Strong Nash Equilibria in Games with the Lexicographical Improvement Property

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Abstract. We provide an axiomatic framework for the the well studied lexicographical improvement property and derive new results on the existence of strong Nash equilibria for a very general class of congestion games with bottleneck objectives. This includes extensions of classical load-based models, routing games with splittable demands, scheduling games with malleable jobs, and more.

1 Introduction

A main criticism about Nash equilibria is the fact that they do not consider deviations of coalitions. To cope with the issue of coordination, we adopt the solution concept of a *strong equilibrium* (SNE for short) proposed in [2]. In a SNE, no coalition (of any size) can deviate and strictly improve the private cost of each of its members. Even though a SNE may rarely exist, it forms a very robust and appealing stability concept.

One of the most successful approaches in establishing existence of PNE (as well as SNE) is the potential function approach initiated in [12]: one defines a real-valued function P on the set of strategy profiles of the game and shows that every improving move of a coalition strictly reduces the value of P. Given that the set of strategy profiles is finite, every sequence of improving moves reaches a SNE. In particular, the global minimum of P is a SNE. For most games, however, it is hard to prove or disprove the existence of such a potential function.

In this paper, we introduce vector-valued potential functions and say that a game has the lexicographic improvement property (LIP) if there is a vector-valued function that lexicographically decreases for every improving move. A game has the π-LIP if the vector of private cost itself constitutes such a vector-valued potential function.

The main contribution of this paper is twofold. We first study desirable properties of arbitrary finite and infinite games having the LIP and π-LIP, respectively. These properties concern the existence of SNE, efficiency and fairness of SNE, and computability of SNE. Secondly, we identify an important class of games that we term bottleneck congestion games for which we can actually prove the π-LIP and, hence, prove that these games possess SNE with the above desirable properties.

Before we outline our results in more detail, we briefly explain the importance of bottleneck objectives in congestion games with respect to real-world applications. Referring to previous work by Keshav, it has been pointed out in [5] that the performance of

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a communication network is closely related to the performance of its bottleneck (most congested) link. This behavior is also stressed by [3], who investigated PNE in routing games with bottleneck objectives. Similar observations are reported in [14], where the applicability of selfish routing models to realistic models of the Internet is investigated.

Our Results. We characterize games having the LIP by means of the existence of a generalized strong ordinal potential function. The proof of this characterization is constructive, that is, given a game G having the LIP for a function ϕ, we explicitly construct a generalized strong ordinal potential P. We then investigate games having the π-LIP with respect to efficiency and fairness of SNE. Our characterization implies that there are SNE satisfying various efficiency and fairness properties, e.g., bounds on the prices of stability and anarchy, Pareto optimality, and min-max fairness.

We establish that bottleneck congestion games possess the π-LIP and thus possess SNE with the above mentioned properties. Moreover, our characterization of games having the LIP implies that bottleneck congestion games possess the strong finite improvement property.

In contrast to most congestion games considered so far, we require only that the cost functions on the facilities satisfy three properties: "non-negativity", "independence of irrelevant choices", and "monotonicity". Roughly speaking, the second and third conditions assume that the cost of a facility solely depends on the *set* of players using the respective facility and that this cost decreases if some players leave this facility. Thus, this framework extends classical *load-based* models in which the cost of a facility depends on the number or total weight of players using the respective facility.

We then study the LIP in *infinite* games, that is, games with infinite strategy spaces that can be described by compact subsets of $\mathbb{R}^p$, $p \in \mathbb{N}$. Infinite games need not admit a strong potential function even if the LIP is satisfied. We prove, however, that continuity of ϕ in the definition of LIP is sufficient for the existence of SNE. Our existence proof (which can be found in [9]) is constructive, that is, we outline an algorithm whose output is a SNE.

