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Reduction of Potential-Based Flow Networks

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Abstract. We consider potential-based flow networks with terminal nodes at which flow can enter or leave the network and physical properties, such as voltages or pressures, are measured and controlled. We study conditions under which such a network can be reduced to a smaller, equivalent network with the same behavior at the terminal nodes. Potential-based flow networks are widely used to model infrastructure networks, such as electricity, gas, or water networks. In contrast to Kron's reduction for electrical networks, we prove that, in general, potential-based flow networks with at least three terminals cannot be reduced to smaller networks whose size only depends on the number of terminals. On the other hand, we show that it is possible to represent a special class of potential-based flow networks by a complete graph on the terminals, and we establish a characterization of networks that can be reduced to a path network. Our results build on fundamental properties of effective resistances proved in this paper, including explicit formulae for their dependence on edge resistances of the network and their metric properties.

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Keywords: potential-based flows • network reduction • effective resistances

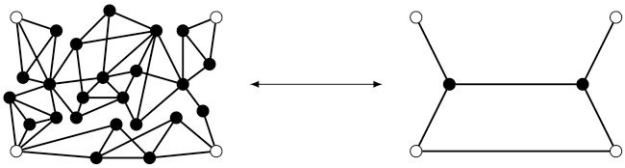
1. Introduction

The behavior of infrastructure networks, such as power, gas, and water networks, is uniquely determined by *potentials*, that is, physical properties, such as voltages, pressures, or hydraulic heads. These values are typically monitored and controlled at *terminals*, that is, nodes at which flow can be injected or discharged from the network. Under reasonable assumptions satisfied by real-world networks, the potentials at the terminals uniquely determine the flow in the network and, thus, the inflows and outflows at the terminals. In mathematical terms, there is a bijection between the potential vector at the terminals and the inflow and outflow at the terminals. For the operation of such networks, one is interested in understanding this bijection. One prominent example is contingency analysis based on injection shift factors. For a linearized model of an electric network, one estimates the effect of a change of injection of flow of a generator on the outflows at different terminals; see, for example, Sauer [24].

Because many real-world networks are very large and complex, it is a natural question whether the bijection of a given network is equal to the bijection of a smaller network with the same set of terminals. More formally, we call two networks *equivalent* (with respect to a given set of terminals) if they exhibit the same bijection between terminal potential vectors and terminal inflows and outflows. For an illustration, see Figure 1. Informally, we call a network a *reduction* of a given network (with respect to a given set of terminals) if it equivalent and has fewer nodes than the original network. Such reductions yield theoretical insights into the complexity and expressivity of a network and can reduce storage space and computation time for network operation simulation and optimization.

For classical network flows, finding reductions (or approximations) of a given network has been an active research area in the past decades; see, for example, Hagerup et al. [15], Chaudhuri et al. [7], and Andoni et al. [1]. Moreover, simple linear model reductions of lossless DC power networks have been obtained (Dorfler and Bullo [11], Kron [20]). However, less is known for more general models of physical networks, in which the flow may be governed by nonlinear equations, such as gas and water networks. A widely used model to describe such networks are potential-based flow networks introduced by Birkhoff and Diaz [4]; see also Rockafellar [22] for a general treatment. These networks are described by a directed graph $G = (V, E)$, where each node $v \in V$ has a potential π_v, representing, for example, the voltage at v for electrical networks or the squared pressure for gas

Figure 1. Example of two equivalent networks with the same behavior at the terminal nodes (white).



networks. The flow x_e through an edge $e = (u, v)$ is uniquely determined by the potential difference between its two incident nodes via the equation $\text{sign}(x_e)\beta_e|x_e|^r = \pi_u - \pi_v$. Here, β_e is an edge-specific *resistance*, and $r > 0$ is the *degree* of the network describing the underlying physics. For instance, a degree of $r = 1$ is used to model lossless DC power networks, whereas $r = 2$ and $r = 1.852$ are used for gas and water flows, respectively; see also Hendrickson and Janson [16] and Groß et al. [14] as well as the references therein.

In this paper, we investigate to what extent reduction results for networks of degree $r = 1$ can be generalized to networks of arbitrary degree. Specifically, we show that, in contrast to Kron's reduction for power networks (Dorfler and Bullo [11], Kron [20]), in general, it is not possible to reduce a gas network with at least three terminals to a smaller equivalent network whose size only depends on the number of terminals. We give, however, a sufficient condition under which potential-based networks of arbitrary degree can indeed be reduced to a graph only containing the terminals and establish a characterization of networks that can be reduced to a path network.

1.1. Results, Techniques, and Paper Outline

In this paper, we study the reduction of potential-based flow networks, generalizing existing results for linear models commonly used to describe lossless DC power networks.

In Section 2, we give a formal definition of potential-based flow networks and describe some of their basic properties. We begin Section 3 by reviewing known reducibility results for potential-based flow networks. Exploiting these results, we give a sufficient condition under which networks can be reduced to equivalent networks defined on the set of terminals only; see Theorem 3. Furthermore, using the concept of *effective resistances*, we establish a characterization of networks that can be reduced to a path network; see Theorem 4.

Generalizing results for linear networks, we study fundamental properties of effective resistances of potential-based flow networks of arbitrary degree in Section 4. Insightful in their own right, these properties turn out to be essential to analyze whether and how potential-based flow networks can be reduced to smaller networks. In Proposition 1, we give explicit formulas for effective resistances and show how to compute them via a convex optimization problem. Moreover, we provide formulas for their derivatives with respect to resistances of the edges of the network. Generalizing the corresponding result by Klein and Randić [18] for the special case of networks of degree $r = 1$, we furthermore show that the effective resistances in networks of arbitrary degree form a metric; see Proposition 2.

Motivated by the reducibility result for linear networks, an immediate question is whether networks of any degree can be reduced to a network only defined on the terminals or at least to a network whose size only depends on the number of terminals. In Section 5, we give a negative answer to this question. Specifically, we prove that there exist networks of degree two (which are widely used to model stationary gas networks) with only three terminals, which cannot be reduced to equivalent networks of constant size. For the proof, we exploit the effect of flow direction changes in parametric flows and apply properties of effective resistances from Section 4. On the one hand, we show that, in networks of arbitrary size, the function mapping a parametric demand vector to the corresponding potentials in the network can have arbitrarily many points in which it is not twice differentiable. On the other hand, using results from semialgebraic geometry, we prove that networks of constant size can only have constantly many such points.

1.2. Related Work

The first graph-theoretic studies of electrical networks go back to Kirchhoff more than 170 years ago. Since then, a great amount of research has been devoted to analyzing fundamental properties of electrical networks. One such property, which is also crucial for our paper, is the effective resistance between two nodes, that is, the potential difference between the two nodes that is needed in order to send a flow of value one from one node to the other. Chandra et al. [6] establish an illuminating relation between effective resistances in an electrical network and the expected length of a random walk from one node to the other in a graph in which the transition probabilities between two nodes are derived from their corresponding edge resistance in the electrical network. Saksena [23] proves conditions under which a given metric on a set of nodes can be represented by the effective

resistances between these nodes in an appropriately chosen electrical network. He further provides an algorithm that allows us to reconstruct any electrical network from its effective resistances. Klein and Randić [18] prove that the effective resistances of an electrical network form a metric, and Ghosh et al. [13] show that the effective resistance between any two nodes is monotone and concave in the edge resistances of the network. All these results are valid for networks of degree $r = 1$, exploiting the linearity of the flow equations in such networks. In contrast, only few results exist for potential-based flow networks of arbitrary degree. One such result is established by Calvert and Keady [5], who prove that the effective resistance between two nodes in a potential-based flow network is monotone in the edge resistances of the network.

Concerning reduction of networks, for classical network flows with edge capacities, Hagerup et al. [15] show that the set of feasible demand vectors of a network with k terminals can equivalently be represented by a network with 2^{2^k} nodes. Chaudhuri et al. [7] reduce the size of such networks for small k. A lower bound of $2^{\Omega(k)}$ nodes is proved by Krauthgamer and Rika [19].

Possibly the most famous reduction result for electrical networks is the so-called Kron [20] reduction, further analyzed, among others, by Dorfler and Bullo [11]. Using the Schur complement of the matrix mapping demand vectors to the corresponding potentials in a given network, a smaller equivalent network can be constructed that only contains the terminal nodes and edges connecting them. Note, however, that the existence of the Kron reduction strongly relies on the linearity of the mapping between demands and potentials in such networks.

For potential-based flow networks of arbitrary degree, the only known result in this direction is from Groß et al. [14], who show that any network with only two terminals can be reduced to a single edge joining the two terminals.

2. Preliminaries

We start by giving a formal definition of potential-based flow networks and review some of their basic properties. Let $G = (V, E)$ be a directed, weakly connected graph. A *flow* in G is a vector $x = (x_e)_{e\in E} \in \mathbb{R}^E$. For any edge $e = (u, v) \in E$, a positive value x_e indicates flow along edge e from u to v, whereas a negative value models flow along this edge in the opposite direction from v to u. The *balance vector* $\mathrm{bal}(x) = (\mathrm{bal}_v(x))_{v\in V} \in \mathbb{R}^V$ of a flow $x \in \mathbb{R}^E$ is defined by

$$\mathrm{bal}_v(x) := \sum_{e\in\delta^+(v)} x_e - \sum_{e\in\delta^-(v)} x_e \qquad \text{for all nodes } v \in V,$$

where $\delta^+(v) \subseteq E$ is the set of edges leaving node v and $\delta^-(v) \subseteq E$ is the set of edges entering v. Let $B := \{b \in \mathbb{R}^V : \sum_{v\in V} b_v = 0\}$ be the set of balanced *demand vectors*. We say that a flow $x \in \mathbb{R}^E$ satisfies demands $b \in B$ if $\mathrm{bal}(x) = b$. Using the (node–edge) incidence matrix $\Gamma \in \mathbb{R}^{V\times E}$ of G defined by

$$\Gamma_{v,e} := \begin{cases} 1 & \text{if } e \in \delta^+(v), \\ -1 & \text{if } e \in \delta^-(v), \\ 0 & \text{else,} \end{cases} \qquad v \in V, e \in E,$$

we observe that the flow x satisfies the demands b if and only if $\Gamma x = b$.

The flows considered in this paper are based on potential vectors $\pi \in \mathbb{R}^V$. For each edge $e = (u, v) \in E$ there is a resistance $\beta_e \in \mathbb{R}_{>0}$. Furthermore, we are given a degree $r \in \mathbb{R}_{>0}$. Setting $\beta := (\beta_e)_{e\in E}$, the tuple $\mathcal{N} = (G, \beta, r)$ is called the potential-based flow network or simply *network* for short. To simplify notation, for an edge $e = (u, v)$, we write $\beta_{u,v} := \beta_{(u,v)} = \beta_e$. The resistances β and the degree r describe the physics of the underlying network in the following way. A flow $x \in \mathbb{R}^E$ is induced by potentials $\pi \in \mathbb{R}^V$ if, for each edge $e = (u, v)$, the difference of the potentials of the end nodes equals the potential loss induced by the flow along e, that is,

$$\mathrm{sign}(x_e)\beta_e|x_e|^r = \pi_u - \pi_v. \tag{1}$$

Because the function $x_e \longmapsto \mathrm{sign}(x_e)\beta_e|x_e|^r$ is bijective, for a given potential vector $\pi \in \mathbb{R}^V$, there is a unique flow $x \in \mathbb{R}^E$ satisfying (1), which we denote by $x(\pi) = (x_e(\pi))_{e\in E}$. For such a flow $x(\pi)$, the corresponding balance vector $\mathrm{bal}(x(\pi))$ can be computed as

$$\begin{aligned} \mathrm{bal}_u(x(\pi)) &= \sum_{e\in\delta^+(u)} x_e(\pi) - \sum_{e\in\delta^-(u)} x_e(\pi) \\ &= \sum_{e\in\delta^+(u)} \sqrt[r]{\frac{|\pi_u - \pi_v|}{\beta_e}}\,\mathrm{sign}(\pi_u - \pi_v) - \sum_{e\in\delta^-(u)} \sqrt[r]{\frac{|\pi_v - \pi_u|}{\beta_e}}\,\mathrm{sign}(\pi_v - \pi_u) \\ &= \sum_{e\in\delta(u)} \sqrt[r]{\frac{|\pi_u - \pi_v|}{\beta_e}}\,\mathrm{sign}(\pi_u - \pi_v) \qquad \text{for all } u \in V, \end{aligned} \tag{2}$$

where $\delta(u) := \delta^+(u) \cup \delta^-(u)$ is the set of edges incident to u and, for each edge $e \in \delta(u)$, v is the node connected to u via edge e.

