Congestion Games with Player-Specific Costs Revisited*

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Abstract. We study the existence of pure Nash equilibria in congestion games with player-specific costs. Specifically, we provide a thorough characterization of the maximal sets of cost functions that guarantee the existence of a pure Nash equilibrium.

For the case that the players are unweighted, we show that it is necessary and sufficient that for every resource and for every pair of players the corresponding cost functions are affine transformations of each other. For weighted players, we show that in addition one needs to require that all cost functions are affine or all cost functions are exponential.

Finally, we construct a four-player singleton weighted congestion game where the cost functions are identical among the resources and differ only by an additive constant among the players and show that it does not have a pure Nash equilibrium. This answers an open question by Mavronicolas et al. [15] who showed that such games with at most three players always have a pure Nash equilibrium.

1 Introduction

The theory of congestion games is an important topic in the operations research and algorithmic game theory literature that has driven the innovation in that field for many years. E.g., the central notions of the price of anarchy and the price of stability were first introduced and studied for special classes of congestion games; see Koutsoupias and Papadimitriou [13] and Anshelevich et al. [3].

In a congestion game, as introduced by Rosenthal [20], we are given a set of resources and each player selects a subset of them. The private cost of each player is the sum of the costs of the chosen resources which depends on the number of players using them. Congestion games appear in a variety of applications ranging from traffic and telecommunication networks to real-world and virtual market places. A fundamental problem in game theory is to characterize conditions that guaranty the existence of a pure Nash equilibrium, a state in which no player can improve by unilaterally changing her (pure, i.e., deterministic) strategy.

Rosenthal proved in a seminal paper [20] that every unweighted congestion game has a pure Nash equilibrium. In contrast to this remarkable positive result,

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it is well known that many natural generalizations of congestion games need not admit a pure Nash equilibrium. This behavior can be observed, e.g., for congestion games with integer-splittable demands (Rosenthal [21], Tran et al. [22]), for weighted congestion games (Fotakis et al. [6], Goemans et al. [9], Libman and Orda [14], Harks and Klimm [10]) and for congestion games with player-specific cost functions (Gairing et al. [7], Milchtaich [16,17]). For these generalizations of congestion games it is NP-hard to decide whether a given instance admits a pure Nash equilibrium, as shown by Dunkel and Schulz [5] for integer-splittable congestion games and weighted congestion games and Ackermann and Skopalik [2] for games with player-specific costs. Milchtaich [18] showed that every finite game is isomorphic to both a weighted congestion game and a congestion game with player-specific costs which makes both classes of games universal.

In light of these results, it is an important problem to find subclasses of these games that are on the one hand narrow enough to guarantee the existence of a pure Nash equilibrium and on the other hand rich enough to model many interesting interactions. Ackermann et al. [1] gave a characterization of the existence of equilibria in terms of the players' strategy space. They showed that games in which the strategy space of each player is the basis of a matroid always possess an equilibrium and that this is basically the maximal property of the strategy space that guarantees the existence of a pure Nash equilibrium in weighted congestion games. They also show that their characterization carries over to games with player-specific costs. For weighted congestion games also the impact of the cost functions on the existence of pure Nash equilibria is now relatively well understood. It is known that games with affine costs or exponential costs always possess a pure Nash equilibrium, and that these are basically the only sets of cost functions that one may allow to guarantee the existence of such an equilibrium point; see Fotakis et al. [6], Panagopoulou and Spirakis [19], and Harks and Klimm [10].

For congestion games with player-specific costs and arbitrary strategy spaces, much less is known. The only known existence result we are aware of is due to Mavronicolas et al. [15] who showed that every weighted congestion game with player-specific costs in which the costs are linear and differ by a player-specific additive constant only, always possess a pure Nash equilibrium.

Voorneveld et al. [23] showed that the class of games considered by Konishi et al. [12] is equivalent to the class of singleton unweighted congestion games with player-specific cost functions. Konishi et al. [12] even proved the existence of a strong equilibrium [4], a strengthening of the pure Nash equilibrium concept that is even robust against coordinated deviations of coalitions of players. Georgiou et al. [8] showed that singleton weighted congestion games with linear player-specific cost functions and three players always admit a pure Nash equilibrium. Also for the case of three players, Mavronicolas et al. [15] showed the existence of a pure Nash equilibrium if the cost functions are non-decreasing and differ only by a player-specific constant.

