A Gallery of Discrete Volumes

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Seminar on
Discrete Convex Geometry

Overview

Polytopes

- A polytope is the convex hull of finitely many points.
- A polyhedron is the solution set of a finite system of linear inequalities.

Theorem (Minkowski,Weyl)

A polytope is a bounded polyhedron and vice versa.

Polytopes and Lattice Points

- A polytope $P \subset \mathbb{R}^n$ is called integral or lattice polytope if every vertex of P lies in $\mathbb{Z}^n$.
- For $t \in \mathbb{N}$, $L_P(t) = \#tP \cap \mathbb{Z}^n$ is the lattice-point enumerator.
- The generating function of $L_P(t)$ is called the Ehrhart series of P and is denoted by $\mathrm{Ehr}_P(z) = 1 + \sum_{t=1}^{\infty} L_P(t) z^t$.

Formal Power Series

- A formal power series is a sequence $(a_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$.
- Write $f = \sum_{n=0}^{\infty} a_n z^n$ with indeterminate z.
- Define $+$ and $\cdot$ according to sum representation. The set of formal power series $\mathbb{C}[[z]]$ with $+$ and $\cdot$ is an integral domain.
- For $f^{-1} \in \mathbb{C}((z)) \setminus \{0\}$ we have $f^{-1} \in \mathbb{C}[[z]]$ iff $a_0 \neq 0$, e.g.

$$(1 - z)^{-1} = \sum_{n=0}^{\infty} z^n.$$

- f is called polynomial if (a_n) is finite.
- f is called rational function if it is the quotient of two polynomials.

The Binomial Coefficient

- For $\alpha \in \mathbb{Z}$, $k \in \mathbb{N}_0$ define $\binom{\alpha}{k} = \prod_{j=0}^{k-1} \frac{\alpha - j}{k - j}$.
- Useful identities:

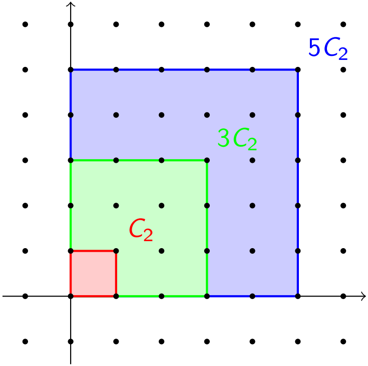
$$\binom{k+\alpha}{k} = \sum_{j=0}^{\alpha} \binom{k-1+j}{k-1}, \qquad \text{if } \alpha \geq 0, k \geq 1, \tag{B1}$$

$$(-1)^k \binom{\alpha}{k} = \binom{k-1-\alpha}{k}, \tag{B2}$$

$$(1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k. \tag{B3}$$

The Unit Cube

- $C_d = [0,1]^d = \{(x_1, \ldots, x_d) \in \mathbb{R}^n : 0 \leq x_i \leq 1\}$.



The Unit Cube

Proposition

Let $C_d = [0,1]^d$ be the unit cube and $t \in \mathbb{N}$.

- $L_{C_d}(t) = (t+1)^d$.
- $(-1)^d L_{C_d}(-t) = (t-1)^d = L_{C_d^\circ}(t)$.
- *The Ehrhart series of C_d is given by*

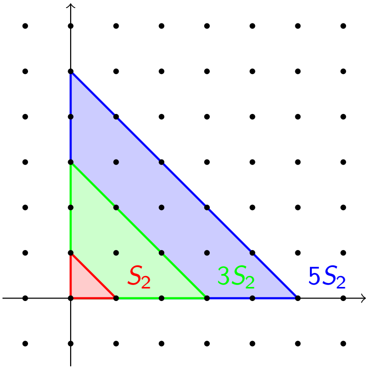
$$\mathrm{Ehr}_{C_d}(z) = \frac{\sum_{k=1}^{d} A(d,k) z^{k-1}}{(1-z)^{d+1}},$$

where $A(d,k) = \sum_{j=0}^{k} (-1)^j \binom{d+1}{j} (k-j)^d$ are the Eulerian numbers. In particular $\mathrm{Ehr}_{C_d}(z)$ is a rational function.

The Standard Simplex

- $S_d = \text{conv}\{0, e_1, \dots, e_d\}$

$$= \left\{ (x_1, \dots, x_d) \in \mathbb{R}^d : x_i \geq 0, \sum_{i=1}^{d} x_i \leq 1 \right\}.$$



The Standard Simplex

Proposition

Let $S_d = \operatorname{conv}\{0, e_1, \ldots, e_d\}$ *be the standard simplex and* $t \in \mathbb{N}$.