Because the right-hand side of (1) is a difference of node potentials, uniformly shifting the entries of a potential vector $\pi \in \mathbb{R}^V$ does not change x. Although, in all applications, potentials are nonnegative, we find it mathematically more convenient to also allow for negative potentials and fix the potential of an arbitrary node $v_0 \in V$ to zero. Let $\Pi_{v_0} := \{\pi \in \mathbb{R}^V : \pi_{v_0} = 0\}$ be the set of such potential vectors. Then, the function $f : \Pi_{v_0} \to B$, mapping potentials to node balances, defined as $f(\pi) := (f_u(\pi))_{u \in V}$ with

$$f_u(\pi) := \sum_{e \in \delta(u)} \sqrt[r]{\frac{|\pi_u - \pi_v|}{\beta_e}} \operatorname{sign}(\pi_u - \pi_v), \tag{3}$$

is bijective and continuous. Moreover, the inverse function $f^{-1} : B \to \Pi_{v_0}$ exists and is also continuous; see, for example, Birkhoff and Diaz [4]. This implies that there is a one-to-one correspondence between node balances $b \in B$ and potentials $\pi \in \Pi_{v_0}$. In particular, for every $b \in B$, the tuple (π, x) of potentials $\pi = f^{-1}(b) \in \Pi_{v_0}$ and their induced flow $x = x(\pi) \in \mathbb{R}^E$ is the unique solution to the following equation system:

$$\begin{aligned} \operatorname{sign}(x) \odot \beta \odot |x|^r &= \Gamma^\top \pi, \\ \Gamma x &= b, \\ \pi_{v_0} &= 0, \end{aligned} \tag{E}$$

where $\odot$ denotes the component-wise product. Moreover, the two operators $\operatorname{sign}(x) := (\operatorname{sign}(x_e))_{e \in E}$ and $|x|^r := (|x_e|^r)_{e \in E}$ are to be understood component-wise. Because we study the dependence of flows and potentials on the network $\mathcal{N}$, we denote the functions mapping demand vectors $b \in B$ to the unique vectors $x \in \mathbb{R}^E$ and $\pi \in \Pi_{v_0}$ fulfilling the equation system (E) by $x^{\mathcal{N}} : B \to \mathbb{R}^E$ and $\pi^{\mathcal{N}} : B \to \Pi_{v_0}$, respectively.

It is shown by Collins et al. [8] and Maugis [21] that, for every demand vector $b \in B$, both the flow $x^{\mathcal{N}}(b)$ and the potentials $\pi^{\mathcal{N}}(b)$ can be obtained by solving a convex program. We state these results in the following lemma.

Lemma 1 (Collins et al. [8], Maugis [21]). *For every demand vector $b \in B$, the following hold:*

i. *The flow $x^{\mathcal{N}}(b)$ is the unique optimal solution to the convex optimization problem*

$$\begin{aligned} \underset{x \in \mathbb{R}^E}{\text{minimize}} \quad & \sum_{e \in E} \beta_e |x_e|^{r+1} \\ \text{subject to} \quad & \Gamma x = b. \end{aligned} \tag{P}$$

ii. *The potential vector $\pi^{\mathcal{N}}(b)$ is the unique optimal solution to the dual of (P) given by the convex program*

$$\underset{\pi \in \Pi_{v_0}}{\text{minimize}} \quad \frac{r}{r+1} \sum_{e=(u,v) \in E} c_e |\pi_u - \pi_v|^{\frac{r+1}{r}} - \sum_{v \in V} b_v \pi_v, \tag{D}$$

where $c_e := \frac{1}{\sqrt[r]{\beta_e}}$ is the conductivity of edge $e \in E$.

iii. *For every $\lambda \in \mathbb{R}$, we have $x^{\mathcal{N}}(\lambda \cdot b) = \lambda \cdot x^{\mathcal{N}}(b)$ and $\pi^{\mathcal{N}}(\lambda \cdot b) = \lambda^r \cdot \pi^{\mathcal{N}}(b)$.*

To conclude this section, we prove the following intuitive result: letting the resistance β_e of an edge e in a network go to infinity, the corresponding potentials $\pi^{\mathcal{N}}$ converge to the potentials of the modified network in which e is removed. A similar lemma is shown by Bienstock and Verma [3, lemma 3.4] for the case of linear networks. Their proof relies on the linearity of the flow equations. In order to be able to also treat networks of higher degree, we here resort instead to the characterization of the flows in Lemma 1. We further make use of the following lemma, which is a special case of a more general theorem of Kanniappan and Sastry [17]. It states that, under certain conditions, if a sequence of strictly convex optimization problems P_k converges to a convex optimization problem P, then also the sequence of optimal solutions x_k^* converges to the optimal solution x^* of P.

Lemma 2 (Kanniappan and Sastry [17]). *Let $S \subset \mathbb{R}^n$ be a nonempty convex compact set, let $f : S \to \mathbb{R}$ be a continuous strictly convex function, and let $(f_k)_{k \in \mathbb{N}}$ with $f_k : S \to \mathbb{R}$ for $k \in \mathbb{N}$ be a sequence of continuous strictly convex functions converging uniformly to f, that is, $\lim_{k \to \infty} \sup_{x \in S} |f(x) - f_k(x)| = 0$. Let $x^* \in S$ and $x_k^* \in S$ be the unique optimal solutions to the optimization problems $\min\{f(x) : x \in S\}$ and $\min\{f_k(x) : x \in S\}$, $k \in \mathbb{N}$, respectively. Then, $\lim_{k \to \infty} x_k^* = x^*$.*

We are now in position to state and prove that a nonexistent edge can be obtained as the limit of an edge as its resistance goes to infinity.

Lemma 3. *Let $\mathcal{N} = (G, \beta, r)$ be a network and let $v_1, v_2 \in V$ be two nodes that are not adjacent in G. For $k \in \mathbb{N}$, let $\mathcal{N}(k)$ be the network that results from adding the edge (v_1, v_2) with resistance $\beta_{v_1,v_2} = k$ to $\mathcal{N}$. Then, we have $\lim_{k\to\infty} \pi^{\mathcal{N}(k)}(b) = \pi^{\mathcal{N}}(b)$ for all $b \in B$.*

Proof. Let $b \in B$ be arbitrary. To simplify the notation, we write $x^* := x^{\mathcal{N}}(b)$, $\pi^* := \pi^{\mathcal{N}}(b)$, $x^{(k)} := x^{\mathcal{N}(k)}(b)$, and $\pi^{(k)} := \pi^{\mathcal{N}(k)}(b)$. Denote by (P_k), (D_k), and (E_k) the convex programs (P) and (D) and the equation system (E) with respect to the network $\mathcal{N}(k)$.

Claim 1. *There exists a constant $M > 0$ such that $|\pi^*_w| \le M$ and $|\pi^{(k)}_w| \le M$ for all $w \in V$ and $k \in \mathbb{N}$.*

Proof. Let x be a feasible solution to (P), and let $z := \sum_{e\in E} \beta_e |x|^{r+1}$ be its objective value. Because x^* is the optimal solution to (P) and x^*, π^* fulfill the equation system (E), we have

$$\frac{|\pi^*_u - \pi^*_v|^{\frac{r+1}{r}}}{\sqrt[r]{\beta_e}} = \beta_e |x^*_e|^{r+1} \le z \quad \text{for all } e = (u,v) \in E.$$

Let $w \in V$ be an arbitrary node and P be a path from v_0 (the node whose potential is fixed to zero) to w in G. It follows that

$$|\pi^*_w| \le \sum_{e=(u,v)\in E[P]} |\pi^*_u - \pi^*_v| \le \sum_{e=(u,v)\in E[P]} (z \sqrt[r]{\beta_e})^{\frac{r}{r+1}} =: M.$$

Similarly, we get $|\pi^{(k)}_w| \le M$ for all $w \in V$, concluding the proof of Claim 1. □

We can, therefore, add the convex constraints $|\pi_w| \le M$ for all $w \in V$ to (D) and (D_k) without changing these programs. Thus, π^* and $\pi^{(k)}$ are the unique optimal solutions to the convex optimization problems $\min\{f(\pi) : \pi \in S\}$ and $\min\{f_k(\pi) : \pi \in S\}$, respectively, where

$$S := \{\pi \in \mathbb{R}^V : \pi_{v_0} = 0, |\pi_w| \le M \text{ for all } w \in V\},$$
$$f(\pi) := \frac{r}{r+1} \sum_{e=(u,v)\in E} \frac{|\pi_u - \pi_v|^{\frac{r+1}{r}}}{\sqrt[r]{\beta_e}} - \sum_{v\in V} b_v \pi_v,$$
$$f_k(\pi) := f(\pi) + \frac{r}{r+1} \frac{|\pi_{v_1} - \pi_{v_2}|^{\frac{r+1}{r}}}{\sqrt[r]{k}}, \quad k \in \mathbb{N}.$$

Because S is nonempty, convex, and compact,

$$\lim_{k\to\infty} \sup_{\pi\in S} |f(\pi) - f_k(\pi)| \le \lim_{k\to\infty} \frac{(2M)^{\frac{r+1}{r}}}{\sqrt[r]{k}} = 0,$$

and f and f_k are continuous and strictly convex for all $k \in \mathbb{N}$, Lemma 2 implies that $\lim_{k\to\infty} \pi^{(k)} = \pi^*$. □

3. Reduction of Potential-Based Flow Networks

In practical applications, flow is typically injected into the network or discharged from the network only at specific nodes $T \subseteq V$, which we call terminals. The remaining nodes in $V \setminus T$ are called *inner nodes*. Potentials are typically measured or controlled by a network operator only at the terminals. When studying the behavior of a network at the terminals, one, therefore, does not necessarily need to know the complete topology G and the resistances β of the network, but only the function, denoted by $\pi^{\mathcal{N}}_T$, mapping demands at the terminals to the corresponding potentials at the terminals fulfilling these demands. In particular, in certain cases, it is possible to represent the same function $\pi^{\mathcal{N}}_T$ by a (much) smaller network. Such reductions to smaller equivalent networks provide important information about fundamental properties of a given network and help characterize the minimum amount of information necessary to represent the function $\pi^{\mathcal{N}}_T$. Furthermore, for practical applications, representing a possibly large network by a considerably smaller equivalent network is very useful in order to reduce computation time and memory requirement.

We point out that our results are also relevant in the more general and realistic case with lower and upper bounds on node potentials: as the maximum and minimum pressure in such networks must always occur at terminal nodes (sources and sinks, respectively), such bounds are fulfilled in one network if and only if they are fulfilled in another equivalent network. Similarly, bounds on the inflow and outflow at terminals are also covered by our results. However, bounds on the flow of edges or potential of inner nodes in the network cannot be covered by our notion of equivalence because they are not preserved under our notion of equivalence.

The goal of this section is to understand which operations on the network $\mathcal{N}$ leave the function $\pi^{\mathcal{N}}_T$ unchanged and under which conditions $\pi^{\mathcal{N}}_T$ can be represented by a network that is smaller than $\mathcal{N}$. To this end, denote by

$B(V,T) := \{b \in B : b_v = 0 \text{ for all } v \in V \setminus T\}$ the set of demand vectors in B with zero demand at inner nodes, and let $B(T) := \{d \in \mathbb{R}^T : \sum_{v\in T} d_v = 0\}$ be the set of balanced demand vectors on the terminals T. Moreover, let $\pi_T^{\mathcal{N}} : B(T) \to \mathbb{R}^T$ be the restriction of the function $\pi^{\mathcal{N}}$ to the set of terminals T; that is, for $d \in B(T)$, define $\pi_T^{\mathcal{N}}(d) := (\pi_v^{\mathcal{N}}(b))_{v\in T}$, where $b = (b_v)_{v\in V} \in B(V,T)$ is the demand vector with $b_v = d_v$ for all terminals $v \in T$ and $b_v = 0$ for all inner nodes $v \in V \setminus T$.

Definition 1. Given two networks $\mathcal{N} = (G,\beta,r)$ and $\mathcal{N}' = (G',\beta',r)$ with $G = (V,E)$ and $G' = (V',E')$ and a set of terminal nodes $T \subseteq V \cap V'$, the two networks $\mathcal{N}$ and $\mathcal{N}'$ are equivalent with respect to T if $\pi_T^{\mathcal{N}} = \pi_T^{\mathcal{N}'}$.

