Optimal Impartial Selection

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We study the problem of selecting a member of a set of agents based on impartial nominations by agents from that set. The problem was studied previously by Alon et al. and by Holzman and Moulin and has important applications in situations where representatives are selected from within a group or where publishing or funding decisions are made based on a process of peer review. Our main result concerns a randomized mechanism that in expectation selects an agent with at least half the maximum number of nominations. Subject to impartiality, this is best possible.

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1. INTRODUCTION

We consider a situation where members of a set of agents nominate other agents from the set for a prize and the goal is to award the prize to an agent who receives a large number of nominations. This situation arises naturally, for example, when representatives are selected from within a group or when publishing or funding decisions are made based on a process of peer review. While nominations are at the discretion of the nominating agents, it is often reasonable to assume that agents are impartial to the selection of others and will nominate who they think should receive the prize as long as they cannot influence their own chances of receiving it. Indeed, the assumption of impartiality was previously made, and justified, in the very same setting [Alon et al. 2011; Holzman and Moulin 2013].

Formally, the situation can be captured by a directed graph with n vertices, one for each agent, in which edges correspond to nominations. A selection mechanism then chooses a vertex for any given graph, and impartiality requires that the chances of a particular vertex to be chosen do not depend on its outgoing edges. It is easy to see that an impartial mechanism cannot always select a vertex with maximum indegree, corresponding to an agent with a maximum number of nominations, even when $n = 2$. We therefore aim at maximizing the indegree of the selected vertex relative to the maximum indegree, and call a mechanism α-optimal, for $\alpha \leq 1$, if for every graph the former is at least α times the latter. We focus here on the selection of a single agent, but note that it is an interesting question whether optimal mechanisms for selecting

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any given number of agents can be obtained directly from mechanisms for selecting a single agent, or whether their design requires additional techniques.

State of the Art. Alon et al. [2011] and Holzman and Moulin [2013] showed independently that deterministic impartial mechanisms are extremely limited, and must sometimes select an agent with zero nominations even though agents are being nominated, or an agent with one nomination when another agent receives $n-1$ nominations.

On the other hand, Alon et al. proposed a simple mechanism that randomly partitions the agents into two sets S_1 and S_2 and selects an agent from S_2 who among agents in this set receives a maximum number of nominations by agents in S_1. By linearity of expectation the mechanism is at least $1/4$-optimal, and a situation with a single nomination shows that it cannot do better. A somewhat closer inspection of situations with one or two nominations shows that no impartial mechanism can be better than $1/2$-optimal. While these bounds are almost trivial, no improvements have been made that hold for general values of n, despite considerable efforts. Moreover, improving the lower bound appears just as difficult in the restricted case considered by Holzman and Moulin, where each agent submits exactly one nomination. This is somewhat embarrassing, as the mechanism of Alon et al. should quite obviously be better than $1/4$-optimal as soon as there is more than just a single nomination.

Our Contribution. Alon et al.'s analysis of the 2-partition mechanism is tight and yields a constant approximation ratio, only a factor of two away from the best possible one. Quite strikingly, however, the analysis does not reveal much of the structure of the problem. It does not lead to stronger bounds for special cases, like the setting with one nomination per agent studied by Holzman and Moulin, and cannot be extended to more complicated mechanisms.

Our first result attempts to close this gap in our understanding of the 2-partition mechanism, by providing a lower bound on its performance relative to the maximum indegree. Specifically, we show that the performance of the 2-partition mechanism monotonically increases with the maximum indegree and converges to $1/2$. As a direct consequence of our more detailed analysis we also obtain a lower bound of $3/8$ for settings without abstentions, i.e., settings where each agent submits at least one nomination. Our analysis uses a novel adversarial argument that allows us to abstract from the underlying graph structure and isolate the critical aspects of difficult problem instances.

More interestingly, our analysis suggests a natural generalization of the 2-partition mechanism that partitions the set of agents into $k > 2$ sets and iteratively considers the nominations submitted by agents in more and more of these sets, to fewer and fewer candidates in the remaining sets. Intuitively this increases the probability of each individual nomination to be counted, which is particularly important for the difficult problem instances where the overall number of nominations is small. Exactly how information from an earlier stage of the mechanism can be used without a negative effect on later stages turns out to be somewhat intricate.

We then generalize the adversarial analysis to show that the k-partition mechanism is $(k-1)/2k$-optimal, which approaches the upper bound of $1/2$ as k tends to infinity. This implicitly provides an analysis of a limiting mechanism, which we term the *permutation mechanism*, in which agents are considered one by one according to a random permutation. We prove that this mechanism is $1/2$-optimal, which is best possible subject to impartiality. The existence of a randomized impartial mechanism with this guarantee was in fact conjectured by Alon et al. [2011, Conjecture 4.4].

We finally give the first non-trivial bounds for settings without abstentions. We show that the permutation mechanism is at least $7/12$-optimal and at most $2/3$-optimal in this case, while no impartial mechanism can be more than $3/4$-optimal.

Related Work and Applications. Impartial decision making was first considered by de Clippel et al. [2008], for the case of a divisible resource to be shared among a set of agents. While the difference between a divisible resource and the indivisible resource considered in this paper disappears for randomized mechanisms, de Clippel et al. studied mechanisms with a more general message space that allows for fractional nominations, and at the same time aimed for weaker requirements to be achieved besides impartiality.

Alon et al. [2011] framed the problem considered here as one of designing approximately optimal strategyproof mechanisms without payments, an agenda proposed by Procaccia and Tennenholtz [2013] and earlier by Dekel et al. [2010]. Strategyproofness requires that an agent maximizes its utility by truthfully revealing its preferences and is equivalent to impartiality if the utility of an agent only depends on its chances of being selected. While this assumption seems somewhat restrictive, Alon et al. pointed out that their results in fact hold for any setting where agents give their own selection priority over that of their nominees.

Strategyproof selection is an important component of the peer review process for scientific articles and project proposals. For its Sensors and Sensing Systems program, the National Science Foundation (NSF) recently introduced a mechanism in which proposals are reviewed by other applicants, and acceptance of an applicant's own proposal depends in part on the extent to which the *reviews* submitted by the applicant agree with other reviews of the same proposals. The specific mechanism used by the NSF was originally proposed by Merrifield and Saari [2009] in the context of allocation of telescope time. Whether the mechanism provides the right incentives in peer review is debatable, but its lack of impartiality, which in this case is deliberate, would make it very hard to show any formal incentive properties. By contrast, our results allow for a separation of preferences regarding an agent's own selection and those regarding the selection of others, and can be combined in a straightforward way with peer prediction techniques [e.g., Witkowski and Parkes 2012] to provide strict incentives for the truthful evaluation of other agents. The exact properties achievable by such hybrid mechanisms and their use in peer review deserve further investigation.

Impartial selection is also more distantly related to work in distributed computing on leader election [e.g., Alon and Naor 1993; Cooper and Linial 1995; Feige 1999; Antonakopoulos 2006] and work on the manipulation of reputation systems [e.g., Friedman et al. 2007]. Leader election seeks to guarantee the selection of a non-malicious agent in the presence of malicious agents trying to manipulate the selection process. Work on reputation systems often considers models with more complex preference and message spaces, where maximization of a one-dimensional objective does not suffice.

The 2-partition mechanism, finally, is reminiscent of random sampling in unlimited-supply auctions [Fiat et al. 2002; Goldberg et al. 2006; Feige et al. 2005] and combinatorial auctions [Dobzinski et al. 2012]. It will be interesting to see whether our more complicated mechanisms and analysis techniques can be applied to these settings in a meaningful way.

