be the normal cone at \mathbf{x} which is generated by all unit vectors \mathbf{u} such that the supporting hyperplanes in direction \mathbf{u} contains \mathbf{x} . If \mathbf{x} is singular then it is at least 2-dimensional.

Now let M be the union of all $(n-2)$ -dimensional affine spaces generated by rational vectors. Let L be the $(n-2)$ -dimensional orthogonal complement of $\text{lin } N_K(\mathbf{x})$ and let $\mathbf{z} \in \text{relint}(\mathbf{x} + N_K(\mathbf{x}))$. Since $\mathbf{z} + L$ can be arbitrarily well approximated by rational planes from M we conclude that there exists a $\mathbf{y} \in M \cap B_n \cap (\mathbf{x} + N_K(\mathbf{x}))$. By the definition of the nearest point map Φ (see Definition 1.5⁵⁴) we have $\Phi(\mathbf{y}) = \mathbf{x}$. Since Φ is Lipschitz (see Lemma 1.10) the singular points of $\text{bd } K$ are covered by countable many compact sets of finite $(n-2)$ -dimensional Hausdorff measure. \square

7.5 Remark. The Gauss map $\eta_K(\cdot) : \text{bd}' K \rightarrow S^{n-1}$ assigning to each $\mathbf{x} \in \text{bd}' K$ its unique unit vector of the supporting hyperplane in \mathbf{x} is well-defined, and for each Borel set $B \subseteq S^{n-1}$ the set $\eta_K^{-1}(B)$ is a measurable set with respect to the $(n-1)$ -Hausdorff measure $\mathcal{H}_{n-1}(\cdot)$ on $\text{bd } K$.

7.6 Definition [Surface area measure]. Let $K \in \mathcal{K}^n$ with $\dim K = n$. Let $\mathcal{B}(S^{n-1})$ be the set of all Borel sets of S^{n-1} . The measure $S_{n-1}^{(1)}(K, \cdot) : \mathcal{B}(S^{n-1}) \rightarrow \mathbb{R}_{\geq 0}$ given by

$$S_{n-1}^{(1)}(K, B) = \mathcal{H}_{n-1}(\eta_K^{-1}(B)) = \int_{\eta_K^{-1}(B)} d\mathcal{H}_{n-1}(\mathbf{u})$$

is called (surface) area measure of K .

7.7 Remark.

- i) For a polytope P with outer normals \mathbf{u}_i and with facet areas α_i , $1 \leq i \leq m$, it is $S_{n-1}^{(1)}(P, \{\mathbf{u}_i\}) = \alpha_i$, and so

$$S_{n-1}^{(1)}(P, B) = \sum_{\mathbf{u}_i \in B} \alpha_i.$$

- ii) Let $K \in \mathcal{K}^n$. Then

$$\text{vol}(K) = \frac{1}{n} \int_{S^{n-1}} h(K, \mathbf{u}) dS_{n-1}^{(1)}(K, \mathbf{u}).$$

7.8 Theorem* [Alexandrov, Fenchel&Jessen, Minkowski]. Let σ be a Borel measure on S^{n-1} which is not concentrated in a great circle. There exists an n -dimensional convex body $K \subset \mathbb{R}^n$ with area measure σ if and only if

$$\int_{S^{n-1}} \mathbf{u} d\sigma(\mathbf{u}) = \mathbf{0}.$$

Moreover, such a K is uniquely determined up to translations.

⁵⁴Skript WS14

7.9 Definition [Wulff-shape]. Let $f : S^{n-1} \rightarrow \mathbb{R}_{>0}$ be continuous. The convex body

$$K_f = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{u}, \mathbf{x} \rangle \leq f(\mathbf{u}) \text{ for all } \mathbf{u} \in S^{n-1}\}$$

is called Wulff-shape with respect to f .

