Counting Lattice Points in Polytopes: The Ehrhart Theory

Seminar on Discrete Convex Geometry

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10. January 2017

Overview

Integer-Point Transform and Ehrhart's Theorem

Theorem (Ehrhart's theorem)

For an integral convex d-polytope $\mathcal{P}$ the lattice-point enumerator $\mathrm{L}_{\mathcal{P}}(t)$ is a polynomial in t of degree d.

- therefore $\mathrm{L}_{\mathcal{P}}$ is called the Ehrhart polynomial of $\mathcal{P}$
- $\mathrm{L}_{\mathcal{P}}(t) = \# t\mathcal{P} \cap \mathbb{Z}^d$ the lattice-point enumerator, $t \in \mathbb{N}$
- $\mathrm{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{t \geq 1} \mathrm{L}_{\mathcal{P}}(t) z^t$ the Ehrhart Series of $\mathcal{P}$

Integer-Point Transform and Ehrhart's Theorem

Recall the integer-point transform of $S \subset \mathbb{R}^d$:

$$\sigma_S(\mathbf{z}) = \sum_{m \in S \cap \mathbb{Z}^d} \mathbf{z}^m \text{, where } \mathbf{z}^m = \prod_{i=1}^{d} \mathbf{z}_i^{m_i}.$$

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Lemma

- $\sigma_S(\mathbb{1}) = \sum_{m \in S \cap \mathbb{Z}^d} \mathbb{1}^m = \sum_{m \in S \cap \mathbb{Z}^d} 1 = \#(S \cap \mathbb{Z}^d) = \mathrm{L}_S(1)$

Integer-Point Transform and Ehrhart's Theorem

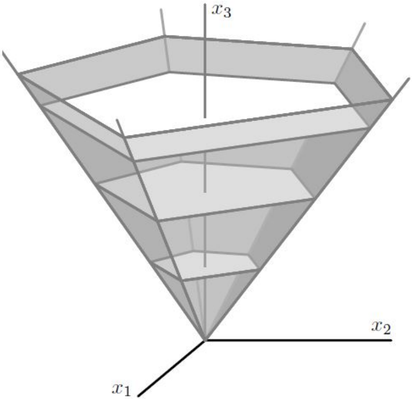

Figure: Dilates of $\mathcal{P}$ in cone($\mathcal{P}$)

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- $\sigma_{cone(\mathcal{P})}(1, \ldots, 1, z) = 1 + \sum_{t \geq 1} \mathrm{L}_{\mathcal{P}}(t) z^t = \mathrm{Ehr}_{\mathcal{P}}(z)$.

The Ehrhart Series for Simplices

Corollary

For an integral d-simplex $\Delta = conv\{v_1, \ldots, v_{d+1}\}$ *it is*

$$\mathrm{Ehr}_\Delta(z) = 1 + \sum_{t \geq 1} \mathrm{L}_\Delta(t) z^t = \frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*}{(1-z)^{d+1}},$$

where h_k^* *equals the number of integer points in* $\{\lambda_1 w_1 + \cdots + \lambda_{d+1} w_{d+1} \,|\, 0 \leq \lambda_1, \ldots, \lambda_{d+1} < 1, w_i = (v_i, 1)\}$ *with last coordinate equal to* k*.*

The Ehrhart Series for Polytopes

Theorem (Stanley's nonnegativity theorem)

Let $\mathcal{P}$ be an integral convex d-polytope with Ehrhart Series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*}{(1-z)^{d+1}}.$$

Then $h_0^*, h_1^*, \ldots, h_d^*$ are nonnegative integers.

The numerator $h_{\mathcal{P}}^*(z)$ is called the h^*-polynomial of $\mathcal{P}$.

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Corollary

In particular, $h_0^* = 1$.

Extracting $L_{\mathcal{P}}$ from its Ehrhart Series

Lemma

Let $\mathcal{P}$ be an integral convex d-polytope with Ehrhart Series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t = \frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*}{(1-z)^{d+1}} .$$

Then

$$L_{\mathcal{P}}(t) = \binom{t+d}{d} + h_1^* \binom{t+d-1}{d} + \cdots + h_{d-1}^* \binom{t+1}{d} + h_d^* \binom{t}{d} .$$

Extracting $\mathrm{L}_{\mathcal{P}}$ from its Ehrhart Series

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Corollary

It is $\mathrm{L}_{\mathcal{P}}(0) = 1$
and $h_1^* = \mathrm{L}_{\mathcal{P}}(1) - d - 1$.

