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Book embeddings of k-framed graphs and k-map graphs ☆

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ABSTRACT

An embedding of a graph in a book, called *book embedding*, consists of a linear ordering of its vertices along the spine of the book and an assignment of its edges to the pages of the book, so that no two edges on the same page cross. The *book thickness* of a graph is the minimum number of pages over all its book embeddings. For planar graphs, a fundamental result is due to Yannakakis, who proposed an algorithm to compute embeddings of planar graphs in books with four pages. Our main contribution is a technique that generalizes this result to a much wider family of nonplanar graphs, namely to k-map graphs. In fact, our technique can deal with any nonplanar graph having a biconnected skeleton of crossing-free edges whose faces have bounded degree. We prove that this family of graphs has book thickness bounded in k, and as a corollary, we obtain the first constant upper bound for the book thickness of optimal 2-planar graphs.

1. Introduction

Book embeddings of graphs form a well-known topic in topological graph theory that has been a fruitful subject of intense research over the years, with seminal results dating back to the 70s [44]. In a *book embedding* of a graph G, the vertices of G are restricted to a line, called the *spine* of the book, and the edges of G are assigned to different half-planes delimited by the spine, called *pages* of the book. From a combinatorial point of view, computing a book embedding of a graph corresponds to finding a *linear ordering* of its vertices and a partition of its edges, such that no two edges in the same part cross; see Fig. 1. The *book thickness* (also known as *stack number* or *page number*) of a graph is the minimum number of pages required by any of its book embeddings, while the *book thickness* of a family of graphs $\mathcal{G}$ is the maximum book thickness of any graph G that belongs to $\mathcal{G}$.

☆ An extended abstract of the research presented in this paper appear in the Proc. of the 36th International Symposium on Computational Geometry (SoCG 2020) [5].

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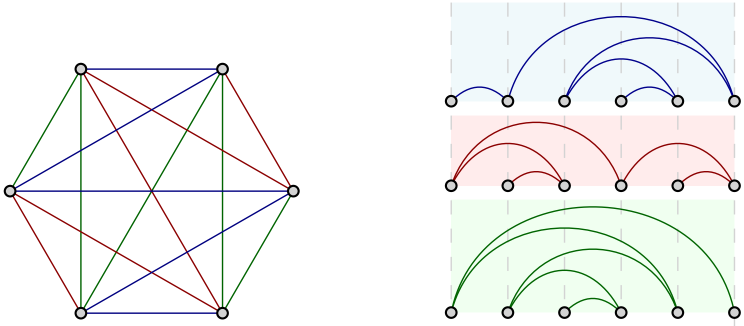


Fig. 1. Graph K_6 and a book embedding of it with the minimum of three pages.

Book embeddings were originally motivated by the design of VLSI circuits [21,48], but they also find applications, among others, in sorting permutations [45,49], compact graph encodings [35,41], graph drawing [11,12,52], and computational origami [1]; for a more complete list, we point the reader to [27]. Unfortunately, determining the book thickness of a graph turns out to be an NP-complete problem even for maximal planar graphs [51]. This negative result has motivated a large body of research devoted to the study of upper bounds on the book thickness of meaningful graph families.

In this direction, there is a very rich literature concerning planar graphs. The most notable result is due to Yannakakis, who back in 1986 exploited a peeling-into-levels technique (a flavor of it is given in Section 3) to prove that the book thickness of any planar graph is at most 4 [53,54], improving upon a series of previous results [17,32,34]. Recently, Yannakakis [55] and Bekos et al. [8] independently proved that 4 pages are sometimes necessary.

Concerning subfamilies of planar graphs, Bernhart and Kainen [10] showed that the book thickness of a graph G is 1 if and only if G is outerplanar, while its book thickness is at most 2 if and only if G is *subhamiltonian*, that is, G is a subgraph of a Hamiltonian planar graph. In particular, several subfamilies of planar graphs are known to be subhamiltonian, e.g., 4-connected planar graphs [43], planar graphs without separating triangles [36], Halin graphs [22], series-parallel graphs [46], bipartite planar graphs [24], planar graphs of maximum degree 4 [7], triconnected planar graphs of maximum degree 5 [33], and maximal planar graphs of maximum degree 6 [29]. In this plethora of results, we should also mention that planar 3-trees have book thickness 3 [32] and that general (i.e., not necessarily triconnected) planar graphs of maximum degree 5 have book thickness at most 3 [31].

In contrast to the planar case, there exist far fewer results for non-planar graphs. Bernhart and Kainen first observed that the book thickness of a graph can be linear in the number of its vertices; for instance, the book thickness of the complete graph K_n is $\lceil n/2 \rceil$ [10]. Improved bounds are usually obtained by meta-theorems exploiting standard parameters of the graph. In particular, Malitz proved that if a graph has m edges, then its book thickness is $O(\sqrt{m})$ [39], while if its genus is g, then its book thickness is $O(\sqrt{g})$ [38]. Also, Ganley and Heath [30] showed that if a graph has treewidth w, then its book thickness is at most $w+1$. This result was reproved by Dujmovic and Wood [28], who also showed that this bound is tight for $w \geq 3$. It is also known that all graphs belonging to a minor-closed family have bounded book thickness [13], while the other direction is not necessarily true. As a matter of fact, the family of 1-planar graphs is not closed under taking minors [42], but it has bounded book thickness [3,4]. We recall that a graph is *h-planar* (with $h \geq 0$), if it can be drawn in the plane such that each edge is crossed at most h times; the reader is referred, e.g., to [25,37] for recent surveys.

Notably, the approaches presented in [3,4] form the first non-trivial extensions of the above mentioned peeling-into-levels technique by Yannakakis [53,54] to graphs that are not planar. Both approaches exploit an important property of 3-connected 1-planar graphs, namely, they can be augmented and drawn so that all pairs of crossing edges are "caged" in the interior of degree-4 faces of a *planar skeleton*, which is defined as the graph consisting of all vertices and of all crossing-free edges of the drawing [47]. A similar property also holds for the optimal 2-planar graphs. Namely, each graph in this family admits a drawing whose planar skeleton is simple, biconnected, and has only degree 5 faces, each containing five crossing edges [9]. The book thickness of these graphs, however, has not been studied yet; the best-known upper bound of $O(\log n)$ is derived from the corresponding one for general h-planar graphs [26].

***k*-map graphs.** Besides h-planar graphs, another well-known generalization of planarity are the *map graphs*, introduced by Chen, Grigni, and Papadimitriou [18]. Roughly speaking, a *k-map graph* is one whose vertices are in correspondence with a set of regions in the sphere (possibly not covering its entire surface) and whose edges correspond to boundary intersections between pair of regions such that at most k regions meet at the same point (see Section 4 for a formal definition). As map graphs find applications in graph drawing, circuit board design and topological inference problems [20], they have been extensively studied in the literature, in particular in terms of characterization and recognition [14,15,18,19,40,50]. For instance, it is known that planar graphs are the 2-map graphs, and that the 4-map graphs are exactly those 1-planar graphs that have a 1-planar drawing Γ such that each pair of crossing edges is caged in a face of the planar skeleton of Γ [15].

Our contribution. We present a technique that further generalizes the result by Yannakakis to k-map graphs. Our main result can be summarized as follows and represents the first nontrivial upper bound for the book thickness of k-map graphs.

Theorem 1. *The book thickness of a k-map graph is at most* $6k + 7$*.*

In fact, the presented technique can deal with a larger family of non-planar graphs, called partial k-framed graphs A graph is *k-framed*, if it admits a drawing having a simple biconnected planar skeleton, whose faces have degree at most $k \geq 3$, and whose crossing edges are in the interiors of these faces. A *partial k-framed* graph is a subgraph of a k-framed graph. Clearly, the book thickness of partial k-framed graphs is lower bounded by $\lceil k/2 \rceil$, as they may contain cliques of size k [10]. We prove an upper bound on the book thickness of partial k-framed graphs that depends linearly *only* on k (but not on n).

Theorem 2. *The book thickness of a partial k-framed graph is at most* $6\lceil \frac{k}{2} \rceil + 7$*.*

Since we can show that every k-map graph is a partial $2k$-framed graph (Theorem 34), Theorem 1 follows immediately from Theorem 2. On the other hand, one can easily show that every partial k-framed graph is a subgraph of a k-map graph.

We also remark that k-framed graphs have been recently studied in the context of the graph product structure theory. Namely, Bekos et al. [6] proved that these graphs are subgraphs of the strong product of a path, of a planar graph of treewidth at most 3, and of a clique of size $O(k)$, which implies the existence of improved bounds on the queue number, non-repetitive chromatic number, and p-centered chromatic number of these graphs. Therefore, Theorem 2 can be considered as a result of independent interest.

Concerning h-planarity, note that the partial 3-framed graphs are exactly the (simple) planar graphs. Also, it is known that 3-connected 1-planar graphs are partial 4-framed [2], while general 1-planar graphs can be augmented to 8-framed. In fact, every two crossing edges can be caged inside a cycle of length (at most) 8 passing through the endpoints of such crossing edges; the faces of the resulting planar skeleton that do not contain any crossing edge can be triangulated. Hence, Theorem 2 implies constant upper bounds for the book thickness of these families of graphs. Since optimal 2-planar graphs are 5-framed, the next corollary guarantees the first constant upper bound on the book thickness of this family.

Corollary 3. *The book thickness of an optimal* 2*-planar graph is at most* 25*.*

More in general, each partial k-framed graph is h-planar for $h = (\frac{k-2}{2})^2$, and hence for this family of h-planar graphs we prove that the book thickness is $O(\sqrt{h})$, while the best-known upper bound for general h-planar graphs is $O(h \log n)$ [26].

After submitting the paper we became aware of a manuscript by Brandenburg [16], in which a better bound for the book thickness of k-map graphs is shown, namely $6\lfloor \frac{k}{2} \rfloor + 5$. This result also implies a better bound of 17 for the book thickness of optimal 2-planar graphs.

Paper organization. In Section 2, we give basic definitions and notation. Section 3 is devoted to the proof of Theorem 2: We start by recalling the peeling-into-level decomposition, and we proceed with an inductive proof based on the resulting leveling of the graph. The base case is described in Section 3.1 and corresponds to graphs consisting of two levels only, while the inductive case is described in Section 3.2 and deals with general (i.e., multi-level) graphs. The proof of Theorem 34 is given in Section 4. Finally, Section 5 contains conclusions and open problems that stem from our research.

2. Preliminaries

Drawings and planar embeddings. A graph is *simple*, if it contains neither self-loops nor parallel edges. A *drawing* of a graph G is a mapping of the vertices of G to distinct points of the plane, and of the edges of G to Jordan arcs connecting their corresponding endpoints. A drawing is *planar*, if no two edges intersect, except possibly at a common endpoint. A graph is *planar*, if it admits a planar drawing. A planar drawing partitions the plane into topologically connected regions, called *faces*. The infinite region is called the *unbounded face*; any other face is a *bounded face*. The *degree* of a face is the number of occurrences of its edges encountered in a clockwise traversal of its boundary (counted with multiplicity). Note that if G is biconnected, then each of its faces is bounded by a simple cycle. A *planar embedding* of a planar graph is an equivalence class of topologically-equivalent (i.e., isotopic) planar drawings. A planar graph with a given planar embedding is a *plane graph*.

k-framed graphs. Let Γ be a drawing of a graph G. The *planar skeleton* $\sigma(G)$ of G in Γ is the plane subgraph of G induced by the crossing-free edges of G in Γ (where the embedding of $\sigma(G)$ is the one induced by Γ). The edges of $\sigma(G)$ are called *crossing-free*, while the edges that belong to G but not to $\sigma(G)$ are *crossing* edges. A *k-framed drawing* of a graph is one such that its crossing-free edges determine a planar skeleton, which is simple, biconnected, spans all the vertices, and has faces of degree at most $k \geq 3$. A graph is *k-framed*, if it admits a k-framed drawing; refer to Fig. 2. A *partial k-framed* graph is a subgraph of a k-framed graph. Clearly, if a k-framed graph has book thickness at most b, then the book thickness of any of its subgraphs is at most b. Thus, in the remainder of the paper, we will only consider k-framed graphs. Further, w.l.o.g., we will also assume that each pair of vertices that belongs to a face f of $\sigma(G)$ is connected either by a crossing-free edge (on the boundary of f) or by a crossing edge (drawn inside f). In other words, the vertices on the boundary of f induce a clique of size at most k. Under this assumption, graph G may contain parallel crossing edges connecting the same pair of vertices, but drawn in the interior of different faces of $\sigma(G)$; see, e.g., the dashed edges of Fig. 2.

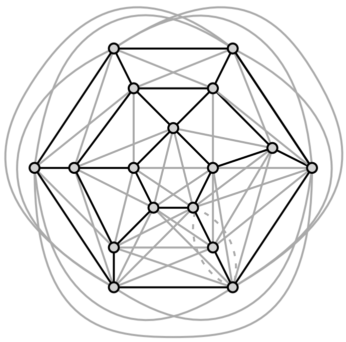


Fig. 2. A drawing of a 6-framed graph, whose crossing-free (crossing) edges are black (gray).

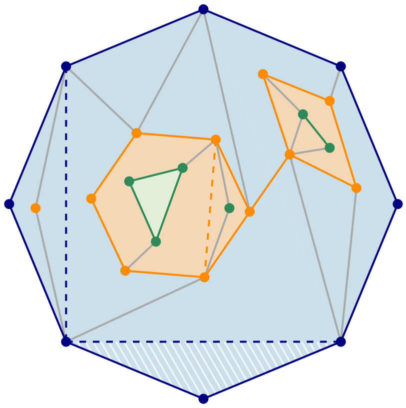


Fig. 3. The peeling-into-levels decomposition of an 8-framed graph without its crossing edges. The vertices and level-edges of level L_0 (L_1; L_2, resp.) are blue (orange; green, resp.) and induce $\sigma_0(G)$ ($\sigma_1(G)$; $\sigma_2(G)$, resp.). Chords are drawn dashed; binding edges are drawn gray. The blue (orange; green, resp.) faces are the intra-level faces of $\sigma_1(G)$ ($\sigma_2(G)$; $\sigma_3(G)$, resp.). Graph $\sigma_0(G)$ ($\sigma_1(G)$; $\sigma_2(G)$, resp.) without the dashed chords forms $C_0(G)$ ($C_1(G)$; $C_2(G)$, resp.). The striped blue face is an intra-level face of $\sigma_1(G)$, whose boundary exists exclusively of L_0-level edges. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Book embeddings. A *book embedding* of a graph G consists of a linear ordering $\prec$ of the vertices of G along a line, called the *spine* of the book, and an assignment of the edges of G to different half-planes delimited by the spine, called *pages* of the book, such that no two edges of the same page *cross*, that is, no two edges (u,v) and (w,z) of the same page with $u \prec v$ and $v \prec w$ are such that $u \prec w \prec v \prec z$. We further say that (u,v) and (w,z) of the same page with $u \prec v$ and $v \prec w$ *nest*, if $u \prec w \prec z \prec v$. The *book thickness* of G is the minimum integer k, such that G has a book embedding on k pages.

3. Proof of Theorem 1

Our approach adopts some ideas from the seminal work by Yannakakis on book embeddings of planar graphs. In particular, we refer to the algorithm which embeds any (internally-triangulated) plane graph in a book with five pages [54], not four. The main challenges of our generalization are posed by the crossing edges and by the fact that we cannot augment the input graph so that its underlying planar skeleton is internally-triangulated. In the following, we explain the basic ideas of Yannakakis' algorithm and recall basic definitions and properties from [54], which we generalize and exploit to introduce new ones.

Our technique is based on the so-called *peeling-into-levels* decomposition. Let G be an n-vertex k-framed graph with a k-framed drawing Γ. We classify the vertices of G as follows: (i) vertices on the unbounded face of $\sigma(G)$ are at level 0, and (ii) vertices that are on the unbounded face of the subgraph of $\sigma(G)$ obtained by deleting all vertices of levels $\leq i-1$ are at level i ($0 < i < n$); see, e.g., Fig. 3. Denote by $\sigma_i(G)$ the subgraph of $\sigma(G)$ induced by the vertices of L_i. Observe that $\sigma_i(G)$ is outerplane, but not necessarily connected. Next, we consider $\sigma_i(G)$ and delete any edge that is not incident to the unbounded face. The resulting spanning subgraph of $\sigma_i(G)$ is denoted by $C_i(G)$. By definition, each connected component of $C_i(G)$ is a cactus. Also, the only edges that belong to $\sigma_i(G)$ but not to $C_i(G)$ are the chords of $\sigma_i(G)$. Finally, we denote by G_i the subgraph of G induced by the vertices of $L_0 \cup \ldots \cup L_i$ containing neither chords of $\sigma_i(G)$ nor the crossing edges that are in the interior of the unbounded face of $\sigma(G)$.

Consider an edge e that belongs to $\sigma(G)$. If the endpoints of e are assigned to the same level, e is a *level* edge; otherwise, e connects vertices of consecutive levels and is called a *binding* edge; see Fig. 3. By the definition of the level-partition, there is no edge $e \in E$, that connects two vertices of levels i and j, such that $|i-j|>1$. Another consequence of the level-partition is that any vertex of level $i+1$ lies in the interior of a cycle of level i. Next, we give a characterization for bounded faces of $\sigma(G)$. A bounded face of $\sigma(G)$ is an *intra-level face* of $\sigma_i(G)$ if it is incident to at least one vertex of L_{i-1} but to no vertex of L_{i-2}. We denote by $\mathcal{F}_i$ the set of all the intra-level faces of $\sigma_i(G)$. By definition, the unbounded face of $\sigma_i(G)$ is not an intra-level face. Also, each intra-level face of $\sigma_i(G)$ has either at least one binding edge between L_{i-1} and L_i on its boundary, or it consists exclusively of edges of level L_{i-1}.

Overview. We give an short overview of how our algorithm embeds a k-framed graph G with a given k-framed drawing Γ on $6 \cdot \left\lceil \frac{k}{2} \right\rceil + 7$. In a high level description, we will inductively compute a book embedding of G_{i+1}, assuming that we have already computed a book embedding of G_i. For this inductive strategy to work, the computed book embeddings satisfy particular invariants, which we define subsequently. We first focus on the base case, in which G consists of only two levels L_0 and L_1 under some additional assumptions (see Section 3.1). Afterwards, we consider the inductive case, in which G consists of more than two levels (see Section 3.2).

3.1. Base case: two-level instances

A *two-level instance* is a k-framed graph G consisting of two levels L_0 and L_1, such that there is no crossing edge in the unbounded face of $\sigma_0(G)$, and either $L_1 = \emptyset$ or $\sigma_1(G) = C_1(G)$, i.e., $\sigma_1(G)$ is chord-less; refer to Fig. 4 for an illustration of a two-level instance. Since $\sigma(G)$ is biconnected, $C_0(G)$ is a simple cycle. Let $u_0, u_1, \dots, u_{s-1}$ with $s \geq 3$ be the vertices of L_0 in the order that they appear in a clockwise traversal of $C_0(G)$ starting from u_0. An edge (u_i, u_j) of $\sigma_0(G)$ is *short* if $i - j = \pm 1$; otherwise it is *long*. By definition, (u_0, u_{s-1}) is long. In the following, we will refer to the intra-level faces of $\sigma_1(G)$ simply as intra-level faces, and we will further denote $\mathcal{F}_1$ as $\mathcal{F}$. Consider now the graph $C_1(G)$. Each of its connected components is a cactus; thus, its biconnected components, called *blocks*, are either single edges or simple cycles (that are chordless, as $\sigma_1(G) = C_1(G)$). A connected component of $C_1(G)$ may degenerate into a single vertex, and this vertex itself is a *degenerate* block. A block that consists of more than one vertex is called *non-degenerate*.

