Dedekind sums

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Outline

Dedekind sums - Definition

Let a, b relatively prime integers, $b > 0$. Define the *Dedekind sums*

$$s(a,b) = \sum_{k=1}^{b-1} \left(\left(\frac{ka}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right),$$

where $((x))$ denotes the saw-tooth function.

Dedekind sum - Properties

$$s(a, b) = \sum_{k=1}^{b-1} \left(\left(\frac{ka}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right)$$

- periodic in a with period b

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our first goal: $s(a, b) + s(b, a) = \textit{some simple function}$

Fourier-Dedekind sums - Definition

Let $a_1, a_2, \ldots, a_d$ pairwise relatively prime, positive integers. Let n some integer. Define the *Fourier-Dedekind sums*

$$s_n(a_1, \ldots, a_d; b) = \frac{1}{b} \sum_{k=1}^{b-1} \frac{\xi_b^{nk}}{\prod_{i=1}^{d}(1 - \xi_b^{ka_i})},$$

where $\xi_b = e^{2\pi i/b}$ denotes the b-th root of unity.

Fourier-Dedekind sums - Properties

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- commutative in a_i arguments

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  $\rightarrow$ write $s_n(A; b)$ with $A = \{a_1, a_2 \ldots a_d\}$
- periodic in each a_i with period b
- $s_0(a, 1; b) = -s(a, b) + \frac{b-1}{4b}$

Restricted partition function

Let $A = \{a_1, a_2 \ldots a_d\} \subset \mathbb{N}^d$, where the a_i are pairwise relatively prime. For a positive integer n, the *restricted partition function* is defined to be

$$p_A(n) = \#\{(k_1 \ldots k_d) \in \mathbb{Z}^d_{\geq 0} | \sum_{i=1}^{d} k_i a_i = n\}.$$

Polynomial part of $p_A(n)$

$$p_A(n) = \sum_{i=1}^{d} (-1)^i B_i + \sum_{i=1}^{d} s_{-n}(\hat{A}_i; a_i)$$

with B_i coefficients at $z = 1$ of the partial fraction expansion of the shifted generating function of $p_A(n)$. B_i are polynomials in n. Here $\hat{A}_i = A \setminus \{a_i\}$.

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$$poly_A(n) = \sum_{i=1}^{d} (-1)^i B_i$$

the polynomial part of the restricted partition function.

Zagier reciprocity

Theorem 8.4 (Zagier reciprocity)
For $A = \{a_1, a_2 \ldots a_d\} \subset \mathbb{N}^d$, with the a_i are pairwise relatively prime,

$$\sum_{i=1}^{d} s_0(\hat{A}_i; a_i) = 1 - poly_A(0),$$

where $\hat{A}_i = A \setminus \{a_i\}$.

Dedekind's reciprocity law

Corollary 8.5 (Dedekind's reciprocity law)
For all a, b relatively prime, positive integers we have

$$s(a,b) + s(b,a) = \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right) - \frac{1}{4}.$$

Equalities for Dedekind sums

- periodicity

$$s(a,b) = s(a \mod b, b)$$

- Dedekind's reciprocity law

$$s(a,b) + s(b,a) = \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right) - \frac{1}{4}.$$

- special values

$$s(1,k) = -\frac{1}{4} + \frac{1}{6k} + \frac{k}{12}$$

Mordell-Pommersheim tetrahedron

For positive integers a, b, c call

$$P = \left\{ (x, y, z) \in \mathbb{R}^3_{\geq 0} | \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1 \right\}$$

the *Mordell-Pommersheim tetrahedron*.

Mordell-Pommersheim tetrahedron

For positive integers a, b, c call

$$tP = \left\{(x, y, z) \in \mathbb{R}^3_{\geq 0} | \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq t\right\}$$

the *t-th dilate of the Mordell-Pommersheim tetrahedron*.

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the *t-th dilate of the Mordell-Pommersheim tetrahedron*. Its Lattice-point enumerator is

$$L_P(t) = \#\left\{(k, l, m) \in \mathbb{Z}^3_{\geq 0} \middle| \frac{k}{a} + \frac{l}{b} + \frac{m}{c} \leq t\right\}$$

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Lattice-point enumerator of M-P tetrahedron

$$L_P(t) = p_{\{bc,ac,ab,1\}}(abct)$$

Lattice-point enumerator of M-P tetrahedron

$$\begin{aligned} L_P(t) &= p_{\{bc,ac,ab,1\}}(abct) \\ &= const\left(\frac{1}{1-z^{bc}}\frac{1}{1-z^{ac}}\frac{1}{1-z^{ab}}\frac{1}{1-z}\frac{1}{z^{abct}}\right) \end{aligned}$$

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&= const\left(\frac{1}{1-z^{bc}}\frac{1}{1-z^{ac}}\frac{1}{1-z^{ab}}\frac{1}{1-z}\frac{1}{z^{abct}}\right) \\
&= const\left(\frac{z^{-abct}-1}{(1-z^{bc})(1-z^{ac})(1-z^{ab})(1-z)}\right)+1
\end{aligned}
$$

