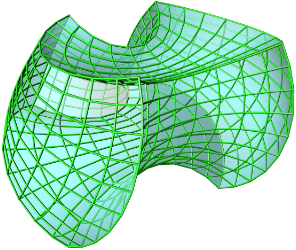
Curvature line parametrized surfaces and orthogonal coordinate systems – Discretization with Dupin cyclides



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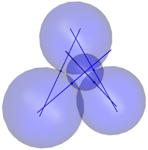
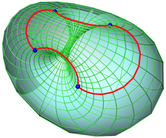
Diploma thesis supervised by A.I. Bobenko

Structure of the talk

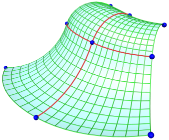
- Curvature line parametrized surfaces and their discretization in terms of contact elements



- Dupin cyclides and patches of them as envelopes of families of spheres



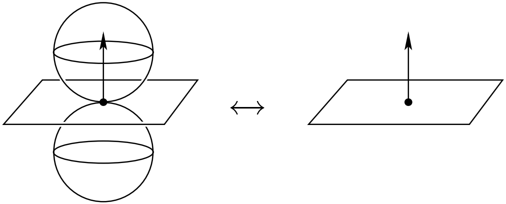
- Cyclidic nets as composition of "cyclidic patches"



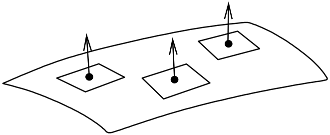
Surfaces composed of contact elements

Definition

A **contact element** in $\mathbb{R}^3$ is a 1-parametric family of oriented spheres which all touch in one point, such that their normals coincide. Such spheres are said to be in **oriented contact**.

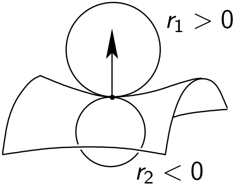


We consider surfaces not as collections of points, but as consisting of contact elements.



Principal Curvatures of a surface

In a non-umbilic point of a C^2 surface f in $\mathbb{R}^3$, the corresponding contact element contains two special oriented spheres s_1 and s_2, called **principal curvature spheres**.

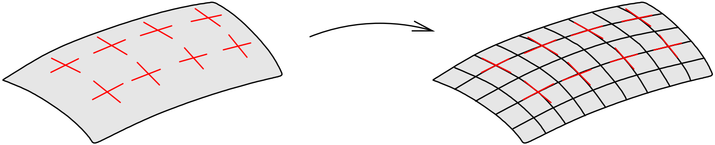


These spheres touch f in the **principal curvature directions**, and the reciprocal values of the signed radii r_1, r_2 of the principal curvature spheres are the **principal curvatures** $\kappa_i = \frac{1}{r_i}, \ i = 1, 2$.

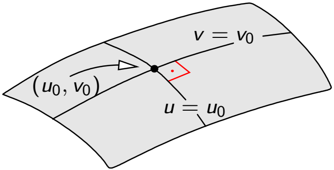
(The principal curvature directions are orthogonal, and the principal curvatures are the minimal/maximal directional curvatures)

Curvature line parametrized surfaces

- The trajectories of the vector fields of principal curvature directions constitute the network of **curvature lines** of a surface



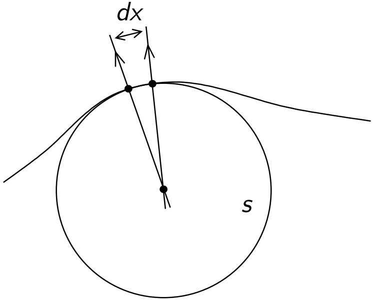
- A parametrization $(u_1, u_2) \mapsto f(u_1, u_2)$ along curvature lines is called a **curvature line parametrization**



- (u_1, u_2) are orthogonal coordinates on the surfaces

Curvature lines in terms of contact elements

Infinitesimally neighbouring contact elements belong to the same curvature line, iff they share a common sphere s



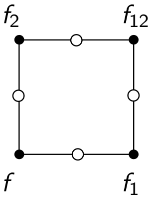
- The sphere s is the corresponding principal curvature sphere
- The normals of the contact elements intersect in the center of s

Principal contact element nets

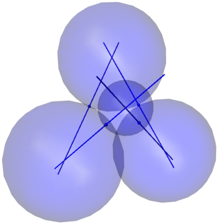
The infinitesimal characterization of curvature lines in terms of contact elements gives a natural discretization of curvature line parametrized surfaces:

Definition (Bobenko & Suris 2007)

A map $f : \mathbb{Z}^2 \to \{\text{contact elements in } \mathbb{R}^3\}$ is called a **principal contact element net**, if for all $u \in \mathbb{Z}^2$ the neighbouring contact elements $f = f(u)$ and $f_i = f(u + e_i),\ i = 1, 2$, have a sphere in common.