We equip $\mathcal{F}$ with a linear ordering $\lambda(\mathcal{F})$ as follows. For $i = 0, \dots, s-1$, the intra-level faces incident to vertex u_i are appended to $\lambda(\mathcal{F})$ as they appear in counterclockwise order around u_i starting from the one incident to (u_{i-1}, u_i) and ending at the one incident to (u_i, u_{i+1}) (indices taken modulo s), unless already present. For a pair of intra-level faces f and f', we write $f \prec_\lambda f'$ if f precedes f' in $\lambda(\mathcal{F})$; similarly, we write $f \preceq_\lambda f'$ if $f = f'$ or $f \prec_\lambda f'$.

Let $C_1, \dots, C_\gamma$ be the connected components of $C_1(G)$ and let $C \in \{C_1, \dots, C_\gamma\}$. In general, several intra-level faces in $\mathcal{F}$ may contain vertices of C on their boundary. Let f_C be the first face in the ordering $\lambda(\mathcal{F})$ that contains a vertex of C. Consider now a counterclockwise traversal of the boundary of f_C starting from the vertex of L_0 with the smallest subscript that belongs to f_C. We refer to the vertex, say v_C, of C that is encountered first in this traversal as the *first vertex* of C. Observe that, by definition, v_C is incident to a binding edge that is on the boundary of f_C. We will further assume that v_C forms a degenerate block r_C of C. The *leader* of a block B of C, denoted by $\ell(B)$, is the first vertex of B that is encountered in any path of C from v_C to B; note that $\ell(B)$ is uniquely defined.

Consider a vertex v of C. If v belongs to only one block of C, then v is *assigned* to that block. Otherwise v is assigned to the block B of C such that v belongs to B and the graph-theoretic distance in C between $\ell(B)$ and v_C is the smallest. It follows that v_C is assigned to the degenerate block r_C, and that for any non-degenerate block B the leader $\ell(B)$ is not assigned to B. We denote by $B(v)$ the block of C that a vertex v is assigned to. Let B be a block of C. Assume first that B is non-degenerate. We refer to the first face in the ordering $\lambda(\mathcal{F})$ containing an edge of B as the face that *discovers* B. Assume now that B is degenerate, i.e., it consists of a single vertex v. We refer to the first face in the ordering $\lambda(\mathcal{F})$ that has v on its boundary as the face that discovers B. In both cases, we denote by $d(B)$ the face in $\mathcal{F}$ that discovers block B.

We extend the notion of discovery to the vertices of G. To this end, let v be a vertex of G (which can be incident to several intra-level faces in $\mathcal{F}$). We distinguish whether v belongs to L_0 or L_1. In the former case, face f of $\mathcal{F}$ *discovers* vertex v if f is the first intra-level face in the ordering $\lambda(\mathcal{F})$ that contains v on its boundary. In the latter case, face f in $\mathcal{F}$ *discovers* vertex v if f is the face that discovers the block vertex v is assigned to. In both cases we denote by $d(v)$ the face in $\mathcal{F}$ that discovers vertex v. This yields $d(v) = d(B(v))$ for any $v \in L_1$. The *dominator* $\mathrm{dom}(B)$ of block B is the vertex of L_0 with the smallest subscript that is on the boundary of $d(B)$. Several blocks of C can be discovered by the same face, and by definition, these blocks have the same dominator. Analogously, we define the dominator $\mathrm{dom}(f)$ of an intra-level face f as the vertex of L_0 with the smallest subscript that is on the boundary of f. This yields $\mathrm{dom}(B) = \mathrm{dom}(d(B))$.

Property 4. *The face $d(B)$ that discovers block B is the first face in $\lambda(\mathcal{F})$ that has a vertex assigned to block B on its boundary.*

Proof. If B is a degenerate block, the property follows by definition. Otherwise, B contains at least one edge on its boundary. The face $d(B)$ is the first intra-level face in $\lambda(\mathcal{F})$ that contains an edge (v, w) of B on its boundary. Since only the leader $\ell(B)$ of B is not assigned to block B and since (v, w) is a boundary edge of B, at least one of v and w is assigned to B. The property follows from the fact that at most one of the endpoints of (v, w) is not assigned to B. $\square$

Consider now two blocks B and B' of $C_1(G)$. Note that B and B' do not necessarily belong to the same connected component of $C_1(G)$. We say that B *precedes* B' if (i) $d(B) \prec_\lambda d(B')$, or (ii) $d(B) = d(B')$ and in a counterclockwise traversal

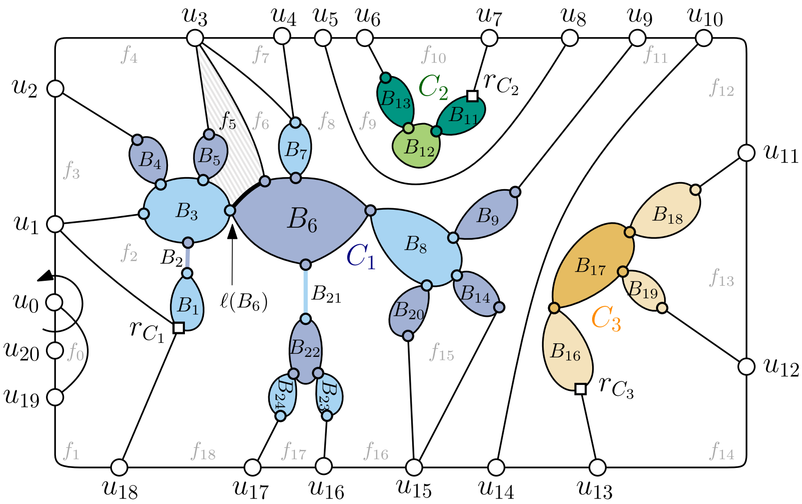


Fig. 4. Illustration of the graph $\sigma_1(G)$ of a two-level instance G: the vertices of L_0 are denoted by $u_0, \ldots, u_{20}$; the vertices of L_1 are the remaining ones; $C_1(G)$ consists of three connected components C_1, C_2 and C_3, whose first vertices are denoted by v_{C_1}, v_{C_2} and v_{C_3}, resp.; the vertices assigned to each block have the same color as the block; C_1 contains two blocks B_2 and B_{21} that are simple edges; the two level edges (u_5, u_6) and (u_5, u_8) are short and long, resp.; edge (u_1, v_{C_1}) is a binding edge; the intra-level faces of $\mathcal{F}$ are all numbered from f_0 to f_{18} according to $\lambda(\mathcal{F})$; the intra-level face that discovers B_6 is the face f_5 tilled gray; f_1, f_9 and f_{12} discover the degenerate blocks.

of $d(B)$ starting from $\mathrm{dom}(d(B))$ block B is encountered before block B'. We denote this relationship between B and B' by $B \prec B'$. Since $\lambda(\mathcal{F})$ is a well-defined ordering, it follows that the relationship "precedes" is also defining a total ordering of the blocks of $C_1(G)$. In the following, we introduce a useful property of $\lambda(\mathcal{F})$.

Property 5. *Let v be a vertex of G and let $f_v \in \mathcal{F}$ be an intra-level face that contains v on its boundary. Then, $d(v) \preceq_\lambda f_v$ holds.*

Proof. If v belongs to L_0, then the property follows by definition. Otherwise, v belongs to L_1, and $d(v)$ is the intra-level face that discovers the block $B(v)$, that is, $d(v) = d(B(v))$. If $B(v)$ is degenerate, then $d(v)$ is the first intra-level face in $\lambda(\mathcal{F})$ that has v on its boundary. Hence, $d(v) \preceq_\lambda f_v$. Otherwise, by Property 4, $d(B(v))$ is the first intra-level face in $\lambda(\mathcal{F})$ that contains a vertex assigned to block B on its boundary. Since $d(v) = d(B(v))$ and since v is assigned to block B, it follows that $d(v) \preceq_\lambda f_v$. $\square$

Next, we introduce the notion of a prime vertex with respect to an intra-level face. We say that a vertex v of L_0 belonging to the boundary of an intra-level face f is *prime* with respect to f if no vertex of L_1 and no long level edge is encountered in the clockwise traversal of f from $\mathrm{dom}(f)$ to v. By definition, $\mathrm{dom}(f)$ is prime with respect to f. We say that a vertex v is *f-prime* if either v is prime with respect to face f or v belongs to L_1. By definition, any vertex of L_1 is g-prime with respect to any intra-level face g. Let u_j be a vertex on L_0 that is not $d(u_j)$-prime with $j \in \{1, \ldots, s-1\}$. Let $f_0^{u_j}, \ldots, f_t^{u_j}$ be the faces that have u_j on their boundary in a counterclockwise traversal of u_j starting from (u_{j-1}, u_j) and ending at (u_j, u_{j+1}) (indices taken modulo s). Let d be the smallest index such that $f_d^{u_j} = d(u_j)$. We say that face $f_l^{u_j}$ is *before* (*after*) $d(u_j)$ *around* u_j if $l < d$ ($l > d$, respectively). Furthermore, the faces $f_0^{u_j}, \ldots, f_{d-1}^{u_j}$ that have u_j as their dominator are called *small*. See Fig. 6 for an illustration of a small face.

3.1.1. Linear ordering

The *linear ordering* of the vertices, denoted by ρ, is computed as follows. First, the vertices of L_0 are embedded in the order $u_0, u_1, \ldots, u_{s-1}$. The remaining vertices of G (i.e., the vertices of L_1) are embedded along the spine based on the blocks that they have been assigned to and according to the following rules:

R.1 For $j = 0, \ldots, s-1$, let $B_0^j, \ldots, B_{t-1}^j$ be the blocks with u_j as dominator such that the faces that discover them are not small (are small, resp.), and $B_i^j \prec B_{i+1}^j$ for $i = 0, 1, \ldots, t-2$. The vertices assigned to these blocks are placed right after (before, resp.) u_j in ρ.

R.2 The vertices assigned to B_i^j are right before those assigned to B_{i+1}^j, for each $i = 0, \ldots, t-2$.

R.3 The vertices assigned to the same block B_i^j are in the order they appear in a counterclockwise traversal of the boundary of B_i^j starting from the leader of B_i^j, for $i = 0, \ldots, t-1$.

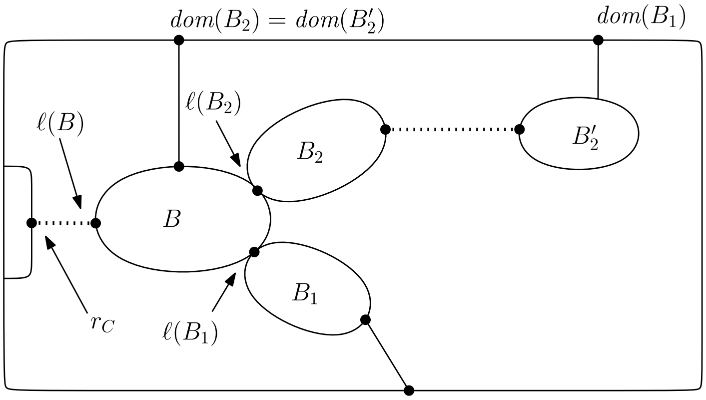


Fig. 5. Illustration for the proof of Property 9.

For a pair of distinct vertices v and w, we write $v \prec_\rho w$ if v precedes w in ρ. By Rule R.1, the vertices of L_1 that are discovered by f and the f-prime vertices of L_0 are right next to each other in ρ. The next property is consequence of Rules R.1–R.3.

Property 6. *The vertices assigned to a block B of L_1 appear consecutively in ρ.*

The order of the blocks together with Rules R.1 and R.2 yields the following property.

Property 7. *Let v and w be two vertices of L_1 assigned to two distinct blocks $B(v)$ and $B(w)$, respectively. Then, $v \prec_\rho w$ if and only if $B(v)$ precedes $B(w)$.*

The next properties will be useful in Section 3.2.

Property 8. *Let C_1 and C_2 be two connected components of $\mathcal{C}_1(G)$ rooted at their first vertices, and let B_1 and B_2 be two non-degenerate blocks of C_1 and C_2, respectively. If there exists a vertex v assigned to B_2 between $\ell(B_1)$ and the vertices assigned to B_1 in ρ, then all vertices assigned to B_2 appear in ρ between $\ell(B_1)$ and the vertices assigned to B_1.*

Proof. Let B_1' be the block that $\ell(B_1)$ is assigned to. Then B_1' is a block of C_1 and $B_1' \neq B_1$. Let w be a vertex assigned to block B_1. Then we have $\ell(B_1) \prec_\rho v \prec_\rho w$ with $\ell(B_1)$ assigned to B_1', v assigned to B_2, and w assigned to B_1. By Property 6, all vertices assigned to the same block are consecutive in ρ, and the claim follows. $\square$

Property 9. *Let C be a connected component of $\mathcal{C}_1(G)$ rooted at its first vertex, and let B be a non-degenerate block of C with two children B_1 and B_2. If $\ell(B_1) \preceq_\rho \ell(B_2)$ and $B_2 \prec B_1$, then all vertices assigned to descendant blocks of B_2 (including B_2) precede in ρ all vertices assigned to descendant blocks of B_1 (including B_1).*

Proof. First, observe that for a block B and any descendant block B' of B, we have the order $B \prec B'$. Therefore, any vertex assigned to B precedes any vertex assigned to B' in ρ. Hence, let B_2' be a descendant of B_2. It remains to show that if B_1 and B_2 are children of the same block, $\ell(B_1) \preceq_\rho \ell(B_2)$, and $B_2 \prec B_1$, then $v \prec_\rho w$ for any vertex v assigned to B_2' and any vertex w assigned to B_1. Since $\sigma(G)$ is planar and biconnected, we get $d(B_2') \prec_\lambda d(B_1)$; see Fig. 5. Hence, $\mathrm{dom}(d(B_2')) \preceq_\rho \mathrm{dom}(d(B_1))$ holds. Now the claim follows by Rules R.1 and R.2. $\square$

Property 10. *Let C be a connected component of $\mathcal{C}_1(G)$, and let B_1 and B_2 be two distinct non-degenerate blocks of C. If there is a vertex v assigned to a block B_1 between $\ell(B_2)$ and the remaining vertices of B_2 such that $\ell(B_1) \prec_\rho \ell(B_2)$, then $\ell(B_2)$ is assigned to B_1.*

Proof. Assume for a contradiction that $\ell(B_2)$ is assigned to a different block, say B_2'. Let also B_1' be the block that $\ell(B_1)$ is assigned to. By Property 7, we obtain the order of the blocks: $B_1' \preceq B_2' \prec B_1 \prec B_2$. We distinguish two cases based on whether (a) $B_1' \prec B_2'$ or (b) $B_1' = B_2'$ holds. First, consider Case (a), that is $B_1' \prec B_2'$. Since B_1 is a child of B_1' and $B_1 \prec B_2' \prec B_1$, it follows that either B_2' is also a child of B_1' which precedes B_1 in the ordering of the blocks, or it is a descendant of another child of B_1' which precedes B_1 in the ordering of the blocks. In both cases, it follows by Property 9 that $B_2 \prec B_1$; a contradiction. Consider now Case (b). Since $\ell(B_1) \prec_\rho \ell(B_2)$, and both vertices are assigned to the same block, it follows that $B_2 \prec B_1$; a contradiction. $\square$

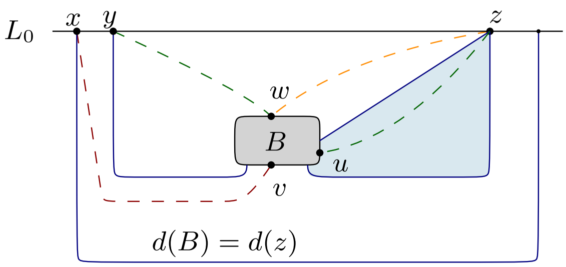


Fig. 6. An illustration of the different types of dominator edges. Crossing edges are drawn dashed. The red edge (x, v) is a primary dominator edge. The green edges (y, w) and (z, u) are secondary dominator edges. The orange edge (z, w) is a tertiary dominator edge. The blue face is a small face with dominator z.

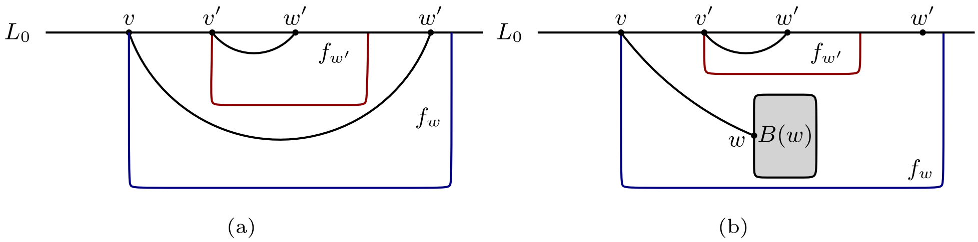


Fig. 7. Illustration for the proof of Lemma 14.

Property 11. *Let v be a $d(v)$-prime vertex of L_0. Then v is f-prime for any intra-level face f that has v on its boundary. Also, $v = dom(f)$, except possibly for $f = d(v)$.*

Proof. Let f be an intra-level face that is different from $d(v)$ such that f has v on its boundary. By planarity, vertex v is the dominator of face f. Thus, v is f-prime. □

Property 12. *Let w be a $d(w)$-prime vertex. For any vertex v with $v \prec_\rho w$, $d(v) \preceq_\lambda d(w)$.*

Proof. Since w is $d(w)$-prime, w precedes any vertex discovered by a face f with $d(w) \prec_\lambda f$. Assuming to the contrary that $d(w) \prec_\lambda d(v)$, we get $w \prec_\rho v$; a contradiction. □

By contraposition the following corollary is a direct consequence of Property 12.

Corollary 13. *Let v be a $d(v)$-prime vertex. For any vertex w, $d(v) \prec_\lambda d(w)$ implies $v \prec_\rho w$.*

3.1.2. Edge-to-page assignment

With the linear ordering ρ at hand, we now describe how to perform the edge-to-page assignment which concludes the construction of our book embedding. We start with some particular types of edges defined as follows. A binding edge (v, w) of G with $v \in L_0$ and $w \in L_1$ is a *primary dominator* edge if v is the dominator of the intra-level face $d(w)$ that discovers w. Note that edge (v, w) belongs to a face f_{vw} with $d(w) \preceq_\lambda f_{vw}$ whose dominator is vertex v. Also, $v \prec_\rho w$ unless $d(w)$ is a small face. A binding edge (v, w) of G with $v \in L_0$ and $w \in L_1$ is a *secondary dominator* edge if (i) (v, w) is not a primary dominator edge, and (ii) v and w are on the boundary of a face f_{vw} such that v is the dominator of f_{vw}. Finally, a binding edge (v, w) of G with $v \in L_0$ and $w \in L_1$ is a *tertiary dominator* edge if (i) (v, w) is not a primary or secondary dominator edge, (ii) v is not $d(v)$-prime, and (iii) v and w are on the boundary of a face f_{vw} such that $f_{v,w}$ is before $d(v)$ around v. Note that, if (v, w) is a secondary or tertiary dominator edge, $w \prec_\rho v$ holds. See Fig. 6 for an illustration of the different types of dominator edges. In the following lemma, we prove that all primary dominator edges of G can be assigned to a single page. We note that the proof is reminiscent of a corresponding one by Yannakakis [54] for similarly-defined backward edges.

