Congestion Games with Higher Demand Dimensions

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Abstract. We introduce a generalization of weighted congestion games in which players are associated with k-dimensional demand vectors and resource costs are k-dimensional functions $c : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$ of the aggregated demand vector of the players using the resource. Such a cost structure is natural when the cost of a resource depends on different properties of the players' demands, e.g., total weight, total volume, and total number of items. A complete characterization of the existence of pure Nash equilibria in terms of the resource cost functions for all $k \in \mathbb{N}$ is given.

1 Introduction

The study of selfish resource allocation problems is one of the core topics of the algorithmic game theory and operations research literature and has proven to be an important source for innovations in the field. E.g., central notions like the price of anarchy or the price of stability have been defined and studied first for such games, see Koutsoupias and Papadimitriou [11] and Anshelevich et al. [3].

In a congestion game, as introduced by Rosenthal [15], we are given a set of resources and players choose subsets of this set. Each player strives to minimize her private cost which is defined as the sum of the costs of the chosen resources, where the cost of each resource is a function of the number of players using it. Congestion games model a wide range of applications including road traffic, animal behavior, and telecommunication networks. In most cases, the resources correspond to the edges of a graph and each player aims to establish a path connecting her source and target vertex. In practical applications it is desirable that the system converges to a stable point because unstable behavior often leads to inefficiency. This can be observed, e.g., in telecommunication networks with distance vector routing protocols where route flapping is a major issue.

The most important stability concept is that of a pure Nash equilibrium, a deterministic state from which no user of the system can improve. Rosenthal showed that unweighted congestion games, in which each user controls one unit of demand, always admit a pure Nash equilibrium. A natural extension of congestion games are *weighted congestion games*, in which an unsplittable demand $d_i > 0$ is assigned to each player i and the cost function $c_r : \mathbb{R}_{\geq 0} \to \mathbb{R}$ of a

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resource r depends on the cumulated demands of all players sharing r rather than the number of players. In contrast to the original unweighted model studied by Rosenthal, weighted congestion games do not always possess a pure Nash equilibrium [6–8, 13]. Consequently, one line of research focused on finding (maximal) subclasses of weighted congestion games for which the existence of a pure Nash equilibrium can be guaranteed.

In a *singleton* game, each strategy of each player consists of a single resource only. Fabrikant et al. [5] remarked that every singleton weighted congestion game with non-decreasing costs admits a pure Nash equilibrium. This existence result can be strengthened towards the existence of a strong equilibrium – a strengthening of the Nash equilibrium concept due to Aumann [4] – see Kukushkin [12] and Andelman et al. [2], Harks et al. [10], and Rozenfeld and Tennenholz [16] for subsequent work in this direction. Ackermann et al. [1] showed that weighted congestion games with non-decreasing costs always admit a pure Nash equilibrium if all strategy spaces are equal to the set of bases of a matroid, and that this is the maximal property defined on the strategy spaces ensuring the existence of a pure Nash equilibrium for arbitrary non-decreasing cost functions. For *arbitrary* strategy spaces, Fotakis et al. [6] showed that every weighted congestion game with affine cost functions always admits a pure Nash equilibrium. Panagopoulou and Spirakis [14] proved the same result for exponential resource costs $c_r(x) = e^x$. Harks et al. [9] additionally confirmed the existence of pure Nash equilibria in weighted congestion games with exponential cost functions of type $a_c e^{\phi x} + b_c$, where $a_c, b_c \in \mathbb{R}$ may depend on c and $\phi \in \mathbb{R}$ is independent of c. There are no further sets of cost functions that guarantee the existence of a pure Nash equilibrium in weighted congestion games [8].