Open Problems. While we completely solve the general case and make significant progress for the special case without abstentions, several interesting directions for future work remain. The most obvious question of course concerns the gap for settings without abstentions between the lower bound of $7/12$ provided by the permutation mechanism and the upper bound of $3/4$. It is unknown whether the lower bound is tight, but an improved upper bound of $2/3$ for the permutation mechanism suggests that the latter may not be optimal. Alon et al. considered the more general problem of selecting any fixed number of agents and gave an α-optimal impartial mechanism with α tending to 1 as the number of agents to be selected tends to infinity. The question of optimal mechanisms for selecting a small number of agents is wide open. We may finally ask whether optimality and anonymity are incompatible. This question arises from the observation that the permutation mechanism considers agents one by one and thus cannot process nominations anonymously. The k-partition mechanism, on the other hand, allows nominations by agents from the same set to be considered simultaneously and thus offers a certain degree of anonymity, but it is not optimal.

2. PRELIMINARIES

For $n \in \mathbb{N}$, let $\mathcal{G}_n = \{(N, E) : N = \{1, \dots, n\}, E \subseteq (N \times N) \setminus \bigcup_{i \in N}(\{i\} \times \{i\})\}$ be the set of directed graphs with n vertices and no loops. Let $\mathcal{G} = \bigcup_{n \in \mathbb{N}} \mathcal{G}_n$. For $G = (N, E) \in \mathcal{G}$, $S \subseteq N$, and $i \in N$, let $\delta^-_S(i, G) = |\{(j, i) \in E : G = (N, E), j \in S\}|$ denote the indegree of vertex i from vertices in S. We use $\delta^-(i, G)$ as a shorthand for $\delta^-_N(i, G)$, denote $\Delta(G) = \max_{i \in N} \delta^-(i, G)$, and write $\delta^-(i)$ instead of $\delta^-(i, G)$ and Δ instead of $\Delta(G)$ if G is clear from the context.

A *selection mechanism* for $\mathcal{G}$ is then given by a family of functions $f : \mathcal{G}_n \to [0, 1]^n$ that maps each graph to a probability distribution on its vertices. In a slight abuse of notation, we use f to refer to both the mechanism and individual functions from the family. Mechanism f is *impartial* on $\mathcal{G}' \subseteq \mathcal{G}$ if on this set of graphs the probability of selecting vertex i does not depend on its outgoing edges, i.e., if for every pair of graphs $G = (N, E)$ and $G' = (N, E')$ in $\mathcal{G}'$ and every $i \in N$, $(f(G))_i = (f(G'))_i$ whenever $E \setminus (\{i\} \times N) = E' \setminus (\{i\} \times N)$. All mechanisms we consider are impartial on $\mathcal{G}$, and we simply refer to such mechanisms as impartial mechanisms. Mechanism f is *α-optimal* on $\mathcal{G}' \subseteq \mathcal{G}$, for $\alpha \leq 1$, if for every graph in $\mathcal{G}'$ the expected indegree of the vertex selected by f differs from the maximum indegree by a factor of at most α, i.e., if

$$\inf_{\substack{G \in \mathcal{G} \\ \Delta(G) > 0}} \frac{\mathbb{E}_{i \sim f(G)}[\delta^-(i, G)]}{\Delta(G)} \geq \alpha.$$

We call a mechanism α-optimal if it is α-optimal on $\mathcal{G}$, and approximately optimal if it is α-optimal for some constant α.

As far as impartiality and approximate optimality are concerned, we can restrict our attention to symmetric mechanisms. Mechanism f is *symmetric* if it is invariant with respect to renaming of the vertices, i.e., if for every $G = (N, E) \in \mathcal{G}$, every $i \in N$, and every permutation $\pi = (\pi_1, \dots, \pi_{|N|})$ of N,

$$(f(G_\pi))_{\pi_i} = (f(G))_i,$$

where $G_\pi = (N, E_\pi)$ with $E_\pi = \{(\pi_i, \pi_j) : (i, j) \in E\}$. For a given mechanism f, denote by f_s the mechanism obtained by applying a random permutation π to the vertices of the input graph, invoking f, and permuting the result by the inverse of π, such that for all $n \in \mathbb{N}$, $G \in \mathcal{G}_n$, and $i \in \{1, \dots, n\}$,

$$\big(f_s(G)\big)_i = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \big(f(G_\pi)\big)_{\pi_i},$$

where $\mathcal{S}_n$ is the set of all permutations $\pi = (\pi_1, \dots, \pi_n)$ of a set of n elements. The following result is straightforward.

LEMMA 2.1. *Consider $\mathcal{G}' \subseteq \mathcal{G}$ such that $G_\pi \in \mathcal{G}'$ for every $G = (N, E) \in \mathcal{G}'$ and every permutation π of N. Let f be a selection mechanism that is impartial and α-optimal on $\mathcal{G}'$. Then f_s is impartial and α-optimal on $\mathcal{G}'$.*

Input: Graph $G = (N, E)$
Output: Vertex $i \in N$
Assign each $i \in N$ independently and uniformly at random to one of two sets A_1 and A_2;
if $A_2 = \emptyset$ **then** return a vertex chosen uniformly at random from N;
Return a vertex chosen uniformly at random from $\arg\max_{i \in A_2} \delta^-_{A_1}(i)$;

Fig. 1. The 2-partition mechanism

Fig. 2. No impartial mechanism is more than $1/2$-optimal

3. THE 2-PARTITION MECHANISM

We begin our investigation with a more detailed analysis of the 2-partition mechanism proposed by Alon et al. [2011]. The mechanism first assigns each vertex independently and uniformly at random to one of two sets A_1 and A_2, i.e., $A_1 \cup A_2 = N$, $A_1 \cap A_2 = \emptyset$, and $\mathbb{P}[i \in A_1] = \mathbb{P}[i \in A_2] = 1/2$ for all $i \in N$. Then it returns a vertex from A_2 that has maximum indegree from vertices in A_1, or a vertex chosen uniformly at random from N in case $A_2 = \emptyset$. A formal description of the mechanism is given in Figure 1.

The 2-partition mechanism is obviously impartial, as the outgoing edges of vertex $i \in N$ can influence the outcome only if $i \in A_1$ and $A_2 \neq \emptyset$, in which case i will never be selected. It is also easy to see that the mechanism is $1/4$-optimal. As noted by Alon et al., for an arbitrary graph G and a particular vertex i^* in G with indegree $\Delta = \Delta(G)$, we have that $\mathbb{P}[i^* \in A_2] = 1/2$ and, by linearity of expectation, we obtain $\mathbb{E}[\delta^-_{A_1}(i^*) \,|\, i^* \in A_2] = \mathbb{E}[\delta^-_{A_1}(i^*)] = \delta^-_N(i^*)/2 = \Delta/2$. The expected indegree of the selected vertex is thus at least $\Delta/2$ with probability at least $1/2$, i.e., at least $\Delta/4$. A graph with a single edge shows that this result is in fact tight. Alon et al. noted further that *no* impartial mechanism can be more than $1/2$-optimal. To this end, consider the two graphs in Figure 2, and the probabilities p_1, p_2, and p_3 with which certain vertices in these graphs are selected. Due to symmetry, which we can assume by Lemma 2.1, $p_1 = p_2$ and thus $p_1 \leq 1/2$. On the other hand, $p_1 = p_3$ by impartiality, so the expected indegree of the vertex selected in the right graph is at most $1/2$ and the claim follows.

This rather straightforward analysis of the 2-partition mechanism does not lead to a tight result, but it is unsatisfactory in particular because it provides no information about the performance of the mechanism on more complicated graphs, and no cues what a better mechanism might look like. We will gain both from the proof of Lemma 3.1 below, which establishes a lower bound on the expected indegree of the selected vertex relative to the maximum indegree $\Delta(G)$.

For the proof of this lemma, we fix an agent i^* with maximum indegree and consider a random partition $\boldsymbol{A} = (A_1, A_2)$ of all agents except i^*. We then let an adversary decide on the maximum number of nominations any agent in A_2 obtains from agents in A_1. Finally, we analyze the performance of the mechanism against the adversary under the uniform distribution over the two partitions $(A_1 \cup \{i^*\}, A_2)$ and $(A_1, A_2 \cup \{i^*\})$. It is easy to see that the performance of the mechanism against any fixed graph will be at least as good as its performance against the adversary. The adversary in turn has to find a tradeoff between two effects. On the one hand, a sufficient number of nominations for an agent in A_2 can prevent agent i^* from being selected in case it is added to A_2. On the other hand, increasing the number of nominations for such an agent improves the performance of the mechanism in the case where agent i^* is added to A_1. To obtain a lower bound on the performance of the 2-partition mechanism, we derive an optimal strategy for the adversary.

