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http://www3.math.tu-berlin.de/geometrie/Lehre/SS12/ComplexAnalysis SS 12

Complex Analysis - Exercise Sheet 1

- holomorphic functions, power series -

due 25-26.04.2012

Tutorial

Exercise 1

Which of the following functions $f_j : \mathbb{C} \to \mathbb{C}$ are holomorphic?

$f_1(z) = \mathrm{Re}(z), \quad f_2(z) = \mathrm{Im}(z), \quad f_3(z) = z^2,$
$f_4(z) = |z|^2, \quad f_5(z) = g(z) + ig(z)$ where $g : \mathbb{C} \to \mathbb{R}$ real differentiable
$f_6(z) = z^2 + 2z + \bar{z}.$

Exercise 2

Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ a holomorphic function such that $f'(z) \neq 0$ for all $z \in U$. According to the lecture holomorphic functions are *conformal* maps, i.e. they preserve angles and orientation.
A good way to visualize a complex map $f : \mathbb{C} \to \mathbb{C}$ is to draw the images of coordinate lines, i.e. $f(x+iy) \in \mathbb{C}$ with $x \in \mathbb{R}$ resp. $y \in \mathbb{R}$ constant. Try this for $f(z) = z^2$ und $f(z) = \cos(z)$.

Exercise 3

On $K := \{\, z \in \mathbb{C} \mid |z-1| < 1 \,\}$ define the function $\lambda : K \to \mathbb{C}$

$$\lambda(z) := -\sum_{n=1}^{\infty} \frac{1}{n}(1-z)^n \; .$$

Show that $\lambda(z)$ converges for all $z \in K$ and that it is holomorphic. What is the derivative of λ?

Homework

Exercise 1 **7 points**

a) Check whether $f : \mathbb{C} \to \mathbb{C}$ with $f(x+iy) = -4xy + 2i(x^2 - y^2 + 3)$ is holomorphic or not. Write f only in terms of z and $\bar{z}$ (without Re and Im).

b) Let $u : \mathbb{R}^2 \to \mathbb{R}$, $u(x,y) = 3xy^2 - x^3 + x$. Show that u is harmonic (i.e. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$) and determine a function $v : \mathbb{R}^2 \to \mathbb{R}$ (harmonic conjugate), such that $f = u + iv$ is holomorphic and $f(0) = 2i$.

c) Draw the images of the coordinate lines $\{ x + iy \mid x = \text{const.} \}$ and $\{ x + iy \mid y = \text{const.} \}$ under the exponential map.

Exercise 2 **6 points**

Let $a \in \mathbb{C}$. Determine the radius of convergence of the following power series:

$$\text{a)} \sum_{n=0}^{\infty} a^n z^n, \qquad \text{b)} \sum_{n=0}^{\infty} a^{n^3} z^n, \qquad \text{c)} \sum_{n=0}^{\infty} n! z^n, \qquad \text{d)} \sum_{n=0}^{\infty} \binom{a}{n} z^n.$$

The generalized binomial for $a \in \mathbb{C}$ is $\binom{a}{0} = 1$ and $\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}$ for $n \geq 1$.

Exercise 3 **7 points**

Consider the following bijective maps from $\mathbb{C}$ to $\mathbb{C}$:

- rotation around the origin $R_\varphi : z \mapsto e^{i\varphi} z$ ($\varphi \in \mathbb{R}$),
- scaling/dilation $S_r : z \mapsto rz$ ($r \in \mathbb{R}, r > 0$), and
- translation $T_b : z \mapsto z + b$ ($b \in \mathbb{C}$).

Let $\mathcal{A}$ be the set of all these maps.

a) For $a, b \in \mathbb{C}$ consider the map $f : \mathbb{C} \to \mathbb{C}, z \mapsto az + b$. Write f as a composition of elements of $\mathcal{A}$. Does the order matter?

b) Show that every composition of elements of $\mathcal{A}$ may be written in the form $z \mapsto az + b$ with suitable $a, b \in \mathbb{C}$.

c) Show that for $a \neq 1$ the map $f : \mathbb{C} \to \mathbb{C}, z \mapsto az + b$ has exactly one fix point $z_0 \in \mathbb{C}$. Show that f is a spiral similarity (i.e. a combination of dilation and rotation) with center z_0 and determine a suitable composition of elements of $\mathcal{A}$. Classify the maps $f(z) = az + b$ geometrically depending on the parameter $a \in C$.

d) (Optional) Extend $\mathcal{A}$ by including the conjugation map: $Z : z \mapsto \bar{z}$, and describe geometrically any new compositions which arise.