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Integration of complex functions and Stokes' Theorem

A very important chapter of complex analysis is the *integration of holomorphic functions along curves*, leading to the central *Cauchy integral theorem.* This theorem, however, is a special case of a prominent theorem in real vector analysis, the *Stokes integral theorem.* I feel that a course on complex analysis should explain this connection. It is almost trivial, if you assume familiarity with Stokes' theorem in the optimal (and only appropriate) setting of integration theory for differential forms on manifolds, a familiarity that many students at this level do not have. Therefore I only assume the knowledge of a special verison of the Stokes, namely Green's theorem, avoiding explicite mention of differential forms.

But there remains a technical difficulty about the curves along which the integrals are formed. Quite often they are closed curves or collections of curves, that form the boundary of "something". Restriction to boundary curves of nice geomeric objects like disks or annuli allows many applications, but misses a point that, I think, must be addressed in the context of complex analysis: the relation between *homology* and *homotopy.* In particular, the notion of homology for curves should be explained. To do this on the background of integration is the second major objective of this chapter.

"One-dimensional" integrals.

Definition 1. Let $P, Q : \mathbb{R}^2 \supset G \to \mathbb{R}$ be continuous functions on a region G, and let $c : [a, b] \to G$ be a continuously differentiable curve with $c(t) = (x(t), y(t))$. We define

$$\int_c P dx + Q dy := \int_a^b \big(P(c(t))\dot{x}(t) + Q(c(t))\dot{y}(t)\big)\, dt. \tag{1}$$

Definition 2. Let $f : \mathbb{R}^2 = \mathbb{C} \supset G \to \mathbb{C}$ be continuous on the region G, and let $c : [a, b] \to G$ be a continuously differentiable curve. We define

$$\int_c f(z)dz := \int_a^b f(c(t))\dot{c}(t)dt. \tag{2}$$

Remember that the integral of a complex-valued function is simply taken separately on real and imaginary part. If we decompose into real and imaginary part, then the two definitions fit together: If

$$f = u + iv, \quad dz = dx + idy,$$

then

$$\int_c f(z)dz = \int_a^b (u(c(t)) + iv(c(t)))(\dot{x}(t) + i\dot{y}(t))dt$$
$$\underset{(1)}{=} \int_c (udx - vdy) + i\int_c (udy + vdx). \tag{3}$$

Example. Let $c(t) = e^{2\pi it}$, $0 \le t \le 1$. Then

$$\int_c \frac{dz}{z} = \int_0^1 \frac{2\pi i e^{2\pi it}}{e^{2\pi it}} dt = 2\pi i. \tag{4}$$

Lemma 3. *If we reverse orientation of the curve, then the integral changes sign. Orientation-preserving re-parametrization does not affect the value of the integral.*

This allows the definition of curve-integrals, where "curve" means a subset of the complex plane that can be "parametrized", and has a prescribed orientation. We shall not discuss this in detail, but shall use it in obvious geometrical situations. For instance, we write

$$\int_{|z-a|=r} f(z)dz := \int_c f(z)dz$$
$$c(t) := a + re^{it}, \quad 0 \le t \le 2\pi$$
$$\text{or, equivalently,}$$
$$c(t) := a + re^{2\pi it}, \quad 0 \le t \le 1.$$

An estimate for complex integrals:

Theorem. *Let $f : \mathbb{C} \supset G \to \mathbb{C}$ be continuous on the region G, $c : [a, b] \to G$ continuously differentiable, and $M \in \mathbb{R}$ with $|f \circ c| \le M$. Then*

$$\left| \int_c f(z)dz \right| \le M \underbrace{\int_a^b |\dot{c}(t)|dt}_{:=L(c)}. \tag{5}$$

$L(c)$ *is the* length of c.

Proof. If $J := \int_c f(z)dz = 0$, then the assertion is obvious. Otherwise

$$1 = \frac{\int_c f(z)dz}{J} = \operatorname{Re}\frac{\int_c f(z)dz}{J} = \operatorname{Re}\int_a^b \frac{f(c(t))\dot{c}(t)}{J}dt$$
$$= \int_a^b \operatorname{Re}\frac{f(c(t))\dot{c}(t)}{J}dt =\le \int_a^b \frac{M|\dot{c}(t)|}{|J|}dt = \frac{M}{|J|}L(c).$$

Sometimes it is convenient to represent curves in $\mathbb{C} \setminus \{0\}$ using polar coordinates. The following theorem guaranties the existence of such.