LEMMA 3.1. *On any graph G with maximum indegree $\Delta = \Delta(G)$, the 2-partition mechanism is $\alpha_2(\Delta)$-optimal, where*

$$\alpha_2(\Delta) = \frac{1}{\Delta 2^\Delta} \sum_{k=0}^{\Delta} \binom{\Delta}{k} \cdot \min\left\{\frac{\Delta}{2}, k\right\}.$$

PROOF. Let $i^* \in N$ such that $\delta^-(i^*) = \Delta$, and denote by X the indegree of the agent selected by the 2-partition mechanism. Then X is a random variable subject to the internal randomness of the mechanism, and we will be interested in its expected value $\mathbb{E}[X]$.

Let $\boldsymbol{A} = (A_1, A_2)$ be the partition selected in Line 1 of the 2-partition mechanism in Figure 1, and consider an arbitrary set $S \subseteq N \setminus \{i^*\}$ of vertices other than i^*. We begin by bounding $\mathbb{E}[X \mid A_1 \setminus \{i^*\} = S]$, i.e., the expected value of X given that $A_1 = S$ or $A_1 = S \cup \{i^*\}$. To this end, let $z(S)$ and $a(S)$ denote the indegree of i^* from S and the maximum indegree of any *other* element of $N \setminus S$ from S, respectively, i.e., $z(S) = \delta_S^-(i^*)$ and $a(S) = \max_{i \in N \setminus (S \cup \{i^*\})} \delta_S^-(i)$.

Assume for now that $S \neq \emptyset$ and $S \neq N \setminus \{i^*\}$. Then, $\mathbb{E}[X \mid A_1 = S] = \Delta$ if $z(S) > a(S)$, $\mathbb{E}[X \mid A_1 = S] \geq a(S)$ if $a(S) \geq z(S)$, and $\mathbb{E}[X \mid A_1 = S \cup \{i^*\}] \geq a(S)$. Note here that the expected value of X only increases if there is an edge from i^* to a vertex for which $a(S)$ is attained. Since the events where $A_1 = S$ and $A_1 = S \cup \{i^*\}$ occur with equal probability,

$$\begin{aligned}\mathbb{E}[X \mid A_1 \setminus \{i^*\} = S] &\geq \frac{\chi\big(z(S) > a(S)\big)\,\Delta + \big(1 - \chi\big(z(S) > a(S)\big)\big)\,a(S)}{2} + \frac{a(S)}{2} \\ &= a(S) + \frac{1}{2}\,\chi\big(z(S) > a(S)\big)\big(\Delta - a(S)\big),\end{aligned}$$

where χ denotes the indicator function on binary events, i.e., $\chi(E) = 1$ if event E takes place and $\chi(E) = 0$ otherwise. Depending on the value of $z(S)$, the right-hand side is minimized either for $a(S) = 0$ or for $a(S) = z(S)$, and it becomes equal to $\chi(z(S) > 0) \cdot \frac{\Delta}{2}$ when $a(S) = 0$ and equal to $z(S)$ when $a(S) = z(S)$. In summary,

$$\mathbb{E}[X \mid S \setminus \{i^*\} = S] \geq \min\left\{\chi\big(z(S) > 0\big) \cdot \frac{\Delta}{2}, z(S)\right\} = \min\left\{\frac{\Delta}{2}, z(S)\right\}. \tag{1}$$

We can now lift the assumption that $S \neq \emptyset$ and $S \neq N \setminus \{i^*\}$. If $S = \emptyset$, then $z(S) = 0$ and (1) holds trivially. If $S = N \setminus \{i^*\}$, then $z(S) = \Delta$, and i^* is in $N \setminus S$ and therefore chosen by the 2-partition mechanism with probability $1/2$. We then obtain that $\mathbb{E}[X \mid A_1 \setminus \{i^*\} = S] \geq \Delta/2 = \min\{\Delta/2, z(S)\}$, and (1) is again satisfied.

By construction of the 2-partition mechanism, each vertex belongs to A_1 with probability $1/2$, so $z(S) = \delta_{A_1}^-(i^*)$ is distributed according to the binomial distribution with Δ

trials and success probability $1/2$. We thus have that

$$\begin{aligned}\mathbb{E}[X] &= \sum_{S\subseteq N} \mathbb{P}[A_1 \setminus \{i^*\} = S] \cdot \mathbb{E}[X \mid A_1 \setminus \{i^*\} = S] \\ &\geq \sum_{k=0}^{\Delta} \sum_{\substack{S\subseteq N\\ z(S)=k}} \mathbb{P}[A_1 \setminus \{i^*\} = S] \cdot \min\left\{\frac{\Delta}{2}, k\right\} \\ &= \frac{1}{2^\Delta} \sum_{k=0}^{\Delta} \binom{\Delta}{k} \cdot \min\left\{\frac{\Delta}{2}, k\right\}.\end{aligned}$$

This finally implies that $\alpha_2(\Delta) \geq \frac{1}{\Delta 2^\Delta} \sum_{k=0}^{\Delta} \binom{\Delta}{k} \cdot \min\{\frac{\Delta}{2}, k\}$, as claimed. □

This result can now be used to derive a closed-form expression for $\alpha_2(\Delta)$. In the interest of space, we defer a formal proof to the full version of the paper.

THEOREM 3.2. *On any graph G with maximum indegree $\Delta = \Delta(G)$, the 2-partition mechanism is $\alpha_2(\Delta)$-optimal, where*

$$\alpha_2(\Delta) = \begin{cases} \frac{1}{4} & \text{if } \Delta = 1, \\ \frac{1}{2} - \frac{1}{2^{\Delta+2}}\binom{\Delta}{\Delta/2} & \text{if } \Delta \geq 2 \text{ and even}, \\ \alpha_2(\Delta-1) & \text{if } \Delta \geq 3 \text{ and odd.} \end{cases}$$

Intuitively, on graphs with an even maximum indegree Δ, the performance of the 2-partition mechanism is bounded by $1/2$ minus half the probability that Δ coin tosses show exactly $\Delta/2$ tails. Given this closed-form expression it is straightforward to show that $\alpha_2(\Delta)$ is non-decreasing in Δ, we again defer the proof to the full version of the paper.

COROLLARY 3.3. *For every $\Delta \in \mathbb{N}$, $\alpha_2(\Delta+1) \geq \alpha_2(\Delta)$ and $\alpha_2(\Delta+2) > \alpha_2(\Delta)$.*

This result implies that a graph with a single edge is in fact the worst case for the 2-partition mechanism, and it also yields the first non-trivial lower bound for settings without abstentions, where each agents nominates at least one other agent. In the absence of abstentions, one of two conditions is always satisfied: either every vertex has indegree exactly one, in which case every mechanism including the 2-partition mechanism is optimal, or $\Delta \geq 2$ and the 2-partition mechanism is at least $\alpha_2(\Delta)$-optimal. Since $\alpha_2(2) = 3/8$, we conclude that the 2-partition mechanism is $3/8$-optimal on all instances without abstentions. We will return to this special case, and show a better bound, in Section 6.

4. THE K-PARTITION MECHANISM

What is perhaps most interesting about the above analysis of the 2-partition mechanism is that the technique we have used can in principle also be applied to a partition of the vertices into more than two sets. Indeed, in this section, we propose a generalization of the 2-partition mechanism to a larger number of sets and then generalize the analysis technique to the new mechanism.

