On the Existence of Pure Nash Equilibria in Weighted Congestion Games

Tobias Harks* and Max Klimm**

Department of Mathematics, TU Berlin, Germany
{harks,klimm}@math.tu-berlin.de

Abstract. We study the existence of pure Nash equilibria in weighted congestion games. Let C denote a set of cost functions. We say that C is *consistent* if every weighted congestion game with cost functions in C possesses a pure Nash equilibrium. We say that C is *FIP-consistent* if every weighted congestion game with cost functions in C has the Finite Improvement Property. Our main results are structural characterizations of consistency for twice continuously differentiable cost functions. More specifically, we show that C is consistent for two-player games if and only if C contains only monotonic functions and for all $c_1, c_2 \in C$, there are constants $a, b \in \mathbb{R}$ such that $c_1 = a\,c_2 + b$. For games with at least 3 players we show that C is consistent if and only if exactly one of the following cases hold: (*i*) C contains only affine functions; (*ii*) C contains only exponential functions such that $c(\ell) = a_c\, e^{\phi \ell} + b_c$ for some $a_c, b_c, \phi \in \mathbb{R}$, where a_c and b_c may depend on c, while ϕ must be equal for every $c \in C$. This characterization is even valid for 3-player games, thus, closing the gap to 2-player games considered above. Finally, we derive various results regarding consistency and FIP-consistency for weighted network congestion games.

1 Introduction

In many situations, the state of a system is determined by a finite number of independent players, each optimizing an individual objective function. A natural framework for analyzing such decentralized systems are noncooperative games. While it is well known that for finite noncooperative games an equilibrium point in mixed strategies always exists, this need not be true for Nash equilibria in pure strategies (PNE for short). One of the fundamental goals in algorithmic game theory is to characterize conditions under which a Nash equilibrium in pure strategies exists. In this paper, we study this question for weighted congestion games.

Congestion games, as introduced by Rosenthal [18], model the interaction of a finite set of strategic agents that compete over a finite set of facilities. A pure strategy of each player is a set of facilities. The cost of facility f is given by a real-valued cost function c_f that depends on the number of players using f and the private cost of every player

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equals the sum of the costs of the facilities in the strategy that she chooses. Rosenthal [18] proved in a seminal paper that such congestion games always admit a PNE by showing these games posses a potential function[1]. In a weighted congestion game, every player has a demand $d_i \in \mathbb{R}_{>0}$ that she places on the chosen facilities. The cost of a facility is a function of the total load on the facility. An important subclass of weighted congestion games are weighted *network* congestion games. Every player is associated with a positive demand that she wants to route from her origin to her destination on a path of minimum cost. In contrast to unweighted congestion games, weighted congestion games do not always admit a PNE. Fotakis et al. [8] and Libman and Orda [15] constructed a single-commodity network instance with two players having demands one and two, respectively. Their instances use different non-decreasing cost values per edge that are defined at the three possible loads $1, 2, 3$. Goemans et al. [11] constructed a two-player single-commodity instance without a PNE that uses different polynomial cost functions with nonnegative coefficients and degree of at most two. On the positive side, Fotakis et al. [8,9] proved that every weighted congestion game with affine cost functions possesses a PNE. While the negative results of [8,11,15] suggest that only affine functions guarantee the existence of PNE, a result of Panagopoulou and Spirakis [17] came as a surprise: PNE always exist for instances with exponential cost functions. It is worth noting that the positive results of [8,9,17] are particularly important as they establish existence of PNE for the respective sets of cost functions *independent* of the underlying game structure, that is, *independent* of the underlying strategy set, the demand vector and the number of players, respectively.