7.10 Lemma. Let $f : S^{n-1} \rightarrow \mathbb{R}_{>0}$ be continuous. Then $h(K_f, \eta_{K_f}(\mathbf{x})) = f(\eta_{K_f}(\mathbf{x}))$ for all $\mathbf{x} \in \text{bd}'K_f$ and thus

$$\text{vol}(K_f) = \frac{1}{n} \int_{S^{n-1}} f(\mathbf{u}) \, dS_{n-1}^{(1)}(K_f, \mathbf{u}).$$

7.11 Definition [p -sum, $p > 0$]. Let $K, L \in \mathcal{K}^n$, $\mathbf{0} \in K \cap L$ and $p > 0$.

i) For $\mathbf{x} \in \mathbb{R}^n$ let

$$f(\mathbf{x}) = [h(K, \mathbf{x})^p + h(L, \mathbf{x})^p]^{1/p},$$

The convex body $K +_p L$ given by the Wulff shape K_f is called the p -sum of K and L , i.e.,

$$K +_p L = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{u}, \mathbf{x} \rangle \leq [h(K, \mathbf{x})^p + h(L, \mathbf{x})^p]^{1/p} \text{ for all } \mathbf{u} \in S^{n-1}\}.$$

ii) Let $\lambda > 0$. $\lambda \cdot_p K = (\lambda \cdot K) +_p K$ is called the p -scalar multiplication.

7.12 Remark. Let $K, L \in \mathcal{K}^n$

i) Let $f(\mathbf{u}) = [h(K, \mathbf{u})^p + h(L, \mathbf{u})^p]^{1/p}$, $\mathbf{u} \in S^{n-1}$, and $p \geq 1$. Then $h(K_f, \mathbf{u}) = f(\mathbf{u})$ for all $\mathbf{u} \in S^{n-1}$.

ii) For $1 \leq p < \infty$ and $\lambda, \mu \geq 0$ we have

$$h(\lambda \cdot K +_p \mu \cdot L, \mathbf{u})^p = \lambda h(K, \mathbf{u})^p + \mu h(L, \mathbf{u})^p.$$

iii)

$$\lim_{\epsilon \searrow 0} \frac{\text{vol}(K +_p \epsilon \cdot B_n) - \text{vol}(K)}{\epsilon} = \frac{1}{n p} \int_{S^{n-1}} h(K, \mathbf{u})^{1-p} \, dS_{n-1}^{(1)}(K, \mathbf{u}).$$

7.13 Definition [L_p -surface area measure]. Let $K \in \mathcal{K}^n$ with $\dim K = n$, $p \geq 0$. Let $\mathcal{B}(S^{n-1})$ be the set of all Borel sets of S^{n-1} . The measure $S_{n-1}^{(p)}(K, \cdot) : \mathcal{B}(S^{n-1}) \rightarrow \mathbb{R}_{\geq 0}$ given by

$$S_{n-1}^{(p)}(K, B) = \int_B h(K, \mathbf{u})^{1-p} \, dS_{n-1}^{(1)}(K, \mathbf{u})$$

is called L_p -(surface) area measure of K .

7.14 Remark.

i) The problem to characterize the L_p -surface area measure of convex bodies among the finite Borel measures on S^{n-1} is called L_p -Minkowski problem. For $p = 1$ one gets the classical Minkowski problem (see Theorems 7.2, 7.8).

ii) For $p = 0$ the L_0 -Minkowski-Problem is also called logarithmic Minkowski problem . It corresponds to the $+_0$ -sum (or geometric sum)

$$\lambda \cdot K +_0 (1 - \lambda) \cdot L := \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{u}, \mathbf{x} \rangle \leq h(K, \mathbf{u})^\lambda h(L, \mathbf{u})^{1-\lambda}\},$$

for $\lambda \in [0, 1]$ and $K, L \in \mathcal{K}^n$, $\mathbf{0} \in K \cap L$.

iii) The measure

$$V_0(K) = \frac{1}{n} S_{n-1}^{(0)}(K, B) = \frac{1}{n} \int_B h(K, \mathbf{u}) \, dS_{n-1}^{(1)}(K, \mathbf{u})$$

is called cone-volume measure of K .

7.15 Definition [Cone data]. Let $P \in \mathcal{P}^n$, $\dim P = n$, with m facets F_1, \dots, F_m and $\mathbf{0} \in \text{int } P$. Let $\mathbf{u}_i \in S^{n-1}$ be the outer unit normal vector of the facet F_i , and let $\gamma_i = \text{vol}(\text{conv}\{\mathbf{0}, F_i\})$, $1 \leq i \leq m$. The data $\{(\mathbf{u}_i, \gamma_i) \in S^{n-1} \times \mathbb{R}_{>0}^n : 1 \leq i \leq m\}$ are called the cone data of P .