For a fixed $k \geq 2$, the new mechanism first assigns each vertex $i \in N$ independently and uniformly at random to one of k sets $A_1, \ldots, A_k$, with the property that $\bigcup_{i=1,\ldots,k} A_i = N$, $A_i \cap A_j = \emptyset$ for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$, and $\mathbb{P}[i \in A_i] = 1/k$ for all $i \in \{1, \ldots, k\}$. The mechanism then proceeds in $k-1$ iterations which we number

Input: Graph $G = (N, E)$
Output: Vertex $i \in N$
1 Assign each $i \in N$ independently and uniformly at random to one of k sets $A_1, \dots, A_k$;
2 Set $\{i^*\} := \emptyset$ and $d^* := 0$;
3 **for** $j := 2, \dots, k$ **do**
4   **if** $\max_{i \in A_j} \delta^-_{A_{<j} \setminus \{i^*\}}(i) \geq d^*$ **then**
5   Choose $i^* \in \arg\max_{i' \in A_j} \delta^-_{A_{<j}}(i')$ uniformly at random; set $d^* := \delta^-_{A_{<j}}(i^*)$;
6   **end**
7 **end**
8 Return i^*;

Fig. 3. The k-partition mechanism

from 2 to k, during which it maintains and updates a candidate vertex that is finally selected after iteration k.

The candidate vertex is updated in iteration j if the maximum indegree among vertices in A_j from vertices in $A_{<j} = \bigcup_{i=1}^{j-1} A_i$ other than the candidate is at least that of the candidate at the time it became the candidate. In that case, the new candidate is chosen uniformly at random from the set of vertices in A_j with maximum indegree from vertices in $A_{<j} = \bigcup_{i=1}^{j-1} A_i$, now including the previous candidate. The mechanism is clearly impartial, because it only takes into account the outgoing edges of vertices that can no longer be selected. The fact that any outgoing edges of the previous candidate are taken into account when selecting the new candidate is somewhat subtle, but it turns out to be crucial for our results. A formal description of the mechanism is given in Figure 3.

For the analysis, consider a graph $G = (N, E) \in \mathcal{G}$ and a vertex $i^* \in N$ with indegree $\Delta = \Delta(G)$. Fix $k \in \mathbb{N}$, and let X be the indegree of the vertex selected from G by the k-partition mechanism. Note that X is a random variable subject to the internal randomness of the mechanism, and that we are interested in its expected value $\mathbb{E}[X]$.

To derive a lower bound on $\mathbb{E}[X]$, we need some notation. For a subset $N' \subseteq N$ of the vertices, let $\mathcal{P}_k(N')$ denote the set of all partitions $\boldsymbol{S} = (S_1, \dots, S_k)$ of N' into k (possibly empty) sets $S_1, \dots, S_k$, i.e.,

$$\mathcal{P}_k(N') = \Big\{ \boldsymbol{S} = (S_1, \dots, S_k) : \bigcup_{j=1}^{k} S_j = N',\, S_i \cap S_j = \emptyset \text{ for } i, j \in \{1, \dots, k\} \text{ with } i \neq j \Big\}.$$

For a partition $\boldsymbol{S} = (S_1, \dots, S_k)$ and $j \in \{1, \dots, k\}$, let $S_{<j} = \bigcup_{i=1}^{j-1} S_i$ denote the union of the sets with index strictly smaller than j. For a partition $\boldsymbol{S} = (S_1, \dots, S_k) \in \mathcal{P}_k(N)$ and $i \in N$, we slightly abuse notation and write $\boldsymbol{S} \setminus \{i\} = (S_1 \setminus \{i\}, \dots, S_k \setminus \{i\})$ for the partition obtained from $\boldsymbol{S}$ by removing i.

Let $\boldsymbol{A}$ be the partition chosen in Line 1 of the k-partition mechanism in Figure 3. The following lemma bounds the expected value of X given that $\boldsymbol{A} = \boldsymbol{S}$ for some given partition $\boldsymbol{S} \in \mathcal{P}_k(N)$ in terms of two parameters a and z. Intuitively, z is the number of nominations that a fixed agent i^* with maximum indegree receives from agents that end up in a set S_i of the partition with a strictly smaller index than that of agent i^*. The parameter a denotes the maximum number of nominations *any* agent receives from agents that end up in a set with a smaller index than the agent itself. The proof crucially relies on the fact that the nomination of the previous candidate is taken into account when selecting the new candidate.

LEMMA 4.1. *Consider a graph $G = (N, E)$ and a vertex i^* with indegree $\Delta = \Delta(G)$. Let $\boldsymbol{S} = (S_1, \dots, S_k) \in \mathcal{P}_k(N)$, and let $j^* \in \{1, \dots, k\}$ such that $i^* \in S_{j^*}$. Then,*

$$\mathbb{E}[X \mid \boldsymbol{A} = \boldsymbol{S}] \geq a + \chi(z > a) \cdot \big(\Delta - a\big),$$

where $a = \max_{j=2,\dots,k} \max_{i \in S_j \setminus \{i^*\}} \delta^-_{S_{<j}}(i)$, $z = \delta^-_{S_{<j^*}}(i^*)$, and χ denotes the indicator function on binary events, i.e., $\chi(E) = 1$ if event E takes place and $\chi(E) = 0$ otherwise.

PROOF. For $j = 2, \dots, k$, let $i^*(j)$ and $d^*(j)$ denote the values of i^* and d^* after iteration j of the mechanism. We show by induction on j that for all $j = 2, \dots, k$, $d^*(j) = \max_{m=2,\dots,j} \max_{i \in S_m} \delta^-_{S_{<m}}(i)$.

First consider the case where $j = 2$. If $S_2 = \emptyset$, there is nothing to show. Otherwise, the mechanism chooses a vertex $i^*(2)$ with $\delta^-_{S_{<2}}(i^*(2)) = \delta^-_{S_1}(i^*(2)) = \max_{i \in S_2} \delta^-_{S_1}(i)$, and thus

$$d^*(2) = \delta^-_{S_1}(i^*(2)) = \max_{i \in S_2} \delta^-_{S_1}(i).$$

Now suppose that $d^*(j-1) = \max_{m=2,\dots,j-1} \max_{i \in S_m} \delta^-_{S_{<m}}(i)$ for some $j \in \{3, \dots, k\}$. If $S_j = \emptyset$, there again is nothing to show. Otherwise we consider iteration j of the mechanism and distinguish two cases.

If $\max_{i \in S_j} \delta^-_{S_{<j} \setminus \{i^*(j-1)\}}(i) \geq d^*(j-1)$, then $i^*(j) \in \arg\max_{i \in S_j} \delta^-_{S_{<j}}(i)$ and

$$d^*(j) = \delta^-_{S_{<j}}(i^*(j)) = \max_{i \in S_j} \delta^-_{S_{<j}}(i).$$

Furthermore,

$$d^*(j) = \delta^-_{S_{<j}}\big(i^*(j)\big) \geq \delta^-_{S_{<j} \setminus \{i^*(j-1)\}}\big(i^*(j)\big) \geq d^*(j-1) = \max_{m=2,\dots,j-1} \max_{i \in S_m} \delta^-_{S_{<m}}(i),$$

where the last equality holds by the induction hypothesis. In summary,

$$d^*(j) = \max_{m=2,\dots,j} \max_{i \in S_m} \delta^-_{S_{<m}}(i).$$

If, on the other hand, $\max_{i \in S_j} \delta^-_{S_{<j} \setminus \{i^*(j-1)\}}(i) < d^*(j-1)$, then $i^*(j) = i^*(j-1)$ and

$$d^*(j) = d^*(j-1) \geq \max_{i \in S_j} \delta^-_{S_{<j} \setminus \{i^*(j-1)\}}(i) + 1 \geq \max_{i \in S_j} \delta^-_{S_{<j}}(i),$$

where the first inequality holds because degrees are integral, and the second inequality because there can be at most one edge from $i^*(j-1)$ to $i^*(j)$. Furthermore,

$$d^*(j) = d^*(j-1) = \max_{m=2,\dots,j-1} \max_{i \in S_m} \delta^-_{S_{<m}}(i),$$

where the second equality holds by the induction hypothesis, so again

$$d^*(j) = \max_{m=2,\dots,j} \max_{i \in S_m} \delta^-_{S_{<m}}(i).$$

Since the mechanism returns $i^*(k)$,

$$\begin{aligned}\mathbb{E}[X \mid \boldsymbol{A} = \boldsymbol{S}] = \delta^-(i^*(k)) = d^*(k) &= \max_{j=2,\dots,k} \max_{i \in S_j} \delta^-_{S_{<j}}(i) \\ &\geq \max_{j=2,\dots,k} \max_{i \in S_j} \delta^-_{S_{<j} \setminus \{i^*\}}(i) = a.\end{aligned}$$