In this paper, we further explore the equilibrium existence problem. Our goal is to precisely characterize, which type of cost functions actually guarantees the existence of PNE. To formally capture this issue, we introduce the notion of *PNE-consistency* or simply *consistency* of a set of cost functions. Let $\mathcal{C}$ be a set of cost functions and let $\mathcal{G}(\mathcal{C})$ be the set of *all* weighted congestion games with cost functions in $\mathcal{C}$. We say that $\mathcal{C}$ is *consistent* if every game in $\mathcal{G}(\mathcal{C})$ possesses a PNE. Using this terminology, the results of [8,9,17] yield that $\mathcal{C}$ is consistent if $\mathcal{C}$ contains either affine or exponential functions.

A natural open question is to decide whether there are further consistent functions, that is, functions guaranteeing the existence of a PNE. We thus investigate the following question: How large is the set $\mathcal{C}$ of consistent cost functions?

1.1 Our Results

We give an almost complete characterization of consistency of cost functions in weighted congestion games. Specifically, we show the following: Let $\mathcal{C}$ be a nonempty set of continuous functions.

1. If $\mathcal{C}$ is consistent, then $\mathcal{C}$ may only contain monotonic functions. We further show that monotonicity of cost functions is necessary for consistency even in singleton games, two-player games, two facility games, games with identical cost functions and games with symmetric strategies.

[1] A real-valued function P on the set of strategies having the property that every improving move by one defecting player strictly reduces the value of P.

2. Let C be a nonempty set of twice continuously differentiable functions. Then, C is consistent for two-player games if and only if C only contains only monotonic functions and for all $c_1, c_2 \in C$, there are constants $a, b \in \mathbb{R}$ such that $c_1 = a\,c_2 + b$. To the best of our knowledge, it was not known before that two-player games possess PNE for the above class of cost functions.
3. Let C be a nonempty set of twice continuously differentiable functions. We prove that C is consistent for games with at least 3 players if and only if exactly one of the following cases hold: (*i*) C contains only affine functions; (*ii*) C contains only exponential functions such that $c(\ell) = a_c\, e^{\phi\,\ell} + b_c$ for some $a_c, b_c, \phi \in \mathbb{R}$, where a_c and b_c may depend on c, while ϕ must be equal for every $c \in C$. This characterization is even valid for 3-player games, thus, closing the gap to 2-player games considered above. We note that the "if" direction follows from [8,9,17].
4. Finally, we study weighted network congestion games. Let C be a non-empty set of strictly increasing, positive and twice continuously differentiable functions. For multi-commodity networks with at least three players, C is consistent if and only if C contains only affine functions or certain exponential functions as specified above. For two-player network games (single or two-commodity networks), we show that C is consistent if and only if for all $c_1, c_2 \in C$, there are constants $a, b \in \mathbb{R}$ such that $c_1 = a\,c_2 + b$. This characterization gives a structural insight, why the instances used in [8,11,15] do not possess a PNE.

   For single-commodity network games with at least three players, we prove that C is FIP-consistent if and only if C contains only affine functions or certain exponential functions.

All proofs missing in the extended abstract are presented in the full version [12].

1.2 Significance and Techniques

Weighted congestion games are one of the core topics in the algorithmic game theory, operations research, and economics literature. In particular, there is a large body of work quantifying the price of anarchy of PNE for different sets of cost functions, see Awerbuch et al. [4], Christodoulou and Koutsoupias [5], and Aland et al. [2]. There are, however, drawbacks in the use of Nash equilibria. While mixed Nash equilibria are guaranteed to exist, their use is unrealistic in many games. On the other hand, pure Nash equilibria as the stronger solution concept may fail to exist in weighted congestion games. Thus, we believe that our work constitutes an important step towards fully characterizing the PNE-existence problem in weighted congestion games.

Our characterizations essentially rely on two ingredients. First, we derive in Section 3 for continuous and consistent cost functions two necessary conditions (Monotonicity Lemma and Extended Monotonicity Lemma). The Monotonicity Lemma states that any continuous and consistent cost function must be monotonic. The Lemma is proved by constructing a sequence of two-player weighted congestion games in which we identify a unique 4-cycle of deviations of two players. Then, we show that for any non-monotonic cost function, there is a weighted congestion game with a unique improvement cycle. By adding additional players and carefully choosing player-weights

and strategy spaces, we then derive the Extended Monotonicity Lemma, which ensures that the set of cost functions contained in a certain *finite integer linear hull* of the considered cost functions must be monotone. The proof of the Extended Monotonicity Lemma provides a template for constructing an instance without a PNE, for any non-affine and non-exponential cost function; we illustrate this template by constructing an instance without PNE using identical cubic cost functions.

