Face Numbers and the Dehn-Sommerville Relations in Ehrhartian Terms

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Content

Lattice

- Lattice consists of a partially ordered set in which every two elements have a unique supremum (also called a least upper bound or join) and a unique infimum (also called a greatest lower bound or meet).

Face lattice

- There is a natural structure on the faces of a polytope $\mathscr{P}$ induced by the containment relation $\mathscr{F} \subset \mathscr{G}$. This relation gives a partial ordering on the set of all faces of $\mathscr{P}$, called the face lattice of $\mathscr{P}$.

Boolean lattice

- Boolean lattice is the partially ordered set formed by all subsets of a finite set, where the partial ordering is again subset containment.

Simple d-polytope

- We call the d-polytope $\mathscr{P}$ simple, if each vertex of $\mathscr{P}$ lies on precisely d edges of $\mathscr{P}$.

Property

Consider an interval $[\mathscr{F}, \mathscr{P}]$ in the face lattice of $\mathscr{P}$; namely, $[\mathscr{F}, \mathscr{P}]$ contains all faces between $\mathscr{F}$ and $\mathscr{P}$. If $\mathscr{P}$ is simple, then every such interval $[\mathscr{F}, \mathscr{P}]$ is isomorphic to a Boolean lattice.

Simple d-polytope

Property

Let $\mathscr{P}$ be a simple d-polytope. The number of k-faces of $\mathscr{P}$ containing a given j-face of $\mathscr{P}$ equals $\binom{d-j}{d-k}$.

Euler relation

- We denote the number of k-dimensional faces of $\mathscr{P}$ by the symbol f_k. As k varies from 0 to d, the face numbers f_k encode intrinsic information about the polytope $\mathscr{P}$.

Theorem

If $\mathscr{P}$ is a convex d-polytope, then

$$\sum_{j=0}^{d} (-1)^j f_j = 1$$

.

Dehn-Sommerville relations

Theorem

If $\mathscr{P}$ is a simple d-polytope and $0 \leq k \leq d$, then

$$f_k = \sum_{j=0}^{k} (-1)^j \binom{d-j}{d-k} f_j$$

.

Dehn-Sommerville relations

- Proof: For each k-face X_i, denote the number of j-face of X_i as $x_{i,j}$.
- Use Euler relation on X_i, we have $\sum_{j=0}^{k}(-1)^j x_{i,j} = 1$.
- Take the sum for all k-faces

$$\sum_{i}\sum_{j=0}^{k}(-1)^j x_{i,j} = f_k$$

.

- Each j-face is counted $\binom{d-j}{d-k}$ times, therefore

$$\sum_{j=0}^{k}(-1)^j \binom{d-j}{d-k} f_j = f_k$$

.

Euler relation

Theorem

If $\mathscr{P}$ is a convex d-polytope, then

$$\sum_{j=0}^{d} (-1)^j f_j = 1$$

.

- $$\mathscr{P} = \bigcup_{\mathscr{F} \subset \mathscr{P}} int(\mathscr{F})$$
  .
- Count the interger points in $t\mathscr{P}$:

$$L_{\mathscr{P}}(t) = \sum_{\mathscr{F} \subset \mathscr{P}} L_{int(\mathscr{F})}(t) = \sum_{\mathscr{F} \subset \mathscr{P}} (-1)^{dim \mathscr{F}} L_{\mathscr{F}}(-t)$$

Dehn-Sommerville relations

- Define

$$F_k(t) = \sum_{\substack{\mathscr{F} \subset \mathscr{P} \\ dim \mathscr{F} = k}} L_{\mathscr{F}}(t)$$

the sum being taken over all k-faces of $\mathscr{P}$.

Theorem

If $\mathscr{P}$ is a simple rational d-polytope and $0 \le k \le d$, then

$$F_k(t) = \sum_{j=0}^{k} (-1)^j \binom{d-j}{d-k} F_j(-t)$$

.

