Information Design for Congestion Games with Unknown Demand

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Abstract

We study a novel approach to information design in the standard traffic model of network congestion games. It captures the natural condition that the *demand* is unknown to the users of the network. A principal (e.g., a mobility service) commits to a signaling strategy, observes the realized demand and sends a (public) signal to agents (i.e., users of the network). Based on the induced belief about the demand, the users then form an equilibrium. We consider the algorithmic goal of the principal: Compute a signaling scheme that minimizes the expected total cost of the induced equilibrium. We concentrate on single-commodity networks and affine cost functions, for which we obtain the following results. First, we devise a fully polynomial-time approximation scheme (FPTAS) for the case that the demand can only take two values. It relies on several structural properties of the cost of the induced equilibrium as a function of the updated belief about the distribution of demands. We show that this function is piecewise linear for any number of demands, and monotonic for two demands. Second, we give a complete characterization of the graph structures for which it is optimal to fully reveal the information about the realized demand. This signaling scheme turns out to be optimal for all cost functions and probability distributions over demands if and only if the graph is series-parallel. Third, we propose an algorithm that computes the optimal signaling scheme for any number of demands whose time complexity is polynomial in the number of supports that occur in a Wardrop equilibrium for some demand. Finally, we conduct a computational study that tests this algorithm on real-world instances.

Introduction

Traffic and congestion are key factors contributing to climate change and air pollution. On the other hand, personal and commercial traffic are fundamental for economic development and the modern way of life. This makes sound traffic planning and improvement an indispensable prerequisite for urban areas around the globe. A popular and successful model for traffic planning are non-atomic congestion games. The road network is represented by a graph $G = (V, E)$ where each edge e has a cost function c_e that models the time needed to traverse the edge and depends on the total flow on that edge. In the single-commodity setting, a continuum of players with travel demand $d > 0$ strives to route from a designated source vertex $s \in V$ (e.g., a residential living area) to a designated destination vertex $t \in V$ (e.g., a city center). Each infinitesimally small player aims to minimize their private cost by choosing a least-cost path from s to t. A so-called *Wardrop equilibrium* is reached when no player has the incentive to deviate from their chosen path since all other paths have either the same or even higher cost. It is a well-known fact that a Wardrop equilibrium does not necessarily minimize the overall travel time, and there is a substantial literature that quantifies the loss in efficiency due to selfish behavior (Dubey 1986; Roughgarden and Tardos 2002; Roughgarden 2003; Roughgarden and Tardos 2004; Correa, Schulz, and Moses 2004, 2008; Dumrauf and Gairing 2006).

In order to achieve better equilibria, interventions through network design (Marcotte 1986; Bhaskar, Ligett, and Schulman 2014; Gairing, Harks, and Klimm 2017; Roughgarden 2006) or mechanism design techniques such as tolls (Harks et al. 2015; Fleischer, Jain, and Mahdian 2004; Hoefer, Olbrich, and Skopalik 2008; Larsson and Patriksson 1999; Hearn and Ramana 1998; Bergendorff, Hearn, and Ramana 1997) have been studied extensively. These approaches, however, usually come with a high cost, e.g., for building or remodeling road segments or for setting up a toll collection system for highways. This paper, therefore, focuses on improving the emerging equilibrium by *information design*.

A significant source of uncertainty in traffic networks concerns the *demand*, i.e., information about the total amount of traffic. Total traffic is highly fluctuating, even during a single day. From a game-theoretic perspective, this implies that the total volume of players in the routing game is not fixed and not common knowledge. Such *games with population uncertainty* were first studied in a systematic way by Myerson (1998), who considered games with atomic players and multiple player types. Cominetti et al. (2022) draw a connection between the Poisson games of Myerson and atomic congestion games where players participate independently at random. Here we adapt the approach to non-atomic games. The total volume of players in the game is drawn from a probability distribution known to all players. Each player observes whether they participate in the game or not, e.g., whether to drive to work in the morning (i.e., has type *active*) or not (type *inactive*, e.g., due to illness or car malfunction). When a player is inactive and does not participate, they re-

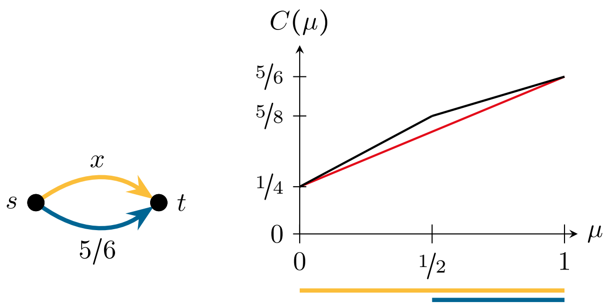

Figure 1: A simple non-atomic congestion game. Left: Instance with two s-t-paths. Right: The cost of the Wardrop equilibrium (black) and of the full information signaling scheme (red) depending on the probability $\mu = \mathbb{P}[d = 1]$. As indicated by the colored lines underneath, the upper path is used for all $\mu \in [0, 1]$ while the lower path is only used for $\mu \in [1/2, 1]$.

ceive a private cost of 0. Otherwise, an active player then fixes a strategy, i.e., an s-t-path in the network, and receives as private cost the cost of the chosen path. This leads to a Bayesian game in the sense of Harsanyi (1967).

Due to technical reasons, in our analysis we treat a slightly less intuitive model variant. In this version, players decide on a route *before* they know whether they are active or not. An inactive player simply discards the route choice made earlier. The two model variants lead to equivalent outcomes in terms of equilibria and cost; see the full version of the paper (Griesbach et al. 2023b, Appendix A). The latter variant allows to avoid a uniform scaling factor in all computations. Hence, despite being less intuitive, we use it throughout the paper for simplicity. For illustration, we start by discussing a simple example.