In Section 4, we give a characterization of the set of functions that arise from affine transformations of a monotonic function and show that the Extended Monotonicity Lemma for two-player games implies that consistent cost functions must be of this form. In Section 5, we characterize the set of affine and exponential functions, and show that the Extended Monotonicity Lemma for games with at least three players implies that consistent cost functions must be either affine or exponential. In Section 6, we discuss implications of the Extended Monotonicity Lemma when applied to weighted network congestion games.

1.3 Related Work

Milchtaich [16] showed that singleton weighted congestion games with player-specific cost functions do not always have a PNE. Milchtaich further characterized topological properties of networks that guarantee the existence of PNE in network congestion games if players control different amounts of flow or cost functions are player specific. This was further elaborated by Gairing et al. [10], who considered different cost functions, slight modifications in the network topology, and both the weighted and the unweighted cases. Harks et al. [13] proved that every weighted congestion game with two-players and uniform (up to affine translations) and strictly monotonic cost functions possesses a PNE.

Ieong et al. [14] proved that in singleton congestion games with non-decreasing cost functions, best response dynamics converge in polynomial time to a PNE. Ackermann et al. [1] extended this result to weighted congestion games with a so called *matroid property*, that is, the strategy of every player forms a basis of a matroid. In the same paper, they also characterized the existence of PNE in weighted congestion games with the same matroid property by proving that for any strategy space not satisfying the matroid property, there is an instance of a weighted congestion game not having a PNE. Fanelli and Moscardelli [7] studied convergence properties of certain improvement dynamics in weighted congestion games with polynomial cost functions of bounded degree. Dunkel and Schulz [6] proved that it is strongly NP-hard to decide whether or not a weighted congestion game with nonlinear cost functions possesses a PNE.

2 Preliminaries

We consider finite strategic games $G = (N, X, \pi)$, where $N = \{1, \ldots, n\}$ is the non-empty and finite set of players, $X = \bigtimes_{i \in N} X_i$ is the non-empty strategy space, and $\pi : X \to \mathbb{R}^n$ is the combined *private cost* function that assigns a private cost vector $\pi(x)$ to each strategy profile $x \in X$. These games are cost minimization games. Unless specified

otherwise, we allow private cost functions to be negative or positive. A strategic game is called *finite* if X is finite.

We use standard game theory notation; for a player $i \in N$ we denote the set $N \setminus \{i\}$ by $-i$. A strategy profile x is a pure Nash equilibrium if there is no player $i \in N$ having an alternative strategy profile $y_i \in X_i$ such that $\pi_i(y_i, x_{-i}) - \pi_i(x) < 0$.