Lemma 14. *Let (v, w) and (v', w') be two primary dominator edges of G. Then, (v, w) and (v', w') do not cross in ρ.*

Proof. We may assume without loss of generality that $v, v' \in L_0$, $w, w' \in L_1$, $v \prec_\rho v'$, and that v, w, v' and w' are four distinct vertices of G (Fig. 7). By the definition of primary dominator edges we have $v = \text{dom}(d(w))$ and $v' = \text{dom}(d(w'))$. Assume first that $v' \prec_\rho w'$. If edges (v, w) and (v', w') cross in ρ, $v \prec_\rho v' \prec_\rho w \prec_\rho w'$ must hold. By definition of the primary dominator edge, the vertex w belongs to a block $B(w)$ dominated by v. By Rule R.1, there is no vertex of L_0

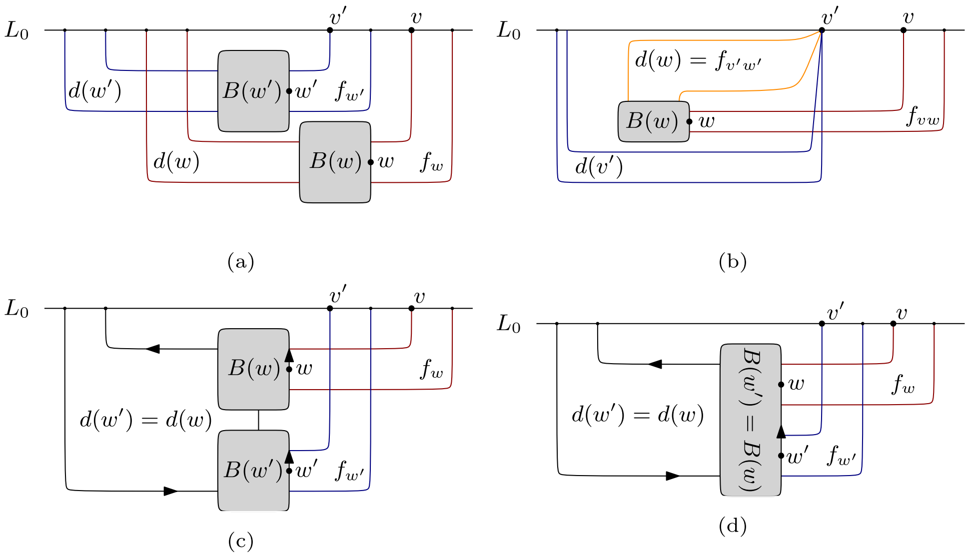


Fig. 8. Illustration for the proof of Lemma 15.

between v and the vertices assigned to $B(w)$ in ρ. Hence, v' cannot appear between v and w in ρ. Thus, $w' \prec_\rho v'$ must hold. Since v' is the dominator of $d(w')$, it follows from Rule R.1, that there is no vertex of L_0 between w' and v'. Therefore, the only way for (v, w) and (v', w') to cross in ρ is if $v \prec w' \prec w \prec v'$. Again, by Rule R.1, w' should belong to a block discovered by v and not v'. $\square$

Next, we prove that all secondary dominator edges can also be assigned to a single page.

Lemma 15. *Let (v, w) and (v', w') be two secondary dominator edges of G. Then, (v, w) and (v', w') do not cross in ρ.*

Proof. We may assume without loss of generality that $v, v' \in L_0$, $w, w' \in L_1$, $v' \prec_\rho v$, and that v, w, v' and w' are four distinct vertices of G. Let f_{vw} ($f_{v'w'}$) be a face that has vertices v and w (v' and w') on its boundary. By the definition of secondary dominator edges, we have $v = \mathrm{dom}(f_{vw})$ and $v' = \mathrm{dom}(f_{v'w'})$. Another immediate consequence of the edges being secondary dominator edges is that $w \prec_\rho v$ and $w' \prec_\rho v'$. It also follows that $f_{v'w'} \prec_\lambda f_{vw}$ since $v' \prec_\rho v$. If $v' \prec_\rho w$, then we have $w' \prec_\rho v' \prec_\rho w \prec_\rho v$, so the edges do not cross. Thus, assume $w \prec_\rho v'$. Then, we have $w' \prec_\rho v' \prec_\rho v$ and $w \prec_\rho v'$, and it remains to show that $w \prec_\rho w'$.

Since (v, w) and (v', w') are not primary dominator edges, we have $d(w) \prec_\lambda f_{vw}$ and $d(w') \prec_\lambda f_{v'w'}$. Note that if $d(w) \prec_\lambda d(w')$, the claim follows from Corollary 13. Thus, we proceed by considering the two subcases, $d(w') \prec_\lambda d(w)$ and $d(w) = d(w')$. In the former case, since $w \prec_\rho v'$ and since v' is the dominator of $f_{v'w'}$, we obtain the order: $d(w') \prec_\lambda d(w) \preceq_\lambda f_{v'w'} \prec_\lambda f_{vw}$. No matter if $d(w) \prec_\lambda f_{v'w'}$ or $d(w) = f_{v'w'}$ holds, in both cases the planarity of $\sigma(G)$ is violated, as illustrated in Figs. 8a and 8b. Consider now the latter case, in which $d(w') = d(w)$. Since w belongs to L_1, vertex w belongs to the boundary of block $B(w)$ discovered by $d(w') = d(w)$. Similarly, vertex w' belongs to the boundary of block $B(w')$ discovered by $d(w') = d(w)$. For the two blocks $B(w')$ and $B(w)$, either $B(w) \neq B(w')$ or $B(w) = B(w')$ holds. Assume first that $B(w) \neq B(w')$. Since $B(w')$ and $B(w)$ are discovered by the same face, and since $f_{w'} \prec_\lambda f_w$, it follows that $B(w)$ precedes $B(w')$ in the counterclockwise traversal of $d(w') = d(w)$. Otherwise the faces $f_{w'}$ and f_w would violate the planarity of $\sigma(G)$, as illustrated in Fig. 8c. Thus, by Property 7, we obtain $w \prec_\rho w'$. To complete the proof, it remains to consider the case in which $B(w') = B(w)$. Similar to the case above, by Rule R.3, in the counterclockwise traversal of $B(w') = B(w)$ starting from its leader, vertex w precedes w' since otherwise the faces $f_{w'}$ and f_w violate the planarity of $\sigma(G)$, as illustrated in Fig. 8d. $\square$

Finally, we prove that also all tertiary dominator edges can be assigned to a single page. For this, we need the following property of tertiary dominator edges.

Property 16. *Let (v, w) be a tertiary dominator edge of G with $v \in L_0$ and $w \in L_1$. Then v is not the dominator of $d(w)$ and $d(w)$ is not a small face.*

Proof. If v is the dominator of $d(w)$, the edge (v, w) is a primary dominator edge and not tertiary. Assume now, that $d(w)$ is a small face. Then, for every intra-level face f_w that contains w on its boundary, $\mathrm{dom}(d(w))$ is the only L_0-vertex on the boundary of f_w. Hence $v = \mathrm{dom}(d(w))$, which is not possible. $\square$

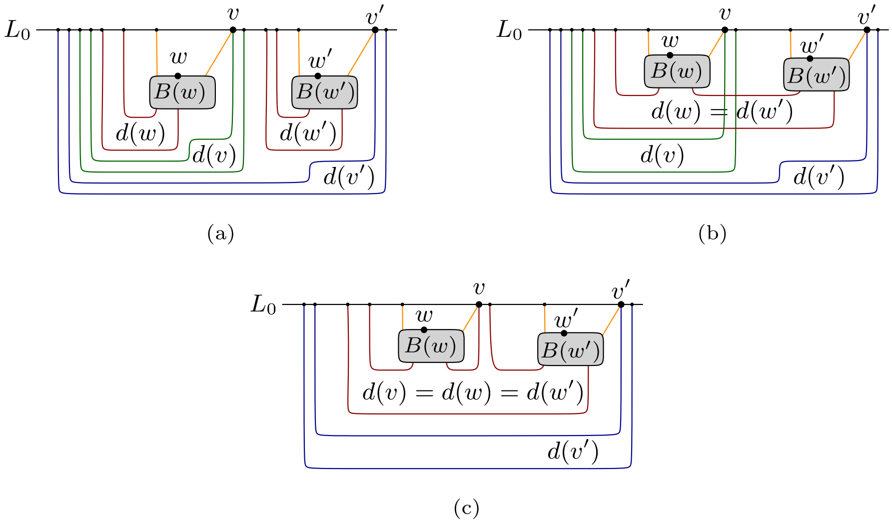


Fig. 9. Illustration for the proof of Lemma 17.

Lemma 17. *Let (v, w) and (v', w') be two tertiary dominator edges of G. Then, (v, w) and (v', w') do not cross in ρ.*

Proof. We may assume without loss of generality that $v, v' \in L_0$, $w, w' \in L_1$, $v \prec_\rho v'$, and that v, w, v' and w' are four distinct vertices of G. Let f_{vw} ($f_{v'w'}$) be a face that has vertices v and w (v' and w') on its boundary. By the definition of tertiary dominator edges, f_{vw} is before $d(v)$ around v and $f_{v'w'}$ is before $d(v')$ around v'. Another immediate consequence of the edges being tertiary dominator edges is that $w \prec_\rho v$ and $w' \prec_\rho v'$. To prove the claim, it remains to show that either $v \prec_\rho w'$ or $w' \prec_\rho w$. We proceed by assuming that $w' \prec_\rho v$ and show that in this case $w' \prec_\rho w$ holds.

With (v, w) and (v', w') being tertiary dominator edges, it follows from Property 16 that $d(w)$ and $d(w')$ are not small and that $v \neq \mathrm{dom}(d(w))$ and $v' \neq \mathrm{dom}(d(w'))$. By Corollary 13, the claim holds if $d(w') \prec_\lambda d(w)$. Hence, we assume that $d(w) \preceq_\lambda d(w')$ and distinguish two cases on whether $d(w) \prec_\lambda d(w')$ or $d(w) = d(w')$ holds. For the former case, note that for f_{vw} to be before $d(v)$ around v, $d(v) \preceq_\lambda d(w)$ must hold. The same argument holds for v' and w'. In total, we have $\mathrm{dom}(f_{vw}) \prec_\rho v$ and $\mathrm{dom}(f_{v'w'}) \prec_\rho v'$ by the definition of tertiary edges. As shown in Fig. 9a, it follows then by planarity of $\sigma(G)$ that $d(v) \preceq_\lambda d(w)$, which yields $d(v) \prec_\lambda d(w')$. Further, in order for vertex w' to be on the boundary of $f_{v'w'}$, it follows by planarity that $v \preceq_\rho \mathrm{dom}(d(w'))$ and in case this holds with equality, $d(w')$ is after $d(v)$ around v. Hence, it follows from Rule R.1 that $v \prec_\rho w'$, which is a contradiction to our initial assumption.

Hence, we focus on the case in which $d(w) = d(w')$ holds. Observe that in this case $d(w) = d(w') = d(v)$ holds, as otherwise the planarity of $\sigma(G)$ is violated, as shown in Fig. 9b. Thus, we assume $d(w) = d(w') = d(v)$, as shown in Fig. 9c. By the planarity of $\sigma(G)$, the vertices w and w' must belong to different blocks $B(w)$ and $B(w')$. With similar arguments as in the proof of Lemma 15, it follows that $B(w')$ precedes $B(w)$ and thus, $w' \prec_\rho w$ follows from Rule R.2, as desired. This finishes the proof. □

In the following, we describe properties that will be useful in the egde-to-page assignment of the non-dominator edges.

Lemma 18. *Let v and w be two vertices of G, such that $v \prec_\rho w$. Also, let f_v and f_w be two intra-level faces containing v and w on their boundaries, respectively, such that $f_v \prec_\lambda f_w$. If the following conditions hold, then $f_v \preceq_\lambda d(w)$.*

(i) v is $d(v)$-prime,
(ii) w is $d(w)$-prime,
(iii) v and w are not the dominators of f_v and f_w, respectively,

Proof. First, observe that by Property 12, we have $d(v) \preceq_\lambda d(w)$. We proceed by considering three cases based on whether v and w belong to L_0 or to L_1 as follows: (a) v belongs to L_0, (b) v belongs to L_1 and w belongs to L_0, and (c) v and w belong to L_1.

- We start with Case (a), in which v belongs to L_0. Since v is $d(v)$-prime, it follows by Property 11 that v is also f_v-prime. However, since v is not the dominator of f_v, it follows that $d(v) = f_v$. Now, the claim $f_v \preceq_\lambda d(w)$ is an immediate consequence of $d(v) \preceq_\lambda d(w)$.

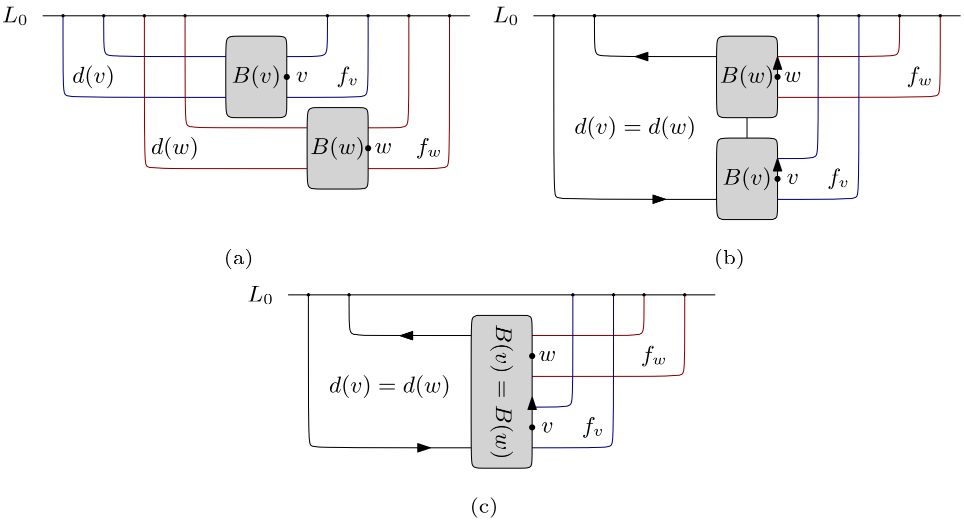


Fig. 10. Illustration for the proof of Lemma 18.

- Consider now Case (b), in which v belongs to L_1 and w belongs to L_0. Analogously to Case (a), it follows by Property 11 that w is also f_w-prime. However, since w is not the dominator of f_w, it follows that $d(w) = f_w$. Now, the claim $f_v \preceq_\lambda d(w)$ is an immediate consequence of the assumption $f_v \prec_\lambda f_w$.
- To complete the proof of the lemma, we consider Case (c), in which v and w belong to L_1. Assume to the contrary that $d(w) \prec_\lambda f_v$. This implies $d(v) \preceq_\lambda d(w) \prec_\lambda f_v \prec_\lambda f_w$. We consider the two subcases, namely, $d(v) \prec_\lambda d(w)$ and $d(v) = d(w)$. In the former case, since v belongs to L_1, vertex v belongs to the boundary of block $B(v)$ discovered by $d(v)$. Similarly, vertex w belongs to the boundary of block $B(w)$ discovered by $d(w)$. Hence, we have $B(v) \neq B(w)$, as $d(v) \prec_\lambda d(w)$; see Fig. 10a. The order $f_v \prec_\lambda f_w$ violates the planarity of $\sigma(G)$; a contradiction. We now consider the case, in which $d(v) = d(w)$. Since v belongs to L_1, vertex v belongs to the boundary of block $B(v)$ discovered by $d(v) = d(w)$. Similarly, vertex w belongs to the boundary of block $B(w)$ discovered by $d(v) = d(w)$. For the two blocks $B(v)$ and $B(w)$ either $B(v) \neq B(w)$ or $B(v) = B(w)$ holds. First, assume that $B(v) \neq B(w)$; see Fig. 10b. $B(v)$ and $B(w)$ are discovered by the same face, and $v \prec_\rho w$. By Rule R.2 it follows that $B(v)$ precedes $B(w)$ in the counterclockwise traversal of $d(v) = d(w)$. With $f_v \prec_\lambda f_w$, the planarity of $\sigma(G)$ is violated; a contradiction. Next, assume $B(v) = B(w)$. Since $v \prec_\rho w$, by Rule R.3, in the counterclockwise traversal of $B(v) = B(w)$ starting from its leader, vertex v precedes w; see Fig. 10c. The order $f_v \prec_\lambda f_w$ violates the planarity of $\sigma(G)$; a contradiction.

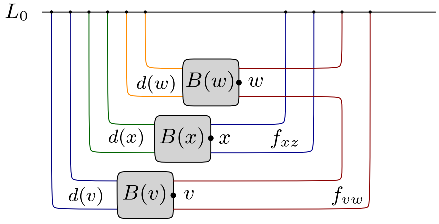
The above case analysis completes the proof. □

The next lemma reveals a relationship between two faces containing two edges that cross in the linear ordering. In the remainder of the paper, we say that an edge (v, w) is *non-dominator*, if (v, w) is neither primary dominator, nor secondary dominator, nor tertiary dominator edge.

Lemma 19. *Let v, w, x and z be four vertices of G, such that (v, w) and (x, z) are two non-dominator edges of G, and $v \prec_\rho x \prec_\rho w \prec_\rho z$. Let f_{vw} be a face with v and w on its boundary, and let f_{xz} be a face with x and z on its boundary such that f_{vw} and f_{xz} are two distinct faces. Moreover, v and w are f_{vw}-prime, whereas x and z are f_{xz}-prime. Then $d(x) = f_{vw}$ or $d(w) = f_{xz}$ holds.*

Proof. We first show that v cannot belong to L_0. Assume the contrary. Vertex v is not the dominator of f_{vw}, as otherwise (v, w) would be a dominator edge. However, with $v \prec_\rho w$, it follows that w also belongs to L_0. Since w is f_{vw}-prime, and $v \prec_\rho w$, the only way for x to lie between v and w in ρ is when $f_{vw} = f_{xz}$ holds; a contradiction. The same argumentation holds for $x \prec_\rho w \prec_\rho z$. Hence, we may assume that both v and x belong to L_1. By Property 12, we have $d(v) \preceq_\lambda d(x)$. We assume to the contrary that $d(x) \neq f_{vw}$ and $d(w) \neq f_{xz}$ hold.

We consider the two cases (a) $f_{vw} \prec_\lambda f_{xz}$ and (b) $f_{xz} \prec_\lambda f_{vw}$. First, consider Case (a). We continue by distinguishing between two subcases based on whether $f_{vw} \prec_\lambda d(x)$ or $d(x) \prec_\lambda f_{vw}$. We start with $f_{vw} \prec_\lambda d(x)$. Let u_j belong to L_0 and be the f_{vw}-prime vertex with the highest index. Then every f_{vw}-prime vertex, except for possibly u_j, precedes any $d(x)$-prime vertex that is discovered by $d(x)$. Since w is f_{vw}-prime and x belongs to L_1, it follows that $w \prec_\rho x$ if $w \neq u_j$; a contradiction. However, if $w = u_j$, then the only way for $x \prec_\rho w$ to hold is when w is not $d(w)$-prime and $d(x)$ is a small face with w as its dominator. But then it follows by planarity and $x \prec_\rho z$, that also f_{xz} is a small face with w as its dominator. This yields $z \prec_\rho w$, a contradiction. Hence, we may focus on the case $d(x) \prec_\lambda f_{vw}$. Our plan is to apply Lemma 18 on vertices v and x for which we know that $v \prec_\rho x$ and $f_{vw} \prec_\lambda f_{xz}$. Since v and x belong to L_1, Conditions i



(a)

Fig. 11. Illustrations for the proof of Lemma 19.