If the set of terminals T is clear from the context, we simply say that $\mathcal{N}$ and $\mathcal{N}'$ are equivalent. Given terminals T, it is well-known that, by applying the so-called Kron reduction (cf. Dorfler and Bullo [11], Kron [20]), any network $\mathcal{N} = (G,\beta,1)$ of degree $r = 1$ can be reduced to an equivalent network $\mathcal{N}'$ consisting only of the terminals T and edges connecting them.

Theorem 1 (Dorfler and Bullo [11], Kron [20]). *Let $\mathcal{N} = (G,\beta,r)$ be a potential-based network of degree $r = 1$ and $T \subseteq V$ be its set of terminals. Then, there exists a set of edges $E' \subseteq T \times T$ with resistances $\beta' \in R_{>0}^{E'}$ such that $\mathcal{N}$ is equivalent to the network $\mathcal{N}' = ((T,E'),\beta',1)$.*

Given Theorem 1, it is a natural question whether networks of degree $r \neq 1$ can also be reduced to an equivalent network of (considerably) smaller size. In Section 5, we give a negative answer to this question. Specifically, we show that, in general, it is not possible to reduce a network of degree $r = 2$ with at least three terminals to a network whose size only depends on the number of terminals. However, under certain additional assumptions, it is possible to reduce the size of a network, which we investigate in the following.

It is shown by Groß et al. [14] that, if a network has only two terminals, it can be reduced to an equivalent network consisting of a single edge only. To make this statement more precise, we introduce the following concepts. For a network $\mathcal{N}$ and two nodes $s, t \in V$, a flow $x \in \mathbb{R}^E$ is called an *s-t flow* if $\mathrm{bal}_v(x) = 0$ for all $v \in V \setminus \{s,t\}$ and $\mathrm{bal}_s(x) = -\mathrm{bal}_t(x) \geq 0$. We call $\mathrm{val}(x) := \mathrm{bal}_s(x)$ the *value* of flow x.

For any node $v \in V$, let $\chi_v \in \mathbb{R}^V$ be the standard unit vector in $\mathbb{R}^V$ corresponding to v, that is, $(\chi_v)_u = 1$ if $u = v$, and $(\chi_v)_u = 0$ otherwise. Furthermore, define $\chi_{u,v} := \chi_u - \chi_v \in B$.

Definition 2. For any two nodes $s, t \in V$, the effective resistance $R_{s,t}^{\mathcal{N}}$ between s and t is defined to be the difference of the potentials at s and t when sending a flow of value one from s to t, that is,

$$R_{s,t}^{\mathcal{N}} := \pi_s^{\mathcal{N}}(\chi_{s,t}) - \pi_t^{\mathcal{N}}(\chi_{s,t}) \geq 0.$$

Theorem 2 (Groß et al. [14]). *Let $\mathcal{N} = (G,\beta,r)$ be a potential-based network and $T = \{s,t\} \subseteq V$ be the set of terminals. Then, $\mathcal{N}$ is equivalent to the network $\mathcal{N}' = (G',\beta',r)$, where the graph $G' = (T,\{(s,t)\})$ consists of a single edge between the two terminals s and t and $\beta'_{s,t} = R_{s,t}^{\mathcal{N}}$.*

In the following, we give a sufficient condition under which a network of arbitrary degree r can be reduced to a network without inner nodes. To this end, for a digraph $G = (V,E)$ and the set of terminals $T \subset V$ let

$$N_T(v) := \{t \in T : \exists\, v\text{-}t\text{-path which does not contain any terminal except } t\},$$

for all inner nodes $v \in V \setminus T$. Notice that the v-t path in the definition of $N_T(v)$ may use edges in a forward or backward direction. We call $N_T(v)$ the *terminal neighborhood* of v. As a preparation, we show that we can restrict our attention to networks with $|N_T(v)| \geq 2$ for all inner nodes $v \in V \setminus T$.

Lemma 4. *Let $\mathcal{N} = (G,\beta,r)$ be a potential-based network and $v \in V \setminus T$ be an inner node with $|N_T(v)| = 1$. Then, for every demand vector $b \in B(V,T)$, we have $x_e^{\mathcal{N}}(b) = 0$ for all edges $e \in \delta(v)$.*

Proof. Let $b \in B(V,T)$ be arbitrary. Let $t \in N_T(v)$ and define $U = \{u \in V \setminus T : N_T(u) = \{t\}\}$. Under the conditions of the lemma, t is a cut-vertex whose removal separates U from the remainder of the graph. It is without loss of generality to assume that $v_0 \notin U$. Let G' be the subgraph induced by $V' = V \setminus U$, and let b' be the restriction of b to V'. There is a unique potential vector $\pi' \in \Pi_{v_0}$ and a corresponding flow x' in G' satisfying b'. Setting $\pi_u = \pi'_t$ for all $u \in U$ and $\pi_v = \pi'_v$ for all $v \in V'$ as well as $x_e = 0$ for all e with both endpoints in $U \cup \{t\}$ and $x_e = x'_e$ otherwise, we obtain a pair (π,x) of potentials and flows satisfying (E) and the balance vector b. Because the mapping from balance vectors to flows is unique, the claim follows. □

As a consequence of Theorem 2, if a part of a network is isolated between two terminals t_1 and t_2, that is, it is not possible to leave this part without passing t_1 or t_2, then this part can be substituted by a single edge between t_1 and t_2.

Lemma 5. *Let $\mathcal{N}$ be a network and $T \subset V$ the set of terminals. Let $t_1, t_2 \in T$ be two distinct terminals, and assume that $U := \{v \in V \setminus T : N_T(v) = \{t_1,t_2\}\}$ is nonempty. Let $\mathcal{N}^*$ be the subnetwork of $\mathcal{N}$ induced by nodes $V^* = \{t_1,t_2\} \cup U$. Then,*

$\mathcal{N}$ is equivalent to the network $\mathcal{N}'$ that results from $\mathcal{N}$ by removing U and all edges incident to nodes in U and adding one edge from t_1 to t_2 with resistance $\beta_{t_1,t_2} = R^{\mathcal{N}^*}_{t_1,t_2}$.

Proof. Let $b \in B(V,T)$ be arbitrary. We show that $\pi^{\mathcal{N}}_v(b) = \pi^{\mathcal{N}'}_v(b)$ for all $v \in T$ implying that $\mathcal{N}$ and $\mathcal{N}'$ are equivalent. Assume without loss of generality that $\pi^{\mathcal{N}}_{t_1}(b) \geq \pi^{\mathcal{N}}_{t_2}(b)$, and let f be the amount of flow sent from t_1 to t_2 via nodes in U. Let π^* and x^* be the restrictions of $\pi^{\mathcal{N}}(b)$ and $x^{\mathcal{N}}_e(b)$ to the subnetwork $\mathcal{N}^*$, and note that, for this subnetwork, we have the balance vector b^* defined as $b^*_{t_1} = f$, $b^*_{t_2} = -f$ and $b^*_v = 0$, otherwise. Because, by Theorem 2, $\mathcal{N}^*$ is equivalent to a network consisting of a single edge between t_1 and t_2 with resistance $R^{\mathcal{N}^*}_{t_1,t_2}$, we have

$$\pi^{\mathcal{N}}_{t_1}(b) - \pi^{\mathcal{N}}_{t_2}(b) = \pi^*_{t_1} - \pi^*_{t_2} = \operatorname{sign}(f) R^{\mathcal{N}^*}_{t_1,t_2} |f|^r. \tag{4}$$

Finally, consider the network $\mathcal{N}'$ with node set $V' = V \setminus U$ and edge set E' that results from $\mathcal{N}$ by removing U and all edges incident to nodes in U and adding one edge e' with resistance $R^{\mathcal{N}^*}_{t_1,t_2}$. Moreover, define potentials $\pi' \in \mathbb{R}^{V'}$ by $\pi'_v = \pi^{\mathcal{N}}_v(b)$ for all $v \in V'$, let $x' \in \mathbb{R}^{E'}$ with $x'_e = x^{\mathcal{N}}_e(b)$ for all $e \in E' \setminus \{e'\}$ and $x'_{e'} = f$, and let $b' \in \mathbb{R}^{V'}$ be defined by $b'_v = b_v$ for all $v \in V'$. Then, because of Equation (4), π', x', and b' fulfill equation system (E), and thus, $\pi' = \pi^{\mathcal{N}'}(b')$. In particular, we have $\pi^{\mathcal{N}}_v(b) = \pi^{\mathcal{N}'}_v(b')$ for all $v \in T$, and hence, $\pi^{\mathcal{N}}_T = \pi^{\mathcal{N}'}_T$, showing that $\mathcal{N}$ and $\mathcal{N}'$ are equivalent with respect to T. □

Applying Lemma 5, we give a sufficient condition under which a network can be reduced to a network without inner nodes.

Theorem 3. *Let $\mathcal{N} = (G,\beta,r)$ be a network and $T \subset V$ be a set of terminals, and assume that $|N_T(v)| \leq 2$ for every inner node $v \in V \setminus T$. Then, $\mathcal{N}$ is equivalent to a network $\mathcal{N}' = (G',\beta',r)$, where $G' = (T,E')$ with $E' \subseteq T \times T$.*

Proof. If $|N_T(v)| = 1$ for an inner node $v \in V \setminus T$, by Lemma 4, there is never any flow on the edges incident to v. Therefore, v and $\delta(v)$ can be removed from the network without changing $\pi^{\mathcal{N}}_T$. We can, thus, assume without loss of generality that $|N_T(v)| = 2$ for all inner nodes $v \in V \setminus T$.

Label the terminals by $T = \{t_1,\dots,t_\ell\}$. Then, partition the set of inner nodes as $V \setminus T = \cup_{i,j\in[\ell]:i<j} V_{i,j}$, where $V_{i,j} := \{v \in V \setminus T : N_T(v) = \{t_i,t_j\}\}$. Therefore, by Lemma 5, for all $i,j \in [\ell]$ with $i < j$ and $V_{i,j} \neq \emptyset$, we can substitute the nodes in $V_{i,j}$ and their incident edges by one single edge from t_i to t_j, yielding an equivalent network without inner nodes. □

We proceed to give two equivalent characterizations of networks that can be reduced to a path. The first characterization is based on the tightness of the triangle inequality of the effective resistances, whereas the second one exploits the structure of the underlying graph. To this end, for three distinct nodes s, t, $v \in V$, we say that v separates s and t if every (undirected) path from s to t contains v.

Theorem 4. *Let $\mathcal{N} = (G,\beta,r)$ be a network and $T = \{t_1,\dots,t_\ell\} \subseteq V$ be the set of $\ell \geq 3$ terminals. Then, the following statements are equivalent:*

a. *$\mathcal{N}$ is equivalent to the network $\mathcal{N}' = (G',\beta',r)$, where $G' = (T,\{(t_i,t_{i+1}) : i \in [\ell-1]\})$ is a path and $\beta'_{t_i,t_{i+1}} = R^{\mathcal{N}}_{t_i,t_{i+1}}$ for all $i \in [\ell-1]$.*

b. *We have $R^{\mathcal{N}}_{t_i,t_k} = R^{\mathcal{N}}_{t_i,t_j} + R^{\mathcal{N}}_{t_j,t_k}$ for all $i,j,k \in [\ell]$ with $i < j < k$.*

c. *t_j separates t_i and t_k for all $i,j,k \in [\ell]$ with $i < j < k$.*

We prove Theorem 4 at the end of Section 4.

It is interesting to recall in this context that any two parallel edges or any two edges in series that are connected by an inner node of degree two can be substituted by a single edge; see Groß et al. [14]. We state these results in the following two lemmas.

Lemma 6 (Groß et al. [14]). *Let $\mathcal{N} = (G,\beta,r)$ be a network and $T \subset V$ be a set of terminals. Assume $v \in V \setminus T$ is an inner node with degree $|\delta(v)| = 2$, say $\delta(v) = \{e_1,e_2\}$, and let u and w be the nodes adjacent to v. Let $\mathcal{N}'$ be the network that results from removing the edges e_1 and e_2 and the node v and adding an edge e from u to w with resistance $\beta'_e = \beta_{e_1} + \beta_{e_2}$. Then, $\mathcal{N}$ and $\mathcal{N}'$ are equivalent.*

Lemma 7 (Groß et al. [14]). *Let $\mathcal{N} = (G,\beta,r)$ be a network, and $T \subseteq V$ be a set of terminals. Let u, $v \in V$ and assume that e_1, $e_2 \in E$ are two parallel or antiparallel edges both connecting u and v. Let $\mathcal{N}'$ be the network that results from $\mathcal{N}$ by substituting the two edges e_1 and e_2 by a single edge e from u to v with edge resistance*

$$\beta'_e = \frac{\beta_{e_1}\beta_{e_2}}{\left(\sqrt[r]{\beta_{e_1}} + \sqrt[r]{\beta_{e_2}}\right)^r}.$$

Then, $\mathcal{N}$ and $\mathcal{N}'$ are equivalent.