A tuple $\mathcal{M} = (N, F, X = \bigtimes_{i \in N} X_i, (c_f)_{f \in F})$ is called a *congestion model*, where F is a non-empty, finite set of facilities, for each player $i \in N$, her collection of pure strategies X_i is a non-empty, finite set of subsets of F and $(c_f)_{f \in F}$ is a set of cost functions. In the following, we will define weighted congestion games similar to Goemans et al. [11]. Let $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$ be a congestion model and $(d_i)_{i \in N}$ be a vector of demands with $d_i \in \mathbb{R}_{>0}$. The corresponding *weighted congestion game* is the game $G(\mathcal{M}) = (N, X, \pi)$, where π is defined as $\pi_i(x) = \sum_{f \in x_i} d_i c_f(\ell_f(x))$ and $\ell_f(x) = \sum_{j \in N : f \in x_j} d_j$ is called the *load* on facility f in strategy x. We will write G as shorthand for $G(\mathcal{M})$. Now, we will define the notion of *consistency* of cost functions. Let C be a set of cost functions and let $\mathcal{G}(C)$ be the set of all weighted congestion games with cost functions in C. Then, C is *consistent* if every $G \in \mathcal{G}(C)$ possesses a PNE. If the set $\mathcal{G}(C)$ is restricted, e.g., two player games etc., we say that C is *consistent w.r.t.* $\mathcal{G}(C)$ if every $G \in \mathcal{G}(C)$ possesses a PNE. A pair $(x, (y_i, x_{-i})) \in X \times X$ is called an *improving move* of player i if $\pi_i(x_i, x_{-i}) - \pi_i(y_i, x_{-i}) > 0$. We denote by I the set of improving moves in G. We call a sequence $\gamma = (x^0, x^1, \ldots)$ an *improvement path* if $(x^k, x^{k+1}) \in I$ for all k. If every improvement path is finite, G has the *Finite Improvement Property (FIP)*. We say that C is *FIP-consistent*, if every $G \in \mathcal{G}(C)$ has the FIP.

3 Necessary Conditions on the Existence of a PNE

Throughout this work, we will assume that C is a set of continuous functions. As a first result, we prove that if C is consistent, then every function $c \in C$ is monotonic. We will first need a (nontrivial) technical lemma, which will be used many times.

Lemma 1. *Let $c : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a continuous function. Then, the following two statements are equivalent:*

1. *c is monotonic on $\mathbb{R}_{\geq 0}$.*
2. *The following two conditions hold:*
   (a) *For all $x \in \mathbb{R}_{>0}$ with $c(x) < c(0)$ there is $\varepsilon > 0$ such that $c(y) \leq c(x)$ for all $y \in (x, x + \varepsilon)$,*
   (b) *For all $x \in \mathbb{R}_{>0}$ with $c(x) > c(0)$ there is $\varepsilon > 0$ such that $c(y) \geq c(x)$ for all $y \in (x, x + \varepsilon)$.*

We proceed by presenting a necessary condition for consistency.

Lemma 2 (Monotonicity Lemma). *Let C be a set of continuous functions. If C is consistent, then every $c \in C$ is monotonic on $\mathbb{R}_{>0}$.*

Besides the continuity of the functions in C, the proof of Lemma 2 relies on rather mild assumptions and, thus, this result can be strengthened in the following way.

Corollary 1. *Let C be a set of continuous functions. Let $\mathcal{G}(C)$ be the set of weighted congestion games with cost functions in C satisfying one or more of the following properties: (i) Each game $G \in \mathcal{G}(C)$ has two players; (ii) Each game $G \in \mathcal{G}(C)$ has two facilities; (iii) For each game $G \in \mathcal{G}(C)$ and each player $i \in N$, the set of her strategies X_i contains a single facility only; (iv) Each game $G \in \mathcal{G}(C)$ has symmetric strategies; (v) Cost functions are identical, that is, $c_f = c_g$ for all $f, g \in F$. If C is consistent w.r.t. $\mathcal{G}(C)$, then, each $c \in C$ must be monotonic.*

In particular, if there is a non-monotonic function $\tilde{c} \in C$, then, there is a symmetric singleton weighted congestion game with identical cost functions, two facilities, two players that does not admit a PNE. However, it is well known that singleton games posses a PNE if cost functions are non-decreasing.

We now extend the Monotonicity Lemma to obtain a stronger necessary condition by regarding more players and more complex strategies. To this end, we consider for finite K those functions that can be written as the sum of K functions in C, possibly with an offset. Formally, we define the *finite integer linear hull* of C as

$$\mathcal{L}_{\mathbb{Z}}(C) = \{c : \mathbb{R}_{\geq 0} \to \mathbb{R} : c(x) = \sum_{k=1}^{K} a_k c_k(x + b_k), K \in \mathbb{N}, a_k \in \mathbb{Z}, b_k \geq 0, c_k \in C\}, \quad (1)$$

and show that consistency of C implies that $\mathcal{L}_{\mathbb{Z}}(C)$ contains only monotonic functions.