To complete the proof, assume that

$$z = \delta^-_{S_{<j^*}}(i^*) > \max_{j=2,\dots,k} \max_{i \in S_j \setminus \{i^*\}} \delta^-_{S_{<j}}(i) = a.$$

Then

$$\delta^-_{S_{<j^*}}(i^*) > \max_{m=2,\dots,j^*-1} \max_{i \in S_m} \delta^-_{S_{<m}}(i) = d^*(j^* - 1)$$

and

$$\delta^-_{S_{<j^*}}(i^*) > \max_{m=j^*+1,\dots,k} \max_{i \in S_m} \delta^-_{S_{<m}}(i),$$

so $i^*(j) = i^*$ for $j = j^*, \dots, k$. The mechanism selects i^*, and $\mathbb{E}[X \mid \boldsymbol{A} = \boldsymbol{S}, z > a] = \Delta$. □

As in our analysis of the 2-partition mechanism, we now proceed to bound the expected value of X given that a partition is fixed for all vertices except i^*, and i^* is then allocated uniformly at random to one of the k sets. The proof uses a generalization of the adversarial argument used for analyzing the 2-partition mechanism.

LEMMA 4.2. *Consider a graph $G = (N, E)$ and a vertex i^* with indegree $\Delta = \Delta(G)$. Let $\boldsymbol{S} = (S_1, \dots, S_k) \in \mathcal{P}_k(N \setminus \{i^*\})$. For $j = 1, \dots, k$, let $z_j = \delta^-_{S_{<j}}(i^*)$. Then,*

$$\mathbb{E}[X \mid \boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}] \geq \min_{j=1,\dots,k} \left\{ z_j + \frac{k-j}{k}(\Delta - z_j) \right\}.$$

PROOF. There are exactly k partitions $\boldsymbol{S}' \in \mathcal{P}_k(N)$ such that $\boldsymbol{S}' \setminus \{i^*\} = \boldsymbol{S}$, and each of them occurs with probability $1/k$, so

$$\mathbb{E}[X \mid \boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}] = \frac{1}{k} \sum_{m=1}^{k} \mathbb{E}[X \mid \boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}, i^* \in A_m].$$

By Lemma 4.1,

$$\mathbb{E}[X \mid \boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}] \geq a(\boldsymbol{S}) + \frac{1}{k} \sum_{m=1}^{k} \chi(z_m > a(\boldsymbol{S})) \cdot (\Delta - a(\boldsymbol{S})),$$

where $a(\boldsymbol{S}) = \max_{j=2,\dots,k} \max_{i \in S_j} \delta^-_{S_{<j}}(i)$. Note that the right-hand side is minimized when $a(\boldsymbol{S}) = z_m$ for some $m \in \{1, \dots, k\}$, so

$$\begin{aligned}\mathbb{E}[X \mid \boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}] &\geq \min_{j=1,\dots,k} \left\{ z_j + \frac{1}{k} \sum_{m=1}^{k} \chi(z_m > z_j) \cdot (\Delta - z_j) \right\} \\ &= \min_{j=1,\dots,k} \left\{ z_j + \frac{1}{k} \sum_{m=j+1}^{k} \chi(z_m > z_j) \cdot (\Delta - z_j) \right\}.\end{aligned}$$

Now observe that whenever $z_j = z_{j+1}$ for some $j = 2, \dots, k$, the respective terms in the minimization are equal as well. This implies that the minimum will always be attained for some j with $z_{j-1} < z_j$. By setting $z_0 = -1$ and simplifying,

$$\begin{aligned}\mathbb{E}[X \mid \boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}] &\geq \min_{\substack{j=1,\dots,k \\ z_{j-1} < z_j}} \left\{ z_j + \frac{1}{k} \sum_{m=j+1}^{k} (\Delta - z_j) \right\} \\ &= \min_{\substack{j=1,\dots,k \\ z_{j-1} < z_j}} \left\{ z_j + \frac{k-j}{k}(\Delta - z_j) \right\}.\end{aligned}$$

Since $z_j + \frac{k-j}{k}(\Delta - z_j)$ attains its minimum for some j with $z_{j-1} < z_j$, we can drop the condition that $z_{j-1} < z_j$ and obtain

$$\mathbb{E}[X \mid \boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}] \geq \min_{j=1,\dots,k} \left\{ z_j + \frac{k-j}{k}(\Delta - z_j) \right\}$$

as claimed. □

To obtain a bound on $\mathbb{E}[X]$, we will now average the expression obtained in Lemma 4.2 over the distribution on partitions of N. Like that expression, the bound we obtain does not depend on the actual partitions, but only on the indegree $\delta^-_{S_j}(i^*)$ of i^* from each set S_j in the partition. For $\Delta, k \in \mathbb{N}$, let $P_k(\Delta) = \{\boldsymbol{v} \in \mathbb{N}^k : \sum_{i=1}^k v_i = \Delta\}$ be the set of partitions of Δ elements into k sets. For a partition $\boldsymbol{v} = (v_1, \ldots, v_k) \in P_k(\Delta)$, let $\binom{\Delta}{\boldsymbol{v}} = \Delta!/(v_1! \cdots v_k!)$ be the number of partitions of a set with Δ elements into k sets of sizes $v_1, \ldots, v_k$. We then have the following result.

LEMMA 4.3. *On any graph G with maximum indegree $\Delta = \Delta(G)$, the k-partition mechanism is $\alpha_k(\Delta)$-optimal, where*

$$\alpha_k(\Delta) = \frac{1}{\Delta k^\Delta} \sum\nolimits_{\boldsymbol{v} \in P_k(\Delta)} \binom{\Delta}{\boldsymbol{v}} \min\nolimits_{j=1,\ldots,k} \left\{ \frac{k-j}{k} \sum_{\ell=1}^{k} v_\ell + \frac{j}{k} \sum_{\ell=1}^{j-1} v_\ell \right\}.$$

PROOF. Consider a vertex i^* with indegree Δ, and note that

$$\mathbb{E}[X] = \sum\nolimits_{\boldsymbol{S} \in \mathcal{P}_k(N \setminus \{i^*\})} \mathbb{P}[\boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}] \cdot \mathbb{E}[X \mid \boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}].$$

For $\boldsymbol{S} \in \mathcal{P}_k(N \setminus \{i^*\})$, let $z_j(\boldsymbol{S}) = \delta^-_{S_{<j}}(i^*)$. Then, by Lemma 4.2,

$$\mathbb{E}[X] \geq \sum\nolimits_{\boldsymbol{S} \in \mathcal{P}_k(N \setminus \{i^*\})} \mathbb{P}[\boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}] \cdot \min\nolimits_{j=1,\ldots,k} \left\{ z_j(\boldsymbol{S}) + \frac{k-j}{k} (\Delta - z_j(\boldsymbol{S})) \right\}.$$