1.1 Our Results

As our main results, we give a complete characterization of the existence of pure Nash equilibria in congestion games with player-specific costs with unweighted and weighted players, respectively. To formally state our results, let N be a finite set of players and let R be a finite set of resources. We say that a collection $\mathcal{C} = (\mathcal{C}_i^r)_{i \in N, r \in R}$ of cost functions is *consistent* if every congestion game with player-specific costs, in which the cost function c_i^r of player i on resource r is an element of $\mathcal{C}_i^r$, possesses a pure Nash equilibrium. Clearly, a player i, with the property that $\bigcup_{r \in R} \mathcal{C}_i^r$ contains constant functions only, has no impact on the existence of pure Nash equilibria since we may let such player choose a best reply and remove her from the game. Such a player will be called *trivial* henceforth.

We first characterize the consistency of cost functions for games with unweighted players. We show that $\mathcal{C}$ is consistent if and only if for each two non-trivial players $i, j \in N$, there is a constant $a_{i,j} \in \mathbb{R}_{>0}$ such that for each $r \in R$ and each $c_i^r \in \mathcal{C}_i^r$ and $c_j^r \in \mathcal{C}_j^r$, there is a constant $b \in \mathbb{R}$ with $c_i^r(x) = a_{i,j}\, c_j^r(x) + b$ for all $x \in \mathbb{N}$.

Based on this result and the characterization of the consistency of cost functions for weighted congestion games obtained in [10], we also give a similar characterization for weighted congestion games with player-specific costs. Specifically, we show that a collection $\mathcal{C}$ of cost functions is consistent if and only if at least one of the following two cases holds: (i) For every player i and every resource r, there is a constant $a_i^r \in \mathbb{R}$ such that the set $\mathcal{C}_i^r$ contains only affine functions of type $c_i^r(x) = a_i^r x + b$ where $b \in \mathbb{R}$ is arbitrary, while the ratio a_i^r / a_j^r is independet of r for each two non-trivial players $i, j \in N$; *or* (ii) There is a constant $\phi \in \mathbb{R}$ and, for every player i and every resource r, a constant a_i^r such that the set $\mathcal{C}_i^r$ contains only exponential functions of type $c_i^r(x) = a_i^r \exp(\phi x) + b$, where $b \in \mathbb{R}$ is arbitrary, while the ratio a_i^r / a_j^r is independent of r for each two non-trivial players $i, j \in N$.

We complement our results constructing an instance of a *singleton* weighted congestion game with costs that differ only by player-specific constants that does not possess a pure Nash equilibrium. Interestingly, this game involves *four* players and thus contrasts a result of Mavronicolas et al. [15] who showed that weighted singleton games with player-specific constants and *three* players always possess a pure Nash equilibrium. To the best of our knowledge, this is the first time that the threshold between existence and non-existence of pure Nash equilibria for a class of games is between three players and four players.

2 Preliminaries

Let $N = \{1, \dots, n\}$ be a non-empty and finite set of n players and let $R = \{1, \dots, m\}$ be a non-empty and finite set of m resources. Each player is associated with a *demand* $d_i \in \mathbb{R}_{>0}$ and a set of strategies $S_i \subseteq 2^R$, where each strategy $s_i \in S_i$ is a non-empty subset of the resources. A tuple of n strategies $s = (s_1, \dots, s_n)$, one for each player, is called a *strategy profile*. The set of all strategy profiles

$S = S_1 \times \cdots \times \ldots S_n$ is called the *strategy space*. The private cost of each player in a strategy profile is defined in terms of a set of *player-specific* cost functions on the resources. Specifically, we are given for each player i and each resource r a *cost function* $c_i^r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. The private cost of each player i in strategy profile s is defined as $\pi_i(s) = \sum_{r \in s_i} c_i^r(x^r(s))$, where $x^r(s) = \sum_{j \in N : r \in s_j} d_j$ is the *aggregated demand* of resource r under strategy profile s. We call the tuple $G = (N, S, (\pi_i)_{i \in N})$ a *weighted congestion game with player-specific costs.* For the special case that $d_i = 1$ for all $i \in N$, we call the game *unweighted*, instead. We call the game simply a weighted (or unweighted) congestion game, if the players' cost functions do not differ, i.e., $c_i^r = c_j^r$ for all $r \in R$ and all $i, j \in N$. If all strategies are singletons, i.e., $|s_i| = 1$ for all $s_i \in S_i$ and all $i \in N$, we call the game a *singleton* game.