Lemma 3 (Extended Monotonicity Lemma). *Let C be a set of continuous functions. If C is consistent, then $\mathcal{L}_{\mathbb{Z}}(C)$ contains only monotonic functions.*

Proof. Let $\tilde{c} \in \mathcal{L}_{\mathbb{Z}}(C)$ be arbitrary. By allowing $c_k = c_l$ for $k \neq l$, we can omit the integer coefficients a_k and rewrite $\tilde{c}$ as $\tilde{c}(x) = \sum_{k=1}^{m_+} c_k(x + b_k) - \sum_{k=1}^{m_-} \bar{c}_k(x + \bar{b}_k)$ for some $c_k, \bar{c}_k \in C, m_+, m_- \in \mathbb{N}$.

We define a congestion model $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$, with $N = N_p \cup N^+ \cup N^-$ and $F = F^1 \cup F^2 \cup F^3 \cup F^4$. The set of players N^+ contains for each c_k, $1 \leq k \leq m_+$, a player with demand b_k and the set of players N^- contains for each $\bar{c}_k$, $1 \leq k \leq m_-$, a player with demand $\bar{b}_k$. We call the players in $N^- \cup N^+$ *offset* players. The set $N_p = \{1, 2\}$ contains two additional (non-trivial) players.

We now explain the strategy spaces and the sets F^1, F^2, F^3, F^4. For each function c_k, $1 \leq k \leq m_+$, we introduce two facilities f_k^2, f_k^3 with cost function c_k. For each function $\bar{c}_k$, $1 \leq k \leq m_-$, we introduce two facilities f_k^1, f_k^4 with cost function $\bar{c}_k$. To model the offsets b_k in (1), for each offset player $k \in N^+$, we define $X_k = \{\{f_k^2, f_k^3\}\}$. Similarly, for each offset player $k \in N^-$, we set $X_k = \{\{f_k^1, f_k^4\}\}$. The non-trivial players in N_p have strategies $X_1 = \{F^1 \cup F^2, F^3 \cup F^4\}$ and $X_2 = \{F^1 \cup F^3, F^2 \cup F^4\}$, where $F^1 = \{f_1^1, \ldots, f_{m_-}^1\}, F^2 = \{f_1^2, \ldots, f_{m_+}^2\}$,
$F^3 = \{f_1^3, \ldots, f_{m_+}^3\}$, and $F^4 = \{f_1^4, \ldots, f_{m_-}^4\}$. We consider a family of weighted congestion games $G_\delta^x(\mathcal{M})$ parameterized by $d_1 = \delta$ and $d_2 = x$ for $1, 2 \in N_p$. For the 4-cycle

$$\gamma = \Big((F^1 \cup F^2, F^1 \cup F^3, \ldots), (F^3 \cup F^4, F^1 \cup F^3, \ldots), (F^3 \cup F^4, F^2 \cup F^4, \ldots), \\ (F^1 \cup F^2, F^2 \cup F^4, \ldots), (F^1 \cup F^2, F^1 \cup F^3, \ldots)\Big),$$