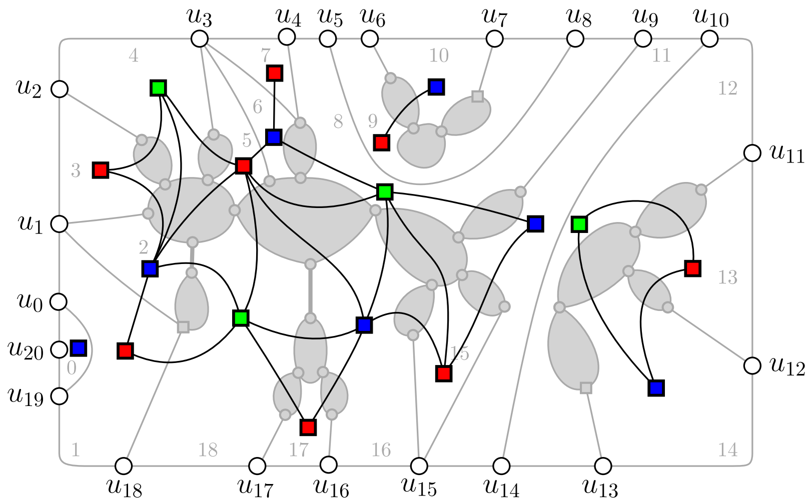


Fig. 12. The conflict graph of the example illustrated in Fig. 4.

and ii of Lemma 18 are satisfied. Furthermore, with v, x belonging to L_1, Condition iii of Lemma 18 holds as well. Hence, by Lemma 18, we obtain $f_{vw} \preceq_\lambda d(x)$. This contradicts the original assumption $d(x) \prec_\lambda f_{vw}$.

Next, consider Case (b), in which $f_{xz} \prec_\lambda f_{vw}$. By Property 5, we have $d(x) \preceq_\lambda f_{xz}$ which together with $d(v) \preceq_\lambda d(x)$ implies $d(v) \preceq_\lambda d(x) \preceq_\lambda f_{xz} \prec_\lambda f_{vw}$. By assumption, $d(w) \neq f_{xz}$. We continue by considering two subcases based on whether $f_{xz} \prec_\lambda d(w)$ or $d(w) \prec_\lambda f_{xz}$. First, assume $f_{xz} \prec_\lambda d(w)$. By Property 5, it follows that $d(z) \preceq_\lambda f_{xz}$. The latter two inequalities imply $d(z) \prec_\lambda d(w)$. We claim that z is $d(z)$-prime. Assume for contradiction that z is not $d(z)$-prime. In particular, z belongs to L_0 and $f_{xz} \neq d(z)$, since this would contradict z being f_{xz}-prime. Since f_{xz} has z on its boundary, f_{xz} either comes before or after $d(z)$ around z. If f_{xz} comes before $d(z)$ around z, the edge (x, z) is a dominator edge, contradicting the original assumption. Hence, we may assume that f_{xz} comes after $d(z)$ around z. However, since (x, z) is not a dominator edge, we have $z \neq \mathrm{dom}(f_{xz})$. By planarity, it follows that $\mathrm{dom}(f_{xz}) \preceq_\rho \mathrm{dom}(d(z))$ but more importantly, $f_{xz} \prec_\lambda d(z)$, which is a contradiction to Property 5. Thus, our claim is true and we may assume that z is $d(z)$-prime. Now, it follows from Corollary 13 and $d(z) \prec_\lambda d(w)$, that $z \prec_\rho w$ which contradicts our assumption $w \prec_\rho z$. Hence, in the following we consider the case $d(w) \prec_\lambda f_{xz}$. We distinguish two subcases based on whether w belongs to L_0 or to L_1. First, consider the case, in which w belongs to L_0. If w is $d(w)$-prime, $d(w) = f_{vw}$ follows by Property 11 since w is not the dominator of f_{vw}. Therefore, we have $d(w) \prec_\lambda f_{xz} \prec_\lambda f_{vw} = d(w)$; a contradiction. Thus, we may assume that w is not $d(w)$-prime which yields $d(w) \prec_\lambda f_{xz} \prec_\lambda f_{vw}$. If f_{vw} comes before $d(w)$ around w, it follows from $d(z) \preceq_\lambda f_{xz} \prec_\lambda f_{vw}$ and z being $d(z)$-prime that $z \prec_\rho w$; a contradiction. Hence, we may assume that f_{vw} comes after $d(w)$ around w. However, with $d(w) \prec_\lambda f_{vw}$, w must be the dominator of f_{vw}, contradicting the assumption that (v, w) is a non-dominator edge. To complete the proof of the lemma, it remains to consider the case, in which w belongs to L_1. Observe that $d(v) \preceq_\lambda d(x) \preceq_\lambda d(w)$ by Property 12. This yields $d(v) \preceq_\lambda d(x) \preceq_\lambda d(w) \prec_\lambda f_{xz} \prec_\lambda f_{vw}$. As illustrated in Fig. 11a, f_{vw} violates the planarity of $\sigma(G)$. Note, that the same violation of planarity occurs, when some of the three discovering faces are the same, since the order of the blocks (or the order of the vertices within the blocks) remains the same due to Rule R.3. □

Observe that in Lemma 19 the edges (v, w) and (x, z) form two non-dominator edges that cannot be assigned to the same page. Lemma 19 translates this conflict into a relationship between the two faces f_{vw} and f_{xz} containing these edges. In the following, we model these conflicts as edges of an auxiliary graph which we call the *conflict graph* and denote by $\mathcal{C}(G)$; see also Fig. 12 for an illustration.

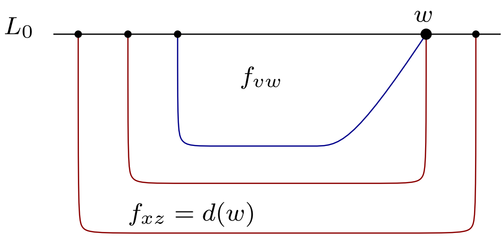


Fig. 13. Illustration for the proof of Lemma 20.

Definition 1. *The* conflict graph $\mathcal{C}(G)$ *of* G *is an undirected graph whose vertices are the faces of* $\mathcal{F}$*. There exists an edge* (f, g) *with* $f \neq g$ *in* $\mathcal{C}(G)$ *if and only if there exists a vertex* w *of level* L_1 *on the boundary of* g *such that* $f = d(w)$*.*

With this definition, we are can restate Lemma 19 as follows.

Lemma 20. *Let* (v, w) *and* (x, z) *be two non-dominator edges of* G *belonging to two distinct faces* f_{vw} *and* f_{xz} *such that* v *and* w *are* f_{vw}*-prime,* x *and* z *are* f_{xz}*-prime,* $v \prec_\rho w$*, and* $x \prec_\rho z$*. If* (v, w) *and* (x, z) *cross in* ρ*, then there is an edge* (f_{vw}, f_{xz}) *in* $\mathcal{C}(G)$*.*

Proof. Without loss of generality, we may assume $v \prec_\rho x \prec_\rho w \prec_\rho z$. As in the proof of Lemma 19, we first show that v and x belong to L_1. Furthermore, by Lemma 19, we have that $f_{vw} = d(x)$ or $f_{xz} = d(w)$ holds. Since x belongs to L_1, it follows that there is an edge (f_{vw}, f_{xz}) in $\mathcal{C}(G)$ if $f_{vw} = d(x)$ holds. Thus, consider $f_{xz} = d(w)$. If w belongs to L_1, it follows that there is an edge (f_{vw}, f_{xz}) in $\mathcal{C}(G)$. Hence, assume that w is on L_0. Recall that $f_{vw} \neq f_{xz}$ holds, vertex w is f_{vw}-prime, and we have $w \neq \mathrm{dom}(f_{vw})$, since (v, w) is not a dominator edge. We split the proof into the two cases (a) $f_{vw} \prec_\lambda f_{xz}$ and (b) $f_{xz} \prec_\lambda f_{vw}$. In Case (a), we get $d(w) \preceq_\lambda f_{vw} \prec_\lambda f_{xz} = d(w)$ by Property 5; a contradiction. In Case (b), we observe that if w is $d(w)$-prime, we have $d(w) = f_{xz} \prec_\lambda f_{vw}$ and thus, $w = \mathrm{dom}(f_{vw})$ by Property 11; a contradiction. Hence, we may assume that w is not $d(w)$-prime. However, since w is f_{vw}-prime and $w \neq \mathrm{dom}(f_{vw})$, there is at least one vertex on L_0 right before w in a clockwise traversal of L_0 that is also on the boundary of f_{vw}. This is illustrated in Fig. 13. Now, recall that by Property 12 and since v and x belong to L_1, we have $d(v) \preceq_\lambda d(x)$. Together with Property 5, we conclude that $d(v) \preceq_\lambda d(x) \preceq_\lambda f_{xz}$. In fact, $d(v) = d(x) = f_{xz}$ has to hold; otherwise not both $d(v)$ and f_{vw} could bound the block $B(v)$ without violating the planarity of $\sigma(G)$. Since $d(v) = f_{xz}$ and since v belongs to L_1, the edge (f_{vw}, f_{xz}) exists in $\mathcal{C}(G)$. $\square$

In the following lemma, we prove an important property of the conflict graph.

Lemma 21. *Graph* $\mathcal{C}(G)$ *is 1-page book embeddable.*

Proof. We order the vertices of $\mathcal{C}(G)$ according to $\lambda(\mathcal{F})$. Suppose for contradiction that two edges (f, g) and (f', g') of $\mathcal{C}(G)$ cross in $\lambda(\mathcal{F})$ such that, without loss of generality, $f \prec_\lambda f' \prec_\lambda g \prec_\lambda g'$. By definition of $\mathcal{C}(G)$, there is either a vertex v of level L_1 on the boundary of f such that $g = d(v)$, or there is a vertex w of level L_1 on the boundary of g such that $f = d(w)$. In the first case, by Property 5, we have $d(v) \preceq_\lambda f$, which contradicts $g = d(v) \preceq_\lambda f \prec_\lambda g$. Now consider the second case. We argue analogously for the edge (f', g'). Hence, there exist two vertices w and w' of level L_1 on the boundaries of g and g', respectively, such that $f = d(w)$ and $f' = d(w')$ hold. This yields $d(w) \prec_\lambda d(w') \prec_\lambda g \prec_\lambda g'$. Since w and w' belong to L_1, they are $d(w)$- and $d(w')$-prime, respectively. By Corollary 13 and since $w \neq w'$, we have $w \prec_\rho w'$. Now we apply Lemma 18 on w and w' with $f_v = g$ and $f_w = g'$, and obtain $g \preceq_\lambda d(w)$, a contradiction to the fact that $d(w) \prec_\lambda g$. $\square$

Since $\mathcal{C}(G)$ is 1-page book embeddable, it is outerplanar [10]. Hence, the following corollary becomes a direct implication of Lemma 21.

Corollary 22. *Graph* $\mathcal{C}(G)$ *admits a vertex coloring with three colors.*

We are now ready to describe how to assign the edges of G to the pages of the book embedding. First, we embed all primary edges in a single page p_0, all secondary edges in a single page p_1, and all tertiary edges in a single page p_2. By Lemmata 14, 15 and 17, this assignment is valid. Next, we assign the remaining edges of G to a total of $3 \cdot \left\lceil \frac{k}{2} \right\rceil$ pages. To ease the description, we partition these pages into three sets R^1, B^1, and G^1, each containing $\left\lceil \frac{k}{2} \right\rceil$ pages as follows: $R^1 = \{r^1_1, \ldots, r^1_{\lceil k/2 \rceil}\}$, $B^1 = \{b^1_1, \ldots, b^1_{\lceil k/2 \rceil}\}$, and $G^1 = \{g^1_1, \ldots, g^1_{\lceil k/2 \rceil}\}$. The actual assignment is done by processing the intra-level faces of $\mathcal{F}$ according to the ordering $\lambda(\mathcal{F})$. Assume that we have processed a certain number of faces in this order and that we have assigned all the non-dominator edges of G that are induced by the vertices of these faces in the pages

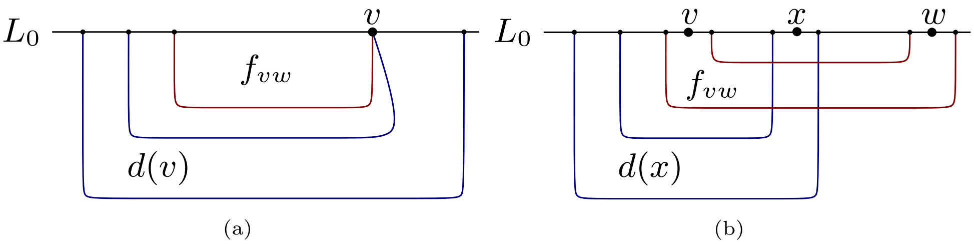


Fig. 14. Illustrations for the proofs of (a) Property 24, and (b) Property 25.

mentioned above. Let f be the next face to process. By Corollary 22, face f has a color out of three available ones, say red, blue, and green. Now, observe that the vertices of f induce at most a k-clique Q_f in G. Also, observe that some of the edges on the boundary of f may have been already assigned to a page. We assign the remaining non-dominator edges of Q_f to the pages of one of the sets R^1, B^1, and G^1 according to the color of f. Since Q_f is at most a k-clique, $\lceil \frac{k}{2} \rceil$ pages are sufficient regardless of the underlying linear order [10].

The remainder of this section is devoted in proving that the (non-dominator) edges assigned to the pages in R^1, B^1, and G^1 do not cross, and thus that the computed book embedding is valid. Consider two non-dominator edges (v, w) and (x, z), and let f_{vw} and f_{xz} be the faces of $\mathcal{F}$ responsible for assigning (v, w) and (x, z) to one of the pages of $R^1 \cup B^1 \cup G^1$. If v and w are f_{vw}-prime, and if x and z are f_{xz}-prime, then by Lemma 20, we know that (v, w) and (x, z) do not cross. Hence, we may assume that the edges (v, w) and (x, z) are incident to vertices that are not prime with respect to the face that belongs to that edge. In this direction, we need a few auxiliary lemmata.

Property 23. *Let v and w be two vertices with $v \prec_\rho w$ on the boundary of a face f_{vw}. If w is f_{vw}-prime, then v is also f_{vw}-prime. If v is not f_{vw}-prime, then w is not f_{vw}-prime.*

Proof. Both claims follow from the fact that all vertices that are f_{vw}-prime precede those that are not f_{vw}-prime. Since $v \prec_\rho w$, the property follows. □

Property 24. *Let v and w be two vertices of G. If the following conditions hold, then $d(v) = f_{vw}$.*

(i) v and w belong to L_0,
(ii) v and w are on the boundary of a face f_{vw}, and
(iii) $dom(f_{vw}) \prec_\rho v \prec_\rho w$.

Proof. Condition i and Property 5 imply that $d(v) \preceq_\lambda f_{vw}$. To prove the property, assume to the contrary $d(v) \prec_\lambda f_{vw}$. Since, by Condition iii, $\mathrm{dom}(f_{vw})$ precedes v, vertex v cannot be prime with respect to face $d(v)$ that discovers v. However, it follows that v is the last vertex on L_0 in the ordering ρ that is on the boundary of f_{vw}; see Fig. 14a. This contradicts the existence of vertex w, which is also on L_0 (by Condition i), on the boundary of f_{vw} and follows v in the ordering ρ (by Condition iii). □

Property 25. *Let v, w and x be three vertices of G. If the following conditions hold, then $f_{vw} \preceq_\lambda d(x)$.*

(i) v and w belong to L_0,
(ii) v and w are on the boundary of a face f_{vw}, and
(iii) $dom(f_{vw}) \prec_\rho v \prec_\rho x \prec_\rho w$.

Proof. Assume to the contrary that $d(x) \prec_\lambda f_{vw}$, therefore $\mathrm{dom}(d(x)) \preceq_\rho \mathrm{dom}(f_{vw})$. Hence, by Condition iii, we obtain $\mathrm{dom}(d(x)) \preceq_\rho \mathrm{dom}(f_{vw}) \prec_\rho v \prec_\rho x \prec_\rho w$. Recall that x is placed between v and w (by Condition iii), both v and w belong to L_0 (by Condition i) and on the boundary of f_{vw} (Condition ii), and neither v nor w is the dominator of f_{vw} (by Condition iii). It follows that either x also belongs to L_0, or x is discovered by a face $d(x)$ with $v \preceq_\rho \mathrm{dom}(d(x))$. The latter case contradicts the fact that $\mathrm{dom}(d(x)) \prec_\rho v$. In the former case, it follows that the faces $d(x)$ and f_{vw} violate planarity of $\sigma(G)$; refer to Fig. 14b for an illustration. Since both cases have been led to a contradiction, the proof follows. □

Property 26. *Let v and x be two vertices of G. If the following conditions hold, then $dom(f_v) \preceq_\rho dom(f_x) \preceq_\rho x \prec_\rho v$.*

(i) v and x belong to L_0,
(ii) v is on the boundary of a face f_v,

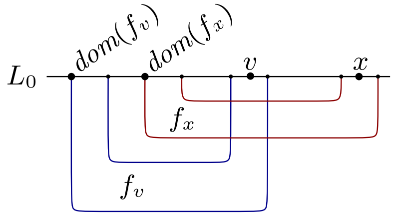


Fig. 15. Illustration for the proof of Property 26.

(iii) x is on the boundary of a face f_x,
(iv) $f_v \prec_\lambda f_x$, and
(v) $dom(f_x) \prec_\rho v$.

Proof. By Condition iv, we obtain $\text{dom}(f_v) \preceq_\rho \text{dom}(f_x)$. Since x is on the boundary of f_x (by Condition iii) and on L_0 (by Condition i), it follows that $\text{dom}(f_x) \preceq_\rho x$. This together with Condition v imply that, in order to prove the property, it suffices to show that $x \prec_\rho v$; recall that we have already shown that $\text{dom}(f_v) \preceq_\rho \text{dom}(f_x)$. Assume to the contrary that $v \prec_\rho x$. By Conditions iv and v, it follows that v is not f_v-prime. Since $v \prec_\rho x$, this leads to the order $\text{dom}(f_v) \preceq_\rho \text{dom}(f_x) \prec_\rho v \prec_\rho x$ and all of these vertices belong to L_0 (by Condition i). Together with Condition iv, this violates the planarity of $\sigma(G)$, as illustrated in Fig. 15. □

Lemma 27. *Let x and z be two vertices of G belonging to the boundary of a face f_{xz} such that $dom(f_{xz}) \prec_\rho x \prec_\rho z$, and let f be a face preceding f_{xz} in $\lambda(\mathcal{F})$, that is, $f \prec_\lambda f_{xz}$. Then, for any vertex y of G with $x \prec_\rho y \prec_\rho z$, we have that y is not on the boundary of f.*

Proof. First, we claim that x is discovered by f_{xz}, that is $f_{xz} = d(x)$. If x belongs to L_1, the claim follows from $\text{dom}(f_{xz}) \prec_\rho x$. Now consider the case in which x belongs to L_0. Since x is preceded by $\text{dom}(f_{xz})$ and followed by vertex z, and both vertices belong to the boundary of f_{xz}, vertex z must belong to L_0 as well. Property 24 concludes the claim. Assume for a contradiction that there exists a vertex y with $x \prec_\rho y \prec_\rho z$ that is on the boundary of f. Note that by assumption $x \neq y \neq z$ holds. We distinguish two cases.