All these games are single-dimensional in the sense that the cost of a resource depends only on a single parameter – the aggregated demand of all players using it. In many situations, however, the cost of a resource depends on different parameters. E.g., in a transportation network, it is reasonable to assume that the cost to ship the player's aggregated goods depends both on the total volume and the total weight of the goods; in a telecommunication network the delay of an arc may depend both on the amount of data and the total number of files handled through the corresponding arc, respectively. To capture such situations more precisely, in this paper, we study *congestion games with k-dimensional demands*, where for $k \in \mathbb{N}_{\geq 1}$ the demand of each player i is represented as a k-dimensional vector $\mathbf{d}_i \in \mathbb{R}^k_{>0}$ and the cost of each resource r is given by a k-dimensional function $c_r : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$. For $k = 1$, we obtain weighted congestion games for which the existence of pure Nash equilibria is reasonably well understood. To formally capture the dependence of the existence of pure Nash equilibria on the underlying cost structure, Harks and Klimm [8] call a set $\mathcal{C}$ of cost functions *consistent*, if all weighted congestion games with cost functions in $\mathcal{C}$ have a pure Nash equilibrium. They also give a complete characterization of the set of consistent cost functions. We extend this terminology to k-dimensional cost functions. For $k \geq 1$, we call a set $\mathcal{C}$ of cost functions $c : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$ k-consistent if each congestion game with k-dimensional demand and cost functions in $\mathcal{C}$ has a pure Nash equilibrium. The main question of this paper is: How large are the sets of k-consistent cost functions?

Our Results. We first give a complete characterization of 2-consistent sets of cost functions. For a set $\mathcal{C}$ of continuous cost functions $c : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$, we show that $\mathcal{C}$ is 2-consistent if and only if there are $\phi_1, \phi_2 \geq 0$ or $\phi_1, \phi_2 \leq 0$ such that $\mathcal{C}$ only contains functions of type $c(x_1, x_2) = a_c(\phi_1 x_1 + \phi_2 x_2) + b_c$ or $\mathcal{C}$ only contains functions of type $c(x_1, x_2) = a_c e^{\phi_1 x_1 + \phi_2 x_2} + b_c$, where $a_c, b_c \in \mathbb{R}$ may depend on c.

We then extend this characterization to congestion games with k-dimensional demands for any $k \in \mathbb{N}_{\geq 3}$. A set $\mathcal{C}$ of continuous cost functions $c : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$ is k-consistent if and only if there is a vector $\mathbf{\Phi} \in \mathbb{R}^k_{\geq 0}$ or $\mathbf{\Phi} \in \mathbb{R}^k_{\leq 0}$ such that $\mathcal{C}$ only contains functions of type $c(\mathbf{x}) = a_c \mathbf{\Phi}^\top \mathbf{x} + b_c$ or $\mathcal{C}$ only contains functions of type $c(\mathbf{x}) = a_c e^{\mathbf{\Phi}^\top \mathbf{x}} + b_c$, where $a_c, b_c \in \mathbb{R}$ may depend on c. A set $\mathcal{C}$ that contains only functions of one of these types is called *degenerate*. Provided that $\mathcal{C}$ is degenerate, our results imply that every congestion game with k-dimensional demands and cost functions in $\mathcal{C}$ is isomorphic to a congestion game with 1-dimensional demands and the same sets of players, resources, strategies and private costs. Our contributions to k-consistency for $k \in \mathbb{N}_{\geq 1}$ generalize the complete characterization of 1-consistent sets for weighted congestion games due to Harks and Klimm [8].

All proofs missing in this extended abstract are deferred to the full version of this paper.

2 Preliminaries

We consider strategic minimization games $G = (N, S, \pi)$, where $N = \{1, \dots, n\}$ is the finite set of players, S_i is the finite set of strategies available to player i, $S = S_1 \times \dots \times S_n$ is the nonempty strategy space and $\pi : S \to \mathbb{R}^n$ is the combined private cost functions assigning a private cost vector $\pi(s) = (\pi_i(s))_{i \in N}$ to each strategy profile $s \in S$. For $i \in N$, we write $S_{-i} = S_1 \times \dots S_{i-1} \times S_{i+1} \times \dots \times S_n$ for the joint strategy set of all players except i. For $s \in S$, we write $s = (s_i, s_{-i})$ meaning that $s_i \in S_i$ and $s_{-i} \in S_{-i}$. A strategy profile is a *pure Nash equilibrium*, if $\pi_i(s) \leq \pi_i(t_i, s_{-i})$ for all $i \in N$ and $t_i \in S_i$.

Let $k \in \mathbb{N}$ with $k \geq 1$ be a demand dimension. We write $[k]$ shorthand for $\{1, \dots, k\}$ and write vectors $\mathbf{x} \in \mathbb{R}^k$ with bold face. We denote the subset of k-dimensional non-negative vectors as $\mathbb{R}^k_{\geq 0}$, i.e., $\mathbb{R}^k_{\geq 0} = \{\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k : x_i \geq 0 \ \forall i \in [k]\}$. When k is clear from the context, we use $\mathbf{0}$ to denote the k-dimensional zero vector and we write $\mathbb{R}^k_{>0}$ for $\mathbb{R}^k_{\geq 0} \setminus \{\mathbf{0}\}$.