we calculate

$$\begin{aligned}
&\pi_1(F^3 \cup F^4, F^1 \cup F^3, \dots) - \pi_1(F^1 \cup F^2, F^1 \cup F^3, \dots) \\
&= d_1 \sum_{k=1}^{m_+} c_k(d_1 + d_2 + b_k) - d_1 \sum_{k=1}^{m_-} \bar{c}_k(d_1 + d_2 + \bar{b}_k) + d_1 \sum_{k=1}^{m_-} \bar{c}_k(d_1 + \bar{b}_k) - d_1 \sum_{k=1}^{m_+} c_k(d_1 + b_k) \\
&= d_1(\tilde{c}(x+\delta) - \tilde{c}(\delta)). \qquad (2)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\pi_2(F^3 \cup F^4, F^2 \cup F^4, \dots) - \pi_2(F^3 \cup F^4, F^1 \cup F^3, \dots) = d_2(\tilde{c}(x) - \tilde{c}(x+\delta)) \\
&\pi_1(F^1 \cup F^2, F^2 \cup F^4, \dots) - \pi_1(F^3 \cup F^4, F^2 \cup F^4, \dots) = d_1(\tilde{c}(x+\delta)) - \tilde{c}(\delta) \qquad (3) \\
&\pi_2(F^1 \cup F^2, F^1 \cup F^3, \dots) - \pi_2(F^1 \cup F^2, F^2 \cup F^4, \dots) = d_2(\tilde{c}(x) - \tilde{c}(x+\delta)).
\end{aligned}$$

If $\tilde{c}(x) > \tilde{c}(0)$, we can find $\varepsilon > 0$ such that $\tilde{c}(x+\delta) > \tilde{c}(\delta)$ for all $0 < \delta < \varepsilon$. For such δ, the values in (2) and (3) are negative and we may conclude that there is an improvement cycle in the game G if $\tilde{c}(x+\delta) - \tilde{c}(x) < 0$. Hence, we have $\tilde{c}(y) - \tilde{c}(x) \geq 0$ for all $y \in (x, x+\varepsilon)$. If $\tilde{c}(0) > \tilde{c}(x)$, considering the 4-cycle in the other direction yields the claimed result by the same argumentation. Using that every strategy combination is contained in γ and applying Lemma 1 delivers the claimed result. □

4 A Characterization for Two-Player Games

We will analyze implications of the Extended Monotonicity Lemma (Lemma 3) for two-player weighted congestion games. First, we remark that if all offsets b_k and $\bar{b}_k$ in (1) are equal to zero, the construction in Lemma 3 only involves two players. For ease of exposition, we additionally restrict ourselves to the case $K = 2$, that is, we only regard those functions that can be written as the difference of two functions in C without offset. Formally, define

$$\mathcal{L}^2_{\mathbb{N}}(C) = \{c : \mathbb{R}_{\geq 0} \to \mathbb{R} \;:\; c(x) = a_1\, c_1(x) - a_2\, c_2(x), a_1, a_2 \in \mathbb{N}, c_1, c_2 \in C\}.$$

Then, we obtain the following immediate corollary of the Extended Monotonicity Lemma.

Corollary 2. *Let C be a set of continuous functions. If C is consistent w.r.t. two-player games, then, $\mathcal{L}^2_{\mathbb{N}}(C)$ contains only monotonic functions.*

Thus, we will proceed investigating sets of functions C that guarantee that $\mathcal{L}^2_{\mathbb{N}}(C)$ contains only monotonic functions. For this purpose, we first need the following technical lemma.

Lemma 4. *Let C be a set of functions that are twice continuously differentiable. Then, the following are equivalent:*

1. *$\mathcal{L}^2_{\mathbb{N}}(C)$ contains only monotonic functions.*
2. *For all $c_1, c_2 \in C$ there are $a, b \in \mathbb{R}$ such that $c_2(x) = a\, c_1(x) + b$ for all $x \geq 0$.*

Using Lemma 4, we are now ready to state our first main result.

Theorem 1. *Let C be a set of twice continuously differentiable functions. Let $\mathcal{G}^2(C)$ be the set of two-player games such that cost functions are in C. Then, the following two conditions are equivalent.*

1. *C is consistent w.r.t. $\mathcal{G}^2(C)$*
2. *C contains only monotonic functions and for all $c_1, c_2 \in C$, there are constants $a, b \in \mathbb{R}$ such that $c_1 = a\,c_2 + b$.*

5 A Characterization for the General Case

We now consider the case $n \geq 3$, that is, we consider weighted congestion games with at least three players. We will show that a set of twice continuously differentiable cost functions is consistent if and only if this set contains either linear or certain exponential functions. Our main tool for proving this result is to analyze implications of the Extended Monotonicity Lemma (Lemma 3) for three-player weighted congestion games. Formally, define

$$\mathcal{L}^3_{\mathbb{N}}(C) = \{c : \mathbb{R}_{\geq 0} \to \mathbb{R} : c(x) = a_1 c_1(x) - a_2 c_2(x+b), a_1, a_2 \in \mathbb{Z}, c_1, c_2 \in C, b \in \mathbb{R}_{>0}\}.$$

Then, we obtain the following immediate corollary of the Extended Monotonicity Lemma.