Partial fraction expansion

Theorem 1.1 (Partial fraction expansion)
Given a rational function

$$f(z) = \frac{p(z)}{q(z)} = \frac{p(z)}{\prod_{k=1}^{m}(z-a_k)^{\alpha_k}}$$

with $\deg q = \sum_{k=1}^{m} \alpha_k \geq \deg p$, there exist unique coefficients $c_{k,i} \in \mathbb{C}$ with

$$f(z) = \sum_{k=1}^{m}\left(\frac{c_{k,1}}{z-a_k} + \frac{c_{k,2}}{(z-a_k)^2} + \cdots \frac{c_{k,\alpha_k}}{(z-a_k)^{\alpha_k}}\right).$$

Assuming a, b, c pairwise relatively prime, the coefficient at ξ_a^k is

$$-\frac{t}{a(1-\xi_a^{kbc})(1-\xi_a^k)}.$$

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Summing over all powers of a-th roots of unity yields

$$-\frac{t}{a}\sum_{k=1}^{a-1}\frac{1}{(1-\xi_a^{kbc})(1-\xi_a^k)}$$

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$$-\frac{t}{a}\sum_{k=1}^{a-1}\frac{1}{(1-\xi_a^{kbc})(1-\xi_a^k)} = -ts_0(bc, 1; a)$$

Lattice-point enumerator of M-P tetrahedron

$$\begin{aligned} L_P(t) &= p_{\{bc,ac,ab,1\}}(abct) \\ &= const\left(\frac{1}{1-z^{bc}}\frac{1}{1-z^{ac}}\frac{1}{1-z^{ab}}\frac{1}{1-z}\frac{1}{z^{abct}}\right) \\ &= const\left(\frac{z^{-abct}-1}{(1-z^{bc})(1-z^{ac})(1-z^{ab})(1-z)}\right)+1 \end{aligned}$$

Back to lattice-point enumeration

$$
\begin{aligned}
L_P(t) = &\frac{abc}{6} t^3 + \frac{ab + ac + bc + 1}{4} t^2 \\
&+ \frac{1}{4}\left(a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{3}\left(\frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c}\right)\right) t \\
&+ \left(s_0(bc, 1; a) + s_0(ac, 1; b) + s_0(ab, 1, c)\right) t \\
&+ 1
\end{aligned}
$$

Ehrhart quasipolynomial of M-P tetrahedron

Theorem 8.11
Let P be the Mordell-Pommersheim tetrahedron with parameters a, b, c pairwise relatively prime. Then its Ehrhart quasipolynomial is given by the formula

$$\begin{aligned} L_P(t) =& \frac{abc}{6}t^3 + \frac{ab+ac+bc+1}{4}t^2 \\ &+ \frac{1}{4}\left(3+a+b+c+\frac{1}{3}\left(\frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c}+\frac{1}{abc}\right)\right)t \\ &- (s(bc,a)+s(ac,b)+s(ab,c))t \\ &+ 1 \end{aligned}$$

Ehrhart series of M-P tetrahedron

Corollary 8.12

Let P be the Mordell-Pommersheim tetrahedron with parameters a, b, c pairwise relatively prime. Then its Ehrhart series is given by

$$Ehr_P(z) = \frac{h_3^* z^3 + h_2^* z^2 + h_1^* z + 1}{(1-z)^4},$$

where

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where

$$\begin{aligned} h_3^* =& \frac{abc}{6} - \frac{ab+ac+bc+a+b+c}{4} - \frac{1}{2} + \frac{1}{12}\left(\frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c}\right) \\ & - \left(s(bc,a) + s(ca,b) + s(ab,c)\right) \end{aligned}$$

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$$h_3^* = \frac{abc}{6} - \frac{ab+ac+bc+a+b+c}{4} - \frac{1}{2} + \frac{1}{12}\left(\frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c}\right) - \left(s(bc,a) + s(ca,b) + s(ab,c)\right)$$

and h_2^* and h_1^* are given by similar formulas as h_3^*.

Summary

- Dedekind sums $s(a, b)$
- Fourier-Dedekind sums $s_n(a_1, \dots a_d; b)$
- Zagier reciprocity and Dedekind's reciprocity law
- how to compute Dedekind sums
- explicit formulas for Mordell-Pommersheim tetrahedron
- geometric interpretation of coefficients of Ehrhart quasipolynomial

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Extra - Dedekind-Rademacher sums

$$r_n(a, b) = \sum_{k=1}^{b-1} \left(\left(\frac{ka+n}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right)$$

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$$r_n(a,b) = \sum_{k=1}^{b-1} \left(\left(\frac{ka+n}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right)$$

Of course, $r_0(a,b) = s(a,b)$.