Let R be a finite set of resources, each endowed with a k-dimensional cost function $c_r : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$. A *k-dimensional congestion game* is given by a set N of players, where for each player i, her set of strategies $S_i \subseteq 2^R \setminus \{\emptyset\}$ is a set of non-empty subsets of R, and her demand vector $\mathbf{d}_i \in \mathbb{R}^k_{>0}$ is a non-negative k-dimensional vector. In the corresponding k-dimensional congestion game the private cost of player i is defined as $\pi_i(s) = \sum_{r \in s_i} c_r(\mathbf{x}_r(s))$, where $\mathbf{x}_r(s) = \sum_{j \in N : r \in s_j} \mathbf{d}_j$ is the k-dimensional aggregated demand of resource r under strategy profile s. For two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ with $\mathbf{x} = (x_1, \dots, x_k)$ and $\mathbf{y} = (y_1, \dots, y_k)$, we write $\mathbf{x} \leq \mathbf{y}$ if and only if $x_i \leq y_i$ for all $i \in [k]$. We call a function $c : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$ *non-decreasing* if $c_r(\mathbf{x}) \leq c_r(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k_{\geq 0}$ with $\mathbf{x} \leq \mathbf{y}$. Non-increasing functions are defined analogously. Functions that are both non-decreasing and non-increasing are called *constant*, functions that are either non-decreasing or non-increasing are called *monotonic*. For a set $\mathcal{C}$ of cost functions $c : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$, we say that $\mathcal{C}$ is k*-consistent* if all k-dimensional congestion games with cost functions contained in $\mathcal{C}$ have a pure Nash equilibrium. For $k = 1$, we obtain the well known class of weighted congestion games for which in previous work Harks and Klimm [8] obtained the following characterization of 1-consistency.

Theorem 1 (Harks and Klimm [8]). *A set C of continuous cost functions $c : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is* 1*-consistent if and only if $\mathcal{C}$ only contains affine functions or $\mathcal{C}$ only contains functions of type $c(x) = a_c e^{\phi x} + b_c$ where $a_c, b_c \in \mathbb{R}$ may depend on c while $\phi \in \mathbb{R}$ is independent of c.*

3 Sufficient Conditions on k-Consistency

To obtain a full characterization of the sets of k-consistent cost functions, we first give sufficient conditions for the consistency of k-dimensional cost functions. We introduce the notion of *degeneracy* of sets of cost functions. In a sense, this notion of degeneracy forecloses the characterization we are going to prove in the remainder of this paper.

Definition 2 (Degeneracy). *A set $\mathcal{C}$ of cost functions $c : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$ is* $\mathbf{\Phi}$-degenerate*, or simply* degenerate*, if there is a vector $\mathbf{\Phi} \in \mathbb{R}^k_{\geq 0}$ or $\mathbf{\Phi} \in \mathbb{R}^k_{\leq 0}$ with one of the following properties:*

1. *For every $c \in C$ there is a function $f_c : \mathbb{R} \to \mathbb{R}$ of type $f(x) = a_c x + b_c$ such that $c(\mathbf{x}) = f(\mathbf{\Phi}^\top \mathbf{x})$ for some $a_c, b_c \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^k_{\geq 0}$.*
2. *For every $c \in C$ there is a function $f_c : \mathbb{R} \to \mathbb{R}$ of type $f(x) = a_c e^x + b_c$ such that $c(\mathbf{x}) = f(\mathbf{\Phi}^\top \mathbf{x})$ for some $a_c, b_c \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^k_{\geq 0}$.*

In case (1) holds, we call $\mathcal{C}$ and every $c \in \mathcal{C}$ affinely $\mathbf{\Phi}$-degenerate*, in case (2) holds, we call $\mathcal{C}$ and every $c \in \mathcal{C}$* exponentially $\mathbf{\Phi}$-degenerate.

As our first result, we show that degeneracy is sufficient for k-consistency. Specifically, we show that any congestion games with k-dimensional demands G for which all cost functions are taken from a degenerate set $\mathcal{C}$ is isomorphic to a congestion game with 1-dimensional demands G' that admits a pure Nash equilibrium.