4. Properties of Effective Resistances

In this section, we prove several basic properties of the effective resistances $R^{\mathcal{N}} = (R^{\mathcal{N}}_{s,t})_{s,t\in V}$, which turn out to be essential when deriving conditions under which two networks $\mathcal{N}$ and $\mathcal{N}'$ are equivalent. To that end, we make use of a special case of Danskin's theorem (see Danskin [10] or Bertsekas [2, section B.5] adapted to our setting.

Lemma 8 (Bertsekas [2], Danskin [10]). *Let $Y \subseteq \mathbb{R}^n$ be an open set and $Z \subset \mathbb{R}^m$ be a convex compact set, and let $\phi : Y \times Z \to \mathbb{R}$ be a differentiable function. Assume that the function $\phi(\cdot, z)$ is linear for every $z \in Z$ and that $\phi(y, \cdot)$ is strictly convex for every $y \in Y$. For $y \in Y$, let $z^*(y) := \arg\min\{\phi(y,z) : z \in Z\}$. Then, the function $f : Y \to \mathbb{R}$, $f(y) = \min\{\phi(y,z) : z \in Z\}$ is differentiable, and it holds that*

$$\frac{\partial f(y)}{\partial y_i} = \left.\frac{\partial \phi(y,z)}{\partial y_i}\right|_{z=z^*(y)} \quad \text{for all } i \in [n].$$

We also need the following lemma, which establishes bounds on node potentials for s-t flows. Recall that all the networks we consider are assumed to be (weakly) connected.

Lemma 9. *Let $\mathcal{N}$ be a network, $x \in \mathbb{R}^E$ an s-t flow, and $\pi \in \mathbb{R}^V$ be such that $\operatorname{sign}(x_e) = \operatorname{sign}(\pi_u - \pi_v)$ for all $e = (u,v) \in E$. Then, $\pi_s \geq \pi_v \geq \pi_t$ for all $v \in V$. Moreover, if $\operatorname{val}(x) > 0$ and $w \in V \setminus \{s,t\}$ lies on an s-t path, then $\pi_s > \pi_w > \pi_t$.*

Proof. Let $\bar{\pi} := \max_{v\in V}\pi_v$ and assume by contradiction that $\pi_s < \bar{\pi}$. Let $\bar{V} := \{v \in V : \pi_v = \bar{\pi}\}$. Because $s \in V \setminus \bar{V}$ and G is connected, there are two nodes $w_1 \in \bar{V}$ and $w_2 \in V \setminus \bar{V}$ that are adjacent to each other. We show that $\operatorname{bal}_{w_1}(x) > 0$, contradicting the fact that x is an s-t flow. By possibly changing the orientation of edges, we can assume without loss of generality that every edge incident to w_1 leaves w_1, that is, $\delta(w_1) = \delta^+(w_1)$. By the maximality of π_{w_1} and $\operatorname{sign}(x_e) = \operatorname{sign}(\pi_u - \pi_v)$ for all $e = (u,v) \in E$, it follows that $x_{(w_1,w_2)} > 0$ and $x_e \geq 0$ for all $e \in \delta^+(w_1)$. Thus,

$$\operatorname{bal}_{w_1}(x) = \sum_{e\in\delta^+(w_1)} x_e \geq x_{(w_1,w_2)} > 0.$$

The lower inequality $\pi_v \geq \pi_t$ follows similarly by considering $\min_{v\in V}\pi_v$.

For the second part, assume that $\operatorname{val}(x) > 0$ and $w \in V \setminus \{s,t\}$ lies on an s-t path. As shown, the node s is contained in $\bar{V}$. Because x has positive value, we have $\pi_s > \pi_t$, and therefore, $V \setminus \bar{V} \neq \emptyset$. Assume by contradiction that $w \in \bar{V}$. Because w lies on an s-t path, there exist two adjacent nodes u and v on this path with $u \in \bar{V} \setminus \{s\}$ and $v \in V \setminus \bar{V}$. By the same argument as earlier, it follows that $\operatorname{bal}_u(x) > 0$, a contradiction. Hence, $\pi_s > \pi_w$. By a similar argument, we get $\pi_t < \pi_w$. □

We are now in a position to prove essential properties of the effective resistances of a network. Specifically, we show that they can be obtained as optimal values of convex optimization problems, and we give formulas for their derivatives with respect to edge resistances.

Proposition 1. *Let $\mathcal{N} = (G, \beta, r)$ be a potential-based flow network. The effective resistances $R^{\mathcal{N}}$ have the following properties:*

a. *For any two nodes $s, t \in V$, it holds that*

$$R^{\mathcal{N}}_{s,t} = \sum_{e\in E} \beta_e |x^*_e|^{r+1},$$

where $x^* := x^{\mathcal{N}}(\chi_{s,t})$. In particular, the effective resistance $R^{\mathcal{N}}_{s,t}$ equals the optimal value of the following optimization problem

$$\begin{aligned} \text{minimize} \quad & \sum_{e\in E} \beta_e |x_e|^{r+1} \\ \text{subject to} \quad & \Gamma x = \chi_{s,t}. \end{aligned} \tag{R}$$

b. *For all edges $e \in E$, we have*

$$\frac{\partial R^{\mathcal{N}}_{s,t}}{\partial \beta_e} = |x^*_e|^{r+1}.$$

c. *Let s, t, v_1, $v_2 \in V$, and let $\mathcal{N}'$ be the network that results from $\mathcal{N}$ by inserting an (additional) edge e' from v_1 to v_2 with an arbitrary resistance $\beta_{e'} > 0$. Then, $R^{\mathcal{N}'}_{s,t} \leq R^{\mathcal{N}}_{s,t}$. Moreover, it holds that $R^{\mathcal{N}'}_{s,t} = R^{\mathcal{N}}_{s,t}$ if and only if $\pi^{\mathcal{N}}_{v_1}(\chi_{s,t}) = \pi^{\mathcal{N}}_{v_2}(\chi_{s,t})$.*

Proof. Let $s, t \in V$. To simplify notation, we set $x^* := x^{\mathcal{N}}(\chi_{s,t})$ and $\pi^* := \pi^{\mathcal{N}}(\chi_{s,t})$. We start by sorting the nodes in decreasing order of their potentials. To that end, let $V = V_1 \cup \cdots \cup V_\ell$ be a partition of V such that $\pi^*_u = \pi^*_v$ for all $u, v \in V_i$, $i \in [\ell]$, and $\pi^*_u > \pi^*_v$ for all $u \in V_i$, $v \in V_j$ with $i < j$. Note that, because x^* is an s-t flow, we have $s \in V_1$ and $t \in V_\ell$ by Lemma 9. Denote by π^*_i the potential of the nodes in V_i, $i \in [\ell]$.

By possibly changing orientations of edges and introducing artificial inner nodes subdividing edges into multiple segments, using Lemma 6 in the opposite direction, we can assume without loss of generality that each edge $e \in E$ is either contained in $V_i \times V_i$ for some $i \in [r]$ or in $V_i \times V_{i+1}$ for some $i \in [\ell - 1]$; see Figure 2 for an illustration. Because x^* and π^* fulfill the equation system (E), $x^*_e = 0$ for all $e \in E \cap (V_i \times V_i)$, $i \in [\ell]$, and the flows in every s-t cut sum up to one, we have

$$\begin{aligned}\sum_{e\in E} \beta_e |x^*_e|^{r+1} &= \sum_{i=1}^{\ell-1} \sum_{e\in E\cap(V_i\times V_{i+1})} \operatorname{sign}(x^*_e)\beta_e |x^*_e|^r x^*_e \\ &= \sum_{i=1}^{\ell-1} (\pi^*_i - \pi^*_{i+1}) \sum_{e\in E\cap(V_i\times V_{i+1})} x^*_e \\ &= \sum_{i=1}^{\ell-1} (\pi^*_i - \pi^*_{i+1}) = \pi^*_s - \pi^*_t = R^{\mathcal{N}}_{s,t}.\end{aligned}$$

Lemma 1(i) then implies that $R^{\mathcal{N}}_{s,t}$ is the optimal value of the convex program (R), establishing part (a).

To prove part (b), we denote $x^*(\beta) := x^{\mathcal{N}}(\chi_{s,t})$ to emphasize that the flow $x^{\mathcal{N}}(\chi_{s,t})$ depends on the edge resistances $\beta \in \mathbb{R}^E_{>0}$. Assume without loss of generality that every edge $e \in E$ is oriented such that $x^*_e(\beta) \geq 0$. Because $x^*(\beta)$ is the optimal solution to (R), $x^*(\beta)$ does not send positive flow along any directed cycle as, otherwise, reducing the flow along this cycle decreases the objective value of this s-t flow, contradicting the optimality of $x^*(\beta)$. Hence, because $x^*(\beta)$ is an s-t flow of value one, we have that $|x^*_e(\beta)| \leq 1$ for all $e \in E$.

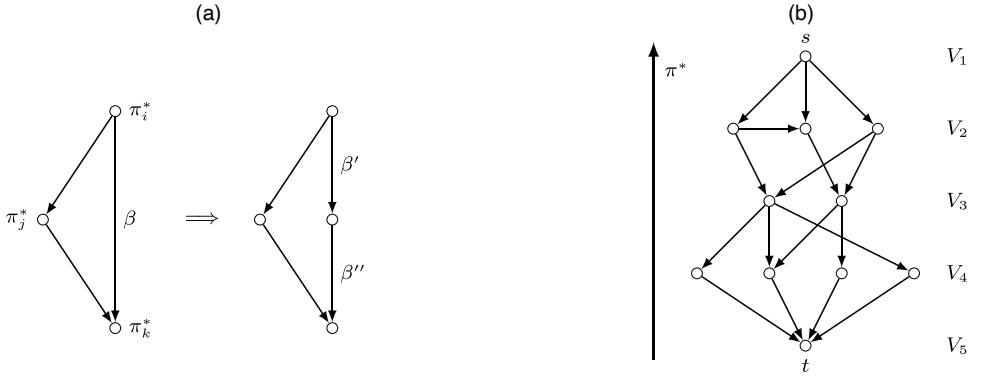
In order to apply Lemma 8, let $Z := \{x \in \mathbb{R}^E : Ax = \chi_{s,t}, |x_e| \leq 1 \ \forall\, e \in E\}$, and $\phi : R^E_{>0} \times Z \to \mathbb{R}$, $\phi(\beta, x) = \sum_{e\in E} \beta_e |x_e|^{r+1}$. Note that Z is convex and compact, $\phi(\cdot, x)$ is linear for every $x \in Z$, and $\phi(\beta, \cdot)$ is strictly convex for every $\beta \in R^E_{>0}$. For all edge resistances $\beta \in R^E_{>0}$ of the network $\mathcal{N}$, we have $x^*(\beta) = \arg\min\{\phi(\beta, x) : x \in Z\}$. Moreover, according to part (a), $R^{\mathcal{N}}_{s,t} = \phi(\beta, x^*(\beta)) = \min\{\phi(\beta, x) : x \in Z\}$. Therefore, Lemma 8 implies that

$$\frac{\partial R^{\mathcal{N}}_{s,t}}{\partial \beta_e} = \frac{\phi(\beta, x)}{\partial \beta_e}\bigg|_{x=x^*(\beta)} = |x^*_e(\beta)|^{r+1}, \qquad \text{for every edge } e \in E.$$

For part (c), let $x := x^{\mathcal{N}}(\chi_{s,t})$, $x' := x^{\mathcal{N}'}(\chi_{s,t})$, $\pi := \pi^{\mathcal{N}}(\chi_{s,t})$, and let $\pi' := \pi^{\mathcal{N}'}(\chi_{s,t})$. By definition, x, π, and $\chi_{s,t}$ fulfill equation system (E) with respect to $\mathcal{N}$, and x', π', and $\chi_{s,t}$ fulfill equation system (E) with respect to $\mathcal{N}'$. Define the flow $\tilde{x} \in \mathbb{R}^{E'}$ by $\tilde{x}_e = x_e$ for all $e \in E$ and $\tilde{x}_{e'} = 0$. Then, $\tilde{x}$ is an s-t flow of value one in the network $\mathcal{N}'$, and thus, by part (a), we have

$$R^{\mathcal{N}'}_{s,t} = \sum_{e=(u,v)\in E'} \beta_e |x'_e|^{r+1} \leq \sum_{e=(u,v)\in E'} \beta_e |\tilde{x}_e|^{r+1} = \sum_{e=(u,v)\in E} \beta_e |x_e|^{r+1} = R^{\mathcal{N}}_{s,t}.$$

Figure 2. Illustration of the proof of Proposition 1. (a) Subdivision of edges. (b) Sorting nodes by their potentials.



