Reciprocity Theorems and a Characterization for Reflexive Polytopes

Matthias Möser

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Chapters

Notation

Let $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d$, $\mathbf{m} := (m_1, \ldots, m_d) \in \mathbb{Z}^d$ and $W = (\mathbf{w}_1, \ldots, \mathbf{w}_n) \in \mathbb{R}^{d \times n}$. Then we define the following notations:

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- $\mathbf{z}^{\mathbf{m}} := \prod_{k=1}^d z_k^{m_k} \in \mathbb{R}$
- We call $H = span(\mathbf{w}_1, \dots, \mathbf{w}_{d-1})$ a **rational** hyperplane iff $\mathbf{w}_1, \dots \mathbf{w}_d \in \mathbb{Q}^d$

Useful Equations

For $\mathbf{z}, \mathbf{v} \in \mathbb{R}^d$, $\mathbf{m} \in \mathbb{Z}^d$, $U \subset \mathbb{R}^d$ and $W = (\mathbf{w}_1, \ldots, \mathbf{w}_n) \in \mathbb{R}^{d \times n}$ we get the following equations:

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- $\Pi(W) = \sum_{k=1}^n \mathbf{w}_k - \Pi(W)$
- $\sigma_{\mathbf{v}+cone(W)} = \frac{\sigma_{\mathbf{v}+\Pi(W)}}{(1-\mathbf{z}^{\mathbf{w}_1})\cdot\ldots\cdot(1-\mathbf{z}^{\mathbf{w}_n})}$ for **simplicial** $cone(W)$.

A First Encounter with Reciprocity

Theorem

Let $W = (\mathbf{w}_1, \ldots, \mathbf{w}_d) \in \mathbb{Z}^d$ *be invertible and* $cone(W)$ *be a* **simplicial** *cone. Then for* $\mathbf{v} \in \mathbb{R}^d$ *with*

$$\partial(v + cone(W)) \cap \mathbb{Z}^d = \{\}$$

we get the reciprocity identity

$$\sigma_{v+cone(W)}(\mathbf{z}^{-\mathbf{1}}) = (-1)^d \sigma_{-v+cone(W)}(\mathbf{z}).$$

Some very handy lemmas

Lemma

The dimension of $(\mathbb{R}, +)$ *as a vector space over the field* $\mathbb{Q}$ *is infinite, i.e.*

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Lemma (of Translation)

Let $\varepsilon > 0$*. Then there exists a* $\mathbf{v} \in \mathbb{R}^d$ *such that we have*

$$H + \mathbf{v} \; \cap \mathbb{Q}^d = \{\} \quad and \quad \|\mathbf{v}\| < \varepsilon$$

for all **rational** *hyperplanes H.*

Lemma (of Distance)

Let $H = span(w_1, \ldots, w_{d-1}) = H(w_d, 0)$ *be a* **rational** *hyperplane,* $w_1 \ldots w_d \in \mathbb{Q}^d$*. Then there exists some* $r > 0$ *such that* $\|v\|_2 < r$ *ensures that*

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$$H_>(w_d, \langle w_d, v\rangle) \cap \mathbb{Z}^d = H_>(w_d, 0) \cap \mathbb{Z}^d$$

or

$$H_<(w_d, \langle w_d, v\rangle) \cap \mathbb{Z}^d = H_<(w_d, 0) \cap \mathbb{Z}^d$$

Lemma (for Shifting Cones)

Let $K = \text{cone}(\mathbf{w}_1, \ldots, \mathbf{w}_n) \subset \mathbb{R}^d$ *be a* **rational** d*-cone, decomposed into* **simplicial** *cones* $K_1, \ldots, K_n$*. Then there exists a* $\mathbf{v} \in \mathbb{R}^d$ *with*

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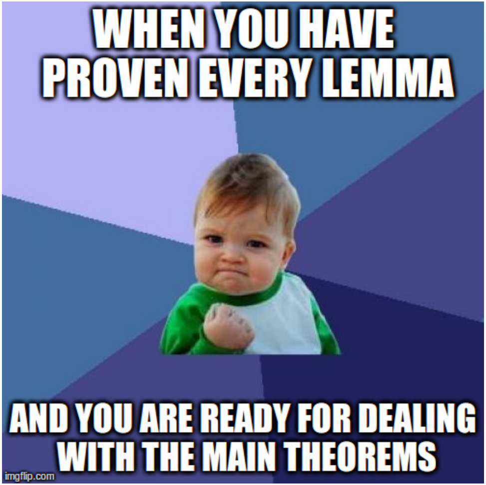
$$
\begin{aligned}
&(a) \quad int(K) \cap \mathbb{Z}^d = (K + \mathbf{v}) \cap \mathbb{Z}^d. \\
&(b) \quad \partial(K_i - \mathbf{v}) \cap \mathbb{Z}^d = \partial(K_i + \mathbf{v}) \cap \mathbb{Z}^d = \{\} \\
&(c) \quad (K - \mathbf{v}) \cap \mathbb{Z}^d = K \cap \mathbb{Z}^d
\end{aligned}
$$


WHEN YOU HAVE PROVEN EVERY LEMMA
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Theorem (Stanley's Reciprocity Theorem)

Let K be a rational d-cone with apex $0 \in \mathbb{R}^d$ and decomposed into **simplicial** *cones $K_1, \ldots K_n$. Then*

$$\sigma_K(\mathbf{z}^{-1}) = (-1)^d \sigma_{int(K)}(\mathbf{z}).$$

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Ehrhart-Macdonald Reciprocity

Definition

Let P be a rational convex polytop. Then we define the **Ehrhart series** of its interior as

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Lemma

Let $P \subset \mathbb{R}^d$ be a rational convex polytop and $K := cone(P)$ it's lifted $d+1$-cone. Then we get

$$\sigma_{int(K)}(1, \ldots, 1, z) = Ehr_{int(P)}(z)$$

Lemma

Let $L : \mathbb{Z} \to \mathbb{C}$ *be a quasipolynomial. Then we get that*

$$\sum_{t \geq 0} L(t) z^t = -\sum_{t<0} L(t) z^t.$$

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Theorem (Ehrhart-Macdonald Reciprocity)

Let P be a rational convex d-polytop. Then we get for the lattice point enumerator $\mathcal{L}_P$ the following equation:

$$\mathcal{L}_P(-t) = (-1)^d \mathcal{L}_{int(P)}(t)$$

Definition (Reflexive Polytope)

Let P be an integral polytop. We then call P a **reflexive** polytop iff there is an $A \in \mathbb{Z}^{n \times d}$ with

$$0 \in int(P) \quad and \quad P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{1}\}.$$

Theorem (Hibi's palindromic theorem)

Let P be an integral d-polytop, $0 \in int(P)$ *and let*

$$Ehr_P(z) = \frac{h_0 z^0 + \ldots + h_d z^d}{(1-z)^{d+1}}.$$

Then P is reflexive iff

$$h_k = h_{d-k} \quad \text{for all } k \in \mathbb{N}_{\leq \frac{d}{2}}.$$

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Example:

Define the cross-polytop $C := conv(\pm e_1, \ldots, \pm e_d)$. Then $0 \in int(C)$ and

$$h_k = \binom{d}{k} = \binom{d}{d-k} = h_{d-k}.$$