We then introduce *infinite bottleneck congestion games*. An infinite bottleneck congestion game arises from a bottleneck congestion game G by allowing players to fractionally distribute a certain *demand* over the pure strategies of G. We prove that these games have the π-LIP provided that the cost functions on the facilities are non-negative and non-decreasing. It turns out, however, that the function π may be discontinuous on the strategy space (even if the cost functions on the facilities are continuous). Thus, the existence of SNE does not immediately follow. We solve this difficulty by generalizing the LIP. As a consequence, we obtain for the first time the existence of SNE for infinite bottleneck congestion games with non-decreasing and continuous cost functions.

In the final section, we show that the methods presented here also apply to a more general framework.

Because of lack of space, the results of this work are presented without proofs. For the proofs, examples and further explanatory notes, we refer to [9].

Related Work. Congestion games were introduced in [16] and further studied in [12]. The existence of SNE in congestion games with monotone increasing cost functions has been studied in [10]. They showed that SNE need not exist in such games and gave a structural characterization of the strategy space for symmetric (and quasi-symmetric) congestion games that admit SNE. Based on the work in [12], they also introduced the concept of a *strong potential function*, that is, a function on the set of strategy profiles that decreases for every profitable deviation of a coalition. The existence of (correlated) SNE in congestion games with non-increasing cost functions is explored in [18].

Several authors studied the existence and efficiency (price of anarchy and stability) of PNE and SNE in various specific classes of congestion games. Referring to an observation of Mehlhorn, a lexicographic argument was used first in [8] in order to establish the existence of PNE in singleton congestion games. Similar arguments have been applied to job scheduling games in [6] and to more general job scheduling games, where the processing time of a machine may depend on the set of jobs scheduled on the respective machine in [7]. Moreover, this line of argumentation has been used in [1] to prove even the existence of SNE in scheduling games on unrelated machines. They further studied differences between PNE and SNE and derived bounds on the (strong) price of anarchy and stability, respectively.

Bottleneck congestion games with network structure have been considered by Banner and Orda [3]. They studied existence of PNE in the unsplittable flow and in the splittable flow setting, respectively. They observed that standard techniques (such as Kakutani's fixed-point theorem) for proving existence of PNE do not apply to bottleneck routing games as the private cost functions may be discontinuous. They proved existence of PNE (but not SNE) by showing that bottleneck games are better reply secure, quasi-convex, and compact. Under these conditions, they recall Reny's existence theorem [15] for better reply secure games with possibly discontinuous private cost functions. In contrast, we show the existence of SNE with direct and constructive methods.

2 Preliminaries

We consider strategic games $G = (N, X, \pi)$, where $N = \{1, \dots, n\}$ is the non-empty and finite set of players, $X = \bigtimes_{i \in N} X_i$ is the non-empty strategy space, and $\pi : X \to \mathbb{R}^n_+$ is the combined *private cost* function that assigns a private cost vector $\pi(x)$ to each strategy profile $x \in X$. These games are cost minimization games and we assume additionally that the private cost functions are non-negative. A strategic game is called *finite* if X is finite. We use standard game theory notation; for a coalition $S \subseteq N$ we denote by $-S$ its complement and by $X_S = \bigtimes_{i \in S} X_i$ we denote the set of strategy profiles of players in S.

A pair $(x, (y_S, x_{-S})) \in X \times X$ is called an *improving move* if $\pi_i(x_S, x_{-S}) - \pi_i(y_S, x_{-S}) > 0$ for all $i \in S$. We denote by $I(S)$ the set of improving moves of coalition S and we set $I := \bigcup_{S \subseteq N} I(S)$. We call a sequence of strategy profiles $\gamma = (x^0, x^1, \dots)$ an *improvement path* if every tuple $(x^k, x^{k+1}) \in I$. A strategy profile x is a *strong Nash equilibrium* (SNE) if $(x, (y_S, x_{-S})) \notin I$ for all $\emptyset \neq S \subseteq N$ and $y_S \in X_S$.