In particular, if $R^{\mathcal{N}'}_{s,t} = R^{\mathcal{N}}_{s,t}$, then

$$\sum_{e=(u,v)\in E'} \beta_e |x'_e|^{r+1} = \sum_{e=(u,v)\in E'} \beta_e |\tilde{x}_e|^{r+1},$$

and by the uniqueness of the optimum solution to (R) with respect to network $\mathcal{N}'$, we have $x' = \tilde{x}$, and thus, $x'_{e'} = 0$ and $\pi'_{v_1} = \pi'_{v_2}$. But, then, $(x'_e)_{e\in E}$, π', $\chi_{s,t}$ fulfill equation system (E) with respect to the network $\mathcal{N}$, and by uniqueness, we obtain $\pi' = \pi$, and thus, $\pi_{v_1} = \pi_{v_2}$.

On the other hand, if $\pi_{v_1} = \pi_{v_2}$, then $\tilde{x}$, π, and $\chi_{s,t}$ fulfill equation system (E) with respect to network $\mathcal{N}'$, and hence, by uniqueness, we get $\tilde{x} = x'$. It follows that

$$R^{\mathcal{N}'}_{s,t} = \sum_{e\in E'} \beta_e |x'_e|^{r+1} = \sum_{e\in E'} \beta_e |\tilde{x}_e|^{r+1} = \sum_{e\in E} \beta_e |x_e|^{r+1} = R^{\mathcal{N}}_{s,t}.$$

This concludes the proof. □

We proceed to show that the effective resistances in potential-based flow networks form a metric, generalizing the corresponding result by Klein and Randić [18] for the special case of networks of degree $r = 1$. Furthermore, we establish a relation between the tightness of the triangle inequality and the graph structure of the underlying network.

Proposition 2. *Let $\mathcal{N}$ be a network. The effective resistances fulfill the following properties:*

i. *The function $V \times V \to \mathbb{R}_{\geq 0}$, $(s,t) \mapsto R^{\mathcal{N}}_{s,t}$, mapping pairs of nodes to the effective resistance between them, is a metric.*

ii. *For any three distinct nodes $s, u, t \in V$, it holds that $R^{\mathcal{N}}_{s,t} = R^{\mathcal{N}}_{s,u} + R^{\mathcal{N}}_{u,t}$ if and only if u separates s and t.*

Proof. We first show that the effective resistances form a metric. The only nontrivial property to show is the triangle inequality. Let $s, t, u \in V$ be three distinct nodes. We need to prove that $R^{\mathcal{N}}_{s,t} \leq R^{\mathcal{N}}_{s,u} + R^{\mathcal{N}}_{u,t}$. As in the proof of Proposition 1(a), let $x^* := x^{\mathcal{N}}(\chi_{s,t})$, $\pi^* := \pi^{\mathcal{N}}(\chi_{s,t})$, and let $V = V_1 \cup \cdots \cup V_\ell$ be the partition of the nodes sorted in decreasing order with respect to the potential vector π^*; see Figure 2. Recall that we can assume without loss of generality that every edge $e \in E$ is contained in $V_j \times V_{j+1}$ for some $j \in [\ell - 1]$ or in $V_j \times V_j$ for some $j \in [\ell]$. By Lemma 9, $s \in V_1$ and $t \in V_\ell$. Let $i \in [\ell]$ be such that $u \in V_i$. If $i = 1$, then $\pi^*_u = \pi^*_s$, and by Lemma 9, there exists no s-t path that contains u, implying that every u-t path contains s. Let E_u be the set of edges that can be reached from u without passing s, and define the flow $\tilde{x} \in \mathbb{R}^E$ by $\tilde{x}_e = 0$ for all $e \in E_u$ and $\tilde{x}_e = x^{\mathcal{N}}_e(\chi_{u,t})$ for all $e \in E \setminus E_u$. Then, $\tilde{x}$ is an s-t flow of value one, and, by Proposition 1(a), we have

$$R^{\mathcal{N}}_{s,t} \leq \sum_{e\in E} \beta_e |\tilde{x}_e|^{r+1} \leq \sum_{e\in E} \beta_e |x^{\mathcal{N}}_e(\chi_{u,t})|^{r+1} = R^{\mathcal{N}}_{u,t}.$$

Similarly, if $i = \ell$, we have $R^{\mathcal{N}}_{s,t} \leq R^{\mathcal{N}}_{s,u}$. Therefore, in the following, we can assume that $1 < i < \ell$. Denote $E_i^- := E \cap (V_{i-1} \times V_i)$ and $E_i^+ := E \cap (V_i \times V_{i+1})$.

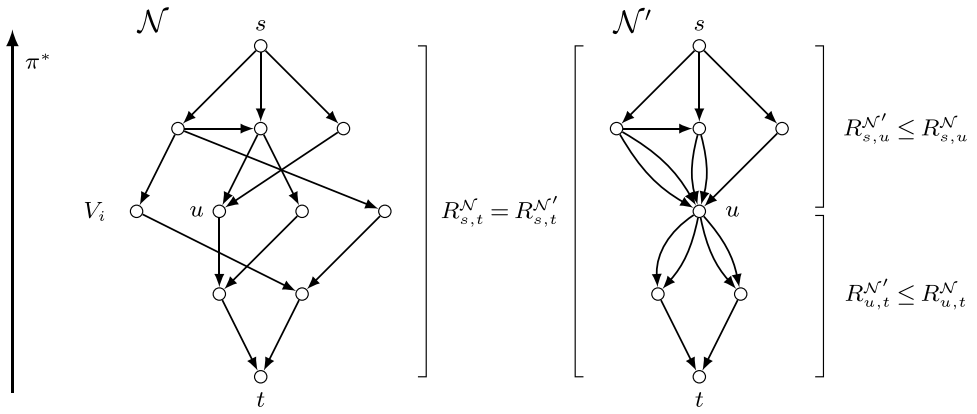
Claim 2. *The effective resistance $R^{\mathcal{N}}_{s,t}$ is the optimal value of the optimization problem*

$$\begin{aligned} \text{minimize} \quad & \sum_{e\in E} \beta_e |x_e|^{r+1} \\ \text{subject to} \quad & \mathrm{bal}_v(x) = (\chi_{s,t})_v \qquad \text{for all } v \in V \setminus V_i, \\ & \sum_{e\in E_i^-} x_e = 1, \\ & \sum_{e\in E_i^+} x_e = 1. \end{aligned} \tag{5}$$

Proof. Let $\mathcal{N}' = (G', \beta, r)$ with $G' = (V', E)$ be the network resulting from $\mathcal{N}$ by merging all nodes in V_i into u; see Figure 3 for an illustration. Thus, we have $V' = (V \setminus V_i) \cup \{u\}$. We first show that $R^{\mathcal{N}}_{s,t} = R^{\mathcal{N}'}_{s,t}$. To this end, let $\pi' \in \mathbb{R}^{V'}$ be defined by $\pi'_v = \pi^*_v$ for all $V \setminus V_i$ and $\pi'_u = \pi^*_u$, and let $\chi'_{s,t} \in \mathbb{R}^{V'}$ be defined by $(\chi'_{s,t})_s = 1$, $(\chi'_{s,t})_t = -1$ and $(\chi'_{s,t})_v = 0$ for all $v \in V' \setminus \{s,t\}$. Then, because x^*, π^*, and $\chi_{s,t}$ fulfill equation system (E) with respect to network $\mathcal{N}$ and all nodes in V_i have the same potential, also the same flow x^* together with π' and $\chi'_{s,t}$ fulfill the equation system (E) with respect to network $\mathcal{N}'$. Hence, by Lemma 1, x^* is the optimal solution to the convex program

$$\begin{aligned} \text{minimize} \quad & \sum_{e\in E} \beta_e |x_e|^{r+1} \\ \text{subject to} \quad & \mathrm{bal}'(x) = \chi'_{s,t}, \end{aligned} \tag{6}$$

Figure 3. Merging all nodes with the same potential into one single node does not change the effective resistance between s and t.



where $\mathrm{bal}'(x) \in \mathbb{R}^{V'}$ is the balance vector with respect to flow x in the network $\mathcal{N}'$. By Proposition 1(a), the optimal value of Program (6) equals

$$R^{\mathcal{N}'}_{s,t} = \sum_{e\in E} \beta_e |x^*_e|^{r+1} = \sum_{e\in E} \beta_e |x^{\mathcal{N}}_e(\chi_{s,t})|^{r+1} = R^{\mathcal{N}}_{s,t}. \tag{7}$$

Because every s-t path in $\mathcal{N}'$ passes through node u, for every s-t flow x of value one, we have $\mathrm{bal}'_u(x) = 0$ if and only if the sum of the flows both entering and leaving u equal one. Therefore, the condition $\mathrm{bal}'_u(x) = 0$ in Program (6) can equivalently be substituted by $\sum_{e\in E_i^-} x_e = \sum_{e\in E_i^+} x_e = 1$. Because, furthermore, $\mathrm{bal}'_v(x) = \mathrm{bal}_v(x)$ and $(\chi'_{s,t})_v = (\chi_{s,t})_v$ for all $v \in V \setminus V_i$, Programs (6) and (5) are equivalent and, by Equation (7), have the optimal value $R^{\mathcal{N}}_{s,t}$. This completes the proof of the claim. □

Let E_s be the set of edges connected to nodes in $V_1 \cup \ldots \cup V_{i-1}$ and let E_t be the set of edges connected to nodes in $V_{i+1} \cup \cdots \cup V_\ell$. Note that $E_s \cap E_t = \emptyset$ and in Program (5) there is no coupling between edges in E_s and edges in E_t. Hence, the optimal value of (5), which is $R^{\mathcal{N}}_{s,t}$, equals the sum of the optimal values of the two programs

$$\begin{aligned} \text{minimize} \quad & \sum_{e\in E_s} \beta_e |x_e|^{r+1} \\ \text{subject to} \quad & \mathrm{bal}_v(x) = (\chi_{s,t})_v \qquad \text{for all } v \in \bigcup_{j=1}^{i-1} V_j, \\ & \sum_{e\in E_i^-} x_e = 1, \end{aligned} \tag{8}$$

and

$$\begin{aligned} \text{minimize} \quad & \sum_{e\in E_t} \beta_e |x_e|^{r+1} \\ \text{subject to} \quad & \sum_{e\in E_i^+} x_e = 1, \\ & \mathrm{bal}_v(x) = (\chi_{s,t})_v \qquad \text{for all } v \in \bigcup_{j=i+1}^{\ell} V_j. \end{aligned} \tag{9}$$

Note that, because $u \in V_i$, the constraints of Program (8) are implied by the constraint $\mathrm{bal}(x) = \chi_{s,u}$, and the constraints of Program (9) are implied by the constraint $\mathrm{bal}(x) = \chi_{u,t}$. It follows that the optimal values of (8) and (9) are lower bounds on the optimal values of the two programs

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{e\in E} \beta_e |x_e|^{r+1} & \qquad \text{minimize} \quad \displaystyle\sum_{e\in E} \beta_e |x_e|^{r+1} \\ \text{subject to} & \mathrm{bal}(x) = \chi_{s,u}, & \qquad \text{subject to} \quad \mathrm{bal}(x) = \chi_{u,t}, \end{array}$$

whose optimal values are $R^{\mathcal{N}}_{s,u}$ and $R^{\mathcal{N}}_{u,t}$, respectively, which follows from Proposition 1(a). We conclude that $R^{\mathcal{N}}_{s,t} \le R^{\mathcal{N}}_{s,u} + R^{\mathcal{N}}_{u,t}$.