Let N and R be given and let $\mathcal{C} = (\mathcal{C}_i^r)_{i \in N, r \in R}$ be a collection of cost functions. We say that $(\mathcal{C}_i^r)_{i \in N, r \in R}$ is *consistent* if there is a pure Nash equilibrium in every congestion game with player-specific costs G that satisfies the constraint that $c_i^r \in \mathcal{C}_i^r$ for all $i \in N$ and $r \in R$. Note that we allow for arbitrarily many copies of a resource in G. Given $\mathcal{C}$, we call player i a *trivial player*, if c_i^r is constant for all $r \in R$.

3 Player-Specific Constants

We start with the positive part of our characterization, i.e., we show that congestion games in which the players' cost function of each resource differ by an (additive) player-specific constant only always have a pure Nash equilibrium. In fact we show the more general result that each such game is isomorphic to a congestion game (without player-specific constants).

Formally, for two strategic games $G = (N, S, \pi)$ and $G = (N, S', \pi')$, we say that G and G' are *isomorphic*, if for each i there is a bijection $B_i : S_i \to S_i'$ such that $\pi(s) = \pi'(B_1(s_1), \ldots, B_n(s_n))$.

To prove the following observation, we model the player-specific constants of each player i by introducing an additional resource that is exclusively used by player i. The proof is omitted due to space constraints.

Proposition 1. *Let G be an unweighted (respectively, weighted) congestion game with player-specific costs such that $c_i^r - c_j^r$ is constant for each resource r and each two players i and j. Then, G is isomorphic to an unweighted (respectively, weighted) congestion game.*

4 A Characterization for Unweighted Players

The technically more challenging part of our characterization is to prove that it is indeed necessary that the cost functions of the players differ by a player-specific constants only in order to guarantee the existence of a pure Nash equilibrium. Before we prove this result, we need the following technical lemma.

Lemma 1. *Let N and R be arbitrary and let $\mathcal{C} = (\mathcal{C}_i^r)_{i\in N, r\in R}$ be a collection of cost functions. If $\mathcal{C}$ is consistent for unweighted congestion games with player-specific costs, then for each two non-trivial players $i, j \in N$ the following two conditions are satisfied:*

1. $\{x \in \mathbb{N} : c_i^r(x+1) = c_i^r(x)\} = \{x \in \mathbb{N} : c_j^r(x+1) = c_j^r(x)\}$ *for all* $r \in R$, $c_i^r \in \mathcal{C}_i^r$ *and* $c_j^r \in \mathcal{C}_j^r$.
2. $\dfrac{c_i^r(x+1) - c_i^r(x)}{c_j^r(x+1) - c_j^r(x)} = \dfrac{c_i^t(y+1) - c_i^t(y)}{c_j^t(y+1) - c_j^t(y)}$ *for all* $r, t \in R$, $c_k^l \in \mathcal{C}_k^l$, $k \in \{i, j\}$, $l \in \{r, t\}$ *and* $x, y \in \mathbb{N}$ *with* $c_j^r(x+1) \neq c_j^r(x)$ *and* $c_j^t(y+1) \neq c_j^t(y)$.

Proof (Sketch). Due to space constraints, we show the claimed results only under the additional assumption that all cost functions are non-decreasing. The general case can be proven with similar arguments.

We start proving the first part of the claim. For a contradiction, let us assume that there are $i, j \in N$, $r \in R$, $c_i^r \in \mathcal{C}_i^r$, $c_j^r \in \mathcal{C}_j^r$ and $x \in \mathbb{N}$ such that $c_i^r(x+1) = c_i^r(x)$ and $c_j^r(x+1) \neq c_j^r(x)$. As player i is non-trivial, there are $t \in R$, $c_j^t \in \mathcal{C}_j^t$ and $y \in \mathbb{N}$ such that $c_i^t(y+1) \neq c_i^t(y)$. Using the additional assumption that the cost functions are non-decreasing, we obtain $c_j^r(x+1) > c_j^r(x)$ and $c_i^t(y+1) > c_i^t(y)$.