Corollary 3. *Let C be a set of continuous functions. If C is consistent w.r.t three-player games, then $\mathcal{L}^3_{\mathbb{N}}(C)$ contains only monotonic functions.*

Note that $\mathcal{L}^3_{\mathbb{N}}(C)$ involves a single offset b, which requires only three players in the construction of the proof of the Extended Monotonicity Lemma.

In order to characterize the sets of functions C such that $\mathcal{L}^3_{\mathbb{N}}(C)$ contains only monotonic functions, we need the following technical lemma.

Lemma 5. *Let c be a twice continuously differentiable function. Then, the following two are equivalent:*

1. *Either $c(\ell) = a\,e^{\phi \ell} + b$ for some $a, b, \phi \in \mathbb{R}$, or $c(\ell) = a\,x + b$ for some $a, b \in \mathbb{R}$.*
2. $\det\begin{pmatrix} c'(x) & c'(y) \\ c''(x) & c''(y) \end{pmatrix} = 0$ *for all $x, y \in \mathbb{R}_{>0}$.*

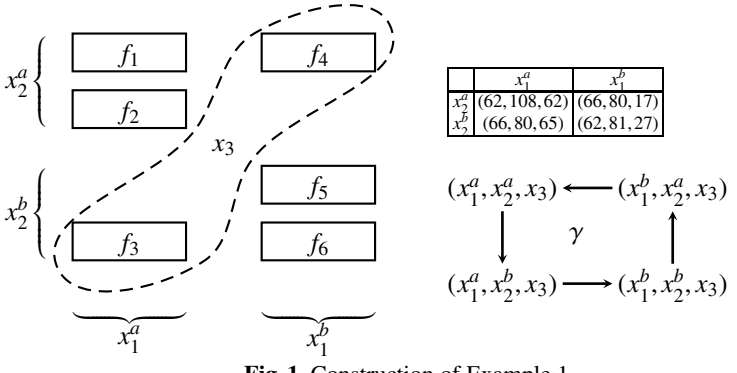
We are now ready to state our second main theorem.

Theorem 2. *Let C be a set of twice continuously differentiable functions. Then, C is consistent if and only if one of the following cases holds*

1. *C contains only affine functions*
2. *C contains only functions of type $c(\ell) = a_c\,e^{\phi \ell} + b_c$ where $a_c, b_c \in \mathbb{R}$ may depend on c while $\phi \in \mathbb{R}$ is independent of c.*

We conclude this section by giving an example that captures the main ideas presented so far. Theorem 2 establishes that for each non-affine and non-exponential cost function $\tilde{c}$, there is a weighted congestion game G with uniform cost function $\tilde{c}$ on all facilities that does not admit a PNE. In the following example, we show how such a game for $c(\ell) = \ell^3$ is constructed.