For $\boldsymbol{S} \in \mathcal{P}_k(N \setminus \{i^*\})$, let $v_j(\boldsymbol{S}) = \delta^-_{S_j}(i^*)$. Then, $z_j(\boldsymbol{S}) = \sum_{m=1}^{j-1} v_m(\boldsymbol{S})$, and

$$\begin{aligned} \mathbb{E}[X] &\geq \sum\nolimits_{\boldsymbol{S} \in \mathcal{P}_k(N \setminus \{i^*\})} \mathbb{P}[\boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}] \cdot \min\nolimits_{j=1,\ldots,k} \left\{ \sum_{m=1}^{j-1} v_m(\boldsymbol{S}) + \frac{k-j}{k} \sum_{m=j}^{k} v_m(\boldsymbol{S}) \right\} \\ &= \sum\nolimits_{\boldsymbol{S} \in \mathcal{P}_k(N \setminus \{i^*\})} \mathbb{P}[\boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}] \cdot \min\nolimits_{j=1,\ldots,k} \left\{ \frac{k-j}{k} \sum_{m=1}^{k} v_m(\boldsymbol{S}) + \frac{j}{k} \sum_{m=1}^{j-1} v_m(\boldsymbol{S}) \right\}. \end{aligned}$$

Since $(z_1(\boldsymbol{S}), \ldots, z_k(\boldsymbol{S}))$ follows a multinomial distribution with Δ trials and success probability $1/k$ for each category, we have that

$$\mathbb{E}[X] \geq \frac{1}{k^\Delta} \sum\nolimits_{\boldsymbol{v} \in P_k(\Delta)} \binom{\Delta}{\boldsymbol{v}} \min\nolimits_{j=1,\ldots,k} \left\{ \frac{k-j}{k} \sum_{\ell=1}^{k} v_\ell + \frac{j}{k} \sum_{\ell=1}^{j-1} v_\ell \right\},$$

and the claim follows. □

In the case of the 2-partition mechanism, we obtained a lower bound on the degree of optimality by deriving a closed-form expression for $\alpha_2(\Delta)$ that turned out to be monotonically non-decreasing in Δ. While the complexity of α_k prevents us from taking the same route for $k > 2$, monotonicity turns out to hold for any value of k. We show this using a projection of samples from $P_{k+1}(\Delta)$ to samples from $P_k(\Delta)$, which allows us to compare $a_k(\Delta)$ with $\alpha_{k+1}(\Delta)$. The proof is deferred to the full version of the paper.

THEOREM 4.4. *For any $k \geq 2$, $\alpha_k(\Delta)$ is non-decreasing in Δ.*

Monotonicity of α_k allows us to obtain a lower bound on the degree of optimality of the k-partition mechanism by bounding $\alpha_k(1)$ from below.

THEOREM 4.5. *The k-partition mechanism for $k \geq 2$ is $\frac{k-1}{2k}$-optimal.*

Input: Graph $G = (N, E)$
Output: Vertex $i \in N$

Choose a permutation $\pi = (\pi_1, \ldots, \pi_{|N|})$ of N uniformly at random;
Set $i^* := \pi_1$, $d^* := 0$;
for $j = 2, \ldots, |N|$ **do**
  if $\delta^-_{\{\pi_1,\ldots,\pi_{j-1}\}\setminus\{i^*\}}(\pi_j) \geq d^*$ **then**
    Set $i^* := \pi_j$, $d^* := \delta^-_{\{\pi_1,\ldots,\pi_{j-1}\}}(\pi_j)$;
  end
end
return i^*;

Fig. 4. The permutation mechanism

PROOF. In light of Theorem 4.4, it suffices to show that $\alpha_k(1) \geq \frac{k-1}{2k}$ for every $k \geq 2$. By Lemma 4.3,

$$\alpha_k(1) = \frac{1}{k} \sum\nolimits_{\boldsymbol{v} \in P_k(1)} \min\nolimits_{j=1,\ldots,k} \left\{ \frac{k-j}{k} \sum_{\ell=1}^{k} v_\ell + \frac{j}{k} \sum_{\ell=1}^{j-1} v_\ell \right\}.$$

Taking the pointwise minimum,

$$\alpha_1(k) \geq \frac{1}{k} \sum\nolimits_{\boldsymbol{v} \in P_k(1)} \left\langle \boldsymbol{v}, \left(\frac{k-1}{k}, \frac{k-2}{k}, \ldots, \frac{1}{k}, 0 \right) \right\rangle.$$

In the sum, every unit vector occurs exactly once, and thus

$$\alpha_k(1) \geq \frac{1}{k} \sum_{i=1}^{k} \frac{k-i}{k} = \frac{1}{k^2} \sum_{i=0}^{k-1} i = \frac{k(k-1)}{2k^2} = \frac{k-1}{2k}.$$

□

5. THE PERMUTATION MECHANISM

We have started from the simple result that no impartial selection mechanism can be more than $1/2$-optimal, and in the previous section have identified a class of mechanisms parametrized by $k \in \mathbb{N}$ that attains this bound in the limit as k goes to infinity. It turns out that the bound can also be achieved exactly, by a limiting mechanism for the above class. This mechanism, which we call the *permutation mechanism*, considers the vertices one by one according to a random permutation $\pi = (\pi_1, \ldots, \pi_n)$ and in each step compares the current vertex π_j to a single candidate vertex π_ℓ with $\ell < j$. In determining the indegree of the candidate vertex π_ℓ it takes into account the outgoing edges of vertices $\pi_1, \ldots, \pi_{\ell-1}$. For the indegree of the current vertex π_j it takes into account the outgoing edges of vertices $\pi_1, \ldots, \pi_{j-1}$, except π_ℓ. If the latter is greater than or equal to the former, π_j becomes the new candidate vertex, and the candidate vertex after the final step is the one selected by the mechanism. A formal description of the mechanism is given in Figure 4.

It is easy to see that this mechanism is impartial, because it only takes into account the outgoing edges of vertices that can no longer be selected. More interestingly, the permutation mechanism is the limit in a certain well-defined sense of the k-partition mechanism as k goes to infinity. We can thus build on our analysis of the k-partition mechanism to obtain the following result. A formal proof is again deferred to the full version of the paper.

THEOREM 5.1. *The permutation mechanism is* $1/2$*-optimal.*

This result is tight in the sense that no impartial mechanism can be more than $1/2$-optimal. The existence of a $1/2$-optimal impartial mechanism was in fact conjectured by Alon et al. [2011, Conjecture 4.4].

A potential disadvantage of the permutation mechanism is that it considers agents one by one and therefore cannot process nominations anonymously. This may be of concern in situations where agents do not want their opinion regarding other agents to be publicly known. In the k-partition mechanism for some fixed value of k, on the other hand, the nominations submitted by agents in block A_j of the partition can be processed simultaneously and thus with partial anonymity. It is an interesting question whether this tradeoff between anonymity and approximate optimality is intrinsic to the problem, or whether there exits a different mechanism that achieves the same degree of optimality as the permutation mechanism but a higher degree of anonymity.

6. NO ABSTENTIONS

Let us finally consider the interesting special case of graphs in which every vertex has outdegree at least 1. This case corresponds to settings without abstentions and in particular includes the setting of Holzman and Moulin [2013], where every agent submits *exactly one* nomination. Formally, let $\mathcal{G}_n^+ = \{(N,E) \in \mathcal{G}_n : \min_{i \in N} \delta^+(i,(N,E)) \geq 1\}$, where $\delta^+(i,(N,E)) = |\{(i,j) \in E : j \in N\}|$, and $\mathcal{G}^+ = \bigcup_{n \in \mathbb{N}} \mathcal{G}_n^+$.

In Section 3 we obtained an improved lower bound for the 2-partition mechanism of $\alpha_2(2) = 3/8$ instead of $\alpha_2(1) = 1/4$ for graphs $G \in \mathcal{G}^+$, using the simple observation that the 2-partition mechanism is optimal on graphs with maximum indegree 1 and that we can therefore focus on graphs with maximum indegree at least 2. The same argument can be applied to the k-partition mechanism as well, but obtaining an expression for $\alpha_k(2)$ is more challenging. The following lemma provides an alternative expression for $\alpha_k(2)$, which will subsequently be used to derive an improved bound.