7.16 Definition [Subspace concentration condition]. The data $\{(\mathbf{u}_i, \gamma_i) \in S^{n-1} \times \mathbb{R}_{>0}^n : 1 \leq i \leq m\}$ satisfy the so called subspace concentration condition if for every subspace $L \subseteq \mathbb{R}^n$

$$\sum_{\mathbf{u}_i \in L} \gamma_i \leq \frac{\dim L}{n} \left(\sum_{i=1}^m \gamma_i \right), \quad (7.16.1)$$

and equality holds for a subspace L if and only if there exists a subspace \bar{L} , complementary to L , such that

$$A = (A \cap L) \cup (A \cap \bar{L}), \quad (7.16.2)$$

where $A = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$.

7.17 Proposition. Let $P \in \mathcal{P}^n$, $\dim P = n$, $\mathbf{0} \in \text{int } P$, with outer unit normals $A = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$. Then there exists complementary subspaces $L, \bar{L} \subset \mathbb{R}^n$ such that $A = (A \cap L) \cup (A \cap \bar{L})$ if and only if

$$P = (P \cap L^\perp) + (P \cap \bar{L}^\perp),$$

i.e., P is the Minkowski sum of two polytopes contained in complementary subspaces.

Proof. Let $P = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{u}_i, \mathbf{x} \rangle \leq b_i, 1 \leq i \leq m\}$ and let $\mathbf{u}_i \in L$, $1 \leq i \leq l$ and $\mathbf{u}_i \in \bar{L}$, $l+1 \leq i \leq m$. Then (observe $\mathbf{0} \in P$)

$$P \cap L^\perp = \{\mathbf{x} \in L^\perp : \langle \mathbf{u}_i, \mathbf{x} \rangle \leq b_i, l+1 \leq i \leq m\},$$

$$P \cap \bar{L}^\perp = \{\mathbf{x} \in \bar{L}^\perp : \langle \mathbf{u}_i, \mathbf{x} \rangle \leq b_i, 1 \leq i \leq l\}.$$

Now let $\mathbf{y} \in P$. Then it can uniquely be written as $\mathbf{y} = \mathbf{v} + \mathbf{w}$ with $\mathbf{v} \in L^\perp$ and $\mathbf{w} \in \bar{L}^\perp$, and

$$\langle \mathbf{u}_i, \mathbf{y} \rangle = \langle \mathbf{u}_i, \mathbf{w} \rangle, \quad 1 \leq i \leq l, \quad \langle \mathbf{u}_i, \mathbf{y} \rangle = \langle \mathbf{u}_i, \mathbf{v} \rangle, \quad l+1 \leq i \leq m. \quad (7.17.1)$$

Hence, $\mathbf{v} \in P \cap L^\perp$ and $\mathbf{w} \in P \cap \bar{L}^\perp$. On the other hand, given those \mathbf{v}, \mathbf{w} we also see by (7.17.1) that $\mathbf{v} + \mathbf{w} \in P$.

Now assume $P = (P \cap L^\perp) + (P \cap \bar{L}^\perp)$ and let $\dim L^\perp = k$. Since $P \cap L^\perp$ is a k -dimensional polytope and $(P \cap \bar{L}^\perp)$ is an $(n - k)$ -dimensional polytope, the facets of P are given by

$$F + (P \cap \bar{L}^\perp) \text{ or } \bar{F} + (P \cap L^\perp),$$

where F is a facet, i.e., a $(k-1)$ -face of $P \cap L^\perp$ and \bar{F} is a facet, i.e., a $(n-k-1)$ -face of $P \cap \bar{L}^\perp$. In the first case the outer unit normal of such a facet is contained in $\bar{L}^{\perp\perp} = \bar{L}$, and in the latter case in $L^{\perp\perp} = L$. Hence $A = (A \cap L) \cup (A \cap \bar{L})$. \square

7.18 Theorem. *The cone data of a centered polytope, i.e., with centroid at the origin, satisfy the subspace concentration condition.*

The proof is based on several lemmas which we prove first.

7.19 Proposition. *Let $f : C \rightarrow \mathbb{R}_{>0}$ be a positive log-concave function on an open convex set $C \subseteq \mathbb{R}^n$. For all $\mathbf{x}, \mathbf{y} \in C$ there exists a subgradient $g(\mathbf{y}) \in \mathbb{R}^n$ such that*

$$\ln f(\mathbf{x}) - \ln f(\mathbf{y}) \leq \langle g(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle.$$

If f is differentiable at \mathbf{y} , the subgradient is the gradient of $\ln f$ at \mathbf{y} , i.e., $g(\mathbf{y}) = \nabla \ln f(\mathbf{y}) = \frac{1}{f(\mathbf{y})} \nabla f(\mathbf{y})$.