Theorem. *Let $c : [0,1] \to \mathbb{C} \setminus \{0\}$ be continuously differentiable, and assume $c(0) = r_0 e^{i\phi_0}$ with $r_0 > 0$ and $\phi_0 \in \mathbb{R}$. Then*

$$\phi(t) := \phi_0 + \operatorname{Im} \int_{c|_{[0,t]}} \frac{dz}{z} = \phi_0 + \operatorname{Im} \int_0^t \frac{\dot{c}(\tau)}{c(\tau)} d\tau$$

defines a continuously differentiable function such that

$$\phi(0) = \phi_0, \tag{6}$$

$$c(t) = |c(t)|\, e^{i\phi(t)} \qquad \textit{for all } t. \tag{7}$$

ϕ is the only continuous function with (6), (7) Equation (7). determines it up to an additve constant of the form $2\pi i k$, $k \in \mathbb{Z}$.

Proof. We compute

$$\frac{d}{dt}\left(c(t) e^{-\int_0^t \frac{\dot{c}(\tau)}{c(\tau)} d\tau}\right) = e^{-\int_0^t \frac{\dot{c}(\tau)}{c(\tau)} d\tau} \left(\dot{c}(t) - c(t) \frac{\dot{c}(t)}{c(t)}\right) = 0.$$

Hence

$$\begin{aligned} c(t) &= const\, e^{\int_0^t \frac{\dot{c}(\tau)}{c(\tau)} d\tau} = c(0)\, e^{\int_0^t \frac{\dot{c}(\tau)}{c(\tau)} d\tau} \\ &= r_0 e^{i\phi_0}\, e^{\int_0^t \frac{\dot{c}(\tau)}{c(\tau)} d\tau} = r_0 e^{\operatorname{Re} \int_0^t \frac{\dot{c}(\tau)}{c(\tau)} d\tau} e^{i\phi_0 + i \operatorname{Im} \int_0^t \frac{\dot{c}(\tau)}{c(\tau)} d\tau} \\ &= r_0 e^{\operatorname{Re} \int_0^t \frac{\dot{c}(\tau)}{c(\tau)} d\tau} e^{i\phi(t)} = |c(t)| e^{i\phi(t)}. \end{aligned}$$

The rest follows from the $2\pi i$-periodicity and continuity of the exponential function.

"Two-dimensional" integrals.

Similar to curves, i.e. mappings of a one-dimensional interval, we consider "surfaces", let us call them *patches*, as mappings of a rectangle into the plane. (The official terminology is *singular rectangle*.)

Definition 4. Let $R : \mathbb{C} \supset G \to \mathbb{C}$ be continuous on the region G, and let

$$C : [a_1.b_1] \times [a_2, b_2] \to G$$

be a continuously differentiable patch. Then we define

$$\int_C R dx dy := \int_{a_1}^{b_1} \int_{a_2}^{b_2} R(C(s,t)) \det \left(\frac{\partial C}{\partial s}, \frac{\partial C}{\partial t} \right) dt ds. \tag{8}$$

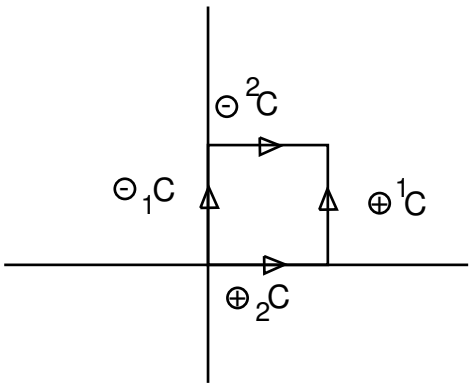
If C is an orientation-preserving imbedding (think of a linear map, that transforms the rectangle into a parallelogram), and $R = 1$, then the left-hand side should produce the area of the image of C: which it does, according to the transformation rule. This is the origin of the functional determinant or Jacobian in (8). If makes the definition invariant with respect to orientation-preserving re-parametrizations.