LEMMA 6.1. *On any graph G with maximum indegree $\Delta(G) = 2$, the k-partition mechanism for $k \geq 2$ is $\alpha_k(2)$-optimal, where*

$$\alpha_k(2) = 1 - \frac{1}{k^3} \sum_{x_1,x_2 \in \{1,\ldots,k\}} \max\left\{\frac{\max\{x_1,x_2\}}{2}, \min\{x_1,x_2\}\right\}.$$

PROOF. Let $k \geq 2$. Consider a graph $G = (N,E)$ with $\Delta(G) = 2$, and note that there must exist $i_1, i_2, i^* \in N$ such that $i_1 \neq i_2$ and $\{(i_1,i^*),(i_2,i^*)\} \subseteq E$. Consider the partition $\boldsymbol{A} = (A_1,\ldots,A_k)$ chosen by the k-partition mechanism and let $x_1, x_2, y \in \{1,\ldots,k\}$ be such that $i_1 \in A_{x_1}$, $i_2 \in A_{x_2}$, and $i^* \in A_y$. Note that x_1, x_2, and y are independent random variables distributed uniformly on $\{1,\ldots,k\}$. Denote $\check{x} = \max\{x_1,x_2\}$ and $\hat{x} = \min\{x_1,x_2\}$. Then, by Lemma 4.2,

$$\begin{aligned}
\alpha_k(2) &\geq \frac{1}{2} \cdot \frac{1}{k^3} \sum_{x_1,x_2,y \in \{1,\ldots,k\}} \Big(2\chi\big(y > \check{x}\big) + \chi\big(y \leq \check{x}\big) \cdot \chi\big(\check{x} \leq 2(\check{x} - \hat{x})\big) + \\
&\qquad\qquad 2\chi\big(\hat{x} \leq y \leq \check{x}\big) \cdot \chi\big(2(\check{x} - \hat{x}) < \check{x}\big)\Big) \\
&= \frac{1}{k^2} \sum_{x_1,x_2 \in \{1,\ldots,k\}} \left(\frac{k - \check{x}}{k} + \frac{1}{2} \cdot \frac{\check{x}}{k} \cdot \chi\big(2\hat{x} \leq \check{x}\big) + \frac{\check{x} - \hat{x}}{k} \cdot \chi\big(2\hat{x} > \check{x}\big)\right) \\
&= \frac{1}{k^2} \sum_{x_1,x_2 \in \{1,\ldots,k\}} \left(1 - \frac{\check{x}}{2k} \cdot \chi\big(2\hat{x} \leq \check{x}\big) - \frac{\hat{x}}{k} \cdot \chi\big(2\hat{x} > \check{x}\big)\right) \\
&= 1 - \frac{1}{k^3} \sum_{x_1,x_2 \in \{1,\ldots,k\}} \max\left\{\frac{\check{x}}{2}, \hat{x}\right\}.
\end{aligned}$$

□

Building on Lemma 6.1, we are now in a position to give the main result of this section.

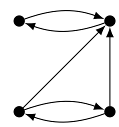

Fig. 5. A graph on which the permutation mechanism is $2/3$-optimal

THEOREM 6.2. *The permutation mechanism is* $7/12$*-optimal on* $\mathcal{G}^+$.

PROOF. Let $G = (N, E) \in \mathcal{G}^+$. If $\Delta(G) = 1$, then $\delta^-(i, G) = 1$ for all $i \in N$ and every mechanism, including the permutation mechanism, is optimal on G. If $\Delta(G) \geq 2$, then by Lemma 4.3, Theorem 4.4, and Lemma 6.1, the k-partition mechanism is $\alpha_k(2)$-optimal on G, where $\alpha_k(2) = 1 - \beta_k/k^3$ with

$$\beta_k = \sum\nolimits_{x_1,x_2\in\{1,\dots,k\}} \max\left\{\frac{\max\{x_1,x_2\}}{2}, \min\{x_1,x_2\}\right\}.$$

Now,

$$\begin{aligned}\beta_k = &\sum\nolimits_{x_1,x_2\in\{1,\dots,k-1\}} \max\left\{\frac{\max\{x_1,x_2\}}{2}, \min\{x_1,x_2\}\right\} + \\ &\sum\nolimits_{x_2\in\{1,\dots,k-1\}} \max\left\{\frac{\max\{k,x_2\}}{2}, \min\{k,x_2\}\right\} + \\ &\sum\nolimits_{x_1\in\{1,\dots,k-1\}} \max\left\{\frac{\max\{x_1,k\}}{2}, \min\{x_1,k\}\right\} + k \\ = &\,\beta_{k-1} + k + 2\sum\nolimits_{x_1\in\{1,\dots,k-1\}} \max\left\{\frac{k}{2}, x_1\right\} \\ = &\,\beta_{k-1} + \frac{5}{4}k^2 + o(k^2)\end{aligned}$$

Since $\beta_1 = 1$,

$$\beta_k = 1 + \sum_{\ell=1}^{k}\left(\frac{5}{4}k^2 + o(k^2)\right) = \frac{5}{12}k^3 + o(k^3)$$

and thus

$$\alpha_k(2) = \frac{7}{12} + \frac{o(k^3)}{k^3}.$$

This expression tends to $7/12$ as k tends to infinity, and the claim can be established analogously to Theorem 5.1. □

One may wonder whether this bound is tight, for the permutation mechanism or even in general. We leave this as an open question, but conclude by giving upper bounds of $2/3$ and $3/4$, respectively, on possible values of α for the permutation mechanism and any impartial mechanism.

To see that the permutation mechanism cannot be more than $2/3$-optimal, consider the graph of Figure 5. The unique vertex with indegree 3 in this graph is selected by the permutation mechanism if and only if it appears in the last two positions of the permutation, which happens with probability $1/2$. Indeed, when it appears in one of the first two positions it has indegree at most 1 at the time it is considered by the mechanism. At the same time, one of the vertices in the last two positions has indegree 1 when it is considered and consequently gets selected. The expected indegree

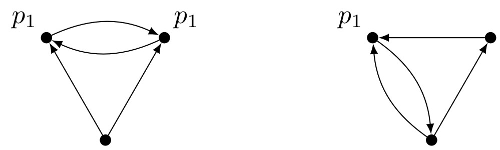

Fig. 6. Impartial probability assignment for two graphs with $n = 3$

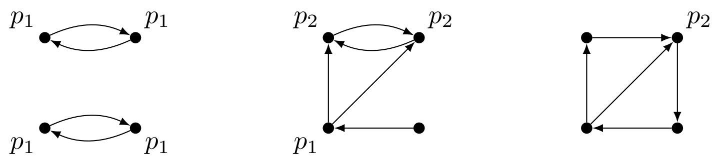

Fig. 7. Impartial probability assignment for three graphs with $n = 4$

of the selected vertex is thus $3 \cdot 1/2 + 1 \cdot 1/2 = 2$, compared to a maximum indegree of 3. What is interesting about this bound is that it is attained for a graph with maximum indegree 3. This suggests that a matching lower bound may not be achievable by a monotonicity result like that of Theorem 4.4.

The same upper bound of $2/3$ holds asymptotically for the more restricted case considered by Holzman and Moulin [2013], where every vertex has outdegree one. To see this, consider the graph $G = (N, E)$ with $N = \{1, \ldots, n\}$ and $E = \{(i, i+1) : i = 1, \ldots, n-2\} \cup \{(n-1, 1), (n, 1)\}$, and observe that the permutation mechanism selects vertex 1, the unique vertex with indegree 2, with significant probability only for permutations in which vertices $n-1$ and n both occur before 1. Since the latter happens with probability exactly $1/3$, the expected indegree of the selected vertex is not significantly greater than $2 \cdot 1/3 + 1 \cdot 2/3 = 4/3$, compared to a maximum indegree of 2.