For the backward direction of part (ii), assume that u separates s and t. Let $E_s \subset E$ be the union of the edges of all s-u paths, and let $E_t \subset E$ be the union of the edges of all u-t paths. Then, $E_s \cap E_t = \emptyset$ and the flow $x^{\mathcal{N}}(\chi_{s,t})$ can

be decomposed into the sum of the s-u flow $x^{(s)}$ and the u-t flow $x^{(t)}$, where

$$x_e^{(s)} = \begin{cases} x_e^{\mathcal{N}}(\chi_{s,t}), & e \in E_s, \\ 0, & e \in E \setminus E_s, \end{cases} \quad \text{and} \quad x_e^{(t)} = \begin{cases} x_e^{\mathcal{N}}(\chi_{s,t}), & e \in E_t, \\ 0, & e \in E \setminus E_t. \end{cases}$$

By Proposition 1(a), it follows that

$$\begin{aligned} R_{s,t}^{\mathcal{N}} &= \sum_{e \in E} \beta_e |x_e^{\mathcal{N}}(\chi_{s,t})|^r \\ &= \sum_{e \in E_s} \beta_e |x_e^{(s)}|^r + \sum_{e \in E_t} \beta_e |x_e^{(t)}|^r \\ &= \sum_{e \in E} \beta_e |x_e^{(s)}|^r + \sum_{e \in E} \beta_e |x_e^{(t)}|^r \\ &\geq \sum_{e \in E} \beta_e |x_e^{\mathcal{N}}(\chi_{s,u})|^r + \sum_{e \in E} \beta_e |x^{\mathcal{N}}(\chi_{u,t})|^r \\ &= R_{s,u}^{\mathcal{N}} + R_{u,t}^{\mathcal{N}}. \end{aligned}$$

The triangle inequality then implies that $R_{s,t}^{\mathcal{N}} = R_{s,u}^{\mathcal{N}} + R_{u,t}^{\mathcal{N}}$.

For the forward direction, assume that $R_{s,t}^{\mathcal{N}} = R_{s,u}^{\mathcal{N}} + R_{u,t}^{\mathcal{N}}$ for three distinct nodes $s, u, t \in V$. Assume by contradiction that there exists an s-t path P that does not contain u. Using Lemma 6 in the opposite direction, subdividing edges into multiple segments, we can assume that P contains a node $w \neq u$ with $\pi_w^{\mathcal{N}}(\chi_{s,t}) = \pi_u^{\mathcal{N}}(\chi_{s,t})$. Let $\mathcal{N}'$ be the network that results from $\mathcal{N}$ by inserting an (additional) edge e' from w to u with an arbitrary resistance $\beta_{e'} > 0$. By Proposition 1(c), we have $R_{s,t}^{\mathcal{N}'} = R_{s,t}^{\mathcal{N}}$. On the other hand, because w is contained in the s-t path P and G is connected, there either exists an s-u path containing w or a u-t path containing w. Without loss of generality, we assume the former. Then, by Lemma 9, we have $\pi_w^{\mathcal{N}}(\chi_{s,u}) > \pi_u^{\mathcal{N}}(\chi_{s,u})$, and hence, by Proposition 1(c), we have $R_{s,u}^{\mathcal{N}'} < R_{s,u}^{\mathcal{N}}$ and $R_{u,t}^{\mathcal{N}'} \leq R_{u,t}^{\mathcal{N}}$. Applying the triangle inequality, we obtain

$$R_{s,t}^{\mathcal{N}'} = R_{s,t}^{\mathcal{N}} = R_{s,u}^{\mathcal{N}} + R_{u,t}^{\mathcal{N}} > R_{s,u}^{\mathcal{N}'} + R_{u,t}^{\mathcal{N}'} \geq R_{s,t}^{\mathcal{N}'},$$

a contradiction. □

Having deepened our understanding of effective resistances and conductivities, we are ready to prove Theorem 4.

Proof of Theorem 4. First, note that (b) and (c) are equivalent by Proposition 2. Assume that (a) holds. Because $\mathcal{N}$ and $\mathcal{N}'$ are equivalent, we have $R_{t_i,t_j}^{\mathcal{N}} = R_{t_i,t_j}^{\mathcal{N}'}$ for all $i, j \in [\ell]$, and because $\mathcal{N}'$ is a path, by Lemma 10, we have

$$R_{t_i,t_k}^{\mathcal{N}'} = R_{t_i,t_j}^{\mathcal{N}'} + R_{t_j,t_k}^{\mathcal{N}'}$$

for all $i, j, k \in [\ell]$ with $i < j < k$, yielding (b).

For the opposite direction, assume that (c) holds. As argued in the proof of Theorem 3, we can assume without loss of generality that $|N_T(v)| \geq 2$ for all inner nodes $v \in V \setminus T$. Then, (c) implies that, for every inner node $v \in V \setminus T$, we have $N_T(v) = \{t_i, t_{i+1}\}$ for some $i \in [\ell - 1]$. Thus, by Lemma 5, for all $i \in [\ell - 1]$, we can remove the set of nodes $V_i := \{v \in V \setminus T : N_T(v) = \{t_i, t_{i+1}\}\}$ and all edges incident to nodes in V_i and instead add one edge from t_i to t_{i+1}, yielding an equivalent network. After removing all parallel edges by applying Lemma 7, we obtain an equivalent path $\mathcal{N}'' = (G'', \beta'', r)$ with $G'' = G'$. By Lemma 10, for every $i \in [\ell - 1]$, we have $\beta''_{t_i,t_{i+1}} = R_{t_i,t_{i+1}}^{\mathcal{N}''} = R_{t_i,t_{i+1}}^{\mathcal{N}}$. Hence, $\mathcal{N}'' = \mathcal{N}'$, showing that (a) holds. □

If the underlying graph of a network $\mathcal{N}$ is a tree, as an immediate consequence of Proposition 1(a), we obtain a simple way to compute the effective resistance between any two nodes in $\mathcal{N}$.

Lemma 10. *Let $\mathcal{N} = (G, \beta, r)$ such that G is a tree. Then, for any two nodes $s, t \in V$, we have $R_{s,t}^{\mathcal{N}} = \sum_{e \in P} \beta_e$, where P is the unique path from s to t.*

5. Nonreducibility of Gas Networks

In this section, we show that, in contrast to networks of degree $r = 1$ (see Theorem 1), networks of degree $r = 2$ with at least three terminals can, in general, not be reduced to a network whose size only depends on the number of terminals. In order to prove this, we exploit properties of the curve $\pi^{\mathcal{N}}(b(t))$, where $b(t)$ is a curve of demand vectors that depends on a one-dimensional parameter t.

We first show that, given a k times continuously differentiable curve of demands $b(t)$, the curve of potentials $\pi^{\mathcal{N}}(b(t))$ is k times continuously differentiable at all parameters t, for which there is a nonzero flow $x_e^{\mathcal{N}}(b(t))$ on every edge e of the network.

Lemma 11. *Let $b : \mathbb{R} \to B$ be a curve that is k times continuously differentiable. Then, at every $t \in \mathbb{R}$ with $x_e^{\mathcal{N}}(b(t)) \neq 0$ for all $e \in E$, the curves $x^{\mathcal{N}}(b(\cdot))$ and $\pi^{\mathcal{N}}(b(\cdot))$ are k times continuously differentiable.*

Proof. Recall that, by Lemma 1, for every $t \in \mathbb{R}$, the vector $\pi^{\mathcal{N}}(b(t))$ is the unique optimal solution to the convex program

$$\begin{aligned} \underset{\pi \in \mathbb{R}^V}{\text{minimize}} \quad & F(\pi, t) := \frac{r}{r+1} \sum_{e=(u,v)\in E} c_e |\pi_u - \pi_v|^{\frac{r+1}{r}} - \sum_{v \in V} b_v(t)\pi_v \\ \text{subject to} \quad & \pi_{v_0} = 0. \end{aligned}$$

Note that, by eliminating the variable π_{v_0}, the problem becomes unconstrained. Let $t^* \in \mathbb{R}$ with $x_e^{\mathcal{N}}(b(t^*)) \neq 0$, and thus, because of Equation (1), $\pi_u^{\mathcal{N}}(b(t^*)) \neq \pi_v^{\mathcal{N}}(b(t^*))$ for all $e = (u,v) \in E$. Then, in a neighborhood of $(\pi^{\mathcal{N}}(b(t^*)), t^*)$, the function F is smooth in π and the gradient of F with respect to π is k times continuously differentiable in t. Furthermore, F is strictly convex in π. It follows from Fiacco [12, corollary 3.2.5] that the function $\pi^{\mathcal{N}}(b(\cdot))$ is k times continuously differentiable at t^*. Because

$$x_e^{\mathcal{N}}(b(t^*)) = c_e |\pi_u^{\mathcal{N}}(b(t^*)) - \pi_v^{\mathcal{N}}(b(t^*))|^{\frac{1}{r}} \operatorname{sign}(\pi_u^{\mathcal{N}}(b(t^*)) - \pi_v^{\mathcal{N}}(b(t^*))) \neq 0,$$

also the function $x^{\mathcal{N}}(b(\cdot))$ is k times continuously differentiable at t^*. □

The condition in Lemma 11 requiring nonzero flow on every edge of the network is essential. As we see later in this section, given a smooth curve of demands $b(t)$, it can happen that the curve of potentials $\pi^{\mathcal{N}}(b(t))$ is not twice continuously differentiable at parameters t at which the flow on one or several edges vanishes. We exploit this fact when proving the nonreducibility of networks of degree $r = 2$. However, as long as the set of edges with zero flow does not contain a cycle, the curve of potentials stays at least once continuously differentiable.

Lemma 12. *Let $b : \mathbb{R} \to B$ be a continuously differentiable curve, and let $t^* \in \mathbb{R}$ be such that the set of edges $\{e \in E : x_e^{\mathcal{N}}(b(t^*)) = 0\}$ does not contain a cycle. Then, the curves $x^{\mathcal{N}}(b(\cdot))$ and $\pi^{\mathcal{N}}(b(\cdot))$ are continuously differentiable at t^*.*

Proof. Recall that $v_0 \in V$ is the node whose potential is fixed to zero. Let $\hat{V} := V \setminus \{v_0\}$, and let $\hat{\pi}^{\mathcal{N}} : B(V,T) \to \mathbb{R}^{\hat{V}}$ be the restriction of $\pi^{\mathcal{N}}$ to $\hat{V}$, that is, $\hat{\pi}^{\mathcal{N}} = (\pi_v^{\mathcal{N}})_{v \in \hat{V}}$. Similarly, let $\hat{b} : \mathbb{R} \to \mathbb{R}^{\hat{V}}$ be the restriction of b to $\pi^{\mathcal{N}}$, and let $\hat{\Gamma} \in \mathbb{R}^{\hat{V} \times E}$ be the incidence matrix Γ of G with the row corresponding to node v_0 removed. Note that, because G is connected, $\hat{\Gamma}$ has full row rank. Because, for every $t \in \mathbb{R}$, the vectors $x^{\mathcal{N}}(b(t))$, $\pi^{\mathcal{N}}(b(t))$, and $b(t)$ fulfill the equation system (E), we have $F(x^{\mathcal{N}}(b(t)), \hat{\pi}^{\mathcal{N}}(b(t)), t) = 0$ for all $t \in \mathbb{R}$, where the function $F : \mathbb{R}^E \times \mathbb{R}^{\hat{V}} \times \mathbb{R} \to \mathbb{R}^E \times \mathbb{R}^{\hat{V}}$ is defined by

$$F(x, \hat{\pi}, t) = \begin{bmatrix} \operatorname{sign}(x) \odot \beta \odot |x|^r - \hat{\Gamma}^\top \hat{\pi} \\ \hat{b}(t) - \hat{\Gamma} x \end{bmatrix}.$$

Consider the partial derivatives

$$J(x) := \begin{bmatrix} \frac{\partial F(x,\hat{\pi},t)}{\partial x} & \frac{\partial F(x,\hat{\pi},t)}{\partial \hat{\pi}} \end{bmatrix} = \begin{bmatrix} r\operatorname{diag}(\beta \odot |x|^{r-1}) & -\hat{\Gamma}^\top \\ -\hat{\Gamma} & 0 \end{bmatrix}.$$

We show that, for $x^* := x^{\mathcal{N}}(b(t^*))$, the matrix $J(x^*)$ is nonsingular. It then follows from the implicit function theorem that the curves $x^{\mathcal{N}}(b(\cdot))$ and $\hat{\pi}^{\mathcal{N}}(b(\cdot))$, and, thus, also $\pi^{\mathcal{N}}(b(\cdot))$ are continuously differentiable at t^*.