Theorem 3. *Let $k \in \mathbb{N}_{\geq 1}$ and let $\mathcal{C}$ be a set of cost functions $c : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$. If $\mathcal{C}$ is degenerate, then $\mathcal{C}$ is k-consistent.*

4 Necessary Conditions on Consistency

In order to show that degeneracy is necessary for the consistency of a set $\mathcal{C}$ of continuous functions, we first derive three necessary conditions that are captured in the Extended Monotonicity Lemma, the Hyperplane Restriction Lemma, and the Line Restriction Lemma. In Sections 5 and 6, we then use these three key lemmas to derive necessary conditions on 2-consistency and k-consistency for $k \geq 3$, respectively.

The Extended Monotonicity Lemma states that any integer linear combination of functions contained in a k-consistent set of cost functions $\mathcal{C}$ is monotonic. A similar lemma has been given by Harks and Klimm [8] for the case $k = 1$. However, the proof of the Extended Monotonicity Lemma turns out to be more intricate as the $\leq$-relation does not give a complete order on $\mathbb{R}^k_{\geq 0}$ which makes the monotonicity of functions defined on $\mathbb{R}^k_{\geq 0}$ somewhat harder to characterize.

Lemma 4 (Extended Monotonicity Lemma). *Let $\mathcal{C}$ be a k-consistent set of continuous cost functions $c : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$ and let*

$$\mathcal{L}_k(\mathcal{C}) = \{c : \mathbb{R}^k_{\geq 0} \to \mathbb{R} : c(x) = \lambda_1 c_1(x) + \lambda_2 c_2(x), c_1, c_2 \in \mathcal{C}, \lambda_1, \lambda_2 \in \mathbb{Z}\}.$$

Then, every $c \in \mathcal{L}_k(\mathcal{C})$ is monotonic.

In order to state the Hyperplane Restriction Lemma, we need some additional notation. For an index $i \in [k]$ and a scalar $\hat{x}_i \geq 0$, the *insertion function* $\tau^i_{\hat{x}_i} : \mathbb{R}^{k-1}_{\geq 0} \to \mathbb{R}^k_{\geq 0}$ maps a $(k-1)$-dimensional vector $\mathbf{x}_{-i} \in \mathbb{R}^{k-1}_{\geq 0}$ to the k-dimensional vector $(x_1, \dots, x_{i-1}, \hat{x}_i, x_{i+1}, \dots, x_k)^\top \in \mathbb{R}^k_{\geq 0}$ by inserting $\hat{x}_i$ at position i. For a function $c : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$ the function composition $c \circ \tau^i_{\hat{x}_i}$ defines a new function $c \circ \tau^i_{\hat{x}_i} : \mathbb{R}^{k-1}_{\geq 0} \to \mathbb{R}$, which can be interpreted as the *restriction of c to the hyperplane* $H^i_{\hat{x}_i} = \{\mathbf{x} \in \mathbb{R}^k_{\geq 0} : x_i = \hat{x}_i\}$ for which the ith component of every vector $\mathbf{x}$ is fixed.

For $k \geq 2$, the following Hyperplane Restriction Lemma establishes a link between k-consistent cost functions and $(k-1)$-consistent cost functions.

Lemma 5 (Hyperplane Restriction Lemma). *Let $\mathcal{C}$ be a k-consistent set of cost functions $c : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$, $k \geq 2$. Then, $\mathcal{C}^i = \{c \circ \tau^i_{\hat{x}_i} : c \in \mathcal{C}, \hat{x}_i \geq 0\}$ is $(k-1)$-consistent for every $i \in [k]$.*

The next lemma establishes a similar connection between k-consistency and 1-consistency. Specifically, we show that given a k-consistent set $\mathcal{C}$ of cost functions $c : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$ and a vector $\mathbf{v} \in \mathbb{R}^k_{>0}$, the restrictions of functions $c \in \mathcal{C}$ to the line $L_{\mathbf{v}} = \{\mathbf{z} \in \mathbb{R}^k_{\geq 0} : \mathbf{z} = x\mathbf{v}, x \geq 0\}$ constitutes a 1-consistent set of cost functions. The proof uses similar ideas as the proof of the Hyperplane Restriction Lemma.