Example 1. Consider the simple game in Fig. 1 with two parallel edges and cost functions x and $5/6$, respectively. We normalize the highest demand to the size of the population and assume that the demands are 1 and $1/2$, each with a probability of $1/2$. For a fixed player, let A be the event that this player is active. If the player chooses the upper edge, their expected cost is given by the flow x_1 on the upper edge when they are active, and 0 otherwise, i.e., $\Pr[A] \cdot \mathbb{E}[x_1 \mid A] + \mathbb{P}[\neg A] \cdot 0$. Either all players are active or half of the players are active, and both cases have probability $1/2$. Hence, the expected cost amounts to the sum of $\mathbb{P}[A \mid d = 1/2] \cdot \mathbb{P}[d = 1/2] \cdot \mathbb{E}[x_1 \mid A \wedge (d = 1/2)]$ and $\mathbb{P}[A \mid d = 1] \cdot \mathbb{P}[d = 1] \cdot \mathbb{E}[x_1 \mid A \wedge (d = 1)]$. When a fraction a of the players are active, we assume each infinitesimal player is active independently with probability a, i.e., $\mathbb{P}[A \wedge (d = a)] = a$, for every $a \in [0, 1]$. Recall that players choose their strategy before the demand is realized. Suppose a fraction of x_1^* players chooses the upper edge. Then the expected cost of a player choosing the upper edge is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{x_1^*}{2} + 1 \cdot \frac{1}{2} \cdot x_1^* = \frac{5}{8} \cdot x_1^*$. A similar calculation shows that the expected cost when choosing the lower edge is $\left(\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1\right) \cdot \frac{5}{6} = \frac{5}{8}$. We conclude that in the unique Wardrop equilibrium $x_1^* = 1$, i.e., all players choose the upper edge. This leads to a total expected cost of $\frac{5}{8} = 0.625$.

Now suppose a traffic service provider like TomTom, Apple, or Google discloses a public signal whether the traffic is low (i.e., $d = 1/2$) or high (i.e., $d = 1$). We call this setting *full information*, since the information about the state of the world is fully disclosed. Now every player updates their belief about the demand and conditions their route choice on this information. In both cases, a corresponding Wardrop equilibrium emerges. For demand $1/2$ we get $x_1^* = \frac{1}{2}$ and $x_2^* = 0$; for demand 1, $x_1^* = \frac{5}{6}$ and $x_2^* = \frac{1}{6}$. The total expected cost with full information is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 \cdot \frac{5}{6} = \frac{13}{24} \approx 0.542$. Hence, full information improves the total expected cost over the case with no signal.

Similarly, consider the no-signal case and the cost of an equilibrium as a function of any prior $\mu = \mathbb{P}[d = 1] \in [0, 1]$ (see the black function in Fig. 1). Inspecting the cost when the provider gives full information (the red function), we see that full information improves the cost over no signal, for any prior $\mu \in [0, 1]$. ■

In this paper, we study how to optimally reveal information about the realized demand in order to induce Wardrop equilibria with low total expected cost. We focus on the case of *public signals* where the information provided is the same for all players. More specifically, we assume that a benevolent traffic service provider has a finite set of public and abstract signals Σ at its disposal. Formally, a signal $\sigma \in \Sigma$ has no a-priori intrinsic meaning. In practice, however, signals may already be biased towards certain information, e.g., that traffic volume is "moderate", "relatively high", or "gridlocked". Before seeing the realized demand, the service provider commits to a signaling scheme that is public knowledge of all players. It fixes the probabilities of emitting the signals for each realization of the demand. Subsequently, the realization of the demand is observed by the service provider (e.g., due to traffic measurements or cell phone data) and signals are sent according to the predefined signaling scheme. Upon receiving a signal, the players update their beliefs about the realization of the demand and react by playing a corresponding Wardrop equilibrium. How can the service provider optimize the public signaling scheme in order to minimize the total expected cost of the induced Wardrop equilibrium?

Contribution. After introducing a formal description of the problem, we derive useful structural properties of equilibria and their cost functions. More precisely, we show that the cost function of the unique Wardrop equilibrium flow is piecewise linear in the belief of the realized state for any finite number of states (Lemma 1). We further show that for two different demands, the cost function is monotonically non-decreasing in the probability that the higher demand is realized (Lemma 2 and Corollary 3). Building upon these properties, we then provide a fully polynomial-time approximation scheme (FPTAS) for optimal signaling with two different states (Theorem 1).

There exist network structures, in which always revealing the realized state is an optimal signaling scheme, no matter which prior belief the players have (c.f. Example 1 above). We call such a signaling scheme *full information revelation*. We show that if the underlying graph is a series-parallel graph, full information revelation is an optimal signaling scheme. We even prove that this characterization is tight – whenever the underlying graph is not series-parallel, there exist some cost functions for the edges and demands such that full information revelation is not optimal (Theorem 2).

In Theorem 3, we provide an LP-based algorithm that computes the optimal signaling scheme that induces Wardrop equilibria only using a distinct set of different supports . The algorithm works for any number of states and runs in time polynomial in the number of states, the number of edges, and the number of given supports. In general, however, there exist networks for which, over all beliefs, the number of different supports used in a Wardrop equilibrium is exponential in the input size, even when there are only two different demands. We conduct a computational study exhibiting that the number of different supports used in a Wardrop equilibrium is small on real-life instances. Therefore, our LP-based algorithm can be implemented in reasonable time in practice. Also, we see that the optimality of *full information revelation* is ubiquitous in these instances, even though they are not series-parallel.