In recent years, much attention has been devoted to games admitting the finite improvement property (FIP), that is, each path of single-handed (one player) deviations is finite. Equivalently, we say that G has the strong finite improvement property (SFIP) if every improvement path is finite. A necessary and sufficient condition for the SFIP is the existence of a generalized *strong* ordinal potential function, that is, a function $P : X \to \mathbb{R}$ such that $P(x) - P(y) > 0$ for all $(x, y) \in I$.

It is known that both the SFIP and the existence of a generalized strong ordinal potential are hard to prove or disprove for a particular game. We define a class of games that we call *games with the Lexicographical Improvement Property (LIP)* and show that such games possess a generalized strong ordinal potential.

Definition 1 (Sorted lexicographical order). *Let $a, b \in \mathbb{R}_+^q$ and denote by $\tilde{a}, \tilde{b} \in \mathbb{R}_+^q$ be the sorted vectors derived from a, b by permuting the entries in non-increasing order. Then, a is* strictly sorted lexicographically smaller *than b (written $a \prec b$) if there exists an index m such that $\tilde{a}_i = \tilde{b}_i$ for all $i < m$, and $\tilde{a}_m < \tilde{b}_m$.*

Definition 2 (Lexicographical improvement property, π-LIP). *A finite strategic game $G = (N, X, \pi)$ possesses the* lexicographical improvement property *(LIP) if there exist $q \in \mathbb{N}$ and a function $\phi : X \to \mathbb{R}_+^q$ such that $\phi(x) \succ \phi(y)$ for all $(x, y) \in I$. G has the π-LIP if G has the LIP for $\phi = \pi$.*

The function ϕ is a generalized strong ordinal potential if $q = 1$. Taking the M-norm of ϕ for a sufficiently large M it is easy to verify that the LIP is equivalent to the existence of a generalized strong ordinal potential, regardless of q.

Theorem 1. *Let $G = (N, X, \pi)$ be a finite strategic game. Then, G has the LIP if and only if there exists $\phi : X \to \mathbb{R}_+^q$, $q \in \mathbb{N}$, and $M \in \mathbb{N}$ such that $P(x) = \sum_{i=1}^{q} \phi_i(x)^M$ is a generalized strong ordinal potential function for G.*

3 Properties of SNE in Games with the π-LIP

As the existence of SNE in games with the LIP is guaranteed, it is natural to ask which properties these SNE may satisfy. In recent years, several notions of efficiency have been discussed in the literature, see [11]. We here cover the price of stability, Pareto optimality and min-max-fairness.

Price of Stability. We study the efficiency of SNE with respect to the optimum of a predefined social cost function. Given a game $G = (N, X, \pi)$ and a social cost function $C : X \to \mathbb{R}_+$, whose minimum is attained in a strategy profile $y \in X$, let $X^{\text{SNE}} \subseteq X$ denote the set of strong Nash equilibria. Then, the strong price of stability for G with respect to C is defined as $\inf_{x \in X^{\text{SNE}}} C(x)/C(y)$. We consider the following natural social cost functions: the sum-objective or L_1-norm defined as $L_1(x) = \sum_{i \in N} \pi_i(x)$, the L_p-objective or L_p-norm, $p \in \mathbb{N}$, defined as $L_p(x) = \big(\sum_{i \in N} \pi_i(x)^p\big)^{1/p}$, and the min-max objective or L_∞-norm defined as $L_\infty(x) = \max_{i \in N}\{\pi_i(x)\}$.

Proposition 1. *Let G be a strategic game with the π-LIP. Then, the strong price of stability w.r.t. L_∞ is 1, and for any $p \in \mathbb{R}$, the strong price of stability w.r.t. L_p is smaller than n.*

Pareto Optimality. Pareto optimality is one of the fundamental concepts studied in economics, see Osborne and Rubinstein [13]. For a strategic game $G = (N, X, \pi)$, a strategy profile x is called *weakly Pareto efficient* if there is no $y \in X$ such that $\pi_i(y) < \pi_i(x)$ for all $i \in N$. A strategy profile x is *strictly Pareto efficient* if there is no $y \in X$ such that $\pi_i(y) \leq \pi_i(x)$ for all $i \in N$, where at least one inequality is strict.