Proof. By definition of concave functions. \square

7.20 Lemma. *Let $C \in \mathcal{K}_o^n$, and let $f : \text{int } C \rightarrow \mathbb{R}_{>0}$ be a log-concave function with $\int_C f(\mathbf{x}) \mathbf{x} \, d\mathbf{x} = \mathbf{0}$. Furthermore, assume that $\nabla f(\mathbf{x})$ exists almost everywhere on $\text{int } C$, and that also $\int_C \langle \mathbf{x}, \nabla f(\mathbf{x}) \rangle \, d\mathbf{x}$ exists. Then*

$$\int_C \langle \mathbf{x}, \nabla f(\mathbf{x}) \rangle \, d\mathbf{x} \leq 0,$$

with equality if and only if there exist $\mathbf{c} \in \mathbb{R}^n$, $\gamma \in \mathbb{R}_{>0}$ with $f(\mathbf{x}) = \gamma e^{\langle \mathbf{c}, \mathbf{x} \rangle}$.

Proof. By Proposition 7.19 we have for all $\mathbf{x}, \mathbf{y} \in \text{int } C$

$$\ln f(\mathbf{x}) - \ln f(\mathbf{y}) \leq \langle g(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle, \quad (7.20.1)$$

where $g(\mathbf{y})$ is a subgradient at \mathbf{y} . Interchanging the role of \mathbf{x} and \mathbf{y} and adding leads to

$$0 \leq \langle g(\mathbf{y}) - g(\mathbf{x}), \mathbf{x} - \mathbf{y} \rangle.$$

Setting $\mathbf{y} = \mathbf{0}$ leads to $\langle g(\mathbf{x}), \mathbf{x} \rangle \leq \langle g(\mathbf{0}), \mathbf{x} \rangle$. For points $\mathbf{x} \in C'$, where $C' \subseteq C$ is the set where $\nabla f(\mathbf{x})$ exists, this gives

$$\langle \nabla f(\mathbf{x}), \mathbf{x} \rangle \leq \langle g(\mathbf{0}), f(\mathbf{x}) \mathbf{x} \rangle.$$

Hence in view of our assumption on ∇f and on the first moment of f on C we get

$$\begin{aligned} \int_C \langle \mathbf{x}, \nabla f(\mathbf{x}) \rangle \, d\mathbf{x} &= \int_{C'} \langle \mathbf{x}, \nabla f(\mathbf{x}) \rangle \, d\mathbf{x} \\ &\leq \int_{C'} \langle g(\mathbf{0}), f(\mathbf{x}) \mathbf{x} \rangle \, d\mathbf{x} \\ &= \left\langle g(\mathbf{0}), \int_C f(\mathbf{x}) \mathbf{x} \, d\mathbf{x} \right\rangle = 0. \end{aligned}$$

If the inequality holds with equality, we must have almost everywhere equality in (7.20.1) for $\mathbf{y} = \mathbf{0}$. Hence, $\ln f(\mathbf{x})$ is an affine function. Hence there exist $\mathbf{c} \in \mathbb{R}^n$, $\gamma \in \mathbb{R}_{>0}$ with $f(\mathbf{x}) = \gamma e^{\langle \mathbf{c}, \mathbf{x} \rangle}$. On the other hand, if f is of this form then $\nabla f(\mathbf{x}) = f(\mathbf{x}) \mathbf{c}$ and so

$$\int_C \langle \mathbf{x}, \nabla f(\mathbf{x}) \rangle \, d\mathbf{x} = \left\langle \mathbf{c}, \int_C f(\mathbf{x}) \mathbf{x} \, d\mathbf{x} \right\rangle = 0.$$

□

7.21 Notation. For a convex body $K \in \mathcal{K}_o^n$ and a k -dimensional subspace L , $0 \leq k \leq n$, let

$$f_L : K|L \rightarrow \mathbb{R}_{\geq 0} \text{ with } \mathbf{x} \mapsto \text{vol}_{n-k}(K \cap (\mathbf{x} + L^\perp)). \quad (7.21.1)$$