Related Work. The question how non-atomic network congestion games behave when the demand changes has been studied thoroughly. Youn, Gastner, and Jeong (2008) and O'Hare, Connors, and Watling (2016) examined empirically how the price of anarchy, i.e., the ratio of the total travel time of a Wardrop equilibrium and the total travel time of a system optimum, changes as a function of the demand. The functional dependence of the price of anarchy as a function of the demand has been studied analytically by Colini-Baldeschi, Cominetti, and Scarsini (2019); Colini-Baldeschi et al. (2020); Cominetti, Dose, and Scarsini (2019), and Wu et al. (2021). Wu et al. (2022) studied a similar question for atomic congestion games. Wang, Doan, and Chen (2014) obtained bounds on the price of anarchy of Wardrop equilibria with stochastic demands depending on parameters of the distribution. Correa, Hoeksma, and Schröder (2019) studied a similar model with the difference that the players perform a Bayesian update of the distributions after observing whether their commodity travels and find that the price of anarchy transfers from the deterministic model; this refines earlier results of Roughgarden (2015) on the price of anarchy of Bayes-Nash equilibria. More generally, the sensitivity of Wardrop equilibria to changes in the demand was studied by (Hall 1978; Josefsson and Patriksson 2007; Fisk 1979; Englert, Franke, and Olbrich 2010; Patriksson 2004; Takalloo and Kwon 2020; Klimm and Warode 2022; Ukkusuri and Waller 2010).

Games with a random number of atomic players were introduced by Myerson (1998). He showed that when the distribution of the number of players follows a Poisson distribution, beliefs about the number of players of an internal player and an external observer coincide. Such Poisson games were further studied by Myerson (2000). Gairing, Monien, and Tiemann (2008) studied atomic congestion games where the weight of a player is their private information and provide bounds on the price of anarchy. Cominetti et al. (2022) studied Bernoulli congestion games, i.e., atomic congestion games where each player participates with an independent probability. They showed that the resource loads converge to a Wardrop equilibrium in the limit when the participation probability vanishes. Cominetti et al. (2019) obtained bounds on the price of anarchy of Bernoulli congestion games with affine costs. Similar models where players participate only with a certain probability were studied in (Angelidakis, Fotakis, and Lianeas 2013; Meir et al. 2012). Ashlagi, Monderer, and Tennenholtz (2006) studied (non-Bayesian) congestion games with unknown number of players. For non-atomic congestion games, the robustness of of social welfare in the face of uncertain demand and risk-averse agents was examined by Meir and Parkes (2018); Nikolova and Stier-Moses (2015).

The potential of information design for non-atomic congestion games was illustrated through examples by Das, Kamenica, and Mirka (2017). Nachbar and Xu (2021) further explored different signaling regimes and study connections with the price of anarchy. Massicot and Langbort (2019) fully characterized the optimal policy for networks consisting of two edges with affine cost where the cost of one edge does not depend on the state. Vasserman, Feldman, and Hassidim (2015) considered a setting with parallel edges with affine costs where the cost functions are permuted and bounded the improvements that can be obtained from private signals. Bhaskar et al. (2016) considered games with affine costs where the offset depends on the state and showed that the problem of computing an optimal signaling scheme cannot be approximated by a factor of $(4/3 - \varepsilon)$ for $\varepsilon > 0$, unless $\mathsf{P} = \mathsf{NP}$. For the same setting, Griesbach et al. (2022) proved that revealing the realized state is always an optimal signaling scheme if and only if the underlying network is a series-parallel graph. They also provided LP-based techniques to compute the optimal signaling schemes. In particular, they can compute optimal signals for parallel links with a constant number of states and commodities. Acemoglu et al. (2018) considered the setting in which players have different knowledge about the available edges in a road network and give a strict characterization of the graph class for which a player cannot obtain higher private cost by gaining additional information. Wu, Amin, and Ozdaglar (2021) characterize the Bayesian Wardrop equilibria that arise when populations of drivers receive multiple signals from heterogeneous information systems. Zhou, Nguyen, and Xu (2022) showed how to compute the optimal public and private signal in an atomic congestion game with constant number of parallel edges. Alon, Meir, and Tennenholtz (2013) demonstrated that information design may improve the social welfare in atomic congestion games. Castiglioni et al. (2021) studied information design for atomic congestion games in the relaxed setting of ex ante persuasion where the players are only persuaded to follow the signaling scheme *before* receiving the signal. They showed that an optimal signal can be computed with LP-based techniques for symmetric players, and show that the problem is NP-hard for asymmetric players. The provision of information in a dynamic model where players have preferences over arrival times was explored by Arnott, de Palma, and Lindsey (1991). Mareček, Shorten, and Yu (2016) considered a dynamic discrete-time model of congestion where a central authority provides a signal for each resource at each time step based on past observations.

Preliminaries

Signaling. We consider a signaling problem in the context of network congestion games. There is a finite set $\Theta = \{\theta_1, \dots, \theta_l\}$ of l *states of nature*, along with a prior distribution μ^*, where $\mu^*_\theta \geq 0$ is the probability that state $\theta \in \Theta$ is realized. We denote by $\Delta(\Theta)$ the space of all distributions over Θ. There is a sufficiently large, finite set of (public) *signals* Σ which can be used by a benevolent principal to influence the information of all players in a congestion game. We study the problem of computing a good signaling scheme, given by a distribution over Σ for each state $\theta \in \Theta$. More formally, a *signaling scheme* is a matrix $\varphi = (\varphi_{\theta,\sigma})_{\theta\in\Theta,\sigma\in\Sigma}$ such that $\varphi_{\theta,\sigma} \geq 0$ for all $\theta \in \Theta, \sigma \in \Sigma$, and $\sum_{\sigma\in\Sigma} \varphi_{\theta,\sigma} = \mu^*_\theta$ for each $\theta \in \Theta$. The value of $\varphi_{\theta,\sigma}$ is the combined probability that state θ is realized and the sender sends signal σ. We define $\varphi_\sigma = \sum_{\theta\in\Theta} \varphi_{\theta,\sigma}$ as the total probability that signal σ gets sent. A signal $\sigma \in \Sigma$ *gets issued* by scheme φ if $\varphi_\sigma > 0$.