- *Vertex y belongs to L_1*: In this case, y is f-prime and assigned to the block $B(y)$. Since y is on the boundary of f, we obtain $d(B(y)) \preceq_\lambda f \prec_\rho f_{xz} = d(x)$. Hence, it follows by Corollary 13 that $y \prec_\rho x$; a contradiction.
- *Vertex y belongs to L_0*: We first observe that $\text{dom}(f_{xz}) \prec_\rho y$ holds, as otherwise we have that $y \preceq_\rho \text{dom}(f_{xz}) \prec_\rho x \prec_\rho z$, which is a clear contradiction. Vertex z either belongs to L_0 or to L_1. First, assume that z belongs to L_0. By Property 26, we obtain $\text{dom}(f) \preceq_\rho \text{dom}(f_{xz}) \prec_\rho z \prec_\rho y$; a contradiction. In the latter case, z is assigned to the block $B(z)$ and with $\text{dom}(f_{xz}) \prec_\rho z$, we get $d(B(z)) = d(z) = f_{xz}$. By Rule R.1, z is placed right after $\text{dom}(f_{xz})$ and to the left of the next vertex on L_0 after $\text{dom}(f_{xz})$. With y belonging to L_0, we obtain $z \prec_\rho y$; a contradiction.

Since each of the cases above have been led to a contradiction, the proof of the lemma follows. □

As a next step, we will consider all cases of crossing non-dominator edges that might arise depending on whether the endpoints are prime or not. In order to reduce the number of cases we show the two following lemmata.

Lemma 28. *Let (v, w) and (x, z) be two non-dominator edges of G belonging to two distinct faces f_{vw} and f_{xz}, respectively, such that $v \prec_\rho w$, $x \prec_\rho z$ and $f_{vw} \prec_\lambda f_{xz}$. If (v, w) and (x, z) cross, then either the edge (f_{vw}, f_{xz}) exists in $\mathcal{C}(G)$, or there exists a non-dominator edge (x', z') in f_{xz} with x' and z' being f_{xz}-prime such that (v, w) and (x', z') cross.*

Proof. Since $v \prec_\rho w$, $x \prec_\rho z$ and since (v, w) and (x, z) cross, either (a) $v \prec_\rho x \prec_\rho w \prec_\rho z$ or (b) $x \prec_\rho v \prec_\rho z \prec_\rho w$ holds. We proceed by distinguishing different cases depending on whether x and z are f_{xz}-prime or not.

We first claim that at least one of the vertices x and z is f_{xz}-prime. For a contradiction, assume that neither x nor z is f_{xz}-prime. In this case, $\text{dom}(f_{xz}) \prec_\rho x \prec_\rho z$ holds. For the partial order of v, w, x and z of Case (a), we apply Property 25 on vertices x, z, and w, and we obtain $f_{xz} \preceq_\lambda d(w)$. By Property 5, we further obtain that $d(w) \preceq_\lambda f_{vw}$. Hence, $f_{xz} \preceq_\lambda d(w) \preceq_\lambda f_{vw}$ must hold, which is a contradiction to the fact that $f_{vw} \prec_\lambda f_{xz}$. For the partial order of Case (b), we obtain a contradiction by applying an argument analogous to the one above in which we interchange the roles of w and v.

By the above claim, we may assume that at least one of the vertices x and z is f_{xz}-prime. Note that if x is not f_{xz}-prime, then, by Property 23, z is not f_{xz}-prime either. Hence, we can conclude that x is f_{xz}-prime, while z is not f_{xz}-prime. We proceed by setting x' to be x (i.e., $x' := x$). Since z is not f_{xz}-prime, z belongs to L_0. It follows that $\text{dom}(f_{xz}) \prec_\rho z$.

We first rule out the case, in which there exists an f_{xz}-prime vertex $\overline{z}$, such that $\text{dom}(f_{xz}) \prec_\rho \overline{z} \prec_\rho z$. By Lemma 27, there is no vertex between $\overline{z}$ and z in ρ that belongs to the boundary of f_{vw}. Hence, edges (v, w) and $(x, \overline{z})$ cross, since (v, w) and (x, z) cross. The proof of the lemma follows by setting z' to be $\overline{z}$.

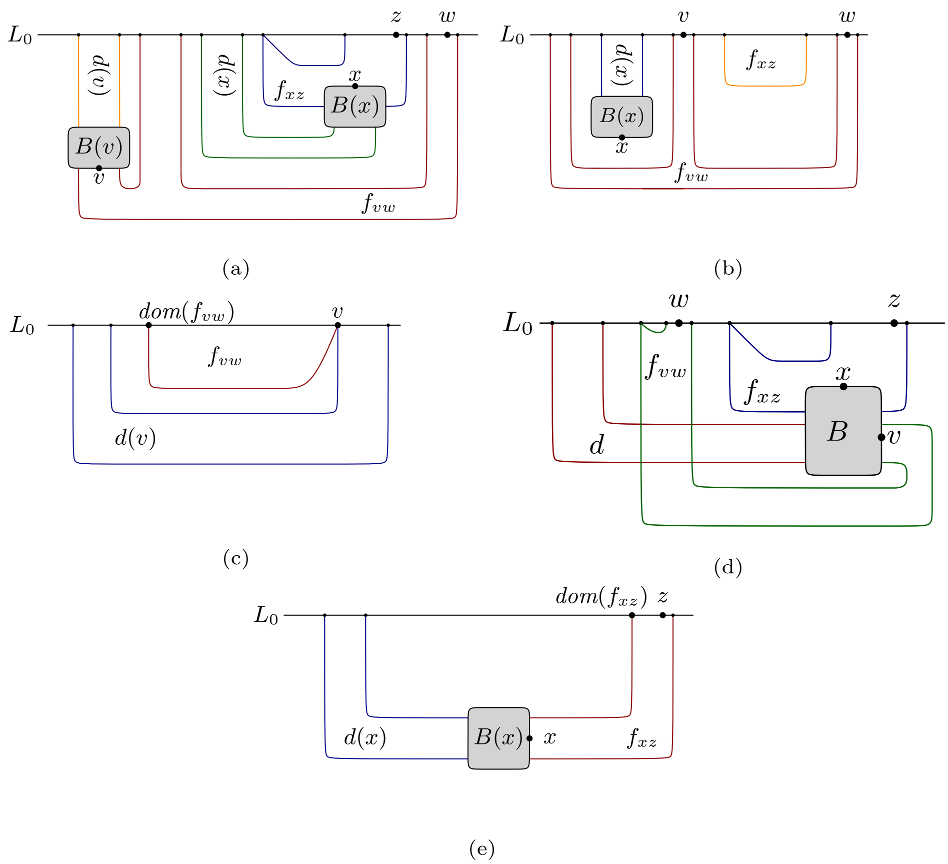


Fig. 16. Illustrations for the proof of Lemma 28.

To complete the proof of the lemma, we have to focus on the case, in which there exists no f_{xz}-prime vertex as defined above. In this case, the dominator of f_{xz} is the only f_{xz}-prime vertex on L_0. Since x is f_{xz}-prime and since x is not the dominator of face f_{xz} (recall that the edge (x, z) is a non-dominator edge and $x \prec_\rho z$), it follows that $x \prec_\rho \mathrm{dom}(f_{xz})$, which in particular implies that x belongs to L_1. Since $x \prec_\rho \mathrm{dom}(f_{xz})$, either the face $d(x)$ that discovers x strictly precedes f_{xz} in $\lambda(\mathcal{F})$ or $d(x)$ is identified with f_{xz} and f_{xz} is small.

We first prove that the latter case does not apply. To see this, assume for a contradiction that f_{xz} is small. Then, $\mathrm{dom}(f_{xz})$ is the only L_0-vertex on the boundary of f_{xz}. Since (x, z) is a non-dominator edge, it follows that $\mathrm{dom}(f_{xz}) \neq z$, which is a contradiction since z belongs to L_0. From the discussion above it follows that $d(x) \prec_\lambda f_{xz}$. We next argue that $d(x) = f_{vw}$ holds, which implies that the edge (f_{vw}, f_{xz}) exist in $\mathcal{C}(G)$, since we have already proved that x belongs to L_1. Hence, the proof of this property also concludes the proof of this lemma.

We assume for a contradiction that $d(x) \neq f_{vw}$ holds. We distinguish two cases based on whether $f_{vw} \prec_\lambda d(x)$ or $d(x) \prec_\lambda f_{vw}$. First, suppose that $f_{vw} \prec_\lambda d(x)$ and consider the partial order of Case (a). Since $x \prec_\rho w$ and $f_{vw} \prec_\lambda d(x)$, it follows that vertex w is not f_{vw}-prime. Thus, w belongs to L_0 and $\mathrm{dom}(d(x)) \prec_\rho w$. By the planarity of $\sigma(G)$, it follows that $z \prec_\rho w$ (see Fig. 16a); a contradiction. Consider now the partial order of Case (b). By Property 5, we obtain $d(v) \preceq_\lambda f_{vw} \prec_\lambda d(x)$. Since $x \prec_\rho v$, it follows that v belongs to L_0 and is not f_{vw}-prime. By Property 23, w is also not f_{vw}-prime; see Fig. 16b for an illustration. For $v \prec_\rho z \prec_\rho w$ to hold, we must have $v \preceq_\rho \mathrm{dom}(d(z)) \prec_\rho w$ and $d(z)$ cannot be small or $v \prec_\rho \mathrm{dom}(d(z)) = w$ and $d(z))$ is small. In both cases x and z cannot both be on the boundary of f_{xz} without violating the planarity of $\sigma(G)$; a contradiction.

Suppose now that $d(x) \prec_\lambda f_{vw}$ and consider first the partial order of Case (a). Since x belongs to L_1, it is $d(x)$-prime. We apply Property 12 with $v \prec_\rho x$, and we get $d(v) \preceq_\lambda d(x)$. Hence, the order is $d(v) \preceq_\lambda d(x) \prec_\lambda f_{vw} \prec_\lambda f_{xz}$. If v belongs to L_0, vertex v cannot be $d(v)$-prime, since otherwise $v = \mathrm{dom}(f_{vw})$ follows by Property 11; a contradiction. Since v is not the dominator of f_{vw}, it follows that v is the last L_0 vertex on the boundary of f_{vw}, and hence, we get that $w \prec_\rho v$, as illustrated in Fig. 16c. Thus, we may assume that both v and x belong to L_1. Furthermore, $d(x)$ and f_{xz} are both incident to $B(x)$ while $d(v)$ and f_{vw} are both incident to $B(v)$. By the planarity of $\sigma(G)$, we have $B(x) = B(v)$ which we abbreviate with B. Thus, $d(x) = d(v)$, which we abbreviate with d. The three faces d, f_{vw}, and f_{xz} are incident to block B and by the fact that $d \prec_\lambda f_{vw} \prec_\lambda f_{xz}$, they appear in this counterclockwise order around B. This violates the planarity of $\sigma(G)$, as illustrated in Fig. 16d.

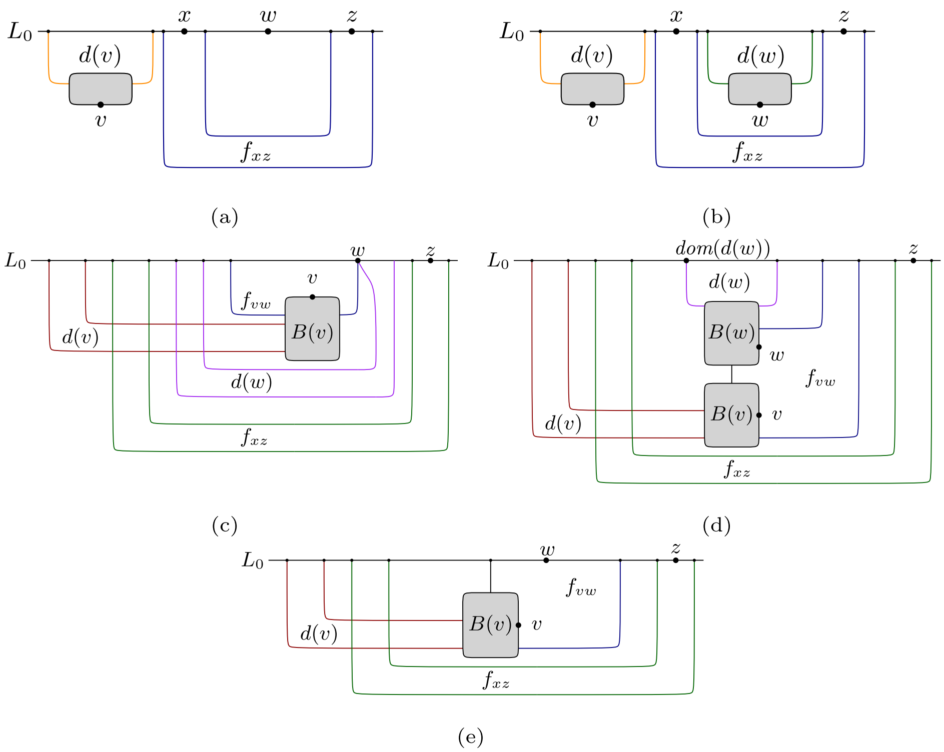


Fig. 17. Illustrations for the proof of Lemma 29.

Next, consider the partial order of Case (b). We have $d(x) \prec_\lambda f_{vw} \prec_\lambda f_{xz}$. Recall that z is not f_{xz}-prime and therefore an L_0-vertex different from $\mathrm{dom}(f_{xz})$; see Fig. 16e. Now vertex w either belongs to L_0 or to L_1. In the first case we have the order $\mathrm{dom}(d(x)) \preceq_\rho \mathrm{dom}(f_{vw}) \preceq_\rho \mathrm{dom}(f_{xz}) \prec_\rho z \prec_\rho w$ of vertices on L_0. Together with $d(x) \prec_\lambda f_{vw} \prec_\lambda f_{xz}$, and in order for f_{vw} to bound vertex w, the planarity of $\sigma(G)$ is violated. Assume the second case, that is w belongs to L_1. By $z \prec_\rho w$, we have $z \preceq_\rho \mathrm{dom}(d(w))$ on L_0. With Property 5, we obtain $\mathrm{dom}(d(w)) \preceq_\rho \mathrm{dom}(f_{vw})$. However, then we have $\mathrm{dom}(f_{vw}) \preceq_\rho \mathrm{dom}(f_{xz}) \prec_\rho z \preceq_\rho \mathrm{dom}(d(w)) \preceq_\rho \mathrm{dom}(f_{vw})$; a contradiction. □

Lemma 29. *Let (v, w) and (x, z) be two non-dominator edges of G belonging to two distinct faces f_{vw} and f_{xz}, respectively, such that v and w are f_{vw}-prime, x is f_{xz}-prime, and z is not f_{xz}-prime. If (v, w) and (x, z) cross such that $v \prec_\rho x \prec_\rho w \prec_\rho z$, then the edge (f_{vw}, f_{xz}) exists in $\mathcal{C}(G)$.*

Proof. First, we rule out the case, in which $f_{vw} \prec_\lambda f_{xz}$. Similar to the proof of Lemma 19, we argue that v cannot belong to L_0. To see this, assume the contrary. Since v is not the dominator of f_{vw} and since $v \prec_\rho w$, it follows that w also belongs to L_0. Since w is also f_{vw}-prime and since $v \prec_\rho w$, the only way for x to appear between v and w in ρ, is if $f_{vw} = f_{xz}$, which is a contradiction to the fact that f_{vw} and f_{xz} are distinct. Next, we claim that x belongs to L_1 as well. Assume the contrary. Since z is not f_{xz}-prime, z also belongs to L_0. Since (x, z) is a non-dominator edge, it follows that $\mathrm{dom}(f_{xz}) \prec_\rho x \prec_\rho z$. We apply Lemma 27 to $\mathrm{dom}(f_{xz}) \prec_\rho x \prec_\rho w \prec_\rho z$ with $f_{vw} \prec_\lambda f_{xz}$ and obtain that w cannot be on the boundary of f_{vw}, which is a contradiction. Thus, we may assume that both v and x belong to L_1. By Property 12, it follows that $d(v) \preceq_\lambda d(x)$. Observe that if $d(x) = f_{vw}$, then the edge (f_{vw}, f_{xz}) exists in $\mathcal{C}(G)$, as desired, since we have already shown that x belongs to L_1.

In order to prove the lemma for the case, in which $f_{vw} \prec_\lambda f_{xz}$, it suffices to show that the case, in which $d(x) \neq f_{vw}$, does not apply. Our proof is by contradiction. First, assume $f_{vw} \prec_\lambda d(x)$. This implies that every vertex that is f_{vw}-prime precedes any vertex that is $d(x)$-prime and that is discovered by $d(x)$. The only exception to this can occur if $w = \mathrm{dom}(d(x))$ and $d(x)$ is small. However, in that case w must also be the dominator of $d(z)$ due to planarity, while $d(z)$ is small as well. This would yield $z \prec_\rho w$; a contradiction. Hence, we may assume that every vertex that is f_{vw}-prime precedes any vertex that is $d(x)$-prime and that is discovered by $d(x)$. Since w is f_{vw}-prime and since x belongs to L_1, it follows that $w \prec x$, which is a contradiction. Hence, we may focus on the case, in which $d(x) \prec_\lambda f_{vw}$. Since $d(v) \preceq_\lambda d(x)$ and since $f_{vw} \prec_\lambda f_{xz}$, it follows that $d(v) \preceq_\lambda d(x) \prec_\lambda f_{vw} \prec_\lambda f_{xz}$. By Property 5, we obtain $d(w) \preceq_\lambda f_{vw}$. Our plan is to apply Lemma 18 on vertices v and x for which we know that $v \prec_\rho x$ and $f_{vw} \prec_\lambda f_{xz}$. Since v and x belong to L_1, Conditions i and ii of Lemma 18 are satisfied. Also, since (v, w) and (x, z) are non-dominator edges, Condition iii of Lemma 18 is satisfied. Hence, by Lemma 18, we have $f_{vw} \preceq_\lambda d(x)$. This contradicts the previous assumption that $d(x) \prec_\lambda f_{vw}$.

To complete the proof of the lemma, we now consider the case, in which $f_{xz} \prec_\lambda f_{vw}$. Our aim is to apply Lemma 18 on $x \prec_\rho w$. For Condition i of Lemma 18 to hold, we prove an even stronger argument, namely that x belongs to L_1. Assume to the contrary that x is on L_0. Since z is not f_{xz}-prime, z also belongs to L_0. Since $v \prec_\rho x \prec_\rho w \prec_\rho z$, and since (v, w) and (x, z) are non-dominator edges, we obtain the following order of vertices on L_0: $\mathrm{dom}(d(v)) \prec_\rho x \prec_\rho w \prec_\rho z$ or $\mathrm{dom}(d(v)) \prec_\rho x \preceq_\rho \mathrm{dom}(d(w)) \prec_\rho z$, depending on whether w belongs to L_0 or to L_1. However, in both cases face f_{vw} violates the planarity of $\sigma(G)$ as shown in Figs. 17a and 17b. Hence, x belongs to L_1 and Condition i of Lemma 18 is satisfied. We now claim that v belongs to L_1 as well. To prove the claim, assume the contrary. Since v and w are on the boundary of the same face and since $v \prec_\rho w$, it follows that w belongs to L_0, too. Since (v, w) is a non-dominator edge, we get $\mathrm{dom}(f_{vw}) \prec_\rho v \prec_\rho w$. By applying Lemma 27 on $\mathrm{dom}(f_{vw}) \prec_\rho v \prec_\rho x \prec_\rho w$, we conclude that x cannot be on the boundary of f_{xz}, which is a contradiction. Hence, v belong to L_1, as desired. Next, we prove Condition ii of Lemma 18, that is, w is $d(w)$-prime. For a contradiction, assume that w is not $d(w)$-prime, which yields that w belongs to L_0. Since, by assumption, w is f_{vw}-prime, we get $d(w) \neq f_{vw}$. In particular, by Property 5, we have that $d(w) \prec_\lambda f_{vw}$. Since v and x belong to L_1, by Properties 5 and 12, it follows that $d(v) \preceq_\lambda d(x) \preceq_\lambda f_{xz}$. If $d(v) = f_{xz}$ holds, then the edge (f_{vw}, f_{xz}) exists in $\mathcal{C}(G)$, since we have already shown that v belongs to L_1. Thus, assume $d(v) \neq f_{xz}$ which yields $d(v) \prec_\lambda f_{xz}$. We illustrate these relationships in Fig. 17c and observe that in order for $d(v)$ and f_{vw} to be incident to block $B(v)$, the planarity of $\sigma(G)$ is violated. Hence, we may assume that w is $d(w)$-prime and therefore Condition ii of Lemma 18 is satisfied. Finally, Condition iii of Lemma 18 holds trivially by the assumption that we only consider non-dominator edges which ensures that neither x nor w is the dominator of f_{xz} or f_{vw}, respectively. Hence, we can apply Lemma 18 on $x \prec_\rho w$ yielding $f_{xz} \preceq_\lambda d(w)$.