Approximating the Volume by Counting Lattice Points

Lemma

Let $S \subset \mathbb{R}^d$ be d-dimensional, then

$$\operatorname{vol} S = \lim_{t\to\infty} \frac{1}{t^d} \cdot \# \left(S \cap \left(\frac{1}{t}\mathbb{Z} \right)^d \right) = \lim_{t\to\infty} \frac{1}{t^d} \cdot \#(tS \cap \mathbb{Z}^d) .$$

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Corollary

Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex d-polytope with Ehrhart polynomial $\mathrm{L}_{\mathcal{P}}(t) = c_d t^d + \cdots + c_1 t + 1$ and $h^*_{\mathcal{P}}(z)$ its h^*-polynomial. Then

$$\operatorname{vol} \mathcal{P} = c_d = \frac{1}{d!}(h^*_d + h^*_{d-1} + \cdots + h^*_1 + 1) \,.$$

Computing vol $\mathcal{P}$ and $\mathrm{L}_{\mathcal{P}}$ from Discrete Data

A polynomial $p(x) = c_d x^d + \cdots + c_1 x + c_0$ of degree d is determined by $d+1$ points $(x, p(x)) \in \mathbb{R}^2$:

$$\begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_{d+1}) \end{pmatrix} = \underbrace{\begin{pmatrix} x_1^d & x_1^{d-1} & \cdots & x_1 & 1 \\ x_2^d & x_2^{d-1} & \cdots & x_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_{d+1}^d & x_{d+1}^{d-1} & \cdots & x_{d+1} & 1 \end{pmatrix}}_{= \text{Vandermondematrix } V} \begin{pmatrix} c_d \\ c_{d-1} \\ \vdots \\ c_0 \end{pmatrix}$$

Computation of $\mathrm{L}_{\mathcal{P}}$ via

$$\begin{pmatrix} \mathrm{L}_{\mathcal{P}}(x_1) - 1 \\ \mathrm{L}_{\mathcal{P}}(x_2) - 1 \\ \vdots \\ \mathrm{L}_{\mathcal{P}}(x_d) - 1 \end{pmatrix} = \begin{pmatrix} x_1^d & x_1^{d-1} & \cdots & x_1 \\ x_2^d & x_2^{d-1} & \cdots & x_2 \\ \vdots & \vdots & & \vdots \\ x_d^d & x_d^{d-1} & \cdots & x_d \end{pmatrix} \begin{pmatrix} c_d \\ c_{d-1} \\ \vdots \\ c_1 \end{pmatrix}$$

Computing vol $\mathcal{P}$ and $L_{\mathcal{P}}$ from Discrete Data

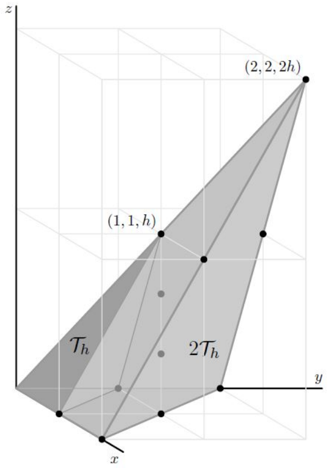

Figure: Reeve's Tetrahedron $\mathcal{T}_h$ and $2\mathcal{T}_h$

Rational Polytopes and Ehrhart Quasipolynomials

Theorem (Ehrhart's theorem for rational polytopes)

For a rational convex d-polytope $\mathcal{P}$ the lattice-point enumerator $\mathrm{L}_{\mathcal{P}}(t)$ is a quasipolynomial in t of degree d.
Its period devides the least common multiple of the denominators of the coordinates of the vertices of $\mathcal{P}$.

- Recall: A quasipolynomial q is an expression of the form
  $$q(t) = c_n(t)t^n + c_{n-1}(t)t^{n-1} + \cdots + c_0(t)\,,$$
  where $c_i(\cdot)$ is a periodic function on $\mathbb{N}$, $i = 0, \ldots, n$.