The scenario proceeds as follows. First, the principal commits to a signaling scheme φ and communicates this to all players. Hence, the prior μ^* and the signaling scheme φ are public knowledge. Then the state of nature is realized. The principal sees the realized state θ and sends a public signal σ chosen according to φ. All agents receive the signal, update their beliefs about the state of nature θ and the resulting costs in the congestion game, and then choose equilibrium strategies as a result of (unilaterally) minimizing their individual expected cost. The goal of the principal is to choose φ to minimize the total expected cost of the resulting equilibrium.

Network Congestion Games. We now describe the network congestion game, the (individual) expected cost of the agents, and the total expected cost. There is a directed graph $G = (V, E)$ with a designated source $s \in V$ and destination $t \in V$. For every edge $e \in E$ there is a *cost function* $c_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is convex and non-decreasing. In this paper, we focus on *affine costs*, i.e., all functions c_e are of the form $c_e(x) = a_e x + b_e$, where $a_e \in \mathbb{R}_{>0}$ and $b_e \in \mathbb{R}_{\geq 0}$ for every $e \in E$. We concentrate on single-commodity games, in which all players want to route from s to t. The player population consists of a continuum of infinitesimally small players of total volume $d > 0$, the *available demand*. For simplicity, we normalize to $d = 1$. The *actual demand*, however, is uncertain and depends on the realized state of nature. Formally, each state $\theta \in \Theta$ is associated with a realization $d_\theta \leq d$ of actual demand. We assume w.l.o.g. $0 < d_{\theta_1} < d_{\theta_2} < \dots < d_{\theta_l} = d = 1$. Intuitively, in state θ, each infinitesimal player participates in the game independently with probability $d_\theta / d = d_\theta$.

The set of feasible strategies $\mathcal{P} \subseteq 2^E$ for each player is the set of directed s-t-paths in G. A *strategy distribution* or *path flow* is a distribution of the players on the paths $P \in \mathcal{P}$. Such a path flow is represented by a vector $x = (x_P)_{P\in\mathcal{P}}$ satisfying the three properties (1) $\sum_{P\in\mathcal{P}} x_P = 1$, (2) $x_P \geq 0$ for all $P \in \mathcal{P}$, and (3) $x_P = 0$ for all $P \notin \mathcal{P}$. Let $\mathcal{X}$ denote the set of those vectors. Every path flow $x \in \mathcal{X}$ induces a *load* x_e on every edge $e \in E$ given by $x_e = \sum_{P\in\mathcal{P}:e\in P} x_P$.

Upon receiving the public signal σ (and knowing φ), all players perform a Bayes update of μ^* to a conditional distribution $\mu_\sigma \in \Delta(\Theta)$. The conditional probability of $\theta \in \Theta$ when receiving a signal σ with $\varphi_\sigma > 0$ is given by $\mu_{\theta,\sigma} = \varphi_{\theta,\sigma} / \varphi_\sigma$. Indeed, every signaling scheme φ can be seen as a convex decomposition of μ^* into distributions μ_σ, i.e., for every $\theta \in \Theta$, $\mu^*_\theta = \sum_\sigma \varphi_{\theta,\sigma} = \sum_\sigma \varphi_\sigma \mu_{\theta,\sigma}$. After this update, the players choose a path flow $x \in \mathcal{X}$. The next definitions apply for every distribution $\mu \in \Delta(\Theta)$. Given μ, suppose a player chooses a path including edge e. In state θ, the player is present in the system with probability d_θ, otherwise the private cost of this player is 0. Conditioned on the presence of this (infinitesimal) player, the expected cost that player will experience on e is $c_e(d_\theta x_e)$. Hence, the *expected cost of an edge* $e \in E$ is $c_e(x_e \mid \mu) = \sum_{\theta\in\Theta} \mu_\theta d_\theta c_e(d_\theta x_e)$. The *(individual) expected cost of a player* on path $P \in \mathcal{P}$ is given by $c_P(x \mid \mu) = \sum_{e\in P} c_e(x_e \mid \mu)$. A path flow $x \in \mathcal{X}$ is a *Wardrop equilibrium* if no player has an incentive to change their chosen strategy. Formally, given μ, no player shall improve their expected cost by deviating to another strategy, i.e., $c_P(x \mid \mu) \leq c_Q(x \mid \mu)$ for all $P, Q \in \mathcal{P}$ with $x_P > 0$.

The next result extends a characterization for non-atomic games with a single state, cf. (Beckmann, McGuire, and Winsten 1956). It directly carries over to the scenario considered in this paper as follows.