Let $k \in \mathbb{N}$ be such that $k\left(c_j^r(x+1) - c_j^r(x)\right) > c_j^t(y+1) - c_j^t(y)$. We introduce $2k$ copies $r_1, \ldots, r_k, r_1', \ldots, r_k'$ of resource r. On all those resources the players have cost functions c_i^r and c_j^r, respectively. Moreover there are two resources t and t' with cost functions $c_i^t = c_i^{t'}$ and $c_j^t = c_j^{t'}$, respectively .

Player i has two strategies. She chooses either $\{r_1 \ldots, r_k, t\}$ or $\{r_1', \ldots, r_k', t'\}$. Player j chooses either $\{r_1, \ldots, r_k, t'\}$ or $\{r_1', \ldots, r_k', t\}$. Furthermore, we introduce $x-1$ additional players with a single strategy only that always choose $\{r_1, \ldots, r_k, r_1', \ldots, r_k'\}$ and $y-1$ additional players that always choose $\{t, t'\}$.

We claim that the thus defined game does not have a pure Nash equilibrium. To see this claim, note that in any strategy profile the two players either share k resources of type r or one resource of type t. Now assume we are in a strategy profile in which the players share one resource of type t. Then, player i may deviate to her other strategy since sharing k resources of type r doesn't increase her costs as we have $c_i^r(x+1) = c_i^r(x)$. On the other hand, her cost is strictly decreased since $c_i^t(y+1) > c_i^t(y)$. For player j, however, the situation is exactly converse. She prefers not to share k resources of type r since $k\left(c_j^r(x+1) - c_j(x)\right) > c_j^t(y+1) - c_j^t(y)$. This observation finishes the first part of the proof.

For the second part of the claim, let us assume for a contradiction, that there are $i, j \in N$, $r, t \in R$, $x, y \in \mathbb{N}$, and $c_k^l \in \mathcal{C}_k^l$ with $k \in \{i, j\}$, $l \in \{r, t\}$ such that $c_j^r(x+1) \neq c_j^r(x)$, $c_j^t(y+1) \neq c_j^t(y)$ and

$$\frac{c_i^r(x+1) - c_i^r(x)}{c_j^r(x+1) - c_j^r(x)} > \frac{c_i^t(y+1) - c_i^t(y)}{c_j^t(y+1) - c_j^t(y)}. \tag{1}$$

Using the addditional assumption that all cost functions are non-decreasing, we obtain that the denominators $c_j^r(x+1) - c_j^r(x)$ and $c_j^t(y+1) - c_j^t(y)$ are strictly positive and together with the first part of the statement of the lemma, this implies that the nominators $c_i^r(x+1) - c_i^r(x)$ and $c_i^t(y+1) - c_i^t(y)$ are strictly positive as well. For $\alpha = c_j^t(y+1) - c_j^t(y)/(c_j^r(x+1) - c_j^r(x))$, we obtain

$$\alpha\big(c_i^r(x+1) - c_i^r(x)\big) > c_i^t(y+1) - c_i^t(y).$$

As this expression is continuous in α, we may find $\alpha' < \alpha$ with $\alpha' \in \mathbb{Q}$ such that we still have

$$\alpha'\big(c_i^r(x+1) - c_i^r(x)\big) > c_i^t(y+1) - c_i^t(y).$$

On the other hand for player j, we derive

$$\alpha'\big(c_j^r(x+1) - c_j^r(x)\big) < \alpha\big(c_j^r(x+1) - c_j^r(x)\big) = c_j^t(y+1) - c_j^t(y).$$

Writing $\alpha' = k/l$ for some $k, l \in \mathbb{N}$, we obtain the following inequalities:

$$k\big(c_i^r(x+1) - c_i^r(x)\big) > l\big(c_i^t(y+1) - c_i^t(y)\big), \tag{2a}$$
$$k\big(c_j^r(x+1) - c_j^r(x)\big) < l\big(c_j^t(y+1) - c_j^t(y)\big). \tag{2b}$$

Next, we will use these inequalities to construct a congestion game with player-specific costs that does not have a pure Nash equilibrium.