- $L_{S_d}(t) = \binom{d+t}{d}$.
- $(-1)^d L_{S_d}(-t) = (-1)^d \binom{d-t}{d} = L_{S_d^\circ}(t)$.
- *The Ehrhart series of* S_d *is given by*

$$\operatorname{Ehr}_{S_d}(z) = \frac{1}{(1-z)^{d+1}}.$$

In particular $\operatorname{Ehr}_{S_d}(z)$ *is a rational function.*

Pyramids

- Let $Q \subset \mathbb{R}^{d-1}$ be an integral polytope. Then $P = \text{conv}((Q \times \{0\}) \cup \{e_d\})$ is called the pyramid over Q.

Proposition

Let P be the pyramid over $Q \subset \mathbb{R}^{d-1}$ and $t \in \mathbb{N}$.

- $L_P(t) = 1 + \sum_{k=1}^{t} L_Q(k)$.
- *The Ehrhart series of P is given by*

$$\text{Ehr}_P(z) = \frac{\text{Ehr}_Q(z)}{1 - z}.$$

Bipyramids

- Let $Q \subset \mathbb{R}^{d-1}$ be an integral polytope, $0 \in Q$. Then $P = \operatorname{conv}((Q \times \{0\}) \cup \{\pm e_d\})$ is called the bipyramid over Q.

Proposition

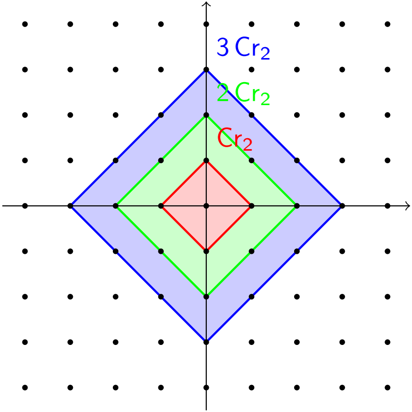
Let P be the bipyramid over $Q \subset \mathbb{R}^{d-1}$ and $t \in \mathbb{N}$.

- $L_P(t) = 2 + 2\sum_{k=1}^{t-1} L_Q(k) + L_Q(t)$.
- *The Ehrhart series of P is given by*

$$\mathrm{Ehr}_P(z) = \frac{1+z}{1-z}\,\mathrm{Ehr}_Q(z).$$

The Cross-Polytope

- $\mathrm{Cr}_d = \mathrm{conv}\{\pm e_1, \ldots, \pm e_d\} = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d : \sum_{i=1}^{d} |x_i| \leq 1 \right\}.$



The Cross-Polytope

Proposition

Let $\mathrm{Cr}_d = \mathrm{conv}\{\pm e_1, \ldots, \pm e_d\}$ *be the cross-polytope and* $t \in \mathbb{N}$.

- $L_{\mathrm{Cr}_d}(t) = \sum_{k=0}^{d} \binom{d}{k}\binom{d-k+t}{d}$.
- $(-1)^d L_{\mathrm{Cr}_d}(-t) = L_{\mathrm{Cr}_d^\circ}(t)$.
- *The Ehrhart series of* Cr_d *is given by*

$$\mathrm{Ehr}_{\mathrm{Cr}_d}(z) = \frac{(1+z)^d}{(1-z)^{d+1}}.$$

In particular $\mathrm{Ehr}_{\mathrm{Cr}_d}(z)$ *is a rational function.*

Pick's Theorem

Theorem (Pick, 1899)

Let $P \subset \mathbb{R}^2$ be a 2-dimensional, integral, convex polygon, $A(P) = V_2(P)$ is the area of P, $I(P) = \# \operatorname{int} P \cap \mathbb{Z}^2$ the number of lattice points in the interior of P, $B(P) = \# \operatorname{bd} P \cap \mathbb{Z}^2$ the number of lattice points in the boundary of P. Then

$$A(P) = I(P) + \frac{1}{2}B(P) - 1.$$

Pick's Theorem

Theorem (Pick, 1899)

$$A(P) = I(P) + \frac{1}{2}B(P) - 1.$$

Corollary (Ehrhart polynomials in dimension 2)

- $L_P(t) = A(P)t^2 + \frac{1}{2}B(P)t + 1.$
- $L_P(-t) = L_{P^\circ}(t).$
- *The Ehrhart series of P is given by*

$$\mathrm{Ehr}_P(z) = \frac{h_0^\star + h_1^\star z + h_2^\star z^2}{(1-z)^3}$$

for some numbers $h_i^\star = h_i^\star(P)$. *In particular* $\mathrm{Ehr}_P(z)$ *is a rational function.*

Rational Polygons

- A polygon $P \subset \mathbb{R}^2$ is called rational if there is an integral polygon $Q \subset \mathbb{R}^2$ and a number $d \in \mathbb{N}$ such that $dP = Q$.
- A quasipolynomial q is an expression of the form

$$q(t) = c_n(t)t^n + c_{n-1}(t)t^{n-1} + \ldots + c_0(t),$$

where $c_i(\cdot)$ is a periodic function on $\mathbb{N}$, $0 \leq i \leq n$.

Theorem

Let $P \subset \mathbb{R}^2$ be a rational polygon. Then is $L_P(t)$ is a quasipolynomial of degree 2.