Proposition 1. *Given any distribution $\mu \in \Delta(\Theta)$, a strategy distribution $x \in \mathcal{X}$ is a Wardrop equilibrium if and only if $x \in \arg\min\left\{\sum_{e\in E}\int_0^{y_e} c_e(t \mid \mu)\,\mathrm{d}t \;:\; y \in \mathcal{X}\right\}$.*

All cost functions c_e are convex, so their convex combinations $c_e(x \mid \mu)$ are convex as well. As such, the Wardrop equilibrium is unique, since the optimization problem in Proposition 1 is strictly convex and has a unique solution. We use $x^*(\mu)$ to denote the unique Wardrop equilibrium for a distribution $\mu \in \Delta(\Theta)$. The total cost of a path flow x for $\mu \in \Delta(\Theta)$ is given by $C(x \mid \mu) = \sum_{P\in\mathcal{P}} x_P c_P(x \mid \mu) = \sum_{e\in E} x_e c_e(x_e \mid \mu)$. For the Wardrop equilibrium for $\mu \in \Delta(\Theta)$, we use the short notation $C(\mu) = C(x^*(\mu) \mid \mu)$. The goal of the principal is to choose φ in order to minimize the total expected cost of the Wardrop equilibrium for the conditional distributions μ_σ resulting from all signals σ, i.e., $C(\varphi) = \sum_{\sigma\in\Sigma} \varphi_\sigma C(\mu_\sigma)$. We refer to the full version (Griesbach et al. 2023b, Appendix B) for a more detailed example illustrating the problem and its concepts.

Structural Properties

We exhibit useful structural properties of the signaling scenario outlined above. We concentrate on a single probability distribution μ over states of nature, i.e., μ_θ is the probability of state θ (i.e., that demand d_θ in the network is realized). Given μ, the expected cost of edge $e \in E$ is

$$\begin{aligned} c_e(x_e \mid \mu) &= \sum_{\theta\in\Theta} \mu_\theta d_\theta c_e(d_\theta x_e) \\ &= a_e \sum_{\theta\in\Theta} \mu_\theta d_\theta^2 x_e + b_e \sum_{\theta\in\Theta} \mu_\theta d_\theta. \end{aligned} \tag{1}$$

We first show that the cost of the unique Wardrop equilibrium with respect to $c_e(\cdot \mid \mu)$ is piecewise linear in $\mu \in \Delta(\Theta)$. Let x be a flow. For $v \in V$, let ψ_v be the length of a shortest path with respect to $c_e(x_e \mid \mu)$ from s to v. We call an edge $e = (v, w)$ *active* in x if $\psi_w - \psi_v = c_e(x_e \mid \mu)$. Clearly, for every flow x, the set of active edges is connected and such that every vertex v is reached by a path of active edges from s. Let $\mathcal{A} = \{A \subseteq E \mid G = (V, A)$ is connected and contains an (s, v)-path for all $v \in V\}$ be the set containing all sets of edges with that property, and let $A(x)$ be the set of active edges for a flow x. In the following, we call a set $A \in \mathcal{A}$ a *support*. The proof of piecewise linearity is deferred to the full version.

Lemma 1. *For a single-commodity network congestion game, the unique Wardrop equilibrium flow and the cost of the unique Wardrop equilibrium are piecewise linear in* μ. *In particular, for every* $A \in \mathcal{A}$, *there is a possibly empty polytope* $P_A \subseteq \Delta(\Theta)$ *such that* $P_A = \{\mu \in \Delta(\Theta) \mid A(x^*(\mu)) = A\}$, *and* x^* *and* C *are affine on* P_A.

Next, we study conditions in which the cost $C(\mu)$ is monotone in μ. Due to space limitations, we defer the proof of the next lemma and its corollary to the full version.

Lemma 2. *Let* $\mu^{(1)}, \mu^{(2)} \in \Delta(\Theta)$ *be such that we have* $\sum_{\theta\in\Theta} \mu_\theta^{(1)} d_\theta^2 \;/\; \sum_{\theta\in\Theta} \mu_\theta^{(1)} d_\theta < \sum_{\theta\in\Theta} \mu_\theta^{(2)} d_\theta^2 \;/\; \sum_{\theta\in\Theta} \mu_\theta^{(2)} d_\theta$ *and* $\sum_{\theta\in\Theta} \mu_\theta^{(1)} d_\theta < \sum_{\theta\in\Theta} \mu_\theta^{(2)} d_\theta$. *Then* $C(\mu^{(1)}) \le C(\mu^{(2)})$.

Corollary 3. *If* $|\Theta| = 2$, $C(\mu)$ *is non-decreasing in* μ_{θ_2}.

The proof of Lemma 1 has striking similarities to the proof of the same result for the model with (known demand and) affine costs and *uncertain offsets* in (Griesbach et al. 2022, Lemma 1). We have not been able to derive a direct reduction between the two scenarios and discuss why it seems non-obvious to establish. First, Lemma 2 and Corollary 3 do not hold for signaling with uncertain offsets. In more detail, reinspecting the proof of Lemma 2, we can reinterpret our model using deterministic demand $d = 1$ and affine costs with *uncertain slopes and offsets* $c_e(x_e \mid \mu) = \sum_{\theta\in\Theta} \mu_\theta \left(a_e d_\theta^2 x_e + b_e d_\theta\right) = \sum_{\theta\in\Theta} \mu_\theta \left(a_{e,\theta} x_e + b_{e,\theta}\right)$. This scenario has been studied in, e.g., (Bhaskar et al. 2016; Das, Kamenica, and Mirka 2017). The reinterpretation per se does not appear to be very useful – games with uncertain affine costs are not very well-understood and in general do not admit, e.g., the linearity properties of Lemma 1 (in contrast to the case when only offsets are uncertain, c.f. (Griesbach et al. 2022)). For a normalized version of the costs $c_e^n(x_e \mid \mu) = \left(\sum_{\theta\in\Theta} \mu_\theta d_\theta c_e(d_\theta x_e)\right) / \left(\sum_{\theta\in\Theta} \mu_\theta d_\theta^2\right) = a_e x_e + b_e \left(\sum_{\theta\in\Theta} \mu_\theta d_\theta\right) / \left(\sum_{\theta\in\Theta} \mu_\theta d_\theta^2\right)$, a fixed μ yields the same scaling factor throughout for every edge cost. As such, every Wardrop equilibrium w.r.t. costs $c_e(\cdot \mid \mu)$ is also a Wardrop equilibrium w.r.t. costs $c_e^n(\cdot \mid \mu)$ and vice versa. The normalized costs c_e^n indeed might seem like a reduction to an instance of affine costs with *uncertain offsets*. However, defining state-specific constants $b_{e,\theta}^n$ independent of μ such that $\sum_{\theta\in\Theta} \mu_\theta b_{e,\theta}^n = b_e \left(\sum_{\theta\in\Theta} \mu_\theta d_\theta\right) / \left(\sum_{\theta\in\Theta} \mu_\theta d_\theta^2\right)$ for every $\mu \in \Delta(\Theta)$ can be impossible. This reduction would be *non-linear* and, as such, substantially change the cost structure of signaling schemes.