The game has $2k$ resources $r_1, \dots, r_k, r'_1, \dots, r'_k$ with cost functions c_i^r respectively c_j^r and $2l$ resources $t_1, \dots, t_l, t'_1, \dots, t'_l$ with cost function c_i^t respectively c_j^t. Player i has two strategies, she chooses either $\{r_1, \dots, r_k, t_1, \dots, t_l\}$ or $\{r'_1, \dots, r'_k, t'_1, \dots, t'_l\}$. Player j has two strategies as well and chooses either $\{r_1, \dots, r_k, t'_1, \dots, t'_l\}$ or $\{r'_1, \dots, r'_k, t_1, \dots, t_l\}$. Furthermore, there are $x-1$ players with the single strategy $\{r_1, \dots, r_k, r'_1, \dots, r'_k\}$ and $y-1$ players with the single strategy $\{t_1, \dots, t_l, t'_1, \dots, t'_l\}$.

We claim that the thus constructed game does not have a pure Nash equilibrium. To see this note that for the strategy profile $s_i = \{r_1, \dots, r_k, t_1, \dots, t_l\}$ and $s_j = \{r_1, \dots, r_k, t'_1, \dots, t'_l\}$, the two players i and j share k resources of type r together. In that case, player i improves switching to her alternative strategy $s'_i = \{r'_1, \dots, r'_k, t'_1, \dots, t'_l\}$ as

$$\begin{aligned}\pi_i(s'_i, s_j, \dots) - \pi_i(s_i, s_j, \dots) &= k\,c_i^r(x) + l\,c_i^t(y+1) - k\,c_i^r(x+1) - l\,c_i^t(y)\\ &= -k\big(c_i^r(x+1) - c_i^r(x)\big) + l\big(c_i^t(y+1) - c_i^t(y)\big),\end{aligned}$$

which is negative using (2a). This strategy profile, in turn, is not a pure Nash equilibrium, since player j may deviate profitably to $s'_j = \{r'_1, \dots, r'_k, t_1, \dots, t_l\}$ as

$$\begin{aligned}\pi_j(s'_i, s'_j, \dots) - \pi_i(s'_i, s_j, \dots) &= k\,c_j^r(x+1) + l\,c_j^t(y) - k\,c_j^r(x) - l\,c_j^t(y+1)\\ &= k\big(c_j^r(x+1) - c_j^r(x)\big) - l\big(c_j^t(y+1) - c_j^t(y)\big),\end{aligned}$$

which is negative using (2b). By symmetry of the strategy space of the game, the other two strategy profiles are also not a pure Nash equilibrium. □

We are now ready to prove our main result.

Theorem 1. *For a collection* $\mathcal{C} = (\mathcal{C}_i^r)_{i \in N, r \in R}$ *of cost functions the following are equivalent:*

1. $\mathcal{C}$ *is consistent for unweighted congestion games with player-specific costs.*
2. *For each two non-trivial players* i, j*, there is a constant* $a_{i,j} \in \mathbb{R}_{>0}$ *such that for each* $r \in R$ *and each* $c_i^r \in \mathcal{C}_i^r$ *and* $c_j^r \in \mathcal{C}_j^r$*, there is a constant* $b \in \mathbb{R}$ *with* $c_i^r(x) = a_{i,j}\, c_j^r(x) + b$ *for all* $x \in \mathbb{N}$*.*
   (Note that b *may depend on* c_i^r *and* c_j^r *while* $a_{i,j}$ *is equal for all* $r \in R$*,* $c_i^r \in \mathcal{C}_i^r$ *and* $c_j^r \in \mathcal{C}_j^r$*.)*

Proof. 2. $\Rightarrow$ *1.*: Let G be a congestion game with player-specific costs as required in the statement, i.e., for each two non-trivial players i, j there is $a_{i,j} \in \mathbb{R}_{>0}$ and $b_{i,j}^r \in \mathbb{R}$ such that $c_i^r(x) = a_{i,j}\, c_j^r(x) + b_{i,j}^r$ for all $x \in \mathbb{N}$. It is a useful observation that the existence of pure Nash equilibria is invariant under player-specific scaling of the private cost functions. We consider a normalized congestion game $\tilde{G}$ with player-specific costs $\tilde{c}_i^r$, which are defined as

$$\tilde{c}_i^r(x) = \frac{c_i^r(x)}{a_{1,i}} = c_1^r(x) + \frac{b_{1,i}^r}{a_{1,i}}$$

for all $x \in \mathbb{N}$ and $i \in N \setminus \{1\}$. Clearly, $\tilde{G}$ has the same set of pure Nash equilibria as G. Moreover, the set of pure Nash equilibria of $\tilde{G}$ is non-empty, as every game with player-specific constants is isomorphic to an unweighted congestion game (Proposition 1).