So strictly Pareto efficient strategy profiles are those strategy profiles for which every improvement of a coalition of players is to the expense of at least one player outside the coalition. Pareto optimality has also been studied in the context of congestion games, see Chien and Sinclair [4] and Holzman and Law-Yone [10].

Proposition 2. *Let G be a finite strategic game with the π-LIP. Then, there is a SNE that is strictly Pareto optimal.*

Min-Max-Fairness. We next define the notion of min-max fairness, which is a central topic in resource allocation in communication networks, see Srikant [19] for an overview and pointers to the large body of research in this area. While strict Pareto efficiency requires that there is no improvement to the expense of anyone, the notion of min-max-fairness is stricter. Here, it is only required that there is no improvement at the cost of someone who receives already higher costs (while an improvement that increases the cost of a player with smaller original cost is allowed). It is easy to see that every min-max-fair strategy profile constitutes a strict Pareto optimum, but the converse need not hold. A strategy profile x is called *min-max fair* if for any other strategy profile y with $\pi_i(y) < \pi_i(x)$ for some $i \in N$, there exists $j \in N$ such that $\pi_j(x) \geq \pi_i(x)$ and $\pi_j(y) > \pi_j(x)$.

Proposition 3. *Let G be a finite strategic game with the π-LIP. Then, there is a SNE that is min-max fair.*

4 Bottleneck Congestion Games

We now present a rich class of games satisfying the π-LIP. We call these games *bottleneck congestion games*. They are natural generalizations of variants of congestion games. In contrast to standard congestion games, we focus on *makespan-objectives*, that is, the cost of a player when using a set of facilities only depends on the highest cost of these facilities.

Definition 3 (Congestion model). *A tuple $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$ is called a* congestion model *if $N = \{1, \ldots, n\}$ is a non-empty, finite set of players, $F = \{1, \ldots, m\}$ is a non-empty, finite set of facilities, and $X = \bigtimes_{i \in N} X_i$ is the set of strategies. For each player $i \in N$, her collection of pure strategies X_i is a non-empty, finite set of subsets of F. Given a strategy profile x, we define $\mathcal{N}_f(x) = \{i \in N : f \in x_i\}$ for all $f \in F$. Every facility $f \in F$ has a cost function $c_f : \bigtimes_{i \in N} X_i \to \mathbb{R}_+$ satisfying*

Non-negativity: $c_f(x) \geq 0$ *for all* $x \in X$
Independence of Irrelevant Choices: $c_f(x) = c_f(y)$ *for all* $x, y \in X$ *with* $\mathcal{N}_f(x) = \mathcal{N}_f(y)$
Monotonicity: $c_f(x) \leq c_f(y)$ *for all* $x, y \in X$ *with* $\mathcal{N}_f(x) \subseteq \mathcal{N}_f(y)$.

Bottleneck congestion games generalize congestion games in the definition of the cost functions on the facilities. For bottleneck congestion games, the only requirements are that the cost $c_f(x)$ on facility f for strategy profile x only depends on the set of players using f in their strategy profile and that costs are increasing with larger sets.

Definition 4 (Bottleneck congestion game). *Let $\mathcal{M}$ be a congestion model. The corresponding* bottleneck congestion game *is the strategic game $G(\mathcal{M}) = (N, X, \pi)$ in which π is defined as $\pi = \bigtimes_{i \in N} \pi_i$ and $\pi_i(x) = \max_{f \in x_i} c_f(x)$.*

We are now ready to state our main result concerning bottleneck congestion games, providing a large class of games that satisfies the π-LIP.