Assume by contradiction that there exists $\begin{pmatrix} x \\ \hat{\pi} \end{pmatrix} \in (\mathbb{R}^E \times \mathbb{R}^{\hat{V}}) \setminus \{0\}$ such that $J(x^*)\begin{pmatrix} x \\ \hat{\pi} \end{pmatrix} = 0$. Defining $\pi \in \mathbb{R}^V$ by $\pi_v = \hat{\pi}_v$ for all $v \in \hat{V}$ and $\pi_{v_0} = 0$ and setting $D := r\operatorname{diag}(\beta \odot |x^*|^{r-1})$, we obtain $Dx = \hat{\Gamma}^\top \hat{\pi} = \Gamma^\top \pi$ and $\hat{\Gamma} x = 0$. Note that, because $x \in \mathbb{R}^E$, x can be interpreted as a flow in G. Because $\hat{\Gamma} x = 0$ and

$$(\Gamma x)_{v_0} = \operatorname{bal}_{v_0}(x) = -\sum_{v \in \hat{V}} \operatorname{bal}_v(x) = -\sum_{v \in \hat{V}} (\hat{\Gamma} x)_v = 0,$$

it follows that $\Gamma x = 0$, and thus, the flow x is a circulation in G. Note that $x \neq 0$ as, otherwise, $\hat{\Gamma}^\top \hat{\pi} = Dx = 0$, and thus, because $\hat{\Gamma}^\top$ has full column rank, $\hat{\pi} = 0$, contradicting $\begin{pmatrix} x \\ \hat{\pi} \end{pmatrix} \neq 0$. By possibly changing orientations of edges, we can assume without loss of generality that $x \geq 0$. Hence, there exists a directed cycle $C \subseteq E$ with $x_e > 0$ for all $e \in C$. Denote the nodes of the cycle C by $v_1, \ldots, v_k, v_{k+1} = v_1$. Because, by assumption, the set $\{e \in E : x_e^* = 0\}$ does not contain a cycle, by the definition of the matrix D, we have that $D_{e,e} > 0$ for at least one $e \in C$. Thus, we have $\sum_{e \in C} (Dx)_e > 0$. On the other hand, because of $Dx = \Gamma^\top \pi$, it holds that

$$\sum_{e \in C} (Dx)_e = \sum_{e \in C} (\Gamma^\top \pi)_e = \sum_{i=1}^{k} (\pi_{v_i} - \pi_{v_{i+1}}) = \pi_{v_1} - \pi_{v_{k+1}} = 0,$$

a contradiction. □

We are ready to prove that, in general, networks of degree $r = 2$ cannot be reduced to smaller networks whose size only depends on the number of terminals. Recall that, for a network $\mathcal{N} = (G, \beta, r)$ and a set of terminals $T \subseteq V$, the function $\pi_T^{\mathcal{N}} : B(T) \to \mathbb{R}^T$ is the restriction of $\pi^{\mathcal{N}}$ to the terminals T, and $B(T) = \{d \in \mathbb{R}^T : \sum_{v \in T} d_v = 0\}$. For a curve $d : \mathbb{R} \to B(T)$ define the set

$$U_T^{\mathcal{N}}(d) := \{t \in \mathbb{R} : \pi_T^{\mathcal{N}}(d(\cdot)) \text{ is not twice continuously differentiable at } t\}.$$

By Definition 1, if two networks $\mathcal{N}$ and $\mathcal{N}'$ are equivalent with respect to T, then $\pi_T^{\mathcal{N}} = \pi_T^{\mathcal{N}'}$, and thus, in particular, $U_T^{\mathcal{N}}(d) = U_T^{\mathcal{N}'}(d)$ for all curves $d : \mathbb{R} \to B(T)$.

The following proof consists of two steps. We first show that, for general networks, the set $U_T^{\mathcal{N}}(d)$ may be arbitrarily large even if T contains only three terminals. In a second step, we prove that, for networks of constant size, the set $U_T^{\mathcal{N}}(d)$ has constant size. Combining these two insights, we then conclude that, for any number of nodes n, there exists a network $\mathcal{N}$ with three terminals such that no network $\mathcal{N}'$ with at most n nodes is equivalent to $\mathcal{N}$.

Lemma 13. *For every $n \in \mathbb{N}$, there exists a network $\mathcal{N} = (G, \beta, r)$ of degree $r = 2$ with $n + 3$ nodes, a set of terminals $T \subset V$ with $|T| = 3$, and an affine linear curve $d : \mathbb{R} \to B(T)$ of the form $d(t) = d^{(0)} + td^{(1)}$, $d^{(0)}, d^{(1)} \in B(T)$ such that $|U_T^{\mathcal{N}}(d)| \geq n$.*

Proof. As preparation, consider the network $\mathcal{N} = (G, T, \beta, 2)$ depicted in Figure 4(a) with $T = \{u_0, v, u_{n+1}\}$, $V = T \cup \{u_1, \dots, u_n\}$, $E = E_1 \cup E_2$, where $E_1 = \{(u_0, v), (v, u_{n+1})\}$, $E_2 = \{(u_i, u_{i+1}) : i = 0, \dots, n\}$ with resistances $\beta_e = n + 1$ for all $e \in E_1$, and $\beta_e = 1$ for all $e \in E_2$. Let u_0 be the node whose potential is fixed to zero. Define the curve $b : \mathbb{R} \to B(V, T)$, $b(t) = b^{(0)} + tb^{(1)}$ with $b^{(0)} = \chi_{u_0} - \chi_{u_{n+1}} \in B(V, T)$ and $b^{(1)} = \chi_{u_0} - \chi_v \in B(V, T)$. To simplify the notation, for $t \in \mathbb{R}$ we write $\pi^{\mathcal{N}}(t) := \pi^{\mathcal{N}}(b(t))$ and $x^{\mathcal{N}}(t) := x^{\mathcal{N}}(b(t))$.

Claim 3. *Let $t_1 := -1/2$ and $t_2 := 1$. We have*

$$\pi_v^{\mathcal{N}}(t_1) = \pi_{u_0}^{\mathcal{N}}(t_1) \quad \text{and} \quad \pi_v^{\mathcal{N}}(t_2) = \pi_{u_{n+1}}^{\mathcal{N}}(t_2).$$

Furthermore, for every $i \in [n+1]$, we have $\pi_{u_i}^{\mathcal{N}}(t) - \pi_{u_{i-1}}^{\mathcal{N}}(t) \geq 1/4$ for all $t \geq t_1$.

Proof. Note that, because $u_1, \dots, u_n$ are all inner nodes with degree two, by Lemma 6, the edges in E_2 could equivalently be substituted by one edge from u_0 to u_{n+1} with resistance $n + 1$ yielding a symmetric triangle. Because $b_v(t_1) = b_{u_0}(t_1) = 1/2$ and $b_v(t_2) = b_{u_{n+1}}(t_2) = -1$, the first two assertions of the claim follow by symmetry. Furthermore, it follows that, for all $e \in E_2$, we have $x_e^{\mathcal{N}}(t_1) = 1/2$.

To prove the last part, let $t \geq t_1$ be arbitrary, and define $\Delta\pi := \pi^{\mathcal{N}}(t) - \pi^{\mathcal{N}}(t_1)$ and $\Delta x := x^{\mathcal{N}}(t) - x^{\mathcal{N}}(t_1)$. Because $b(t) - b(t_1) = (t - t_1)(\chi_{u_0} - \chi_v)$, the flow Δx is a u_0-v flow. Moreover, by equation system (E), for all $e = (u, w) \in E$,

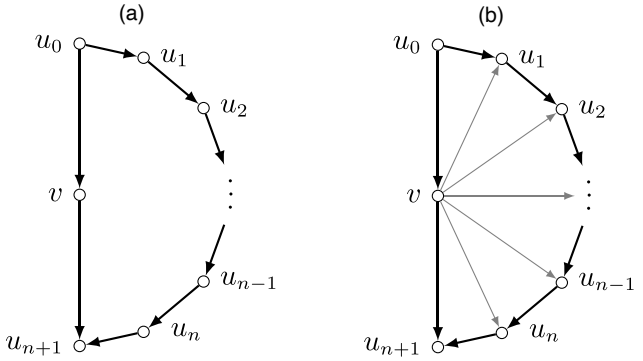
$$\begin{aligned}\operatorname{sign}(\Delta x_e) &= \operatorname{sign}(x_e^{\mathcal{N}}(t) - x_e^{\mathcal{N}}(t_1)) \\ &= \operatorname{sign}((\pi_u^{\mathcal{N}}(t) - \pi_w^{\mathcal{N}}(t)) - (\pi_u^{\mathcal{N}}(t_1) - \pi_w^{\mathcal{N}}(t_1))) \\ &= \operatorname{sign}(\Delta\pi_u - \Delta\pi_w).\end{aligned}$$

Therefore, by Lemma 9, we have $\Delta\pi_{u_0} \geq \Delta\pi_{u_{n+1}}$. As a consequence, we get $\pi_{u_0}^{\mathcal{N}}(t) - \pi_{u_{n+1}}^{\mathcal{N}}(t) \geq \pi_{u_0}^{\mathcal{N}}(t_1) - \pi_{u_{n+1}}^{\mathcal{N}}(t_1)$, and thus, $x_e^{\mathcal{N}}(t) \geq x_e^{\mathcal{N}}(t_1) = 1/2$ for all edges $e \in E_2$. We obtain

$$\pi_{u_i}^{\mathcal{N}}(t) - \pi_{u_{i-1}}^{\mathcal{N}}(t) = \beta_e [x_e^{\mathcal{N}}(t)]^2 \geq \frac{1}{4} \quad \text{for all } i \in [n+1],$$

which completes the proof of the claim. □

Figure 4. Construction of a network $\mathcal{N}'$ with $|U_T^{\mathcal{N}'}(d)| \geq n$. (a) The graph $G = (V, E)$. (b) The graph $G' = (V, E \cup E_3)$.



We proceed to construct a network $\mathcal{N}'$ with $|U_T^{\mathcal{N}'}(d)| \geq n$, where $d : \mathbb{R} \to B(T)$ is the restriction of the curve b to T, that is, $d = (b_v)_{v \in T}$. To that end, let $\mathcal{N}' = (G', \beta', 2)$ with $G' = (V, E')$ and $E' = E \cup E_3$ be the network that results from $\mathcal{N}$ by adding the edges $E_3 = \{(v, u_i) : i \in [n]\}$ to $\mathcal{N}$, each with a very large resistance $\beta'_e = B$, $e \in E_3$; see Figure 4(b) for an illustration. The resistances of the edges in E are unchanged; that is, $\beta'_e = \beta_e$ for all $e \in E$. Denote the edges in E_3 by $e_i := (v, u_i)$, $i \in [n]$. To simplify the notation, for $t \in \mathbb{R}$, we write $\pi(t) := \pi^{\mathcal{N}'}(b(t))$ and $x(t) = x^{\mathcal{N}'}(b(t))$. Note that, by Lemma 3, letting the resistance B go to infinity, the potential vector $\pi(t)$ converges to $\pi^{\mathcal{N}}(t)$. Therefore, according to Claim 3, by choosing B large enough, we can ensure that $\pi_{u_0}(t) > \cdots > \pi_{u_{n+1}}(t)$ for all $t \geq t_1$ as well as $\pi_v(t_1) > \pi_{u_1}(t_1)$ and $\pi_v(t_2) < \pi_{u_n}(t_2)$.

Claim 4. *For every $u \in V \setminus \{v\}$, the functions $t \longmapsto \pi_v(t) - \pi_u(t)$ are strictly decreasing.*

Proof. Let $\tau_1, \tau_2 \in \mathbb{R}$ with $\tau_2 > \tau_1$ be arbitrary. Similarly to the proof of Claim 3, let $\Delta x := x(\tau_2) - x(\tau_1)$ and $\Delta\pi := \pi(\tau_2) - \pi(\tau_1)$. Then, Δx is a u_0-v flow of positive value and $\operatorname{sign}(\Delta x_e) = \operatorname{sign}(\Delta\pi_u - \Delta\pi_w)$ for all $e = (u, w) \in E'$. By Lemma 9, it follows that $\Delta\pi_v < \Delta\pi_u$, and thus, $\pi_v(\tau_2) - \pi_u(\tau_2) < \pi_v(\tau_1) - \pi_u(\tau_1)$ for all $u \in V \setminus \{v\}$. □

It follows that there exist $\tau_i \in (t_1, t_2)$, $i \in [n]$ with $\tau_1 < \cdots < \tau_n$ such that, for every $i \in [n]$,

$$x_{e_i}(t) \begin{cases} > 0, & \text{for all } t < \tau_i, \\ = 0, & \text{for } t = \tau_i, \\ < 0, & \text{for all } t > \tau_i, \end{cases} \tag{10}$$

and $x_e(\tau_i) \neq 0$ for all $e \in E' \setminus \{e_i\}$. We conclude our proof by showing that the function π_v is not twice continuously differentiable at τ_i for every $i \in [n]$. In particular this implies $|U_T^{\mathcal{N}'}(d)| \geq n$.