FPTAS for Two States

The main result of this section is the following.

Theorem 1. *For a single-commodity network G with unknown demands, affine costs, and two states, there is an FPTAS for optimal signaling.*

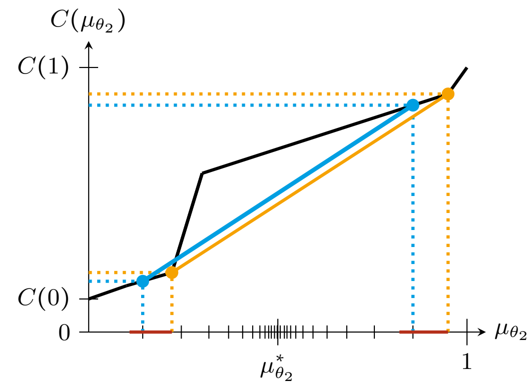

Figure 2: Illustration for the proof sketch of Theorem 1.

For a formal proof, see the full version; in the following we sketch the main arguments. With Lemma 1 and Corollary 3, the cost function $C(\mu)$ is piecewise-linear and monotone, see Fig. 2 where the orange line shows the cost function induced by the optimal signaling scheme. The algorithm computes polynomially many sample points for C with exponentially decreasing step size towards the prior as indicated by the ticks on the abscissa. The algorithm uses an LP to compute the best signaling scheme restricted to the sampling points and the alternative of revealing no information at all. Using Cramer's rule and Hadamard's theorem, it can be shown that a polynomial number of sample points suffice, implying that the algorithm runs in polynomial time. The approximation ratio of $(1 + \varepsilon)$ for any $\varepsilon > 0$ is obtained by proving that for any potential optimal conditional belief, there exists a sampling point within an ε-distance (red area) that has smaller cost. The cost function induced by these signals guarantees a $(1 + \varepsilon)$-approximation and is shown in blue in Fig. 2.

Full Information Revelation

As our main result in this section, we show that for a single-commodity network congestion game an optimal signaling scheme always reveals the true state of nature if and only if the underlying network is a series-parallel graph.

Formally, a graph $G = (V, E)$ with two designated vertices $s, t \in V$ is a series-parallel graph if either it consists only of a single edge $E = \{\{s, t\}\}$, or it is obtained by a parallel or serial composition of two series-parallel graphs. For two series-parallel graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with designated vertices $s_1, t_1 \in V_1$ and $s_2, t_2 \in V_2$ the *parallel composition* is the graph $G = (V, E)$ created from the disjoint union of graphs G_1 and G_2 by merging the vertices s_1 and s_2 into a new vertex s, and merging t_1 and t_2 into a new vertex t. The *serial composition* of G_1 and G_2 is the graph created from the disjoint union of graphs G_1 and G_2 by merging the vertices t_1 and s_2, and renaming s_1 to s and t_2 to t. We treat series-parallel graphs as directed graphs by directing every edge in the orientation as it appears in any path from s to t. This is well-defined since in a series-parallel graph, there is a global order on the vertices such that every path only visits vertices in increasing order.

The proofs of this section are very similar to the ones in (Griesbach et al. 2022). Due to space limitations, we defer them to the full version. Broadly speaking, we first recall from Lemma 1 that the cost of the unique Wardrop equilibrium C is a piecewise linear function on $\Delta(\Theta)$. The result then follows from showing that C is concave in $\Delta(\Theta)$ for all cost functions and probability distributions over demands if and only if the underlying network is series-parallel.

The general idea of the proof of the concavity of C on $\Delta(\Theta)$ is as follows. In Lemma 1, we have shown that C is affine on P_A for all $A \in \mathcal{A}$. Let $x^*_A : \Delta(\Theta) \to \mathbb{R}^E$ be an affine function such that $x^*(\mu) = x^*_A(\mu)$ for all $\mu \in P_A$. For x to be a Wardrop equilibrium flow, it must satisfy a system of equations and inequalities. For $\mu \in P_A$, x^*_A is the unique solution x for the system of equations. While $x^*(\mu) = x^*_A(\mu)$ for all $\mu \in P_A$, the vector $x^*_A(\mu)$ with $\mu \in \Delta(\Theta) \setminus P_A$ will not be a Wardrop equilibrium or not even a feasible flow at all depending on which of the inequalities is violated.

Next, we show that the pointwise minimum $\min_{A \in \mathcal{A}} C_A(\mu)$ of all Wardrop equilibria costs always corresponds to a support that is feasible. More specifically, we show in Lemma 4 that when for some $\mu \in \Delta(\Theta)$, there is a support $A \in \mathcal{A}$ with $\mu \notin P_A$, then there is another support $A' \in \mathcal{A}$ with either lower cost or the same cost but fewer edges. As a consequence, Lemma 5 shows that a Wardrop equilibrium is given by the flows x^* that correspond to the pointwise minimum $\min_{A \in \mathcal{A}} C_A(\mu)$, where ties are broken in favor of smaller supports. Finally, our main result is Theorem 2.