Recall that if $f_{xz} = d(w)$ holds and w belongs to L_1, then the edge (f_{vw}, f_{xz}) exists in $\mathcal{C}(G)$. For a contradiction, we may assume that f_{xz} and f_{vw} do not induce an edge in $\mathcal{C}(G)$. Thus, either $f_{xz} \neq d(w)$ holds or w belongs to L_0. However, if $f_{xz} \neq d(w)$, we obtain $d(v) \preceq_\lambda d(x) \preceq_\lambda f_{xz} \prec_\lambda d(w) \preceq_\lambda f_{vw}$, and $d(v) \neq f_{xz}$ implies $d(v) \prec_\lambda f_{xz} \prec_\lambda d(w) \preceq_\lambda f_{vw}$. Since z is not f_{xz}-prime, it belongs to L_0. Since $w \prec_\rho z$, we obtain the order $\mathrm{dom}(d(w)) \prec_\rho z$ or $w \prec_\rho z$ on L_0 depending on whether w belongs to L_0 or to L_1. However, Figs. 17d and 17e show that in both cases the planarity of $\sigma(G)$ is violated. Finally, assume $f_{xz} = d(w)$, but w belongs to L_0. Since w is f_{vw}-prime, we have that v, x and w are $d(v)$-, $d(x)$- and $d(w)$-prime, respectively. This yields $d(v) \preceq_\lambda d(x) \preceq_\lambda d(w)$ by Property 12. From $f_{xz} \prec_\lambda f_{vw}$ we obtain $d(v) \preceq_\lambda d(x) \preceq_\lambda d(w) = f_{xz} \prec_\lambda f_{vw}$. However, by Property 11, w is the dominator of f_{vw}; a contradiction to the fact that (v, w) is a non-dominator edge. □

The edge-to-page assignment. We embed all primary edges in page p_0, all secondary edges in page p_1, and all tertiary edges in page p_2. We next assign the remaining edges of G to three sets R^1, B^1, and G^1, each containing $\lceil \frac{k}{2} \rceil$ pages. We process the intra-level faces of $\mathcal{F}$ according to $\lambda(\mathcal{F})$. Let f be the next face to process. By Corollary 22, face f has a color in $\{r, b, g\}$. The vertices of f induce at most a k-clique C_f in G. We assign the non-dominator edges of C_f to the pages of one of the sets R^1, B^1 and G^1 depending on whether the color of f is r, b, or g, respectively. This is possible since C_f is at most a k-clique [10]. In the following, we prove that this assignment is valid, which is the main result of this section.

Theorem 30. *The book thickness of a two-level k-framed graph G is at most* $3 \cdot \left\lceil \frac{k}{2} \right\rceil + 3$.

Proof. Consider two non-dominator edges (v, w) and (x, z), and assume without loss of generality that $v \prec_\rho w$ and $x \prec_\rho z$ in ρ. For a contradiction, assume (v, w) and (x, z) have been assigned to the same page p and that either $v \prec_\rho x \prec_\rho w \prec_\rho z$ or $x \prec_\rho v \prec_\rho z \prec_\rho w$, i.e., (v, w) and (x, z) cross in the same page. By Lemmata 14, 15 and 17, $p \notin \{p_0, p_1, p_2\}$. Hence, $p \in R^1 \cup B^1 \cup G^1$. Let f_{vw} and f_{xz} be the two faces of $\mathcal{F}$ responsible for assigning (v, w) and (x, z) to one of the pages of $R^1 \cup B^1 \cup G^1$. Assume without loss of generality that $f_{vw} \prec_\lambda f_{xz}$. If v and w are f_{vw}-prime, and x and z are f_{xz}-prime, then by Lemma 20, (v, w) and (x, z) cannot cross. Also, by Lemma 28, we may assume that x and z are f_{xz}-prime. On the other hand, each of v and w can be f_{vw}-prime or not. In the following, we distinguish cases based on the relative order of x, z, v and w and on the types of the vertices v and w.

Assume first that the relative order of the vertices is $v \prec_\rho x \prec_\rho w \prec_\rho z$. Since $x \prec_\rho z$, since both vertices are on the boundary of f_{xz}, and since (x, z) is non-dominator, it follows that if x belongs to L_0, then z also belongs to L_0, in which case the order on L_0 is $\mathrm{dom}(f_{xz}) \prec_\rho x \prec_\rho w \prec_\rho z$. However, by Lemma 27, this contradicts the fact that $f_{vw} \prec_\lambda f_{xz}$. Thus, x necessarily belongs to L_1. Next, we distinguish cases based on the types of vertices v and w.

- *Vertex v is not f_{vw}-prime*, which, by Property 23, implies that w is also not f_{vw}-prime. Hence, both v and w belong to L_0, and as result $\mathrm{dom}(f_{vw}) \prec_\rho v \prec_\rho w$. Since $w \prec_\rho z$ and since z is f_{xz}-prime, it follows that $w \preceq_\rho \mathrm{dom}(f_{xz})$. By Property 12 and since v and w belong to L_0, we get $v \preceq_\rho \mathrm{dom}(d(x)) \preceq_\rho w \preceq_\rho \mathrm{dom}(f_{xz})$. If $\mathrm{dom}(d(x)) = w$, then $d(x)$ has to be small, since otherwise $w \prec_\rho x$. Therefore, regardless of whether $\mathrm{dom}(d(x)) = w$ or $\mathrm{dom}(d(x)) \prec_\rho w$ holds, a situation as the one illustrated in Fig. 18a arises; recall that $f_{vw} \prec_\lambda f_{xz}$. If $w = \mathrm{dom}(f_{xz})$ holds, then f_{xz} has to be small, as otherwise the planarity of $\sigma(G)$ is violated. However, the fact that $z \prec_\rho w$ contradicts the fact that f_{xz} is small. Hence, $w \prec_\rho \mathrm{dom}(f_{xz})$ must hold. In this case, v cannot be on the boundary of the intra-level face f_{xz} without violating the planarity of $\sigma(G)$, which is again a contradiction.

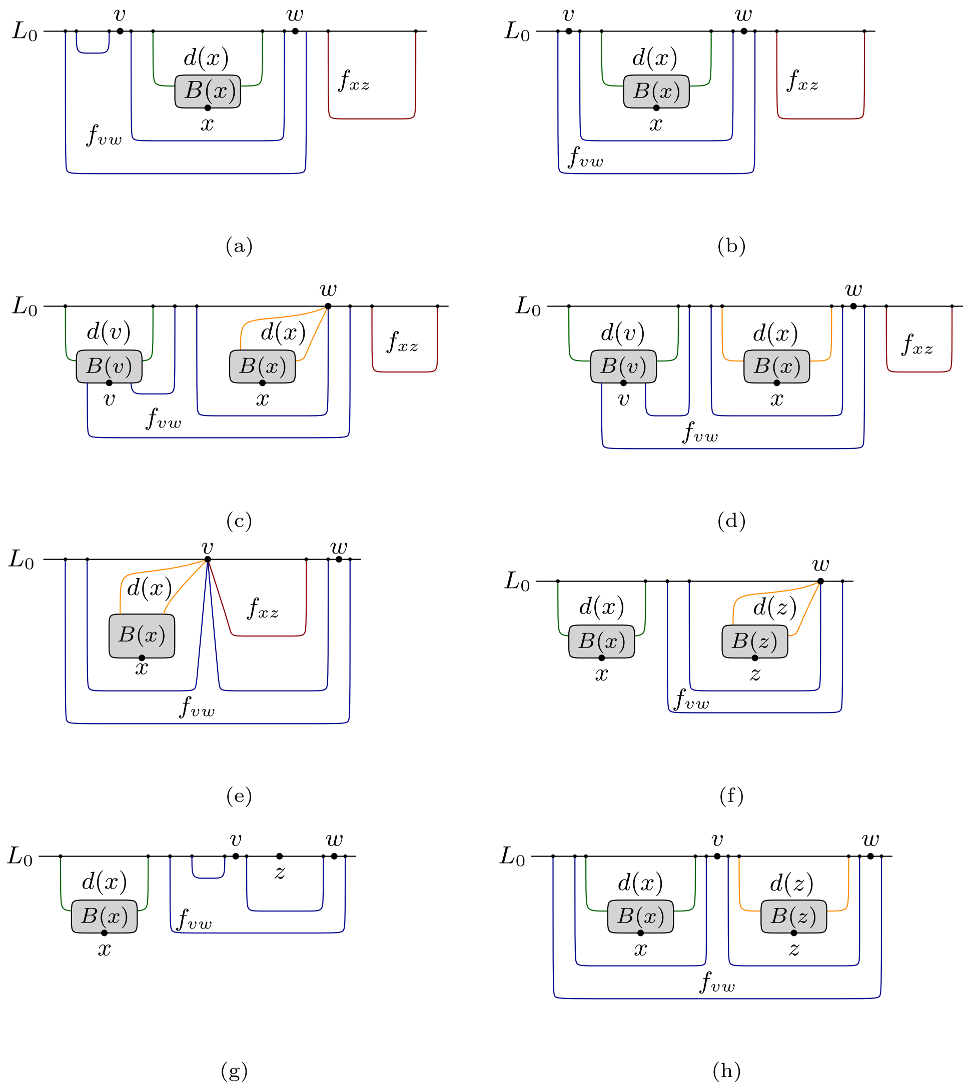


Fig. 18. Illustrations for the proof of Theorem 30.

– *Vertex v is f_{vw}-prime and w is not f_{vw}-prime.* First, we show that vertex v belongs to L_1. To this end, we assume to the contrary that v belongs to L_0. Since v belongs to L_0 and v is f_{vw}-prime, it follows that $v \neq \text{dom}(f_{vw})$. Since w also belongs to L_0, we have $\text{dom}(f_{vw}) \prec_\rho v \prec_\rho w$. We apply Properties 23 and 24 which yields $d(v) = f_{vw} \preceq_\lambda d(x)$. Observe that since z is f_{xz}-prime, we have $w \preceq_\rho \text{dom}(f_{xz})$, as otherwise $z \prec_\rho w$, which is a contradiction. Similarly, if f_{xz} is small, it follows again that $z \prec_\rho w$, which is the same contradiction. Hence, f_{xz} cannot small. Hence, f_{xz} follows $d(w)$ in a counterclockwise traversal of w starting from (u_{j-1}, u_j) and ending at (u_j, u_{j+1}) with $u_j = w$. Thus, we arise at a situation as the one illustrated in Fig. 18b, which shows that x cannot be on the boundary of f_{xz} without violating the planarity of $\sigma(G)$; a contradiction. Thus, v belongs to L_1, as desired. Now all conditions of Lemma 18 for vertices v and x are satisfied, which implies that $f_{vw} \preceq_\lambda d(x)$. We are now ready to show that the (f_{vw}, f_{xz}) exist in graph $\mathcal{C}(G)$, which completes the proof this case, since it also implies that (u, v) and (x, z) have been assigned to different pages. Assume for a contradiction that there exists no edge (f_{vw}, f_{xz}) in $o\mathcal{C}(G)$. Since x belongs to L_1, it follows that $f_{vw} \neq d(x)$. In total, we have $d(v) \preceq_\lambda f_{vw} \prec_\lambda d(x) \preceq_\lambda f_{xz}$. Since $x \prec_\rho w$, we have either that $\text{dom}(d(x)) \prec_\rho w$ or that $\text{dom}(d(x)) = w$ and $d(x)$ is small. If $d(x)$ is small, then we arise at a situation as the one illustrated in Fig. 18c. In order for $w \prec_\rho z$ to hold, either $w \prec_\rho \text{dom}(f_{xz})$ or $w = \text{dom}(f_{xz})$ and f_{xz} is not small. However, in both cases face f_{xz} violates the planarity of $\sigma(G)$; a contradiction. Thus, we may assume $\text{dom}(d(x)) \prec_\rho w$, as illustrated in Fig. 18d. Since $w \prec_\rho z$ and since z is f_{xz}-prime, we have $w \preceq_\rho \text{dom}(f_{xz})$. If equality holds, f_{xz} cannot be small, since otherwise it follows that $z \prec_\rho w$. Hence, according to the definition of small faces, f_{xz} follows $d(w)$ in a counterclockwise traversal of w starting from (u_{j-1}, u_j) and ending at (u_j, u_{j+1}) with $u_j = w$. Now, f_{xz} cannot have vertex x on its boundary without violating the planarity of $\sigma(G)$; a contradiction.

– *Vertices v and w are f_{vw}-prime*. By Lemma 20, the edge (f_{vw}, f_{xz}) exists in $\mathcal{C}(G)$, which implies that (u, v) and (x, z) have been assigned to different pages.

Consider now the case, in which the relative order of x, z, u and w is $x \prec_\rho v \prec_\rho z \prec_\rho w$. We proceed as above by considering subcases based on the types of vertices v and w.

– *Vertex v is not f_{vw}-prime*, which, by Property 23, implies that w is also not f_{vw}-prime. Hence, both v and w belong to L_0 and since (v, w) is a non-dominator edge, we obtain $\mathrm{dom}(f_{vw}) \prec_\rho v \prec_\rho w$. Observe that by Property 24, vertex v is discovered by f_{vw}. On the other hand, we have $f_{vw} \preceq_\lambda d(x)$ by Property 25. We claim that x belongs to L_1. Assume the contrary. Since x precedes z and both vertices are on the boundary of f_{xz}, it follows that z also belongs to L_0. Therefore, all four vertices belong to L_0 and their order is $x \prec_\rho v \prec_\rho z \prec_\rho w$. Since v and w are on the boundary of f_{vw} and x and z on the boundary of f_{xz}, the two faces f_{vw} and f_{xz} clearly violate the planarity of $\sigma(G)$. Thus, we may assume that x belongs to L_1, as we initially claimed. We are now ready to show that the (f_{vw}, f_{xz}) exist in graph $\mathcal{C}(G)$, which completes the proof this case. Assume for a contradiction that there exists no edge (f_{vw}, f_{xz}) in o$\mathcal{C}(G)$. Since x belongs to L_1, we have $f_{vw} \neq d(x)$. Therefore, we get $d(v) = f_{vw} \prec_\lambda d(x)$. In order for $x \prec_\rho v$ to hold, the dominator of $d(x)$ either precedes v on L_0 or the dominator of $d(x)$ is v and $d(x)$ is small. By applying the same arguments on vertices x and z, we can similarly conclude that the dominator of $d(z)$ either precedes w on L_0 or the dominator of $d(z)$ is w and $d(z)$ is small. This gives rise to three subcases to consider.
  – *$d(x)$ is small and $\mathrm{dom}(d(x)) = v$*. Since $v \prec_\rho z$ and z is f_{xz}-prime, we have $v \preceq \mathrm{dom}(f_{xz})$. If $v = \mathrm{dom}(f_{xz})$ holds, then f_{xz} is not small since otherwise it follows that $z \prec_\rho v$; a contradiction. Thus, f_{xz} is not small. However, Fig. 18e shows that in this case the face f_{xz} cannot have x on its boundary without violating the planarity of $\sigma(G)$.
  – *$d(z)$ is small and $\mathrm{dom}(d(z)) = w$*. Having ruled out the case above, we may further assume that $d(x)$ is not small. Since $d(x)$ is not small and since $x \prec_\rho v$, we get $\mathrm{dom}(d(x)) \prec_\rho v$. As illustrated in Fig. 18f, face f_{xz} cannot have x and z on its boundary without violating the planarity of $\sigma(G)$; a contradiction.
  – *Neither $d(x)$ nor $d(z)$ is small*. This yields $\mathrm{dom}(d(x)) \prec_\rho v$ and $\mathrm{dom}(d(z)) \prec_\rho w$ on L_0. We claim that $v \preceq_\rho \mathrm{dom}(d(z))$. Assume the contrary, that is, $\mathrm{dom}(d(z)) \prec_\rho v$. Since $v \prec_\rho z$, vertex z cannot be $d(z)$-prime and therefore z belongs to L_0. We obtain the order $\mathrm{dom}(d(x)) \prec_\rho v \prec_\rho z \prec_\rho w$ on L_0. As shown in Fig. 18g, face f_{vw} violates the planarity of $\sigma(G)$. Thus, we conclude that $\mathrm{dom}(d(x)) \prec_\rho v \preceq_\rho \mathrm{dom}(d(z)) \prec_\rho w$. With $f_{vw} \prec_\lambda d(x)$, we get the situation illustrated in Fig. 18h, in which face f_{xz} violates the planarity of $\sigma(G)$.

– *Vertex v is f_{vw}-prime but w is not f_{vw}-prime*. By Lemma 29, the edge (f_{vw}, f_{xz}) exists in $\mathcal{C}(G)$, which implies that (u, v) and (x, z) have been assigned to different pages.
– *Vertices v and w are f_{vw}-prime*. Again, by Lemma 20, the edge (f_{vw}, f_{xz}) exists in $\mathcal{C}(G)$, which implies that (u, v) and (x, z) have been assigned to different pages.

From the above case analysis, we can conclude that edges (v, w) and (x, z) cannot be assigned to the same, which concludes the proof. □

3.2. Inductive step: multi-level instances

In this section, we consider the general instances, which we call *multi-level instances*, in which the input k-framed graph G consists of $q \geq 3$ levels $L_0, L_1, \ldots, L_{q-1}$. We refer to Fig. 19 for a schematic representation of a multi-level instance. Initially, we assume that the unbounded face of $\sigma(G)$ contains no crossing edges in its interior; we will eventually drop this assumption. Recall that G_i denotes the subgraph of G induced by the vertices of $L_0 \cup \ldots \cup L_i$ containing neither chords of $\sigma_i(G)$ nor the crossing edges that are in the interior of the unbounded face of $\sigma(G)$. We will further denote by $\hat{G}_i$ the subgraph of G_i that is induced by the vertices of $L_{i-1} \cup L_i$. Observe that $\hat{G}_i$ is not necessarily connected; however, its maximal biconnected components, referred to as *bicomponents* in the following, form two-level instances. To ease the description, we refer to the blocks of all bicomponents of $\hat{G}_i$ simply as the blocks of $\hat{G}_i$. In a book embedding of G_i, we say that two vertices of the level L_j (with $j \leq i$) are *sequential* if there is no other vertex of level L_j between them along the spine. We say that a set U of vertices of level $L_{j'}$ is *j-delimited*, with $j' \neq j$, if either: (a) there exist two sequential vertices of level L_j such that all vertices of U appear between them along the spine, or (b) all vertices of U are preceded or followed along the spine by all vertices of L_j.

A book embedding $\mathcal{E}_i$ of G_i is *good* if it satisfies the following properties[1]:

P.1 The left-to-right order of the vertices on the boundary of each non-degenerate block B of $\hat{G}_i$ in $\mathcal{E}_i$ complies with the order of these vertices in a counterclockwise (clockwise) traversal of the boundary of B, if i is odd (even).
P.2 All vertices of each block B of $\hat{G}_i$, except possibly for its leftmost vertex, are consecutive and $(i-1)$-delimited.