1. $\Rightarrow$ *2.*: Lemma 1 implies that for two non-trivial players $i, j \in N$, each resource $r \in R$, and each $c_i^r \in \mathcal{C}_i^r$ and $c_j^r \in \mathcal{C}_j^r$ the sets $\{x \in \mathbb{N} : c_i^r(x+1) = c_i(x)\}$ and $\{x \in \mathbb{N} : c_j^r(x+1) = c_j(x)\}$ coincide and that $\frac{c_i^r(x+1)-c_i^r(x)}{c_j^r(x+1)-c_j^r(x)}$ is constant for all resources $r \in R$ and all cost functions $c_i^r \in \mathcal{C}_i^r, c_j^r \in \mathcal{C}_j^r$ and $x \in \mathbb{N}$ for which this ratio is defined. Let us call this constant $a_{i,j}$. This implies that

$$c_i^r(x+1) - c_i^r(x) = a_{i,j}\big(c_j^r(x+1) - c_j^r(x)\big).$$

for all $r \in R$, $c_i^r \in \mathcal{C}_i^r$, $c_j^r \in \mathcal{C}_j^r$ and $x \in \mathbb{N}$. Using telescoping sums, we obtain

$$c_i^r(x) - c_i^r(1) = a_{i,j}\big(c_j^r(x) - c_j^r(1)\big),$$

or equivalently

$$c_i^r(x) = a_{i,j} c_j^r(x) + \big(c_i^r(1) - a_{i,j} c_j^r(1)\big),$$

for all $r \in R$, $c_i^r \in \mathcal{C}_i^r$, $c_j^r \in \mathcal{C}_j^r$ and $x \in \mathbb{N}$. Setting $b = c_i^r(1) - a_{i,j} c_j^r(1)$, the claimed result follows. □

5 A Characterization for Weighted Players

Combining the results obtained in the last section and the characterization of consistency for weighted congestion games obtained in [10], we can also give a complete characterization of consistency for games with weighted players.

Theorem 2. *For a collection $\mathcal{C} = (\mathcal{C}_i^r)_{i \in N, r \in R}$ of continuous utility functions the following are equivalent:*

1. *$\mathcal{C}$ is consistent for weighted congestion games with player-specific costs.*
2. *One of the following two conditions is satisfied:*
   (a) *For every non-trivial player i and every resource r, there is a constant $a_i^r \in \mathbb{R}$ such that the set $\mathcal{C}_i^r$ contains only affine functions of type $c_i^r(x) = a_i^r x + b$ where $b \in \mathbb{R}$ is arbitrary, while the ratio a_i^r / a_j^r is independent of r for each two non-trivial players $i, j \in N$.*
   (b) *There is a constant $\phi \in \mathbb{R}$ and, for every non-trivial player i and every resource r, a constant $a_i^r \in \mathbb{R}$ such that the set $\mathcal{C}_i^r$ contains only exponential functions of type $c_i^r(x) = a_i^r \exp(\phi x) + b$, where $b \in \mathbb{R}$ is arbitrary, while the ratio a_i^r / a_j^r is independent of r for each two non-trivial players $i, j \in N$.*

Proof. 1. ⇒ 2.: Since the set of unweighted congestion games is a subset of the set of weighted congestion games our characterization of consistency for unweighted games obtained in Theorem 1 implies that for each two non-trivial players i, j, there is a constant $a_{i,j} \in \mathbb{R}_{>0}$, such that for all $c_i^r \in \mathcal{C}_i^r$ and $c_j^r \in \mathcal{C}_j^r$ we have $c_i^r(x) = a_{i,j}\, c_j^r(x) + b$ for all $x \in \mathbb{N}$ and some $b \in \mathbb{R}$. Regarding games in which the demand of each players is equal to an arbitrary but fixed $\epsilon > 0$, we obtain along the same lines that this statement holds for all $x \in \mathbb{R}_{\geq 0}$ which are an integer multiple of ϵ. Letting ϵ go to zero and using the continuity of all cost functions in $\mathcal{C}$, we conclude that $c_i^r(x) = a_{i,j}\, c_j^r(x) + b$ for all $x \in \mathbb{R}_{\geq 0}$ and some $b \in \mathbb{R}$. As we already argued in the proof of Theorem 1, it is without loss of generality to assume that $a_{i,j} = 1$ for all $i, j \in N$, i.e., $c_i^r(x) = c_j^r(x) + b$ for all $x \in \mathbb{R}_{\geq 0}$ and some $b \in \mathbb{R}$.