Example 1. As $c(\ell) = \ell^3$ is neither affine nor exponential, there are $a_1, a_2 \in \mathbb{N}$ and $x \in \mathbb{R}_{>0}$ such that $\tilde{c}(\ell) = a_1\, c(\ell) - a_2\, c(\ell + x)$ has a strict extremum. In fact, we can choose $a_1 = 2, a_2 = 2$ and $x = 1$, that is, the function $\tilde{c}(\ell) = 2c(\ell) - c(\ell + 1) = 2\ell^3 - (\ell + 1)^3$ has a strict local minimum at $\ell = 1 + \sqrt{2}$. In particular, we can choose $d_1 = 1$ and $d_2 = 2$ such that $\tilde{c}(d_1) = -6 > \tilde{c}(d_2) = -11 < \tilde{c}(d_1 + d_2) = -10$. The weighted congestion game without PNE is now constructed as follows: We introduce $2(a_1 + a_2)$ facilities $f_1, \ldots, f_6$ and the following strategies $x_1^a = \{f_1, f_2, f_3\}$, $x_1^b = \{f_4, f_5, f_6\}$, $x_2^a = \{f_1, f_2, f_4\}$, $x_2^b = \{f_3, f_5.f_6\}$, and $x_3 = \{f_3, f_4\}$. We then set $X_1 = \{x_1^a, x_1^b\}$, $X_2 = \{x_2^a, x_2^b\}$, and $X_3 = \{x_3\}$. The so defined game has four strategy profiles, namely (x_1^a, x_2^a, x_3), (x_1^a, x_2^b, x_3), (x_1^b, x_2^a, x_3), (x_1^b, x_2^b, x_3). As Player 3 is an offset player, she has a single strategy only, thus, the players' private costs depend only on the choice of Players 1 and 2 as indicated in the table in Fig. 1. We derive that the 4-cycle γ depicted in Fig. 1 is a best-reply cycle in G. As there are no strategy profiles outside γ, G has no PNE.



	x_1^a	x_1^b
x_2^a	(62, 108, 62)	(66, 80, 17)
x_2^b	(66, 80, 65)	(62, 81, 27)

Fig. 1. Construction of Example 1

6 Weighted Network Congestion Games

We discuss the implications of our characterizations to the important subclass of weighted network congestion games. We start with multi-commodity networks.

Multi-commodity Networks. Our characterization of consistent cost functions given in Theorem 2 crucially relies on the construction used in the Extended Monotonicity Lemma. By assuming that cost functions are strictly increasing and positive, we can transform this construction to a weighted network congestion game, see the full version [12] for details.

Theorem 3. *Let C be a set of strictly increasing, positive and twice continuously differentiable functions. Then, C is consistent for multi-commodity network games with*

at least three players if and only if one of the following cases holds: (*i*) *C contains only affine functions;* (*ii*) *C contains only exponential functions $c(\ell) = a_c\, e^{\phi \ell} + b_c$ where $a_c, b_c, \phi \in \mathbb{R}$ and ϕ is independent of c.*

For multi-commodity weighted network games with two players, we obtain the following.

Theorem 4. *Let C be a set of strictly increasing, positive and twice continuously differentiable functions. Let $\mathcal{G}^2(C)$ be the set of two-player (single or two)-commodity network games such that cost functions are in C. Then, C is consistent w.r.t. $\mathcal{G}^2(C)$ if and only if C contains monotonic functions such that for all $c_1, c_2 \in C$, there are constants $a, b \in \mathbb{R}$ with $c_1 = a\, c_2 + b$.*

Single-commodity Networks. Considering weighted single-commodity network congestion games we conclude with a result concerning the FIP.

Theorem 5. *Let C be a set of strictly increasing, positive and twice continuously differentiable functions. Then, C is FIP-consistent for single-commodity network games with at least three players if and only if one of the following cases holds:* (*i*) *C contains only affine functions;* (*ii*) *C contains only exponential functions $c(\ell) = a_c\, e^{\phi \ell} + b_c$ where $a_c, b_c, \phi \in \mathbb{R}$ and ϕ is independent of c.*

7 Conclusions

We obtained an almost complete characterization of consistency of cost functions in weighted congestion games. The following issues are open. We required that cost functions are twice-continuously differentiable. Although almost all practically relevant functions satisfy this condition, it would be interesting to weaken this assumption.

For single-commodity games with at least three players, we were only able to characterize the FIP, not consistency. The single-commodity case, however, behaves completely different as, for instance, Anshelevich et al. [3] have shown that for positive and strictly decreasing cost functions, there is always a PNE.

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