Our final result establishes upper bounds on α for *any* mechanism that is impartial and α-optimal on $\mathcal{G}_n^+$, for a given value of n. We obtain these bounds from a class of linear optimization problems characterizing the α-optimal impartial mechanisms for the maximum value of α. These optimization problems have a number of constraints that is exponential in n, but can be solved optimally for $n \leq 7$. A generalization of the corresponding dual solutions to arbitrary values of n yields the upper bounds, and these bounds are tight for $n \leq 7$.

THEOREM 6.3. *Consider an impartial selection mechanism that is α-optimal on $\mathcal{G}_n^+$. Then,*

$$\alpha \leq \begin{cases} 3/4 & \textit{if } n = 3, \\ (3n-1)/4n & \textit{otherwise.} \end{cases}$$

PROOF. By Lemma 2.1 we can restrict our attention to symmetric mechanisms. We distinguish the cases where $n = 3$, $n \geq 4$ and even, $n = 5$, and $n \geq 7$ and odd.

First assume that $n = 3$, and consider the two graphs shown in Figure 6. It is easily verified that any impartial mechanism must assign probabilities as shown, and it must therefore be the case that $p_1 \leq \frac{1}{2}$. In the graph on the right, the agent assigned probability p_1 is the unique agent with the maximum indegree of 2, and thus

$$\alpha \leq \frac{2p_1 + (1 - p_1)}{2} = \frac{p_1 + 1}{2} \leq \frac{3}{4}.$$

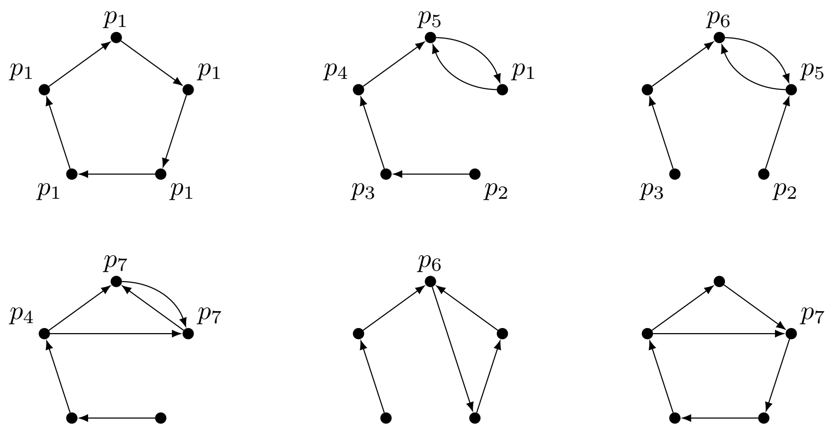

Fig. 8. Impartial probability assignment for six graphs with $n = 5$

Now assume that $n \geq 4$ and even, and consider the set of three graphs on n agents where agents 1 to 4 nominate as in Figure 7 and the remaining $n-4$ agents are grouped in pairs such that agent $2i$ nominates agent $2i-1$ and vice versa. It is easily verified that any impartial mechanism must assign probabilities as in Figure 7, and thus $np_1 = 1$ and $p_1 + 2p_2 \leq 1$. Moreover, the agent assigned probability p_2 in the last graph is the unique agent with indegree 2 in that graph, and thus

$$\alpha \leq \frac{2p_2 + (1-p_2)}{2} = \frac{p_2+1}{2} \leq \frac{\frac{n-1}{2n}+1}{2} = \frac{3n-1}{4n}.$$

Now assume that $n = 5$, and consider the six graphs shown in Figure 8. It is easily verified that any impartial mechanism must assign probabilities as in Figure 8, so

$$p_1 = 1/5, \tag{2}$$

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1, \tag{3}$$

$$p_2 + p_3 + p_5 + p_6 \leq 1, \tag{4}$$

$$p_4 + 2p_7 \leq 1. \tag{5}$$

By adding (2), (4), and (5) and subtracting (3),

$$p_6 + 2p_7 \leq \frac{6}{5} \qquad \text{and thus} \qquad \min(p_6, p_7) \leq \frac{2}{5}.$$

The agents assigned probabilities p_6 and p_7 in the last two graphs on the bottom row of Figure 8 are the unique agents with indegree 2 in those graphs, so

$$\alpha \leq \frac{2p_6 + (1-p_6)}{2} = \frac{p_6+1}{2} \qquad \text{and} \qquad \alpha \leq \frac{2p_6 + (1-p_6)}{2} = \frac{p_6+1}{2},$$

and thus

$$\alpha \leq \frac{\min(p_6,p_7)+1}{2} \leq \frac{\frac{2}{5}+1}{2} = \frac{7}{10} = \frac{3n-1}{4n}.$$

Finally assume that $n \geq 7$ and odd, and consider the set of five graphs on n agents where agents 1 to 7 nominate as in Figure 9 and the remaining $n-7$ agents are grouped in pairs such that agent $2i$ nominates agent $2i-1$ and vice versa. It is easily verified

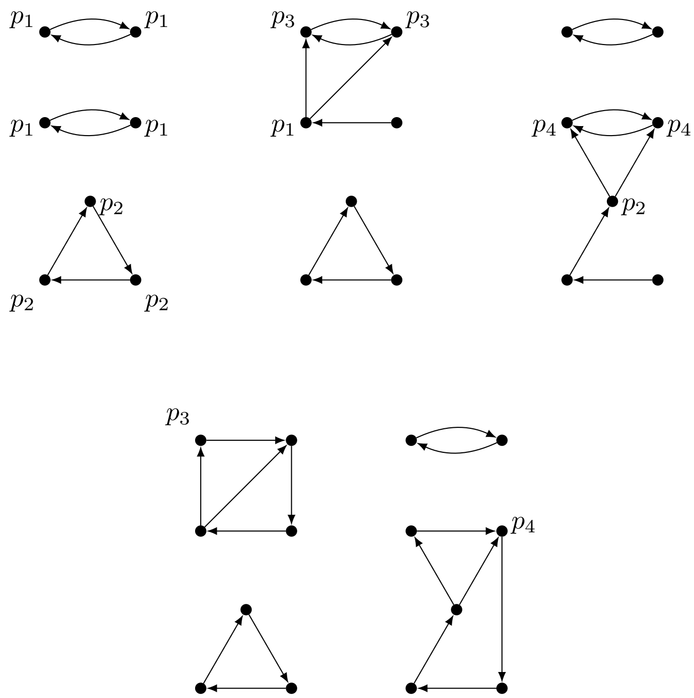

Fig. 9. Impartial probability assignment for five graphs with $n = 7$

that any impartial mechanism must assign probabilities as in Figure 9, so

$$\begin{aligned}(n-3)p_1 + 3p_2 &= 1,\\ p_1 + 2p_3 &\leq 1,\\ p_2 + 2p_4 &\leq 1.\end{aligned}$$

The agents assigned probabilities p_3 and p_4 in the two graphs on the bottom row are the unique agents with indegree 2 in those graphs, so

$$\alpha \leq \frac{2p_3 + (1-p_3)}{2} = \frac{p_3+1}{2} \leq \frac{\frac{1-p_1}{2}+1}{2} = \frac{3-p_1}{4},$$
$$\alpha \leq \frac{2p_4 + (1-p_4)}{2} = \frac{p_4+1}{2} \leq \frac{\frac{1-p_2}{2}+1}{2} = \frac{3-p_2}{4},$$

and thus

$$\alpha \leq \frac{3-\max(p_1,p_2)}{4} \leq \frac{3-\frac{1}{n}}{4} = \frac{3n-1}{4n},$$

where the second inequality holds because $\max(p_1, p_2) \geq 1/n$. □

Somewhat surprisingly, restricting the set of graphs even further, by requiring that every vertex has outdegree *exactly* 1, does not enable significantly better impartial mechanism. Using similar arguments as in the proof of Theorem 6.3, it can be shown that in this case any impartial and α-optimal mechanism must satisfy $\alpha \leq 5/6$ if $n = 3$, $\alpha \leq (6n-1)/8n$ if $n \geq 6$ and even, and $\alpha \leq 3/4$ otherwise. These bounds are tight for $n \leq 9$.

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