[1] We stress at this point that even though Properties P.7c, P.7d and P.7e might be a bit difficult to be parsed, they formalize the main idea of Yannakakis' algorithm for reusing the same set of pages in a book embedding. Notably, this formalization in the original seminal paper [54] is not present.

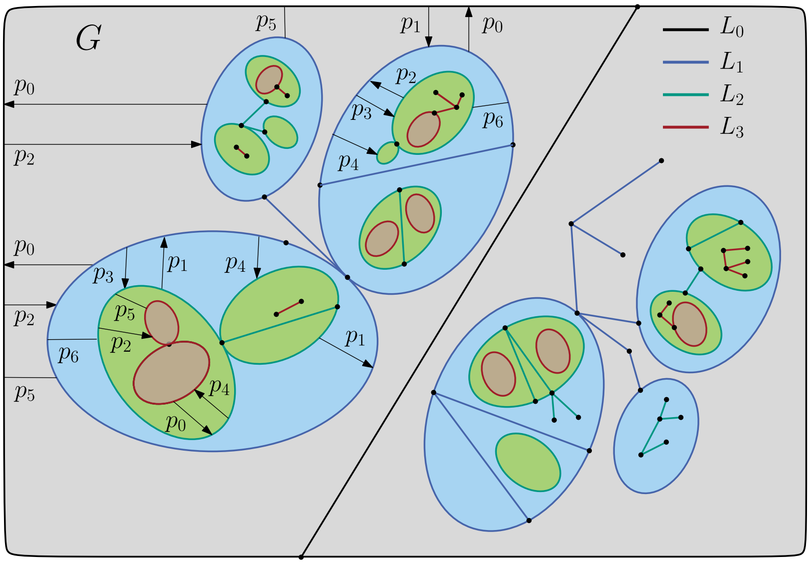


Fig. 19. A multi-level instance G with four levels of vertices, such that the bicomponents of $\hat{G}_2$ (which are shaded blue) form two connected components. The incoming edge and the two outgoing edges incident to the components are used to indicate the page to which the primary edges and the two sets of secondary edges of each bicomponent are assigned to, respectively. The undirected binding edges are used to indicate to which pages the tertiary edges are assigned to.

P.3 If between the leftmost vertex $\ell(B)$ of a block B of $\hat{G}_i$ and the remaining vertices of B there is a vertex v of L_i that belongs to a block B' of $\hat{G}_i$ in the same connected component as B, such that the leftmost vertex $\ell(B')$ of B' is to the left of $\ell(B)$, then B and B' share $\ell(B)$.

P.4 Let B and B' be two blocks of $\hat{G}_i$ for which P.3 does not apply, and let $\ell(B)$ and $\ell(B')$ be their leftmost vertices. If $\ell(B)$ precedes $\ell(B')$, then either $\ell(B')$ precedes all remaining vertices of B or all remaining vertices of B' precede all remaining vertices of B.

P.5 For any $j \leq i-2$, all the vertices of each block of $\hat{G}_i$ are j-delimited.

P.6 The edges of G_i are assigned to $6\lceil k/2\rceil + 7$ pages partitioned as (i) $P = \{p_0, \dots, p_6\}$, and (ii) $R^j = \{r^j_1, \dots, r^j_{\lceil k/2\rceil}\}$, $B^j = \{b^j_1, \dots, b^j_{\lceil k/2\rceil}\}$, $G^j = \{g^j_1, \dots, g^j_{\lceil k/2\rceil}\}$, $j \in \{0, 1\}$.

P.7 The edges of G_i are classified as primary, secondary, tertiary, or non-dominator in such a way that the following hold:

- **a** For $\zeta \leq i$, the non-dominator edges of $\hat{G}_\zeta$ are assigned to $R^j \cup B^j \cup G^j$ with $j = \zeta \mod 2$.
- **b** The edges that are incident to the leftmost vertex of a bicomponent of $\hat{G}_i$ and that are in its interior are primary.
- **c** Let $\mathcal{B}_i$ be a bicomponent of $\hat{G}_i$. The primary edges of $\hat{G}_i$ in the interior of $\mathcal{B}_i$ are assigned to a single page $p(\mathcal{B}_i)$. The secondary edges are assigned to two pages $s_1(\mathcal{B}_i)$ and $s_2(\mathcal{B}_i)$ of P different from $p(\mathcal{B}_i)$, and the tertiary edges are assigned to a single page $t(\mathcal{B}_i)$ of P different from $p(\mathcal{B}_i), s_1(\mathcal{B}_i)$, and $s_2(\mathcal{B}_i)$; refer to Fig. 19.
- **d** Let $\mathcal{B}_{i-1}$ be a bicomponent of $\hat{G}_{i-1}$. The blocks $B^1_{i-1}, \dots, B^\mu_{i-1}$ of $\mathcal{B}_{i-1}$ are the boundaries of several bicomponents of $\hat{G}_i$. Then, the secondary edges of $\hat{G}_{i-1}$ incident to B^j_{i-1}, with $j = 1, \dots, \mu$, are either all assigned to $s_1(\mathcal{B}_{i-1})$ or to $s_2(\mathcal{B}_{i-1})$.
- **e** Let $\langle p'_0, \dots, p'_6\rangle$ be a permutation of P. Assume that the primary edges of $\hat{G}_{i-2}$ that are in the interior of a bicomponent $\mathcal{B}_{i-2}$ of $\hat{G}_{i-2}$ have been assigned to p'_0 (in accordance with P.7c), while the secondary edges of $\hat{G}_{i-2}$ that are in the interior of $\mathcal{B}_{i-2}$ have been assigned to p'_1 and p'_2 (in accordance to P.7c and P.7d), and the tertiary edges of $\hat{G}_{i-2}$ that are in the interior of $\mathcal{B}_{i-2}$ have been assigned to p'_5. The blocks of $\mathcal{B}_{i-2}$ are the boundaries of several bicomponents $\mathcal{B}^1_{i-1}, \dots, \mathcal{B}^\mu_{i-1}$ of $\hat{G}_{i-1}$. Consider now a bicomponent $\mathcal{B}^j_{i-1}$ with $1 \leq j \leq \mu$ of $\hat{G}_{i-1}$. Assume w.l.o.g. that the secondary edges of $\mathcal{B}_{i-2}$ incident to $\mathcal{B}^j_{i-1}$ are assigned to p'_1. Then, the primary edges of $\mathcal{B}^j_{i-1}$ (which are incident to its blocks, and thus to the bicomponents of $\hat{G}_i$) are assigned to p'_2, while its secondary edges to p'_3 and p'_4. The tertiary edges are then assigned to p'_6.

We next argue that the book embeddings computed by the algorithm of Section 3.1 can be easily adjusted to become good.

Lemma 31. *Any two-level instance admits a good book embedding.*

Proof. To prove the lemma, we show that the book embedding $\mathcal{E}$ of a two-level instance G computed by the algorithm of Section 3.1 can be slightly modified to satisfy the properties of a good book embedding. We first observe that Properties P.5 and P.7e are clearly satisfied, since G consists of only two levels. Regarding the remaining properties, we argue as follows. Property P.1 holds by construction. Property P.2 directly follows from Property 6 of Section 3.1.1 and Rule R.1 of the constructed linear order. Property P.3 follows from Property 10 of Section 3.1.1. Property P.4 follows from Property 8 of Section 3.1.1. Property P.6 follows from the page assignment described in Section 3.1.2; in particular, since G consists of only two levels, its primary edges can be assigned to page p_0 by Lemma 14, while its non-dominator edges can be assigned to pages in $R^1 \cup B^1 \cup G^1$. Hence, Property P.7a holds. Property P.7b holds by the definition of primary edges. Finally, as already discussed, the primary edges of G are assigned to a single page p_0 of P in $\mathcal{E}$. Further, by Lemma 15 all secondary edges of G can be embedded in a single page of P in $\mathcal{E}$. However, in order to satisfy Property P.7d, we reassign the secondary edges to two pages of P in $\mathcal{E}$ as follows. Assume that each connected component of the blocks of G is rooted at the degenerate block corresponding to its first vertex. We assign the secondary edges towards the blocks that are at odd (even) distance from such a root block to $p_1 = f_1(\mathcal{B}_1)$ ($p_2 = f_2(\mathcal{B}_1)$, resp.) of P, where $G = \mathcal{B}_1$. By Lemma 17, all tertiary edges can be assigned to a single page. $\square$

Finally, the next lemma deals with good book embeddings of multi-level instances.

Lemma 32. *Any multi-level instance admits a good book embedding.*

Proof. Assume that we have recursively computed a good book embedding $\mathcal{E}_i$ of G_i. We next show how to extend $\mathcal{E}_i$ to a good book embedding $\mathcal{E}_{i+1}$ of G_{i+1}. Note that G_{i+1} is the union of G_i and $\hat{G}_{i+1}$, which share the vertices of L_i and the edges of $C_i(G)$.

Consider the set $\mathcal{H}$ of bicomponents $\mathcal{B}_1, \ldots, \mathcal{B}_\chi$ of $\hat{G}_{i+1}$. As already mentioned, each of the bicomponents in $\mathcal{H}$ forms a two-level instance. Consequently, the vertices delimiting the unbounded faces of $\mathcal{B}_1, \ldots, \mathcal{B}_\chi$ form blocks $B_1, \ldots, B_\chi$ of $\hat{G}_i$, which in turn form a set of cacti in $\sigma_i(G)$. We assume that each connected component in this set is rooted at one of its blocks. This allows as to associate each bicomponent $\mathcal{B}_i$ out of the initial ones with a root bicomponent denoted by $r(\mathcal{B}_i)$, $i = 1, \ldots, \chi$. This further allows us to also associate each bicomponent $\mathcal{B}_i$ with a parity bit $\epsilon(\mathcal{B}_i)$ that expresses whether the distance between $\mathcal{B}_i$ and $r(\mathcal{B}_i)$ is odd or even.

We process the bicomponents of $\mathcal{H}$ one by one as follows. Assume now that we have processed the first $x-1 < \chi$ bicomponents $\mathcal{B}_1, \ldots, \mathcal{B}_{x-1}$ of $\mathcal{H}$ and that we have extended $\mathcal{E}_i$ to a good book embedding $\mathcal{E}_i^{x-1}$ of G_i together with $\mathcal{B}_1, \ldots, \mathcal{B}_{x-1}$. Consider the next bicomponent $\mathcal{B}_x$ of $\hat{G}_{i+1}$ in $\mathcal{H}$. Observe that the boundary of $\mathcal{B}_x$ is a simple cycle consisting of vertices of level L_i. As a result, the vertices and the edges of this cycle are present in G_i and therefore they have been embedded in $\mathcal{E}_i$ and thus in $\mathcal{E}_i^{x-1}$.

In the following, we show how to extend $\mathcal{E}_i^{x-1}$ to a good book embedding $\mathcal{E}_i^{x}$ of G_i together with $\mathcal{B}_1, \ldots, \mathcal{B}_x$. Once all blocks in $\mathcal{H}$ have been processed, the obtained book embedding $\mathcal{E}_i^{\chi}$ is the desired good book embedding $\mathcal{E}_{i+1}$ of G_{i+1}. The vertices that delimit the unbounded face of $\mathcal{B}_x$ form a block B_x of $\hat{G}_i$. By Property P.1, their left to right order in $\mathcal{E}_i^{x-1}$ (say $u_0, \ldots, u_{s-1}$) complies with the order in which these vertices appear in either a counterclockwise or in a clockwise traversal of the boundary of B_x, depending on whether i is odd or even, respectively. We proceed by computing a good book embedding $\mathcal{E}_x$ of $\mathcal{B}_x$ which exists by Lemma 31, such that the left-to-right order of the vertices of $\mathcal{B}_x$ is $u_0, \ldots, u_{s-1}$ in $\mathcal{E}_x$. Note that this can be achieved by flipping $\mathcal{B}_x$, if i is even. Further, note that $\mathcal{E}_x$ is good by Lemma 31. We extend $\mathcal{E}_i^{x-1}$ to a good book embedding $\mathcal{E}_i^{x}$ in two steps as follows.

In the first step, for $j = 0, 1, \ldots, s-2$, the vertices of $\mathcal{B}_x$ that appear between u_j and u_{j+1} in $\mathcal{E}_x$, if any, are embedded right before u_{j+1} in $\mathcal{E}_i^{x-1}$ in the same left-to-right order as in $\mathcal{E}_x$; also, the vertices of $\mathcal{B}_x$ that appear after u_{s-1} in $\mathcal{E}_x$, if any, are embedded right after u_{s-1} in $\mathcal{E}_i^{x-1}$ in the same left-to-right order as in $\mathcal{E}_x$. Let $\mathcal{E}_i^{x}$ be the resulting embedding (which still does not contain all the edges of $\mathcal{B}_x$). Since $\mathcal{E}_x$ is a good book embedding and since we do not change relative order of the vertices of $\mathcal{B}_x$ in $\mathcal{E}_x$ and in $\mathcal{E}_i^{x}$, Properties P.1 and P.2 hold for $\mathcal{E}_i^{x}$. Since Property P.2 holds for block B_x in $\mathcal{E}_i^{x-1}$, it follows that there is no vertex of level L_j, with $j \leq i-1$, in $\mathcal{E}_i^{x-1}$ between any two vertices of $\{u_1, \ldots, u_{s-1}\}$. This and the fact that we have placed the remaining vertices of $\mathcal{B}_x$ either right before or right after any of $u_1, \ldots, u_{s-1}$ implies that there exists no vertex of level L_j, with $j \leq i-1$, between the vertices of $\mathcal{B}_x$ along the spine, which proves Property P.5 for $\mathcal{E}_i^{x}$.

In the second step, we assign the internal edges of $\mathcal{B}_x$ to the already existing pages of $\mathcal{E}_i^{x}$ to complete the embedding, which also implies that Property P.6 will not be deviated. This step will complete the extension of $\mathcal{E}_i^{x-1}$ to $\mathcal{E}_i^{x}$. The assignment is done in a straight-forward manner. The primary, secondary, tertiary, and non-dominator edges of $\mathcal{E}_x$ that are internal in $\mathcal{B}_x$ will be classified as primary, secondary, tertiary, and non-dominator, respectively, also in $\mathcal{E}_i^{x}$, which guarantees Property P.7. To guarantee that Property P.7a holds for $\mathcal{E}_x$, we proceed as follows. The non-dominators edges of $\mathcal{E}_x$ that are internal in $\mathcal{B}_x$ and are assigned to $r_1^1, \ldots, r_{\lceil k/2 \rceil}^1$, $b_1^1, \ldots, b_{\lceil k/2 \rceil}^1$, $g_1^1, \ldots, g_{\lceil k/2 \rceil}^1$ in $\mathcal{E}_x$ are assigned to $r_1^j, \ldots, r_{\lceil k/2 \rceil}^j$, $b_1^j, \ldots, b_{\lceil k/2 \rceil}^j$, $g_1^j, \ldots, g_{\lceil k/2 \rceil}^j$ in $\mathcal{E}_i^{x}$, respectively, where $j = i+1 \mod 2$. Hence, Property P.7a holds for $\mathcal{E}_x$, as desired.

We now show that no two edges assigned to any of these pages cross. Assume for a contradiction that there is a crossing in page $p \in R^j \cup B^j \cup G^j$ with $j = i+1 \mod 2$. Since $\mathcal{E}_i^{x-1}$ is a good book embedding, this crossing must necessarily involve an edge e of $\mathcal{B}_x$. Let e' be the second edge involved in the crossing. We distinguish two cases: (i) e' belongs to one of

$\mathcal{B}_1, \ldots, \mathcal{B}_x$, and (*ii*) e' belongs to some previously embedded graph $\hat{G}_\zeta$ with $\zeta < i+1$. In Case (*i*), we first observe that e' cannot belong to $\mathcal{B}_x$, as otherwise e and e' would also cross in $\mathcal{E}_x$, contradicting the fact that $\mathcal{E}_x$ is a good book embedding of $\mathcal{B}_x$. Hence, we may assume that e' belongs to $\mathcal{B}_j$ with $j < x$. Since $e \in \mathcal{B}_x$ and $e' \in \mathcal{B}_j$, by Property P.2, at least one of e and e' must be incident to the leftmost vertex of the blocks B_x and B_j that delimit the unbounded faces of $\mathcal{B}_x$ and $\mathcal{B}_j$, respectively, which, by Property P.7b, implies that at least one of them is primary; a contradiction. Consider now Case (*ii*) and recall that in this case e' belongs to some graph $\hat{G}_\zeta$ with $\zeta < i+1$. Since e and e' cross in p, it follows that $\zeta \equiv i+1$ mod 2. The latter property further implies that $\zeta \leq i-1$. In this case, however, Property P.5 implies the endpoints of edge e are $(i-1)$-delimited, which in turn implies that e and e' nest, which contradicts our initial assumption. In fact, the same argumentation can be used to show, that no two tertiary edges that are assigned to the same page (either p_5 or p_6) can cross.

By Lemma 31, all primary edges of $\mathcal{E}_x$ have been assigned to page p_0 in $\mathcal{E}_x$, while its secondary edges have been assigned to p_1 and p_2, and the tertiary edges have been assigned to p_5; also, recall that no edge of $\mathcal{E}_x$ has been assigned to pages p_3, p_4 and p_6. To guarantee Property P.7c in $\mathcal{E}_i^x$, the primary edges of $\mathcal{E}_x$ that are interior to B_x will be assigned to $\mathcal{E}_i^x$ to a common page p of P (i.e., not necessarily to p_0), while the corresponding secondary edges assigned to p_1 and p_2 in $\mathcal{E}_x$ will be reassigned to two pages s_1 and s_2, respectively. Finally, the tertiary edges will be assigned to page $t \in \{p_5, p_6\}$.

To determine pages p, s_1, s_2, and t, we have to take into account Properties P.7d and P.7e that hold for $\mathcal{E}_i^{x-1}$. Assume first that $i \geq 3$; the case $i = 2$ is immediate. Then, there is a bicomponent $\mathcal{B}_{i-2}$ of $\hat{G}_{i-2}$, whose boundary vertices form a cycle that, in G_{i+1}, contains the bicomponent $\mathcal{B}_x$ in its interior. Assume w.l.o.g. that the primary edges of $\mathcal{B}_{i-2}$ are assigned to page $p'_0 \in P$, in accordance to P.7c. It follows by P.7e that we may further assume w.l.o.g. that all the primary edges of the bicomponents of $\hat{G}_{i-1}$, whose boundaries are blocks of $\mathcal{B}_{i-2}$, have been assigned to pages p'_1 and p'_2 different from p'_0. Assume also, w.l.o.g., that the secondary edges of $\mathcal{B}_{i-2}$ incident to $\mathcal{B}_x$ have been assigned to p'_1. By Property P.7e, this implies that the primary (secondary) edges of bicomponent $\mathcal{B}_x$ must be assigned to page p'_2 (to p'_3 and p'_4, respectively). Note that also of all the previously processed bicomponents of $\hat{G}_{i+1}$ in $\mathcal{H}$ make use of these three pages plus the page p'_1. Hence, both Properties P.7c and P.7e are satisfied. The choice between the two pages p'_3 and p'_4 is done based on the parity bit $\epsilon(\mathcal{B}_x)$, so that, all secondary edges of all bicomponents in $\mathcal{H}$ having the same parity bit will be assigned to the same page in $\{p'_3, p'_4\}$, thus guaranteeing that Property P.7c holds for $\mathcal{E}_i^x$. Finally, assume w.l.o.g., that the tertiary edges of $\mathcal{B}_{i-2}$ incident to $\mathcal{B}_x$ have been assigned to p'_5. By Property P.7e, this implies that the tertiary edges of bicomponent $\mathcal{B}_x$ must be assigned to page p'_6.