If we consider a continuously differentiable patch $C : [0,1]^2 \to \mathbb{C}$ (where we restrict to the unit square for simplicity), then the intuitive definition of the *boundary curve* of C would be a piecewise continously differentiable curve consisting of four parts, the four *sides* or edges of C, all of which we reparametrize on $[0,1]$:

$$\begin{aligned} t &\mapsto C(1,t) \\ t &\mapsto C(1-t,1) \\ t &\mapsto C(0,1-t) \\ t &\mapsto C(t,1) \end{aligned}$$

We stick to the standard convention using a simpler parametrization, and putting a formal sign mark on each of the four curves that indicates, which sign to choose for the corresponding integral:

$$\begin{aligned} {}^1C &: t \mapsto C(1,t) && \text{with sign } \oplus, \\ {}^2C &: t \mapsto C(t,1) && \text{with sign } \ominus, \\ {}_1C &: t \mapsto C(0,t) && \text{with sign } \ominus, \\ {}_2C &: t \mapsto C(t,1) && \text{with sign } \oplus\,. \end{aligned}$$



We formally write the boundary ∂C of C as a *chain*

$$\partial C = \oplus {}^1C \ominus {}_1C \ominus {}^2C \oplus {}_2C$$

meaning e.g.

$$\int_{\partial C} f dz = + \int_{{}_1C} f dz - \int_{{}_1C} f dz - \int_{{}_2C} f dz + \int_{{}_2C} f dz.$$

Examples.

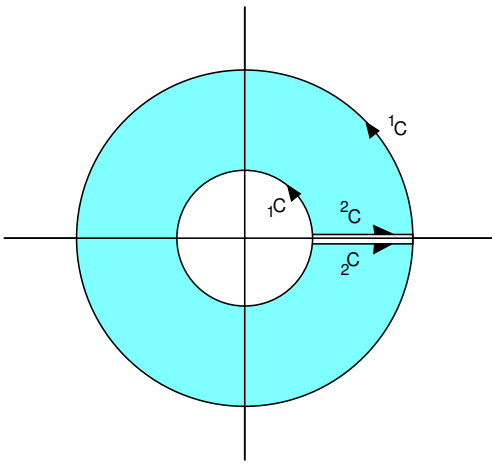
1. **Annulus.** Let $0 < r < R$ and define $C : [0,1]^2 \to \mathbb{C}$ by

   $$C(s,t) := (r + s(R-r))e^{2\pi i t}$$

   Then, for example,

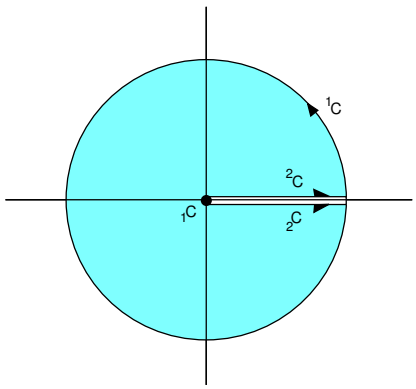
   $${}^1C(t) = (r + 1(R-r))e^{2\pi i t} = Re^{2\pi i t}.$$

   Find the other sides.

   Since ${}_2C = {}^2C$, and in the boundary definition they occur with opposite signs, the "relevant" boundary for integration consist of the two circles that you intuitively consider as the boundary of the annulus. Both are parametrized counter-clockwise, but the outer circle counts positive, the inner negative.

2. **Disk.** If in the preceeding example we put $r = 0$, the image of the unit square under C becomes a disk of radius R around the origin. For integration, again the two radial curves cancel, while the "inner circle" degenerates to a point that does not contribute to the integral.

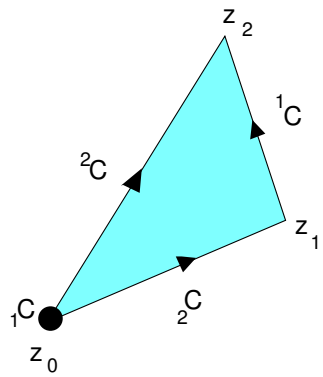


The "relevant" boundary is again the intuitive one, a circle of radius R.

3. **Triangle.** Let $z_0, z_1, z_2 \in \mathbb{C}$. Define

$$C(s,t) := (1-s)z_0 + s((1-t)z_1 + tz_2)$$

The image of C is a triangle with vertices the z_k. The side ${}_1C$ again degenerates to a point, the others form the edges of the triangle. Integration over the boundary chain amounts to the integral taken around the edges from z_0 to z_1 to z_2 and back to z_0.



The Cauchy theorem.

With these definitions we can formulate the simplest form of Stokes' theorem:

Green's theorem. *Let $G \subset \mathbb{C}$ be a region, let $P, Q : G \to \mathbb{R}$ be continuously differentiable functions, and let $C : [0,1]^2 \to G$ be a continuously differentiable patch. Then*

$$\int_C \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right) dxdy = \int_{\partial C} (Pdx + Qdy). \tag{9}$$

For the proof I refer to the regular Analysis class, or to my Skriptum "Komplexe Analysis". The proof is not difficult, the basic ingredients are the Fubini theorem (successive integration), and the fundamental theorem of calculus, which can be considered as the baby version of Stokes' theorem.