7.22 Proposition. f_L is on the interior of $K|L$ a log-concave function, and

$$\int_{K|L} f_L(\mathbf{x}) \mathbf{x} \, d\mathbf{x} = \mathbf{0}.$$

Proof. Left as an exercise. □

7.23 Lemma. Let $P \in \mathcal{P}^n$, $\dim P = n$, $\mathbf{0} \in \text{int } P$. Let A be the set of its outer unit normals, and let $L \subset \mathbb{R}^n$ be a k -dimensional subspace, $0 < k < n$. Then $f_L : P|L \rightarrow \mathbb{R}_{\geq 0}$ is a constant function if and only if there exists a subspace \bar{L} , complementary to L , such that

$$A = (A \cap L) \cup (A \cap \bar{L}).$$

Proof. Suppose $f_L(\mathbf{x}) = f_L(\mathbf{0})$ for all $\mathbf{x} \in P|L$. Then, in particular,

$$f_L((1 - \lambda)\mathbf{x} + \lambda\mathbf{0})^{1/(n-k)} = (1 - \lambda)f_L(\mathbf{x})^{1/(n-k)} + \lambda f_L(\mathbf{0})^{1/(n-k)}$$

for all $\lambda \in [0, 1]$ and $\mathbf{x} \in P|L$. Hence we have equality in the Brunn-Minkowski inequality (Theorem 2.1⁵⁵) applied in the space L^\perp and thus, for every $\mathbf{x} \in P|L$ there exists a uniquely determined $\mathbf{t}(\mathbf{x}) \in \mathbb{R}^n$ such that

$$(\mathbf{x} + L^\perp) \cap P = \mathbf{t}(\mathbf{x}) + (L^\perp \cap P).$$

⁵⁵Skript WS14

Let $\mathbf{t} : P|L \rightarrow \mathbb{R}^n$ be the associated map sending $\mathbf{x} \mapsto \mathbf{t}(\mathbf{x})$. Then $\mathbf{t}(\cdot)$ is injective and convex linear, i.e., $\mathbf{t}((1 - \lambda)\mathbf{x} + \lambda\mathbf{y}) = (1 - \lambda)\mathbf{t}(\mathbf{x}) + \lambda\mathbf{t}(\mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in P|L$ and $\lambda \in [0, 1]$, since by convexity and definition of $\mathbf{t}(\cdot)$ we have

$$(1 - \lambda)\mathbf{t}(\mathbf{x}) + \lambda\mathbf{t}(\mathbf{y}) + (L^\perp \cap P) \subseteq ((1 - \lambda)\mathbf{x} + \lambda\mathbf{y}) + L^\perp \cap P.$$

Together with constant volume property we get equality here. Since $\mathbf{t}(\mathbf{0}) = \mathbf{0}$ we conclude that $\mathbf{t}(\cdot)$ is linear. Hence, $\tilde{L} = \text{lin } \mathbf{t}(P|L)$, i.e., the linear hull of $\mathbf{t}(P|L)$, is a k -dimensional linear subspace and we have

$$P = (P \cap \tilde{L}) + (P \cap L^\perp).$$

Observe, $\mathbf{t}(P|L) = P \cap \tilde{L}$ and in view of 7.17 we get $A = (A \cap L) \cup (A \cap \tilde{L}^\perp)$ with complementary subspaces \tilde{L}^\perp and L .

On the other hand, if we have $A = (A \cap L) \cup (A \cap \bar{L})$ we know by Proposition 7.17 that

$$P = (P \cap L^\perp) + (P \cap \bar{L}^\perp). \quad (7.23.1)$$

In particular, by the complementarity of the subspaces for every $\mathbf{x} \in P|L$ there exists a unique $\mathbf{t}(\mathbf{x}) \in P \cap \bar{L}^\perp$ with $\mathbf{t}(\mathbf{x})|L = \mathbf{x}$. Hence, $P \cap (\mathbf{x} + L^\perp) = \mathbf{t}(\mathbf{x}) + (P \cap L^\perp)$ for every $\mathbf{x} \in P|L$, which gives $f_L(\mathbf{x}) = f_L(\mathbf{0})$. \square

7.24 Proposition*. *Let L be a k -dimensional subspace, $0 < k < n$. The function $f_L : P|L \rightarrow \mathbb{R}_{\geq 0}$ is a piecewise polynomial function of degree at most $n - k$; more precisely, on every k -dimensional polytope of the subdivision $\mathcal{D}(P)_L$ it is a polynomial of degree at most $n - k$.*