We conclude the proof by showing that no two edges assigned to pages in $\{p'_2, p'_3, p'_4, p'_6\}$ cross in $\mathcal{E}_i^x$. We first focus on page p'_2. Clearly, no two edges in p'_2 belonging to $\mathcal{B}_x$ can cross, since $\mathcal{E}_x$ is a good book embedding. Hence, if there is a crossing in p'_2 it must involve an edge e in $\mathcal{B}_x$ and an edge e' of either G_i or of one of the previously embedded bicomponents of $\hat{G}_{i+1}$ in $\mathcal{H}$. We first consider the case, in which e' belongs to G_i. In particular, by Property 5 since all the vertices of B_x are $(i-2)$-delimited, it follows that e' is an edge of $\hat{G}_i$. By Property P.7e, e' must be incident to the leftmost vertex of B_x. Now, observe that the edges of $\mathcal{B}_x$ that are incident to the leftmost vertex of $\mathcal{B}_x$ in $\mathcal{E}_x$ are by definition primary; thus, they are not assigned to p'_2. Since by Property P.2 the remaining vertices of $\mathcal{B}_x$ are $(i-1)$-delimited, it follows that if there exists a crossing in page p_2, this should involve a previously embedded bicomponent of $\hat{G}_{i+1}$ in $\mathcal{H}$. As a result, we can assume that e' belongs to $\mathcal{B}_j$, with $j < x$. Let B_x and B_j be the blocks that delimit the unbounded faces of $\mathcal{B}_x$ and $\mathcal{B}_j$, respectively. Since $e \in \mathcal{B}_x$ and $e' \in \mathcal{B}_j$, by Property P.4, it follows that B_x and B_j belong to the same connected component C formed by the blocks of $\hat{G}_i$. By Property P.2, at least one of e and e' must be incident to the leftmost vertex of B_x or B_j in $\mathcal{E}_i^x$, respectively. Since B_x and B_j belong to C, by Property P.3, B_x and B_j must share a common vertex, which implies that $\mathcal{B}_x$ and $\mathcal{B}_j$ have different parity bits, i.e. $\epsilon(\mathcal{B}_x) \neq \epsilon(\mathcal{B}_j)$. However, since e is assigned to p'_2, edge e' is assigned to p'_1, contradicting our assumption that e and e' cross. Hence, we can conclude that there is no two edges assigned to p'_2 that cross in $\mathcal{E}_i^x$.

We now focus on the edges of $\{p'_3, p'_4\}$. Assume w.l.o.g. that e is assigned to p'_3. As above, we argue that e' either belongs to G_i (in particular, to $\hat{G}_i$) or to one of the previously embedded bicomponents of $\hat{G}_{i+1}$ in $\mathcal{H}$. The former case is actually not possible, since by Property P.7e there is no edge of $\hat{G}_i$ assigned to p'_3 that is incident to $\mathcal{B}_x$. So, we may focus on the latter case, in which e' belongs to $\mathcal{B}_j$, with $j < x$. As above, we can conclude that $\mathcal{B}_x$ and $\mathcal{B}_j$ should belong to the same connected component C formed by the blocks of $\hat{G}_i$, and in particular, the corresponding blocks B_x and B_j that delimit their unbounded faces share a common vertex, which implies that $\mathcal{B}_x$ and $\mathcal{B}_j$ have different parity bits. In this case, however, the involved edges e and e' are assigned to p'_1 and p'_2, and thus they cannot cross in p'_3.

From the discussion above, we can conclude that $\mathcal{E}_i^x$ is in fact a good book embedding. However, recall that we initially assumed that the unbounded face of $\sigma(G)$ contains no crossing edges in its interior, to support the recursive strategy. We complete the proof by dropping this assumption as follows. We assign these edges to the pages of $R^0 \cup B^0 \cup G^0$, which results in a good book embedding of G, since the endvertices of the edges already assigned to these pages are 0-delimited. □

Altogether, Lemma 32 in conjunction with Lemma 31 completes the proof of Theorem 2.

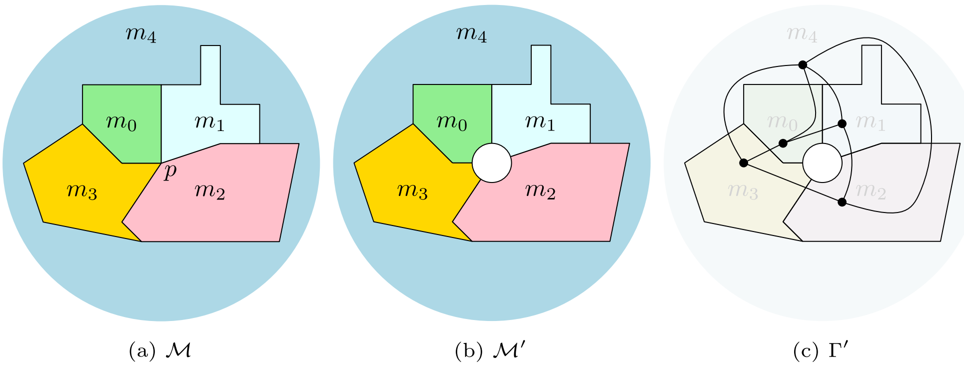


Fig. 20. Illustration for the proof of Lemma 33. (a) A well-formed hole-free 4-map graph G with one h-point with $h > 3$, denoted by p. (b) The well-formed 3-map graph G' obtained by deleting p. (c) A drawing of the planar skeleton of G'.

4. Application of Theorem 2 to map graphs

We begin by formally defining map graphs (refer also to [18]). A *map graph* G is one that admits a *map* $\mathcal{M}$, i.e., a bijection that puts in correspondence each vertex v of G with a region $\mathcal{M}(v)$ of the sphere homeomorphic to a closed disk, called *nation*, in such a way that the following properties hold: (i) the interiors of any two distinct nations are disjoint, and (ii) two vertices u and v are adjacent in G if and only if the boundaries of $\mathcal{M}(u)$ and $\mathcal{M}(v)$ intersect. The points of the sphere that are not covered by any nation fall into open connected regions; the closure of each such region is a *hole* of $\mathcal{M}$. A *k-map* graph (with $k > 1$) is a graph that admits a map $\mathcal{M}$ such that at most k nations intersect in a single point. Also, G is *well-formed* if for every edge (u, v) of G the intersection of $\mathcal{M}(u)$ and $\mathcal{M}(v)$ is either a single point or a single curve segment. Moreover, if $\mathcal{M}$ does not contain holes, G is a *hole-free k-map*.

In order to prove Theorem 1, we first deal with a simpler case. Namely, we prove that well-formed hole-free k-map graphs are k-framed, which, by Theorem 2, implies they have book thickness at most $6\lceil \frac{k}{2} \rceil + 5$.

Lemma 33. *Every well-formed hole-free k-map graph is k-framed.*

Proof. Let $\mathcal{M}$ be a well-formed hole-free k-map of a graph G and refer to Fig. 20 for an illustration. A point p of $\mathcal{M}$ is an *h-point*, if $h > 1$ nations intersect in p. Let p be an h-point of $\mathcal{M}$ (if any) with $4 \le h \le k$. The operation of *deleting* the h-point p works as follows. Denote by $V_p = \{m_0, m_1, \dots, m_{h-1}\}$, the set of h nations that intersect in p. Consider now a small open disk D in $\mathcal{M}$ centered at p such that any point in D is either p, an interior point of a nation in V_p, or a point where exactly two nations of V_p intersect. We shall indeed assume that, excluding point p and up to a relabeling of the nations, the only adjacencies realized in D are those between m_i and m_{i+1}, for $i = 0, 1, \dots, h-1$ (indices taken modulo h). Clearly, for a sufficiently small radius, such disk always exists. Removing the parts of the nations in D from $\mathcal{M}$ introduces a hole in the map, removes p, and does not introduce any new h'-point with $h' > 3$. Let $\mathcal{M}'$ be the well-formed 3-map obtained by deleting all h-points of $\mathcal{M}$ with $h > 3$, and let G' be the corresponding map graph. We aim at proving that G admits a k-framed drawing Γ having G' as planar skeleton.

First of all note that G' is simple, because $\mathcal{M}'$ is well-formed, and spanning, because we do not destroy any nation. Recall that, by definition of well-formed, any two adjacent vertices u and v are such that $\mathcal{M}'(u)$ and $\mathcal{M}'(v)$ intersect in either a single point p_{uv} or in a curve segment s_{uv}. In the latter case, we denote by p_{uv} an arbitrary 2-point of $\mathcal{M}'$ along s_{uv}.

Let Γ' be a drawing of G' obtained by representing each vertex u as an interior point p_u of $\mathcal{M}'(u)$, and each edge (u, v) as a Jordan arc that starts a p_u, traverses $\mathcal{M}'(u)$ until p_{uv}, and finally traverses $\mathcal{M}'(v)$ ending in p_v. The circular order of the edges around a vertex u is kept the same as the circular order of the corresponding points p_{uv} around $\mathcal{M}'(u)$; this ensures that no two arcs intersect in an interior point of a nation. On the other hand, the only point where two Jordan arcs may intersect is a 3-point (if it exists). In such a case it suffices to slightly perturb the curves around such h-point so to avoid any crossing. Thus Γ' does not contain any crossing. Note that Γ' is a spherical drawing, in what follows we consider its stereographic projection onto the plane, i.e., we view Γ' as a planar drawing. Concerning the size of the largest face of G', observe that the maximum degree of a face of G' (including the unbounded face) cannot be larger than the greatest number of nations that intersect the same hole of $\mathcal{M}'$, which is at most k by construction.

It remains to prove that: (a) G' is biconnected, and (b) all edges of $\overline{E} = G \setminus G'$ can be drawn entirely inside faces of G' and are all crossed.

Concerning (a), if there existed a vertex v whose removal disconnects G', this would imply that the original map $\mathcal{M}'$ contains a hole that intersects $\mathcal{M}'(v)$ at least twice and (at least) two nations whose corresponding vertices in G' are connected only by paths containing v. (Recall that any nation is homeomorphic to a closed disk, i.e., it intersects neither

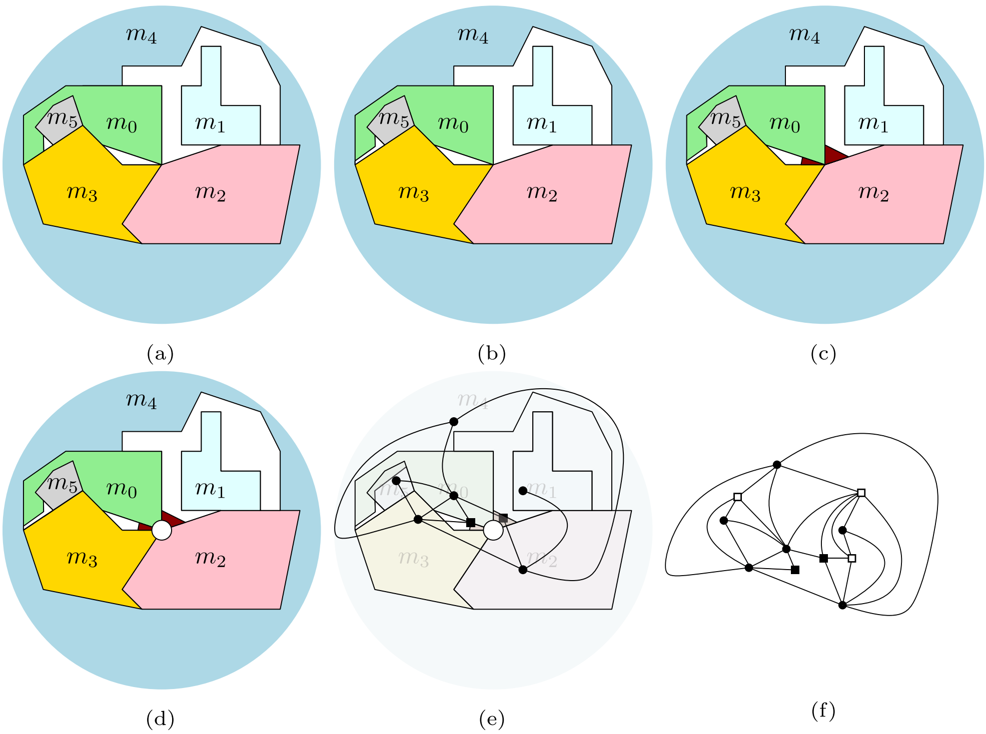


Fig. 21. Illustration for the proof of Theorem 34. (a) A 3-map graph that is not well-formed, in particular m_0 and m_3 have two distinct adjacencies. (b) Removing multiple adjacencies by introducing holes. The resulting graph is a well-formed 3-map graph G with only one h-point with $h = 3$ that intersects two holes. (c) The well-formed 5-map graph G' obtained by adding a dummy nation such that no h-point with $h > 2$ intersects a hole. (d) Deleting h-points with $h > 3$. (e) A drawing of the planar skeleton Γ'; vertices representing dummy nations are black squares. (f) Augmenting the planar skeleton to make it biconnected; the inserted dummy vertices are white squares.

holes nor other nations in its interior.) However, such hole does not exist, because by construction any hole of $\mathcal{M}'$ intersects a set of nations $m_0, m_1, \ldots, m_{h-1}$ whose induced graph contains a cycle.

Concerning (b), recall that any edge (u, v) in $\overline{E}$ connects two nations that intersect in $\mathcal{M}$ and do not intersect anymore in $\mathcal{M}'$. In particular, there exists at least one hole in $\mathcal{M}'$ intersecting $\mathcal{M}'(u)$ and $\mathcal{M}'(v)$. When constructing Γ' from $\mathcal{M}'$, such hole yields a face in Γ' having both u and v on its boundary. Thus, we can draw a copy of (u, v) inside each such face. This concludes the proof. $\square$

The next theorem extends the proof of Lemma 33 and, together with Theorem 2, implies Theorem 34.

Theorem 34. *Every k-map graph is partial $2k$-framed.*

Proof. Refer to Fig. 21 for an illustration. Let $\mathcal{M}_0$ be a k-map of a graph G. We first aim at turning $\mathcal{M}_0$ into a *nearly well-formed* k-map $\mathcal{M}$ of G, i.e., a k-map in which multiple adjacencies occur only in presence of h-points with $h > 2$ Recall that each intersection between two nations is either a single h-point ($h \geq 2$) or a curve segment. If any two nations intersect at most once, then $\mathcal{M} = \mathcal{M}_0$. Else, let m and m' be two nations that intersect $r > 1$ times, and consider any intersection that is a 2-point or a curve segment, excluding from this segment possible h-points with $h > 2$. We can remove each such intersection between m and m' by locally retracting m (or m'). Such operation introduces a hole (which can possibly merge with some other holes) in place of the intersection between m and m' and does not destroy any other intersection because we avoided h-points with $h > 2$. The resulting k-map $\mathcal{M}$ may not be well-formed yet, however it is nearly well-formed.

Under this assumption, the proof of Lemma 33 can be adjusted as follows. Observe that an h-point p in $\mathcal{M}$ touches at most h holes. For each h-point p in $\mathcal{M}$ that touches $\chi \leq h$ holes and with $h > 2$, we introduce a sufficiently small *dummy* nation such that p becomes an $(h + \chi)$-point that does not touch holes anymore (this operation does not introduce new h-points with $h > 2$). After this preliminary operation, we let $\mathcal{M}^*$ be the resulting map and $G^* \supseteq G$ be the corresponding map graph. We remark that G^* does not contain any new edge connecting two vertices of G. Moreover, $\mathcal{M}^*$ is a $2k$-map, which is still nearly well-formed. We then apply the procedure in the proof of Lemma 33. Namely, we first delete all h-points with $h > 3$, which implies that the resulting map is now well-formed. We then compute a drawing Γ' of the planar skeleton of the resulting graph G'. The proof of Lemma 33 ended by showing how to reinsert the edges in $\overline{E} = G^* \setminus G'$ inside their corresponding faces of Γ' so to create a $2k$-framed drawing Γ of G. Before applying this last step, we observe that the absence of holes was used in the proof to guarantee that the size of each face of Γ' is at most $2k$ and that G' is biconnected. We show that, after a suitable augmentation of G', both properties still hold.

A face f of Γ' is *large* if the size of f is greater than 3 and f does not contain crossing edges in the final drawing Γ of G^* (i.e., f is generated by a disk inserted to eliminate an h-point with $h > 2$). Let f be a large face; the *stellation operation* of f works as follows. We insert a vertex v_f inside f and connect it to all vertices on the boundary of f by drawing the new edges inside f without creating edge crossings; if a vertex of f is a cut-vertex, we connect it to v_f only once. The stellation operation removes f and creates new faces of size strictly smaller than the size of f. By repeatedly applying this operation until there are no large faces we obtain a planar skeleton G'' such that the boundary of each face is a simple cycle, which implies that G'' is biconnected. Also, the size of a face of G'' is at most 3 if it does not contain crossing edges in Γ, and at most $2k$ otherwise. By finally reintroducing the crossing edges inside the faces of G'' of size (at most) $2k$, we obtain a $2k$-framed drawing of a $2k$-framed graph, which is a super graph of the input graph G. This proves that every k-map graph is partial $2k$-framed. □

We conclude this section by giving the following simple result, which implies that the book thickness of k-framed graphs (and hence of partial k-framed graphs) is bounded by the book thickness of k-map graphs.

Theorem 35. *Every k-framed graph is a k-map graph, under the assumption that each face of the planar skeleton induces a clique of size k.*

Proof. Consider a k-framed drawing Γ of a k-framed graph G. Let Γ' be the planar skeleton of Γ. As already said, we shall assume that each face of Γ' induces a clique of size k in Γ. We replace each vertex v of Γ with a sufficiently thin star-shaped nation that includes each curve representing an each edge (u, v) up to the midpoint of such curve. Since Γ' is planar, this operation transforms Γ' into a 2-map $\mathcal{M}'$. Observe that a face of size $h \le k$ in Γ' corresponds to a hole in $\mathcal{M}'$. Thus, the crossing edges of Γ can be easily reintroduced by creating an h-point inside each such hole. □

5. Conclusions and open problems

Our research generalizes a fundamental result by Yannakakis in the area of book embeddings. To achieve $O(k)$ pages for k-map graphs and, more in general, for partial k-framed graphs, we exploit the special structure of these graphs which allows us to model the conflicts of the crossing edges by means of a graph with bounded chromatic number (thus keeping the unavoidable relationship with k low).

Even though our result only applies to a subclass of h-planar graphs, it provides useful insights towards a positive answer to the intriguing question of determining whether the book thickness of (general) h-planar graphs is bounded by a function of h only.

Another natural question that stems from our research is whether k-map graphs are partial k-framed, and in particular, whether Theorem 34 can be improved.

A third direction for extending our result is to drop the biconnectivity requirement of partial k-framed graphs.

We conclude by mentioning that the time complexity of our algorithm is $O(k^2 n)$, assuming that a k-framed drawing of the considered graph is also provided. It is of interest to investigate whether (partial) k-framed graphs can be recognized in polynomial time. The question remains valid even for the class of optimal 2-planar graphs, which exhibit a quite regular structure. In relation to this question, Brandenburg [15] provided a corresponding linear-time recognition algorithm for the class of optimal 1-planar graphs, while Da Lozzo et al. [23] showed that the related question of determining whether a graph admits a planar embedding whose faces have all degree at most k is polynomial-time solvable for $k \le 4$ and NP-complete for $k \ge 5$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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