The following **Integral Theorem of Cauchy** is the most important theorem of complex analysis, though not in its strongest form, and it is a simple consequence of Green's theorem. We shall later give an independent proof of Cauchy's theorem with weaker assumptions.

Cauchy's Theorem (Version 0). *Let $f : \mathbb{C} \supset G \to \mathbb{C}$ be holomorphic in the region G, and let $C : [0,1]^2 \to G$ be continuously differentiable. Assume moreover that*

$$f' \text{ is continuous.} \tag{10}$$

Then

$$\int_{\partial C} f(z)dz = 0. \tag{11}$$

Proof. We decompose $f = u + iv$ into real and imaginary part.

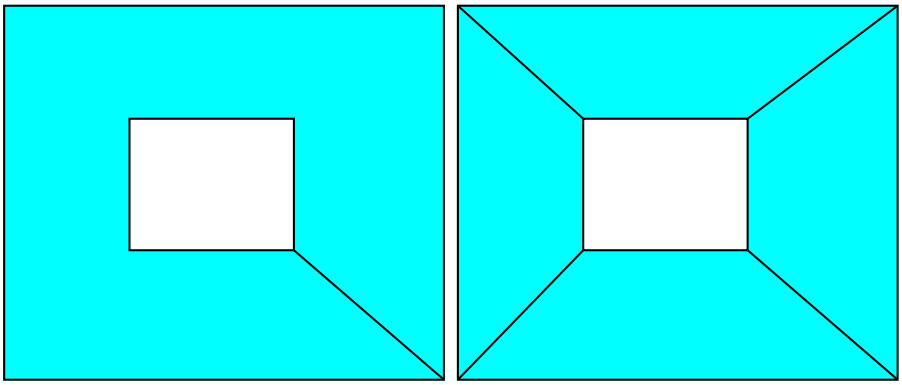
$$\begin{aligned}
\int_{\partial C} f(z)dz &= \int_{\partial C} (udx - vdy) + i \int_{\partial C} (udv + vdx) \\
&= \int_C \left(-\frac{\partial u}{\partial y} + \frac{\partial(-v)}{\partial x} \right) dxdy + i \int_C \left(-\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right) dxdy \\
&= 0
\end{aligned}$$

by the Cauchy-Riemann equations. □

Generalizations. Homology.

When we consider integration over the boundary of a square or rectangle, it is recommended not to work out a continuously differentiable parametrization of the complete boundary quadrangle (which is possible), but use the idea of a *chain* consisting of four linearly parametrized line segments with proper signs for the integral. Similarly, the parametrization of complicated *areas* in the

plane is often simpler, when using several patches instead of one: There exists a continuously differentiable parametrization of the "rectangular annulus" by the unit square, which is similar to the parametrization of the circular annulus above, but nobody ever wants to explicitely write it down. Much simpler is a parametrization using four trapezoidal patches, which can be obtained conveniently by linear maps, see the figures.



This leads to the concept of *chains of patches*, which by definiton are formal linear combinations of patches defined on the unit square with integer coefficients (mostly, but not necessarily, 0 or ± 1). For brevity, we call them *2-chains*. It is helpful to also generalize the concept of chains of curves, or *1-chains*, admitting formal linear combinations of curves (defined on $[0, 1]$) with arbitrary integer coefficient.

Integration over a chain is done by integrating over its curves or patches and counting each integral with the multiplicity given by the integer coefficient of that patch or curve.

A chain in G may look as follows:

$$3C_1 \oplus C_2 \ominus 5C_3,$$

where the C_j are curves or patches, respectively.

Remark. Chains are "formal" linear combinations. The signs and coefficients are only used as information how to weigh the integrals. Since the values of the C_j lie in the vector space $\mathbb{C}$, you could be tempted to take the *linear combination of the values*, and obtain again a map $[0, 1]^2 \to \mathbb{C}$ (not necessarily with values in G). This is not meant(!), and therefore we speak of "formal" linear combinations.

Using integer coefficients other than ± 1, which reflect the "orientation", seems strange. The motivation is given by the fact, that with the obvious "formal" addition and subtraction the 1-chains or the 2-chains in G form an abelian group, allowing the use of algebraic methods. In particular, for a fixed continuous function $f : G \to \mathbb{C}$

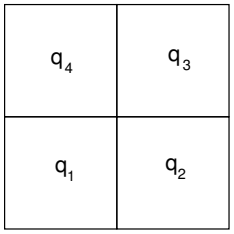
$$c \mapsto \int_c f(z)dz$$

is a homomorphism of the 1-chain group in G to (the additive group) $\mathbb{C}$. In other words, the integral operator is *extended as a homomorphism* to the (free) abelian group of chains generated by the curves in G. Similarly the boundary operator is extended to a homomorphism of the 2-chain group in G to the 1-chain group in G.