7.25 Remark. *Let $P \in \mathcal{P}^n$ be an n -dimensional polytope and let L be a k -dimensional subspace, $0 \leq k \leq n - 1$. The union of all k -faces of P is called the k -skeleton of P . The orthogonal projection of the $(k - 1)$ -skeleton onto a k -dimensional plane L induces a polytopal subdivision $\mathcal{D}(P)_L$ of $P|L$, i.e., $\mathcal{D}(P)_L$ is a collection of k -dimensional polytopes having pairwise disjoint interiors, the intersection of any two of them is a face of both, the union covers $P|L$ and the preimage of the boundary of a polytope in $\mathcal{D}(P)_L$ is contained in a $(k - 1)$ -face of P .*

7.26 Notation. *For n linearly independent unit vectors $V = \{v_1, \dots, v_n\}$ and a k -subset $I \subset \{1, \dots, n\}$, $0 < k < n$, we denote by $L_I(V) = \text{lin } \{v_j : j \in I\}$ the k -dimensional subspace generated by this selection of vectors.*

7.27 Lemma*. *Let $P \in \mathcal{K}_o^n$ be a polytope and let $0 < k < n$. There exist n linearly independent unit vectors $V = \{v_1, \dots, v_n\}$ such that the function $f_{L_I(V)} : P|L_I(V) \rightarrow \mathbb{R}_{\geq 0}$ is constant for every k -subset $I \subset \{1, \dots, n\}$ if and only if P is a parallelotope.*

7.28 Notation. *Let $F_L : P|L \rightarrow L$ be the vector field given by*

$$F_L(\mathbf{x}) = f_L(\mathbf{x}) \mathbf{x}. \quad (7.28.1)$$

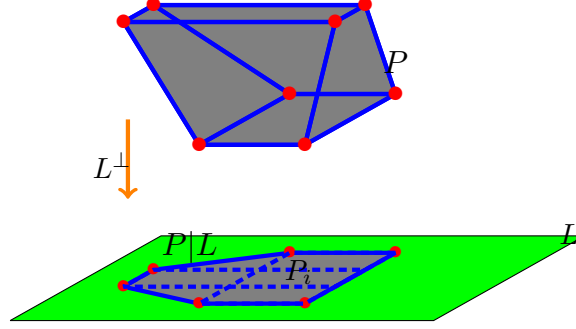


Figure 1: The polytopal subdivision induced by the orthogonal projection of the $(k - 1)$ -skeleton of P onto L

7.29 Lemma. Let P be a polytope, and let $L \subset \mathbb{R}^n$ be a k -dimensional linear subspace, $0 < k < n$. Let $\mathcal{D}(P)_L = \{P_1, \dots, P_r\}$ be the polytopal subdivision induced by the orthogonal projection of the $(k - 1)$ -skeleton S of P onto L . Let $F : P|L \rightarrow L$ be a vector field which is a polynomial in each component and on each polytope $P_i \in \mathcal{D}(P)_L$. Then ⁵⁶

$$\int_{(P|L) \setminus (S|L)} \operatorname{div} F(\mathbf{x}) \, d\mathbf{x} = \int_{\operatorname{bd}'(P|L)} \langle F(\mathbf{x}), \mathbf{u}(\mathbf{x}) \rangle \, d\mathcal{H}^{k-1}(\mathbf{x}).$$

Here $\mathbf{u}(\mathbf{x}) \in L$ is the unique outer normal of the polytope $P|L$ in the boundary point $\mathbf{x} \in \operatorname{bd}'(P|L)$.

Proof. Since F is a polynomial vector field on each P_i , F can canonically be extended to an open neighborhood of P_i , and hence we can use the divergence theorem of Gauss and get

$$\int_{P_i} \operatorname{div} F(\mathbf{x}) \, d\mathbf{x} = \int_{\operatorname{bd}'(P_i)} \langle F(\mathbf{x}), \mathbf{u}_i(\mathbf{x}) \rangle \, d\mathcal{H}^{k-1}(\mathbf{x}),$$

with $\mathbf{u}_i(\mathbf{x}) \in L$ being the unique outer normal of P_i in $\mathbf{x} \in \operatorname{bd}'(P_i)$. Thus, in particular, $\int_{P_i \setminus (S|L)} \operatorname{div} F(\mathbf{x}) \, d\mathbf{x}$ is well defined and since $S|L$ is a set of measure 0, we have

$$\int_{(P|L) \setminus (S|L)} \operatorname{div} F(\mathbf{x}) \, d\mathbf{x} = \sum_{i=1}^r \int_{\operatorname{bd}'(P_i)} \langle F(\mathbf{x}), \mathbf{u}_i(\mathbf{x}) \rangle \, d\mathcal{H}^{k-1}(\mathbf{x}).$$