Lemma 4. *For a single-commodity network congestion game on a series-parallel graph, let $A \in \mathcal{A}$ and $\mu \in \Delta(\Theta) \setminus P_A$. Then, there is another support $A' \in \mathcal{A}$ with $C_{A'}(\mu) < C_A(\mu)$ or $C_{A'}(\mu) = C_A(\mu)$ and $|A'| < |A|$.*

Lemma 5. *We have $C(\mu) = \min_{A \in \mathcal{A}} C_A(\mu) \forall \mu \in \Delta(\Theta)$.*

Theorem 2. *For a single-commodity network G with unknown demands and affine costs, full information revelation is always an optimal signaling scheme if and only if G is series-parallel.*

Computing Optimal Schemes

We consider the computation of optimal signaling schemes. Towards this end, we first investigate the unique Wardrop equilibria for a fixed set of active edges. We again use the term *support* for a set of active edges. Our approach is generally similar to (Griesbach et al. 2022), but there are notable differences in the analysis to establish the result. Suppose we are given a set of k distinct supports, which we denote by $A_1, \dots, A_k$. Consider the set of signaling schemes φ with the following properties: φ sends k signals (where for simplicity we assume $\sigma \in [k] = \{1, \dots, k\}$), and each signal $\sigma \in [k]$ shall result in a Wardrop equilibrium x_σ with support A_σ. The main result in this section shows that we can efficiently optimize over this set of signaling schemes. The proof is deferred to the full version.

Theorem 3. *Given k distinct support vectors $(A_\sigma)_{\sigma \in [k]}$, we can compute the best signaling scheme that induces Wardrop equilibria with supports $(A_\sigma)_{\sigma \in [k]}$ in time polynomial in $|\Theta|$, $|E|$ and k.*

Network	$\|V\|$	$\|E\|$	$\|Z\|$	d_{θ_2}
Sioux Falls (SF)	24	76	24	360,600
Eastern Massachusetts (EM)	74	258	74	65,576
Berlin-Friedrichshain (BF)	224	523	23	11,205
Berlin-Pr.-Berg-Center (BP)	352	749	38	16,660
Berlin-Tiergarten (BT)	361	766	26	10,755
Berlin-Mitte-Center (BM)	398	871	36	11,482

Table 1: Network instances in the computational studies.

This reduces optimizing the signaling scheme to an optimal choice of supports. Suppose for some optimal signaling scheme φ^* we know (a superset of) all supports A_σ used in the Wardrop equilibrium resulting from each signal $\sigma \in \Sigma$ issued in φ^*. We inspect the conditions of optimal schemes φ^* a bit more closely. Indeed, we need to consider at most $k \leq |\Theta|$ signals, and each signal σ can be assumed to have a distinct support vector A_σ. The proof of the following proposition is deferred to the full version.

Proposition 2. *There is an optimal signaling scheme φ^* such that at most $|\Theta|$ signals get issued in φ^*, and there is no pair of signals $\sigma \neq \sigma'$ that both get issued in φ^* and $A_\sigma \subseteq A_{\sigma'}$. In particular, every signal σ that gets issued in φ^* has a distinct support vector A_σ.*

For a given support A and two states, we can optimize efficiently to find the largest and smallest value of μ_{θ_2} such that the distribution has a Wardrop equilibrium with support A. Similar to (Griesbach et al. 2022, Proposition 3), this property can be used to compute *all* supports of Wardrop equilibria for all $\mu \in \Theta(\Delta)$.

Proposition 3. *The set of all supports of Wardrop equilibria for all $\mu \in \Theta(\Delta)$ in games with two states can be computed in output-polynomial time.*

When the distributions in $\Delta(\Theta)$ generate at most a polynomial number of different supports in the resulting Wardrop equilibrium, we can compute these supports and, hence, even an optimal signaling scheme in polynomial time. However, there also exist games, in which an exponential number of supports can arise. The instances are nested Braess graphs and emerge as a straightforward adaptation of the constructions in (Klimm and Warode 2022; Griesbach et al. 2022).

Corollary 6. *For every number $n \in \mathbb{N}$, there is a single-commodity game with two states, $O(n)$ vertices, $O(n)$ edges, and $O(n)$ source-target paths, in which $2^{\Theta(n)}$ different supports arise in the Wardrop equilibria for all $\mu \in \Delta(\Theta)$.*

Computational Studies

In the face of Corollary 6, the goal of our study was to investigate i) if instances of our model on realistic networks generate a small number of different supports in the Wardrop equilibrium, and ii) by how much public signaling can improve the total cost in these networks. We considered non-atomic network congestion games with affine costs and uncertain demand on real-world networks for a single commodity and two possible states of nature $\Theta = \{\theta_1, \theta_2\}$. Table 1 shows the six different networks we examined. The network data was obtained from the Transportation Networks for Research Core Team (2022). The data set includes a model for each network, i.e., it specifies nodes V and links E which correspond to crossings and roads in the real world, respectively. It also defines a partition of the nodes into *zones* Z. The size of the networks ranges from smaller ones (SF, EM) to larger ones (BF, BP, BT, BM). The first two are frequently considered in the traffic assignment literature; the latter were used, e.g., by Jahn et al. (2005).