Dehn-Sommerville relations

- $$L_{\mathscr{F}}(t) = \sum_{\mathscr{G} \subset \mathscr{F}} L_{int(\mathscr{G})}(t)$$
- $$= \sum_{\mathscr{G} \subset \mathscr{F}} (-1)^{dim\mathscr{G}} L_{\mathscr{G}}(-t) = \sum_{j=0}^{k} (-1)^j \sum_{\substack{\mathscr{G} \subset \mathscr{F} \\ dim\mathscr{G}=j}} L_{\mathscr{G}}(-t)$$
- $$F_k(t) = \sum_{\substack{\mathscr{F} \subset \mathscr{P} \\ dim\mathscr{F}=k}} L_{\mathscr{F}}(t)$$
- $$= \sum_{\substack{\mathscr{F} \subset \mathscr{P} \\ dim\mathscr{F}=k}} \sum_{j=0}^{k} (-1)^j \sum_{\substack{\mathscr{G} \subset \mathscr{F} \\ dim\mathscr{G}=j}} L_{\mathscr{G}}(-t)$$

Dehn-Sommerville relations

- $$= \sum_{j=0}^{k} (-1)^j \sum_{\substack{\mathscr{F} \subset \mathscr{P} \\ dim \mathscr{F} = k}} \sum_{\substack{\mathscr{G} \subset \mathscr{F} \\ dim \mathscr{G} = j}} L_{\mathscr{G}}(-t)$$
- $$= \sum_{j=0}^{k} (-1)^j \sum_{\substack{\mathscr{G} \subset \mathscr{F} \\ dim \mathscr{G} = j}} L_{\mathscr{G}}(-t) \binom{d-j}{d-k}$$
- $$= \sum_{j=0}^{k} (-1)^j \binom{d-j}{d-k} F_j(-t)$$

Coefficients of an Ehrhart Polynomial

- $$L_{\mathscr{P}}(t) = F_d(t) = \sum_{j=0}^{d} (-1)^j F_j(-t)$$
- $$L_{\mathscr{P}}(t) - L_{int(\mathscr{P})}(t) = \sum_{j=0}^{d-1} (-1)^j F_j(-t)$$
- Write $L_{\mathscr{P}}(t) = c_d t^d + c_{d-1} t^{d-1} + \ldots + c_0$.
- $$L_{\mathscr{P}}(t) - L_{int(\mathscr{P})}(t) = 2c_{d-1} t^{d-1} + 2c_{d-3} t^{d-3} + \ldots$$

Coefficients of an Ehrhart Polynomial

Theorem

Suppose $L_{\mathscr{P}}(t) = c_d t^d + c_{d-1} t^{d-1} + \ldots + c_0$ *is the Ehrhart polynomial of* $\mathscr{P}$. *Then*

$$c_{d-1} t^{d-1} + c_{d-3} t^{d-3} + \ldots = \frac{1}{2} \sum_{j=0}^{d-1} (-1)^j F_j(-t)$$

Coefficient of an Ehrhart Polynomial

- Write $F_j(t) = c_{j,j}t^j + c_{j,j-1}t^{j-1} + \ldots + c_{j,0}$, then

Property

If k and d are of different parties, then

$$c_k = \frac{1}{2}\sum_{j=0}^{d-1}(-1)^{j+k}c_{j,k}$$

- c_{d-1} equals $1/2$ times the sum of the leading coefficients of the Ehrhart polynomials of the facets of $\mathscr{P}$.

Relative volume

- Recall: if $S \subset \mathbb{R}^d$ is d-dimensional, then $vol(S) = \lim_{t\to\infty} \frac{1}{t^d}\#(tS \cap \mathbb{Z})$.
- Say, $S \subset \mathbb{R}^d$ is of dimension $m < d$, and let $span(S) = \{\boldsymbol{x} + \lambda(\boldsymbol{y} - \boldsymbol{x}) : \boldsymbol{x}, \boldsymbol{y} \in S, \lambda \in \mathbb{R}\}$, the affine span of S.
- We count the volume relative to the sublattice $(spanS) \cap \mathbb{Z}^d$; we call it the relative volume of S.

Relative volume

- Suppose $\mathscr{P} \subset \mathbb{R}^d$ is an integral m-polytope with Ehrhart polynomial

$$L_{\mathscr{P}}(t) = c_m t^m + c_{m-1} t^{m-1} + \ldots + c_1 t + 1$$

- Similarly,

$$vol(\mathscr{P}) = \lim_{t\to\infty} \frac{1}{t^m} L_{\mathscr{P}}(t) = c_m$$

Theorem

Suppose $L_{\mathscr{P}}(t) = c_m t^m + c_{m-1} t^{m-1} + \ldots + c_1 t + 1$ is the Ehrhart polynomial of the integral polytope $\mathscr{P}$. Then

$$c_{d-1} = \frac{1}{2} \sum_{\mathscr{F} \text{ facet}} vol(\mathscr{F})$$