Note that, because, for every $t \in (t_1, t_2)$, the set $\{e \in E' : x_e(t) = 0\}$ contains at most one edge, by Lemma 12, the functions x and π are continuously differentiable in (t_1, t_2). For $t \in (t_1, t_2)$, by definition, the vectors $x(t)$, $\pi(t)$, and $b(t)$ fulfill the equation system (E) with respect to network $\mathcal{N}' = (G', \beta', 2)$ with incidence matrix $\Gamma' \in \mathbb{R}^{V \times E'}$. Taking the derivative of this system with respect to t yields

$$\begin{aligned} 2\beta \odot |x(t)| \odot \dot{x}(t) &= (\Gamma')^\top \dot{\pi}(t), \\ \Gamma' \dot{x}(t) &= \chi_{u_0} - \chi_v, \\ \dot{\pi}_{u_0}(t) &= 0, \end{aligned} \tag{11}$$

where we use the notation $\dot{\pi}(t) := \frac{\mathrm{d}}{\mathrm{d}t}\pi(t)$ and $\dot{x}(t) := \frac{\mathrm{d}}{\mathrm{d}t}x(t)$. As argued, the functions $\dot{x}$ and $\dot{\pi}$ are continuous in (t_1, t_2).

Now, assume that $t \in (t_1, t_2) \setminus \{\tau_1, \ldots, \tau_n\}$. Then, $x_e(t) \neq 0$ for all $e \in E'$, and thus, the values $\bar{\beta}_e(t) := 2\beta_e|x_e(t)| > 0$, $e \in E'$ can be interpreted as edge resistances, and thus, by (11), the flow $\dot{x}(t)$ is a u_0-v flow of value one in the network $\mathcal{N}(t) := (G', \bar{\beta}(t), 1)$ of degree one with corresponding node potentials $\dot{\pi}(t)$. Hence, by definition,

$$R^{\mathcal{N}(t)}_{u_0, v} = \dot{\pi}_{u_0}(t) - \dot{\pi}_v(t) = -\dot{\pi}_v(t)$$

is the effective resistance between u_0 and v in $\mathcal{N}(t)$, which, by Proposition 1, is given by

$$-\dot{\pi}_v(t) = \min_z \left\{ \sum_{e \in E'} \bar{\beta}_e(t) z^2 : \Gamma' z = \chi_{u_0} - \chi_v \right\} = \sum_{e \in E'} \bar{\beta}_e(t) [\dot{x}_e(t)]^2.$$

In particular, $\dot{\pi}_v(t)$ is uniquely determined by the resistances $\bar{\beta}(t)$, that is, $\dot{\pi}_v(t) = \dot{\pi}_v(\bar{\beta}(t))$. Property (b) of Proposition 1 then implies

$$-\frac{\partial \dot{\pi}_v(\bar{\beta}(t))}{\partial \bar{\beta}_e(t)} = [\dot{x}_e(t)]^2 \quad \text{for all } e \in E'.$$

Because, by assumption, $x_e(t) \neq 0$ for all $e \in E'$, by Lemma 11, the function π is smooth at t. Applying the chain rule, we obtain

$$\ddot{\pi}_v(t) = \frac{\mathrm{d}}{\mathrm{d}t} \dot{\pi}_v(\bar{\beta}(t)) = \sum_{e \in E} \frac{\partial \dot{\pi}_v(\bar{\beta}(t))}{\partial \bar{\beta}_e(t)} \frac{\mathrm{d}\bar{\beta}_e(t)}{\mathrm{d}t} = -2 \sum_{e \in E} \beta_e [\dot{x}_e(t)]^3 \operatorname{sign}(x_e(t)).$$

By (10) and the fact that $\dot{x}$ is continuous in (t_1, t_2), it follows that, for every $i \in [n]$, the left and right limits of $\ddot{\pi}_v$ at τ_i fulfill

$$\begin{aligned} \lim_{t \to \tau_i^-} \ddot{\pi}_v(t) &= -2 \sum_{e \in E' \setminus \{e_i\}} \beta_e [\dot{x}_e(\tau_i)]^3 \operatorname{sign}(x_e(\tau_i)) - 2\beta_{e_i} [\dot{x}_{e_i}(\tau_i)]^3, \\ \lim_{t \to \tau_i^+} \ddot{\pi}_v(t) &= -2 \sum_{e \in E' \setminus \{e_i\}} \beta_e [\dot{x}_e(\tau_i)]^3 \operatorname{sign}(x_e(\tau_i)) + 2\beta_{e_i} [\dot{x}_{e_i}(\tau_i)]^3. \end{aligned}$$

It remains to show that, for every $i \in [n]$, $\dot{x}_{e_i}(\tau_i) \neq 0$. It then follows that $\lim_{t\to\tau_i^-} \ddot{\pi}_v(t) \neq \lim_{t\to\tau_i^+} \ddot{\pi}_v(t)$, which implies that, for every $i \in [n]$, the function π_v is not twice continuously differentiable at τ_i, and thus, $|U_T^{\mathcal{N}'}(d)| \geq n$. To this end, let $i \in [n]$ and assume by contradiction that $\dot{x}_{e_i}(\tau_i) = 0$. Then, the vectors $\dot{x}(\tau_i)$, $\dot{\pi}(\tau_i)$, and $\dot{b}(\tau_i)$ stay solutions to the equation system (11) even if $\bar{\beta}_{e_i}(\tau_i)$ is substituted by an arbitrary positive resistance. In particular, the flow $\dot{x}(\tau_i)$ is a u_0-v flow of value one with node potentials $\dot{\pi}(\tau_i)$ in the network $(G', \tilde{\beta}, 1)$, where $\tilde{\beta}_e = \bar{\beta}_e(\tau_i) > 0$ for all $e \in E' \setminus \{e_i\}$, and $\tilde{\beta}_{e_i} = 1$. But, then, by Lemma 9, it follows that $\dot{\pi}_v(\tau_i) < \dot{\pi}_{u_i}(\tau_i)$, a contradiction to

$$\dot{\pi}_v(\tau_i) - \dot{\pi}_{u_i}(\tau_i) = \tilde{\beta}_{e_i}\dot{x}_{e_i}(\tau_i) = 0$$

implied by (11). This concludes the proof of the lemma. □

As we have just seen, the set $U_T^{\mathcal{N}}(d)$ may be arbitrarily large even for only three terminals. However, for networks of constant size and integer degree, also the set $U_T^{\mathcal{N}}(d)$ has only constant size.

Lemma 14. *Let $\mathcal{N} = (G, \beta, r)$ be a network of degree $r \in \mathbb{N}$ with m edges, $T \subseteq V$ be the set of terminals, and $d : \mathbb{R} \to B(T)$ be an affine linear curve. Then, we have $|U_T^{\mathcal{N}}(d)| \leq 3^m(2r-1)^{5m}$.*

Proof. Let $d : \mathbb{R} \to B(T)$, $t \mapsto d^{(0)} + td^{(1)}$ with $d^{(0)}, d^{(1)} \in B(T)$ be an arbitrary affine linear curve, and define $b : \mathbb{R} \to B$, $b(t) = b^{(0)} + tb^{(1)}$ by $b_v^{(i)} = d_v^{(i)}$ for $v \in T$ and $b_v^{(i)} = 0$ for $v \in V \setminus T$, $i = 1, 2$. Denote $x^*(t) := x^{\mathcal{N}}(b(t))$ and $\pi^*(t) := \pi^{\mathcal{N}}(b(t))$. Consider the curve $c : \mathbb{R} \to \mathbb{R} \times \mathbb{R}^E \times \mathbb{R}^V$ defined by $c(t) = (t, x^*(t), \pi^*(t))$, which is injective, continuous, and open. It follows that its image $P := \{(t, x^*(t), \pi^*(t)) : t \in \mathbb{R}\}$ is homeomorphic to the real line $\mathbb{R}$. We can decompose P into the disjoint union $P = \cup_{\sigma\in\{-1,0,1\}^E} P_\sigma$, where $P_\sigma := \{(t, x, \pi) \in P : \operatorname{sign}(x_e) = \sigma_e \text{ for all } e \in E\}$. Note that $\operatorname{sign}(x_e) = \sigma_e$ implies that $\operatorname{sign}(x_e)|x_e|^r = (\sigma_e)^{r-1}x_e^r$. Therefore, for every $\sigma \in \{-1, 0, 1\}^E$, the set P_σ is the solution set of the following system of polynomial equations and inequalities:

$$\begin{aligned}
\beta_e(\sigma_e)^{r-1}x_e^r &= \pi_u - \pi_v && \text{for all } e = (u,v) \in E,\\
\sum_{e\in\delta^+(v)} x_e - \sum_{e\in\delta^-(v)} x_e &= b_v^{(0)} + tb_v^{(1)} && \text{for all } v \in V,\\
\pi_{v_0} &= 0,\\
x_e &> 0 && \text{for all } e \in E \text{ with } \sigma_e > 0,\\
x_e &= 0 && \text{for all } e \in E \text{ with } \sigma_e = 0,\\
x_e &< 0 && \text{for all } e \in E \text{ with } \sigma_e < 0.
\end{aligned}$$

It is a well-known result in semialgebraic geometry that the solution set of a system of s polynomial equations and inequalities in k variables of degree at most r has at most $r(2r-1)^{k+s-1}$ connected components; see, for example, Coste [9, proposition 4.13]. Applied to our setting, after eliminating the variable π_{v_0}, P_σ has, thus, at most $r(2r-1)^{(m+n-1)+(2m+n-1)-1} \leq (2r-1)^{5m}$ connected components. Thus, it follows that P is the disjoint union of at most $3^m(2r-1)^{5m}$ connected sets, and within each of these connected sets, $\operatorname{sign}(x_e^*)_{e\in E}$ is constant. Because P is homeomorphic to the real line and any connected subset of the real line is an interval, we have that $\mathbb{R}$ is the disjoint union of at most $3^m(2r-1)^{5m}$ intervals, and within each such interval, $\operatorname{sign}(x_e^*)_{e\in E}$ is constant. Let $I \subseteq \mathbb{R}$ be such an interval, and let $\mathcal{N}'$ be the network that results from $\mathcal{N}$ by deleting all edges $e \in E$ with $x_e^*(t) = 0$ for all $t \in I$. Note that deleting these edges has no influence on the node potentials, that is, $\pi^{\mathcal{N}'}(b(t)) = \pi^{\mathcal{N}}(b(t))$ for all $t \in I$, and because, in the network $\mathcal{N}'$, all remaining edges fulfill $x_e^{\mathcal{N}'}(b(t)) \neq 0$ for all $t \in I$, it follows from Lemma 11 that $\pi^{\mathcal{N}}(b(\cdot))$ is smooth in the interior of I. Consequently, it follows that $\pi^{\mathcal{N}}(b(\cdot))$ is not twice continuously differentiable at at most $3^m(2r-1)^{5m}$ points, and thus, $|U_T^{\mathcal{N}}(d)| \leq 3^m(2r-1)^{5m}$. □

Combining Lemmas 13 and 14 directly implies the main result of this section.

Theorem 5. *For every $m \in \mathbb{N}$, there exists a network $\mathcal{N} = (G, \beta, r)$ of degree $r = 2$ and a set of terminals $T \subseteq V$ with $|T| = 3$ such that no network with at most m edges is equivalent to $\mathcal{N}$.*

Similar to our remark at the end of Section 3, we note that the negative result obtained here is automatically also applicable in the more general and realistic case with bounds on potentials and flow values.

6. Summary and Outlook

In this paper, we investigate the equivalence and reduction of potential-based flow networks. We prove that, in contrast to Kron's reduction for networks of degree $r = 1$, potential-based flow networks of degree $r = 2$ with at

least three terminals cannot be reduced to smaller networks whose size only depends on the number of terminals. On the other hand, we show that it is possible to represent a special class of potential-based flow networks by a complete graph on the terminals, and we establish a characterization of networks that can be reduced to a path network.

It remains an open question if the negative reducibility result for networks of degree $r = 2$ can be generalized to all potential-based flow networks of degree $r \neq 1$. Notice that, for nonintegral network degree r, our approach via polynomial systems as in Lemma 14 is no longer directly applicable. In addition, our proof of Lemma 13 relies on the fact that the second derivative of the potentials do not exist when the flows vanish. This is not the case for integral values $r \geq 3$. It may be possible to obtain similar results for higher order derivatives in this case, but this is beyond the scope of this paper. In addition, by relaxing the concept of equivalence and introducing a notion of approximate equivalence, it might be possible to obtain positive reducibility results for networks of arbitrary degree. These problems are left for future research.

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