Every $\mathbf{x} \in \operatorname{bd}'(P_i) \setminus \operatorname{bd}'(P|L)$ is contained in exactly one more P_j , $j \neq i$, and $\mathbf{u}_j(\mathbf{x}) = -\mathbf{u}_i(\mathbf{x})$. Hence, with $\mathbf{u}_i(\mathbf{x}) = \mathbf{u}(\mathbf{x})$ for $\mathbf{x} \in \operatorname{bd}'(P_i) \cap \operatorname{bd}'(P|L)$ we get

$$\sum_{i=1}^r \int_{\operatorname{bd}'(P_i)} \langle F(\mathbf{x}), \mathbf{u}_i(\mathbf{x}) \rangle \, d\mathcal{H}^{k-1}(\mathbf{x}) = \int_{\operatorname{bd}'(P|L)} \langle F(\mathbf{x}), \mathbf{u}(\mathbf{x}) \rangle \, d\mathcal{H}^{k-1}(\mathbf{x}),$$

which finishes the proof. \square

⁵⁶ $\operatorname{div} F(\mathbf{x}) = \sum_{i=1}^k \partial F_i(\mathbf{x}) / \partial x_i$

Proof. [Proof of Theorem 7.18] Let $P \in \mathcal{K}_o^n$ be a polytope with centroid at the origin. Let F_1, \dots, F_m be the facets of P , and let $\mathbf{u}_i \in S^{n-1}$ be the outer unit normal of the facet F_i , $1 \leq i \leq m$. Let $A = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$, and let $b_i > 0$ be the distance of the facet F_i from the origin. Let L be a k -dimensional subspace with $0 < k < n$. We have to show (cf. Definition 7.16)

$$\sum_{\mathbf{u}_i \in L} V_{n-1}(F_i) b_i \leq k V(P). \quad (7.29.1)$$

with equality if and only if there exists a subspace \bar{L} , complementary to L , so that $A = (A \cap L) \cup (A \cap \bar{L})$.

According to Proposition 7.24, the vector field $F_L(\mathbf{x}) = f_L(\mathbf{x}) \mathbf{x}$ satisfies the assumptions of Lemma 7.29 and on account of

$$\operatorname{div} F_L(\mathbf{x}) = k f_L(\mathbf{x}) + \langle \nabla f_L(\mathbf{x}), \mathbf{x} \rangle$$

we get

$$\begin{aligned} & \int_{\operatorname{bd}'(P|L)} f_L(\mathbf{x}) \langle \mathbf{x}, \mathbf{u}(\mathbf{x}) \rangle \, d\mathcal{H}^{k-1}(\mathbf{x}) \\ &= k \int_{(P|L) \setminus (S|L)} f_L(\mathbf{x}) \, d\mathbf{x} + \int_{(P|L) \setminus (S|L)} \langle \nabla f_L(\mathbf{x}), \mathbf{x} \rangle \, d\mathbf{x} \quad (7.29.2) \\ &= k V(P) + \int_{(P|L) \setminus (S|L)} \langle \nabla f_L(\mathbf{x}), \mathbf{x} \rangle \, d\mathbf{x}, \end{aligned}$$

where in the last step we used again that $S|L$ is a set of measure 0.

Now let $\tilde{\mathbf{u}}_1, \dots, \tilde{\mathbf{u}}_l$ be the outer unit normals of the facets of $P|L$, i.e., the $(k-1)$ -faces of $P|L$, having distance \tilde{b}_i to the origin. Let $\tilde{F}_i = P \cap \{\mathbf{x} \in \mathbb{R}^n : \langle \tilde{\mathbf{u}}_i, \mathbf{x} \rangle = \tilde{b}_i\}$, $1 \leq i \leq l$, be the faces of P projected onto the facets of $P|L$. Taking into account that f_L measures the $(n-k)$ -dimensional volume we have

$$\int_{\operatorname{bd}'(P|L)} f_L(\mathbf{x}) \langle \mathbf{x}, \mathbf{u}(\mathbf{x}) \rangle \, d\mathcal{H}^{k-1}(\mathbf{x}) = \sum_{i=1}^l V_{n-1}(\tilde{F}_i) \tilde{b}_i.$$