Theorem 2. *Let $G(\mathcal{M})$ be a bottleneck congestion game with allocation model $\mathcal{M}$. Then, G fulfills the LIP for the functions $\phi : X \to \mathbb{R}^n_+$ and $\psi : X \to \mathbb{R}^{mn}_+$ defined as*

$$\phi_i(x) = \pi_i(x) \quad \text{for all } i \in N, \qquad \psi_{i,f}(x) = \begin{cases} c_f(x) & \text{if } f \in x_i \\ 0 & \text{else} \end{cases} \qquad \text{for all } i \in N, f \in F.$$

As a corollary of Theorem 2 we obtain that each bottleneck congestion game has the π-LIP and hence possesses the SFIP. In addition, the results on the price of stability, Pareto optimality and min-max-fairness apply.

The class of bottleneck congestion games contains some special cases of particular interest. For example, *scheduling games* on unrelated machines can be seen as bottleneck congestion games in which all strategies are singletons. As the private costs received from each machine may depend on the *set* of players on that machine we cover certain *interference games* as well, see [9] for details. Another special case of bottleneck congestion games are bottleneck routing games. To the best of our knowledge, this work establishes for the first time the existence of the FIP and SFIP in such games. In the full version of this paper [9], we identify some special cases of bottleneck routing games in which the SNE can be computed in polynomial time.

5 Infinite Bottleneck Congestion Games

From $\mathcal{M}$ we derive an *infinite congestion model* $\mathcal{IM} = (N, F, X, d, \Delta, (c_f)_{f \in F})$, where $d \in \mathbb{R}^n_+$, $\Delta = \Delta_1 \times \cdots \times \Delta_n$, and $\Delta_i = \{\xi_i = (\xi_{i1}, \ldots, \xi_{in_i}) : \sum_{j=1}^{n_i} \xi_{ij} = d_i, \xi_{ij} \geq 0, j = 1, \ldots, n_i\}$. The strategy profile $\xi_i = (\xi_{i1}, \ldots, \xi_{in_i})$ of player i can be interpreted as a distribution of non-negative *intensities* over the elements in X_i satisfying $\sum_{j=1}^{n_i} \xi_{ij} = d_i$ for $d_i \in \mathbb{R}_+, i \in N$. Clearly, Δ_i is a compact subset of $R^{n_i}_+$ for all $i \in N$. For a profile $\xi = (\xi_1, \ldots, \xi_n)$, we define the set of used facilities of player i as $F_i(\xi) = \{f \in F : \text{ there exists } j \in \{1, \ldots n_i\} \text{ with } f \in x_{ij} \text{ and } \xi_{ij} > 0\}$. We define the *load* of player i on f under profile ξ by $\xi_i^f = \sum_{x_{ij} \in X_i : f \in x_{ij}} \xi_{ij}$, $i \in N, f \in F$. In contrast to finite bottleneck congestion games, we assume that cost functions $c_f : X \to \mathbb{R}_+$ only depend on the *total load* defined as $\ell_f(\xi) = \sum_{i \in N} \xi_i^f$, and are continuous and non-decreasing.

Definition 5 (Infinite bottleneck congestion game). *Let $\mathcal{IM} = (N, F, X, d, \Delta, (c_f)_{f \in F})$ be an infinite congestion model derived from $\mathcal{M}$. The corresponding* infinite bottleneck congestion game *is the strategic infinite game $G(\mathcal{IM}) = (N, \Delta, \pi)$, where π is defined as $\pi = \bigtimes_{i \in N} \pi_i$ and $\pi_i(\xi) = \max_{f \in F_i(\xi)} c_f(\ell_f(\xi))$.*

Examples of such games are bottleneck routing games with splittable demands.