Lemma 6 (Line Restriction Lemma). *Let $\mathcal{C}$ be a k-consistent set of cost functions $c : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$, $k \geq 1$. Then, $\mathcal{C}^{\mathbf{v}} = \{c^{\mathbf{v}} : \mathbb{R}_{\geq 0} \to \mathbb{R}, x \mapsto c(x\mathbf{v}) : c \in \mathcal{C}\}$ is 1-consistent for every $\mathbf{v} \in \mathbb{R}^k_{>0}$.*

5 A Characterization of 2-Consistency

Combining the the Extended Monotonicity Lemma (Lemma 4), the Hyperplane Restriction Lemma (Lemma 5), the Line Restriction Lemma (Lemma 6) and Theorem 1, we establish the following complete characterization of 2-consistent sets of continuous cost functions.

Theorem 7. *Let $\mathcal{C}$ be a set of continuous cost functions $c : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$. Then, $\mathcal{C}$ is 2-consistent if and only if $\mathcal{C}$ is degenerate, i.e., there are $\phi_1, \phi_2 \in \mathbb{R}_{\geq 0}$ or $\phi_1, \phi_2 \in \mathbb{R}_{\leq 0}$ such that one of the following two cases holds:*

1. *$\mathcal{C}$ only contains functions of type $c(x_1, x_2) = a_c(\phi_1 x_1 + \phi_2 x_2) + b_c$ with $a_c, b_c \in \mathbb{R}$,*
2. *$\mathcal{C}$ only contains functions of type $c(x_1, x_2) = a_c e^{\phi_1 x_1 + \phi_2 x_2} + b_c$ with $a_c, b_c \in \mathbb{R}$.*

6 A Characterization of k-Consistency

In this section, we generalize the characterization of 2-consistency discussed in the previous section to arbitrary dimensions $k \in \mathbb{N}_{\geq 3}$. Specifically, we show that a set $\mathcal{C}$ of continuous cost functions is k-consistent if and only if $\mathcal{C}$ is degenerate.

For the proof, we use that for $k \geq 3$, the intersection of two $(k-1)$-dimensional hyperplanes contains a line. This allows us to consider partial derivatives of the cost functions along this intersection which enables us to establish a connection between $c \circ \tau^i_{\hat{x}_i}$ and $c \circ \tau^j_{\hat{x}_j}$. The following straightforward lemma takes a first step in that direction.

Lemma 8. *Let $k \in \mathbb{N}_{\geq 3}$ and $c : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$ and let $\mathbf{x} \in H^i_{\hat{x}_i} \cap H^j_{\hat{x}_j}$ for $i, j \in [k]$ with $i \neq j$ and some $\hat{x}_i, \hat{x}_j \geq 0$. If for $z \in [k] \setminus \{i, j\}$ the partial derivatives $\partial(c \circ \tau^i_{\hat{x}_i})/\partial x_z(\mathbf{x}_{-i})$ and $\partial(c \circ \tau^j_{\hat{x}_j})/\partial x_z(\mathbf{x}_{-j})$ exist, then $\partial c/\partial x_z(\mathbf{x})$ exists and satisfies $\partial c/\partial x_z(\mathbf{x}) = \partial(c \circ \tau^i_{\hat{x}_i})/\partial x_z(\mathbf{x}_{-i}) = \partial(c \circ \tau^j_{\hat{x}_j})/\partial x_z(\mathbf{x}_{-j})$.*

Using Lemma 8, we can derive the following complete characterization of k-consistency.

Theorem 9. *Let $\mathcal{C}$ be a set of continuous cost functions $c : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$, $k \geq 1$. Then, $\mathcal{C}$ is k-consistent if and only if $\mathcal{C}$ is degenerate, i.e., there is a vector $\mathbf{\Phi} \in \mathbb{R}^k_{\geq 0}$ or $\mathbf{\Phi} \in \mathbb{R}^k_{\leq 0}$ such that one of the following statements holds:*

1. *$\mathcal{C}$ only contains functions of type $c(\mathbf{x}) = a_c \mathbf{\Phi}^\top \mathbf{x} + b_c$ for some $a_c, b_c \in \mathbb{R}$.*
2. *$\mathcal{C}$ only contains functions of type $c(\mathbf{x}) = a_c e^{\mathbf{\Phi}^\top \mathbf{x}} + b_c$ for some $a_c, b_c \in \mathbb{R}$.*

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