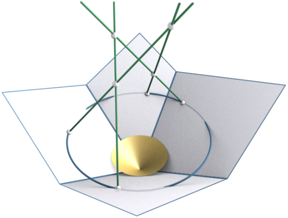
- • contact elements
- ○ common spheres (discrete principal curvature spheres)



Principal contact element nets

PCENs unify the formerly known discretizations of curvature line parametrized surfaces as circular nets and conical nets

- The points contained in a PCEN build a circular net
- The planes contained in a PCEN build a conical net



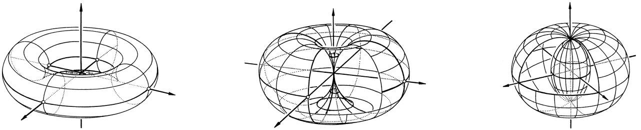
[picture by Stefan Sechelmann]

- Conversely, every circular/conical net is contained in a 2-parameter family of PCENs

PCENs are described by 4D consistent 3D systems
(are discrete integrable)

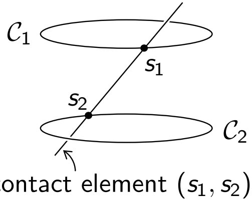
Dupin cyclides

Example (Standard tori)



[pictures from „Mathematische Modelle", Vieweg 1986]

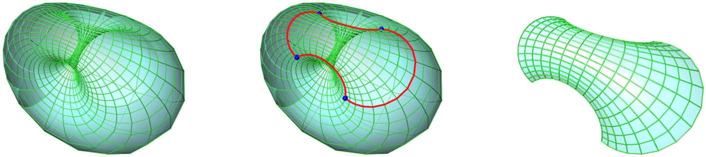
A **Dupin cyclide** is the envelope of two 1-parametric families of spheres $\mathcal{C}_1$ and $\mathcal{C}_2$, where each sphere of the family $\mathcal{C}_1$ is in oriented contact with each sphere of the family $\mathcal{C}_2$.



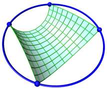
- All curvature lines of Dupin cyclides are circles on the (principal curvature) spheres of the two families (characterizing fact)
- Parametrizations $s_i : S^1 \to \mathcal{C}_i,\ u_i \mapsto s_i(u_i),\ i = 1, 2$, induce a (smooth) curvature line parametrization of the Dupin cyclide

Cyclidic patches

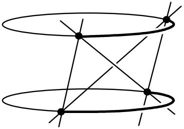
Cutting a Dupin cyclide along curvature lines gives a **cyclidic patch**



- The boundary curves of a cyclidic patch are circular arcs
- Vertices of a cyclidic patch are concircular

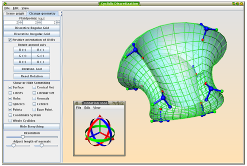


- The contact elements at the vertices of such a patch form an elementary quadrilateral of a principal contact element net!



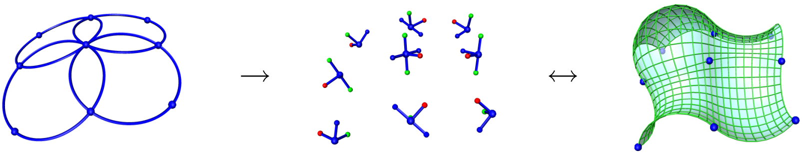
Cyclidic patches for prescribed data

- For an elementary quad of a PCEN (i.e. four spheres in oriented contact), there is a 1-parameter family of cyclidic patches
- For an elementary quad of a circular net (i.e. four concircular points), there is a 3-parameter family of cyclidic patches
- This freedom may be well encoded using frames sitting in the four circular vertices (frames are reflections of each other)

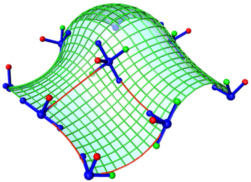


Cyclidic nets - 2D

The construction of cyclidic patches for a circular quad can be used to construct a 3-parameter familiy of C^1 surfaces for a given circular net.