Theorem 3. *Let $G(\mathcal{IM}) = (N, \Delta, \pi)$ be an infinite bottleneck congestion game. Then, $G(\mathcal{IM})$ has the LIP for the functions $\phi : \Delta \to \mathbb{R}^n_+$ and $\psi : X \to \mathbb{R}^{mn}_+$ defined as*

$$\phi_i(\xi) = \pi_i(\xi), \quad \textit{for all } i \in N, \quad \psi_{i,f}(\xi) = \begin{cases} c_f(\ell_f(\xi)), & \textit{if } f \in F_i(\xi) \\ 0, & \textit{else} \end{cases} \quad \textit{for all } i \in N, f \in F.$$

We remark that for infinite games, the LIP does not imply the existence of a strong potential function. However, if the LIP is satisfied for a continuous function and the strategy space is compact, we can prove the existence of a SNE, see [9] for details. Unfortunately, a simple example reveals that the functions used in Theorem 3 for proving the LIP are not necessarily continuous. In order to obtain a the LIP for a continuous function, we generalize the notion of lexicographical ordering to ordered sets that are different from $(\mathbb{R}, \leq)$. To this end, consider a totally ordered set $(\mathcal{A}, \leq_{\mathcal{A}})$. Similar to Definition 1, we introduce a lexicographical order on $\mathcal{A}$-valued vectors. For two vectors $a, b \in \mathcal{A}^q$, let $\tilde{a}$ and $\tilde{b}$ be two vectors that arise from of a and b by ordering them w.r.t $\leq_{\mathcal{A}}$ in non-increasing order. We say that a is $\mathcal{A}$-lexicographically smaller than b, written $a \prec_{\mathcal{A}} b$ if there is $m \in \{1, \ldots, q\}$ such that $\tilde{a}_i =_{\mathcal{A}} \tilde{b}_i$ for all $i < m$ and $\tilde{a}_m <_{\mathcal{A}} \tilde{b}_m$. A game satisfies the $\mathcal{A}$-LIP if there are $q \in \mathbb{N}$ and a function $\phi : X \to \mathcal{A}^q$ such that $\phi(x) \succ_{\mathcal{A}} \phi(y)$ for all $(x, y) \in I$.

The following theorem establishes the $\mathcal{A}$-LIP for infinite bottleneck congestion games, where $(\mathcal{A}, \leq_{\mathcal{A}}) = (\mathbb{R}^2, \leq_{lex})$ and $\leq_{lex}$ denotes the ordinary lexicographical order (that does not involve any sorting of the entries) on $\mathbb{R}^2$, that is, $(a_1, a_2) <_{lex} (b_1, b_2)$ if either $a_1 < b_1$ or ($a_1 = b_1$ and $b_2 < b_2$).

Theorem 4. *Let $(\mathcal{A}, \leq_{\mathcal{A}}) = (\mathbb{R}^2, \leq_{lex})$ and let $G(\mathcal{I}\mathcal{M}) = (N, \Delta, \pi)$ be an infinite bottleneck congestion game. Then, $G(\mathcal{I}\mathcal{M})$ has the $\mathcal{A}$-LIP for $\phi : \Delta \to \mathcal{A}^m$ defined as $\phi_f(\xi) = \big(c_f(\ell_f(\xi)), \ell_f(\xi)\big)$ for all $f \in F$.*

The function ϕ in Theorem 4 is continuous. Hence, we obtain for the first time the existence of SNE for a variety of games such as scheduling games with malleable jobs, bottleneck routing games with splittable demands, etc. Note that, compared with the proof in [3], our result gives also an alternative and constructive proof for the existence of PNE in bottleneck routing games with splittable demands.

6 Extensions

We present three extensions that (as we feel) are the most interesting ones.

Regarding α-approximate strong Nash equilibria it is possible to soften the conditions on the cost functions that establish the existence of an equilibrium point. In fact, we can show that every infinite bottleneck congestion game with bounded cost functions possesses an α-approximate SNE for every $\alpha > 0$.

A natural generalization of bottleneck congestion games can be obtained by assuming that players are *heterogeneous* with respect to the cost of the most expensive facility, that is, they attach different *values* to the cost of the most expensive facility.

Assuming that higher costs on facilities are associated with higher private costs we can show that these games possess the LIP (though not the π-LIP).

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