Weighted congestion games (without a player-specific additive constant) are guaranteed to have a pure Nash equilibrium if and only if one of the following two cases holds: (i) the set of cost functions contains only affine functions $c = a\,x + b$; *or* the set of cost functions contains only exponential functions $c(x) = a \exp(\phi x) + b$, where ϕ is equal for all cost functions [10]. For the proof of this result, one considers three-player games in which two players have two strategies each, and one player has a single strategy only. The two strategies of the first two players have the property that they contain only resources with cost functions of at most two types, and each of the types occurs with the same cardinality, i.e., there are two cost functions c and c' and two integers a, a' such that each strategy of each player consists of exactly a resources of type c and a' resources of type c'. Now imagine that the cost functions c and c', in fact, differ by a player-specific additive constant. Adding these player-specific additive constants, however, shifts the private cost of each player for each of her strategies by a constant value and, thus, does not affect the existence of a pure Nash equilibrium. This observation establishes that the characterization for weighted congestion games obtained in [10] translates to weighted congestion games with player-specific additive constants, which completes the proof of the claim.

2. $\Rightarrow$ *1*: Let G be a game as required in (a) or (b). We will transform G into an equivalent game (with the same set of pure Nash equilibria) for which we then show the existence of a pure Nash equilibrium.

In the first step, we scale the private cost functions of the players such that $a_i^r = a_j^r$ for each two non-trivial players $i, j \in N$ and obtain a weighted congestion game with player-specific costs G'. By Proposition 1, there is a weighted congestion game G'' that is isomorphic to G. Furthermore, if we started from a game G as required in (a) all cost functions in G'' are affine. Weighted congestion games with affine costs admit a potential function and, hence, a pure Nash equilibrium; see [6,11]. If, on the other hand, we started from a game G as required in (b), the game G'' has the property that the (player-independent) cost function of each resource is of type $c^r(x) = a\ \exp(\phi\, x) + b$, where ϕ is a common constant of all cost functions. Under this assumption, a weighted potential function exists [11], implying the existence of a pure Nash equilibrium. □

6 Singleton Games

In this section, we consider singleton congestion games with player-specific costs. Milchtaich [16] proved that a pure Nash equilibrium always exists, if the players are unweighted and the player-specific cost functions are non-decreasing. He also provided a counterexample of a three-player game with weighted players that does not have a pure Nash equilibrium. On the positive side, Mavronicolas et al. [15] showed that each *three-player* game in which the cost functions are non-decreasing and differ by an additive constant only, have a pure Nash equilibrium. It has been open whether such a positive result holds for an arbitrary number of players. As the main result of this section, we answer this question to the negative, i.e., we give a counterexample of a *four-player* singleton weighted congestion game with non-decreasing and concave costs that differ by player-specific constants only but does not have a pure Nash equilibrium.

Proposition 2. *There is a singleton weighted congestion game, in which the cost functions are non-decreasing and concave and differ by player-specific additive constants only, that does not have a pure Nash equilibrium.*

Proof. There are four players $N = \{1, 2, 3, 4\}$ with demands $d_i = i$ for all $i \in N$. Further, we are given four resources $\{t, u, v, w\}$. The players' strategy sets are given as $S_1 = \{\{t\}, \{u\}\}$, $S_2 = \{\{u\}, \{v\}\}$, $S_3 = \{\{t\}, \{v\}\}$, $S_4 = \{\{v\}, \{w\}\}$. We first define player-independent cost functions c^t, c^u, c^v, c^w as

$$c^t(x) = \min\{6x, 24\} \qquad c^u(x) = \min\{20x, 40\}$$
$$c^v(x) = \min\{2x, 14\} \qquad c^w(x) = 0$$

for all $x \in \mathbb{R}_{\geq 0}$. For $i \in N$ and $r \in s_i \in S_i$, we obtain the player-specific cost functions c_i^r by adding a player-specific constant b_i^r to the cost function c^r. The player-specific constants are given as

$$\begin{array}{llll} b_1^t = 15, & b_2^u = 0, & b_3^t = 0, & b_4^v = 0, \\ b_1^u = 0, & b_2^v = 29, & b_3^v = 9, & b_4^w = 13. \end{array}$$