Example. Each of the maps

$$q_1(s,t) = \frac{1}{2}(s,t), \qquad q_2(s,t) = \frac{1}{2}(1+s,t)$$
$$q_3(s,t) = \frac{1}{2}(1+s,1+t), \qquad q_4(s,t) = \frac{1}{2}(s,1+t)$$

maps the unit square onto a quarter of it.



If you compute $\partial(q_1 \oplus q_2 \oplus q_3 \oplus q_4)$, the interior curves cancel, and it remains a chain of eight curves on the boundary of the unit square, such that with

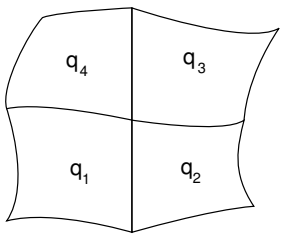
$$q : [0,1]^2 \to \mathbb{C}, \quad (s,t) \mapsto (s,t)$$

we get

$$\sum_{k=1}^{4} \int_{\partial q_k} f(z)dz = \int_{\partial(q_1 \oplus q_2 \oplus q_3 \oplus q_4)} f(z)dz = \int_{\partial q} f(z)dz.$$

More generally, we have for any patch C in some G and continuous $f : G \to \mathbb{C}$

$$\sum_{k=1}^{4} \int_{\partial(C \circ q_k)} f(z)dz = \int_{\partial(C \circ q)} f(z)dz.$$



Definition 5. A 1-chain c in a region $G \subset \mathbb{C}$ is called *0-homologous in G*, if it is the boundary of a 2-chain C <u>in G</u> up to a 1-chain of constant curves.

$$c = \partial C \oplus n_1 c_1 \oplus n_2 c_2 \oplus \ldots \oplus n_k c_k,$$

for some $k \geq 0$, and $n_1, \ldots, n_k \in \mathbb{Z}$ and constant curves $c_1, \ldots, c_k$.

Two 1-chains c_1 and c_2 are called *homologous in G*, if their (formal) difference is 0-homologous in G. We denote this by

$$c_1 \underset{G}{\sim} c_2.$$

Example. If $0 < r < R$, then the 1-chains consisting each of the single curve

$$\begin{aligned} c_1 :&[0,1] \to G, \quad t \mapsto Re^{2\pi it}, \text{ and} \\ c_2 :&[0,1] \to G, \quad t \mapsto re^{2\pi it}, \text{ respectively,} \end{aligned}$$

are homologous to each other in $G = \mathbb{C} \setminus \{0\}$, compare the annulus example above. On the other hand, in $G = \mathbb{C}$ the curve c_1 is homologous to the constant curve of value 0, see the disk example above. Therefore $c_1 \underset{\mathbb{C}}{\sim} 0$.

By the homomorphism property of integration and boundary operation the following generalization is obvious:

Cauchy's Theorem (Version 1). *Let $f : \mathbb{C} \supset G \to \mathbb{C}$ be holomorphic in the region G, and let c be a continuously differentiable 1-chain, which is 0-homologous in G. Assume moreover that f' is continuous. Then*

$$\int_c f(z)dz = 0. \tag{12}$$

Example. $c(t) = e^{2\pi it}$, $0 \le t \le 1$ is a curve (or 1-chain) in $G = \mathbb{C} \setminus \{0\}$. But it is not 0-homologous in this set, because

$$\int_c \frac{dz}{z} \neq 0,$$

compare (4).

The following notion generalizes the notion of a closed curve to chains.

Definition 6. A 1-chain $n_1c_1 \oplus \ldots \oplus n_kc_k$ with curves $c_j : [0,1] \to G$ is called *closed* or a *cycle*, if for all $p \in \mathbb{C}$

$$\sum_{c_j(0)=p} n_j = \sum_{c_j(1)=p} n_j.$$

Any point occurs as often as starting point of a curve as it occurs as endpoint.

Check that ∂C for a patch C is closed. Therefore 0-homologous 1-chains are necessarily closed, while the circle of the preceeding example is closed, but in $G = \mathbb{C} \setminus \{0\}$ it is not 0-homologous.