In addition, the data set provides experimental data on traffic-related properties for each link $e \in E$, such as the capacity C_e and the free-flow travel time t_e (i.e., the time needed to traverse the link in the absence of congestion), and on representative demands between pairs of zones. Originally, the data set is designed for computational studies on the traffic assignment problem with multiple commodities and link cost functions $c_e^{\text{BPR}}(x)$ as defined in the congestion model of the Bureau of Public Roads (1964), $c_e^{\text{BPR}}(x) = t_e\big(1 + \eta(x/C_e)^{\beta}\big)$. Here, $\beta = 4$ and η are dimensionless parameters ($\eta = 0.15$ for SF and EM, $\eta = 1$ else). For our model, we defined the coefficients in the cost function $c_e(x) = a_e x + b_e$ as $a_e = \eta t_e / C_e$ and $b_e = t_e$. These cost functions correspond to a linear variant of c_e^{BPR} (for $\beta = 1$). We set the demand in our single-commodity scenario d_{θ_2} equal to the total demand that is routed through the network for the multi-commodity scenario in the original data (see Table 1). The alternative demand d_{θ_1} was defined relative to d_{θ_2}, i.e., $d_{\theta_1} = \rho \cdot d_{\theta_2}$ for some $\rho \in [0, 1]$. In the following, we show results for $\rho = 0.2$. We performed 40 simulations for each network with varying (s, t)-pairs. For each simulation, the (s, t)-pair was drawn uniformly at random from the set of zones such that $s \neq t$ and no pair was chosen more than once. Thus, each simulation is given one network and one (s, t)-pair. We call such a tuple an *instance*.

The sets of all supports of $C(\mu_{\theta_2})$ over $\mu_{\theta_2} \in [0, 1]$ were computed by implementing the approach underlying Proposition 3, i.e., by recursively computing the support of the emerging Wardrop equilibrium at a mean value for μ_{θ_2} (initially $\mu_{\theta_2} = 1/2$), and then optimize twice: once with the objective of maximizing μ_{θ_2} and once with the objective of minimizing μ_{θ_2}. We used the built-in solver of the SciPy package (v1.8.1). The flow assignments were computed by an implementation of the conjugate Frank-Wolfe algorithm (Frank and Wolfe 1956; Daneva and Lindberg 2003) in Python (v3.10.6) based on the code of Bettini (2022). Experiments were performed on an Intel Core i5 based computer at 3.47 GHz with 8 GB RAM operating on Ubuntu 22.04.1 LTS. More information on used libraries and parameters is provided in the full version (Griesbach et al. 2023b, Appendix D). All source codes and data sets are available on GitHub (Griesbach et al. 2023a).

For each network with instances $i = 1, \dots, 40$, let $\mathcal{A}_i$ be the set of all (distinct) supports of $C(\mu_{\theta_2})$. Table 2 shows averaged results on the properties of $\mathcal{A}_i$. We point out that both the average (AV) and the maximum (MAX) number of used supports turn out to be very small compared to the number of edges in each network, even though the relative difference between d_{θ_1} and d_{θ_2} is rather large. In fact, these quantities decrease even more for larger values of ρ (for $\rho = 0.8$, the maximum number of used supports ranges from three to five across all instances). Moreover, the averaged standard deviation (SD) is small as well. Therefore, these findings imply that computing the optimal signaling scheme in realistic network instances can be done efficiently by solving our approach. The share of instances where $C(\mu_{\theta_2})$ is linear is mainly caused by adjacent sources s and targets t. The share of concave cost functions reported in Table 2 excludes the purely linear cost functions.

Net.	$\lvert\mathcal{A}_i\rvert$			C	
	AV	SD	MAX	conc. [%]	lin. [%]
SF	4.67	2.08	9	80	10
EM	5.15	3.14	12	70	8
BF	5.28	2.76	12	68	10
BP	4.90	1.85	11	88	3
BT	5.10	2.54	11	78	8
BM	5.15	2.38	11	75	3

Table 2: Results for the set of all supports $\mathcal{A}_i$ and the concavity and linearity of $C(\mu_{\theta_2})$ for $\mu_{\theta_2} \in [0, 1]$ averaged over 40 instances for each network instance.

Net.	FI is opt. [%]	$\frac{C(\text{FI})}{C(\text{OPT})}$	$\frac{C(\text{NO})}{C(\text{OPT})}$	$\frac{C(\text{OPT})}{C(\text{PSO})}$	$\frac{C(\text{WE})}{C(\text{PSO})}$
SF	100	1.0000	1.0064	1.0135	1.0200
EM	100	1.0000	1.0052	1.0101	1.0154
BF	98	1.0000	1.0049	1.0106	1.0156
BP	100	1.0000	1.0042	1.0091	1.0134
BT	100	1.0000	1.0051	1.0117	1.0169
BM	100	1.0000	1.0045	1.0108	1.0154

Table 3: Performance of full information revelation (FI), no-signaling (NO), and the optimal signaling scheme (OPT) averaged over 40 instances for each network with $\mu_{\theta_2}^* = 0.5$. The cost of the optimal signaling scheme and the Wardrop equilibrium (WE) are compared to the pointwise social optimum (PSO) defined as $(1 - \mu_{\theta_2}^*)\text{SO}(\mu_{\theta_2} = 0) + \mu_{\theta_2}^*\text{SO}(\mu_{\theta_2} = 1)$.

For the second part of our study, we analyzed the performance of full information revelation, no-signaling, and the optimal signaling scheme, as shown in Table 3. The results are rounded to four decimal places due to numerical precision. Recall that the cost of the Wardrop equilibrium corresponds to the cost of no-signaling. One can see that in most cases full information revelation is optimal. Moreover, even if it is not optimal, it only produces marginal extra costs compared to the optimal signaling scheme (which are not captured within the numerical precision here).

On another note, the study reveals that using optimal signaling schemes results in slight but consistent improvements over no-signaling. However, even with optimal information design there remains a notable gap to the average cost of a pointwise social optimal flow. As a last remark, Tables 2 and 3 suggest that the optimality criterion of full information revelation goes beyond the resulting Wardrop equilibrium being concave, as used for our characterization in Theorem 2, since all networks are not series-parallel.

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