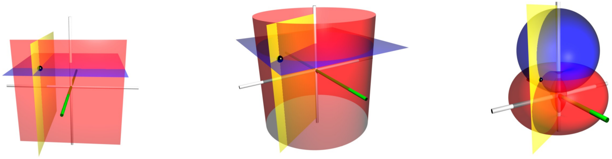


In order to obtain differentiable joins between neighbouring patches, one has to flip vectors in an appropriate way.



Triply orthogonal coordinate systems in $\mathbb{R}^3$

Example (Cartesian, cylindrical & bispherical coordinates)



[pictures from http://en.wikipedia.org/wiki/Orthogonal_coordinates]

Theorem (Dupin)

The coordinate surfaces of a triply orthogonal coordinate system intersect along curvature lines

Coordinate surfaces are curvature line parametrized surfaces

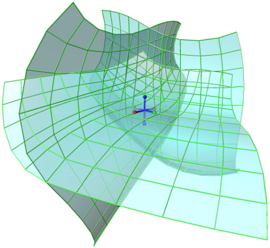
Cyclidic nets - 3D

Use 2D cyclidic nets for the discretization of triply orthogonal coordinate systems as a collection of coordinate surfaces

- such that surfaces contained in different families intersect each other orthogonally

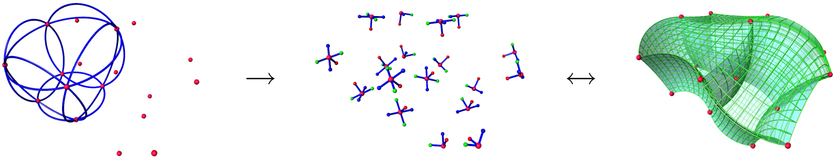
So again we use frames sitting in vertices of a circular net (now 3D), but this time there is no distinguished normal direction

$\rightsquigarrow$ Each vertex is the orthogonal intersection of three different 2D cyclidic nets

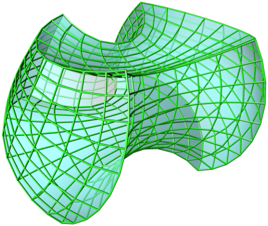


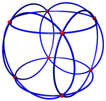
Cyclidic nets - 3D

For a given 3D circular net, there is a 3-parameter family of 3D cyclidic nets with the circular points as vertices



Elementary hexahedron of a 3D cyclidic net





- All shown curves are curvature lines of cyclidic patches
- For each face the vertices are circular
- ALL intersections of curves in the above picture are orthogonal
- The shown circular arcs constitute the boundary curves of cyclidic patches covering the interior of the hexahedron
  - These patches also intersect orthogonally in the interior

Definition of cyclidic nets

A map

$$(x, B) : \mathbb{Z}^N \to \mathbb{R}^N \times \left\{ \text{orthonormal frames } B = (n^{(1)}, \dots, n^{(N)}) \right\},$$

is called an **N-dimensional cyclidic net**, if x is a circular net and the frames B, B_i in neighbouring vertices x, x_i are related as follows:

The $(N-1)$-tuples

$$(n^{(1)}, \dots, n^{(i-1)}, n^{(i+1)}, \dots, n^{(N)})$$

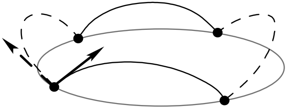
and

$$(n_i^{(1)}, \dots, n_i^{(i-1)}, n_i^{(i+1)}, \dots, n_i^{(N)})$$

are reflections of each other in the perpendicular bisecting hyperplane of the line segment $[x, x_i]$, whereas the vector $n_i^{(i)}$ is obtained from $n^{(i)}$ by first reflecting and afterwards changing the orientation.

Remarks

- The considered frames sitting in vertices of circular nets were already introduced previously for algebraic reasons, in order to prove the C^∞ convergence of circular nets to smooth orthogonal nets (Bobenko, Matthes, Suris 2003).
- When generalizing to higher dimensions, the data for each cyclidic patch is generically four dimensional



But: All the data is contained in a 3-sphere!

$\rightsquigarrow$ project stereographically to $\mathbb{R}^3$, construct the patch and lift back

References

- **Discrete Differential Geometry. Integrable Structure.**
  Book by A.I. Bobenko and Yu.B. Suris,
  Graduate Studies in Mathematics, vol. 98, AMS, 2008.
- **jReality: a Java 3D Viewer for Mathematics**
  `http://www.jreality.de`.