We proceed to show that the thus defined congestion game does not have a pure Nash equilibrium. For the proof, we distinguish between the set of players $N_v(s)$ that uses the critical resource v.

We first note that the cost functions of player 4 are designed so as to ensure that she uses resource v if and only if the load on v is smaller or equal to 6. This rules out the possibilities $\{2\}$, $\{3,4\}, \{2,3,4\}$ for $N_v(s)$ as in these cases player 4 would always prefer to switch. Next, note that the cost functions of player 2 are such that she uses v if and only if the load on v is smaller or equal 5 which rules out the possibilities $\{3\}$, $\{2,4\}$, $\{2,3,4\}$ for $N_v(s)$ as player 2 would prefer to switch in these cases. This leaves use with the following two cases that can occur in equilibrium, which we will consider separately.

First case: $N_v(s) = 4$. This implies that $s_2 = u$ and $s_3 = t$. If $s_1 = t$ as well, the load on t is 4 and thus player 3 would be better of switching to v where the cost for her is at most 23. If, on the other hand, $s_1 = u$, then she would improve switching to t where the cost for her is at most 39.

Second case: $N_v(s) = \{2,3\}$. Note that this implies that player 1 is on u, as she prefers u over t when both resources are not used by other players. From the strategy profile (u, v, v, w), however, player 3 improves switching to t where the cost for her equals 18 which is strictly less than the 19 cost units she experiences on v. □

We can slightly strengthen the negative result showing that even for *identical* cost functions in the presence of player-specific additive constants a pure Nash equilibrium need not exist.

Corollary 1. *There is a singleton weighted congestion game with player-specific constants and identical cost functions that does not have a pure Nash equilibrium.*

Proof. As shown in Proposition 2, there is a weighted singleton congestion game with player-specific additive constants that does not admit a pure Nash equilibrium. We proceed to show how to transform G into an equivalent game $\tilde{G}$ that has the claimed properties and does not admit a pure Nash equilibrium as well. To this end, let N denote the set of players and R the set of resources of G. For ease of exposition, we assume that $R = \{0, \dots, m-1\}$ for some $m \in \mathbb{N}$ and that $c^r(0) = 0$ for all $r \in R$. Let $D = \sum_{i \in N} d_i$ and $M = \max_{r \in R} c^r(D)$. We introduce $m-1$ additional players $i_1, \dots, i_{m-1}$ with demand $d_{i_j} = j \cdot D$ and a single strategy $S_{i_j} = \{j\}$, $j \in \{1, \dots, m-1\}$. The cost function $\tilde{c}$ of all resources in $\tilde{G}$ is defined as

$$\tilde{c}(x) = \begin{cases} c^0(x), & \text{if } x \in [0, D], \\ c^1(x-D) + M, & \text{if } x \in (D, 2D], \\ \vdots & \\ c^r(x-2D) + r \cdot M, & \text{if } x \in \big(r \cdot D, (r+1)D\big], \\ \vdots & \\ c^{m-1}\big(x-(m-1)D\big) + (m-1)M, & \text{if } x \in \big((m-1)D, m \cdot D\big]. \end{cases}$$

Finally, we redefine the player-specific constants as $\tilde{b}_i^r = b_i^r - r \cdot M$.

Next, for every strategy profile s of G we associate the strategy profile $\tilde{s} = (s_1, \dots, s_n, s_{i_1}, \dots, s_{i_{m-1}})$ of $\tilde{G}$ in which the additional players use their unique strategy. Using the particular definitions of D, M and $\tilde{c}$, it is easy to see, that the private costs of each player $i \in N$ in s and $\tilde{s}$ coincide. Using that the additional players have a single strategy only, we derive that $\tilde{G}$ does not have a pure Nash equilibrium. □

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