Hence \tilde{F}_i contributes to the above sum only when it is a facet of P , i.e., $F_j = \tilde{F}_i$, $\mathbf{u}_j = \tilde{\mathbf{u}}_i \in L$ and $b_j = \tilde{b}_i$ for a certain $j \in \{1, \dots, m\}$. Thus we may write (cf. (7.29.2))

$$\sum_{\mathbf{u}_i \in L} V_{n-1}(F_i) b_i = k V(P) + \int_{(P|L) \setminus (S|L)} \langle \nabla f_L(\mathbf{x}), \mathbf{x} \rangle \, d\mathbf{x}. \quad (7.29.3)$$

Since 0 is the centroid of P we have (cf. Remark 7.22)

$$\int_{P|L} f_L(\mathbf{x}) \mathbf{x} \, d\mathbf{x} = \mathbf{0},$$

and since $S|L$ is a set of measure 0 we may apply Lemma 7.20 to f_L . Thus

$$\int_{(P|L) \setminus (S|L)} \langle \nabla f_L(\mathbf{x}), \mathbf{x} \rangle \, d\mathbf{x} \leq 0, \quad (7.29.4)$$

which yields (7.29.1) by (7.29.3).

Now suppose we have equality in (7.29.1). Then we also have equality in (7.29.4) and by Lemma 7.20 there exist $\gamma > 0$, $\mathbf{c} \in \mathbb{R}^n$ such that $f_L(\mathbf{x}) = \gamma e^{\langle \mathbf{c}, \mathbf{x} \rangle}$. Since the $(n - k)$ -th root of $f_L(\mathbf{x})$ is concave we must have $\mathbf{c} = \mathbf{0}$, i.e., $f_L(\mathbf{x})$ is a constant function. Thus, by Lemma 7.23, there exists a complementary subspace \bar{L} with $A = (A \cap L) \cup (A \cap \bar{L})$.

On the other hand, if we have such a partition of A into complementary subspaces L and \bar{L} , $\dim L = k$ and $\dim \bar{L} = n - k$, then we may either apply Lemma 7.23 and then (7.29.3), or we just observe that in this case we may write

$$\begin{aligned} kV(P) + (n - k)V(P) &= nV(P) \\ &= \sum_{\mathbf{u}_i \in A \cap L} V_{n-1}(F_i) b_i + \sum_{\mathbf{u}_i \in A \cap \bar{L}} V_{n-1}(F_i) b_i. \end{aligned}$$

Hence, in view of the validity of the inequality (7.29.1) for L and \bar{L} , we have actually equality in (7.29.1) for L and \bar{L} . \square

7.30 Definition [Cone volume measure]. Let $K \in \mathcal{K}^n$ with $\dim K = n$. The measure $S_0(K, \cdot) : \mathcal{B}(S^{n-1}) \rightarrow \mathbb{R}_{\geq 0}$ given by

$$S_0(K, B) = \int_{\eta_K^{-1}(B)} \langle \mathbf{u}, \eta_K(\mathbf{u}) \rangle d\mathcal{H}_{n-1}(\mathbf{u})$$

is called the cone volume measure of K .

7.31 Remark. If P is a polytope with cone data $\{(\mathbf{u}_i, \gamma_i) : 1 \leq i \leq m\}$ then

$$S_0(P, B) = n \sum_{\mathbf{u}_i \in B} \gamma_i.$$

7.32 Definition [Subspace concentration condition for measures]. A finite Borel measure μ on S^{n-1} is said to satisfy the subspace concentration condition if for every subspace L

$$\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n} \mu(S^{n-1})$$

and equality holds for a subspace L if and only if there exists a subspace \bar{L} , complementary to L , such that

$$\mu(S^{n-1}) = \mu(L \cap S^{n-1}) + \mu(\bar{L} \cap S^{n-1}).$$

7.33 Theorem* [Börözky, K.J., Lutwak, Yang, Zhang, 2013]. An even finite Borel measure is the cone volume measure of an o -symmetric convex body if and only if it satisfies the subspace concentration condition.

7.34 Theorem*. The cone-volume measure of a centered convex body satisfies the subspace concentration condition.

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