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http://www3.math.tu-berlin.de/geometrie/Lehre/SS12/ComplexAnalysis SS 12

Complex Analysis - Exercise Sheet 2

- Möbius transformations -

due 02-03.05.2012

Tutorial

In the following, let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation. A *circle* means either an ordinary circle or a line.

Exercise 1

1. Show that the Möbius transformation $f(z) = \frac{1}{z}$ maps circles to circles. Deduce the same for all Möbius transformations.
2. Determine f such that $f(0) = i$, $f(i) = \infty$, and $f(\infty) = 1$. Draw the images of the four quadrants. What happens to the coordinate lines?

Exercise 2

1. Let $A \in SL(2, \mathbb{C})$ be the matrix associated to f. Show that f maps the upper half-plane onto itself $\iff$ $A \in SL(2, \mathbb{R})$.

2. Let n be the number of fixed points of such an $f \neq Id$ on the real axis. f is said to *hyperbolic* if $n = 2$, *elliptic* if $n = 0$, and *parabolic* if $n = 1$. Show that

$$tr(A) \begin{cases} > & 2 \iff f \text{ is hyperbolic} \\ = & 2 \iff f \text{ is parabolic} \\ < & 2 \iff f \text{ is elliptic} \end{cases} \tag{1}$$

Exercise 3

1. Show that every Möbius transformation has at least one fixpoint.

2. Consider a Möbius transformation f with exactly one fixpoint. Show that there exists a Möbius transformation g and $b \in \mathbb{C}$ such that

$$(g^{-1} \circ f \circ g)(z) = z + b.$$

Homework

Exercise 1 **5 points**

Let $\mathbb{D} = \{\, z \in \mathbb{C} \mid |z| \leq 1 \,\}$ be the unit disc and $\mathbb{H} = \{\, z \in \mathbb{C} \mid \operatorname{Im} z \geq 0 \,\}$ be the upper half-plane. Find a Möbius transformation $f : \mathbb{C} \to \mathbb{C}$ with

$$f(0) = -1, f(i) = 0, \text{ and } f(\infty) = 1.$$

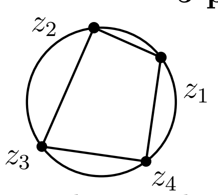
Show that $f(\mathbb{H}) = \mathbb{D}$.

Exercise 2 **10 points**

Let $a, b, c, d \in \mathbb{C}$ and $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $f(z) = \frac{az+b}{cz+d}$ a Möbius transformation.

1. Determine a, b, c, and d if f has at least 3 fixpoints.
2. When does f have exactly two fix points?
3. Let g be a Möbius transformation with fix points 0 and ∞. Show that there exists $s \in \mathbb{C} \setminus \{0, 1\}$ such that $g(z) = sz$.
4. Let f be a Möbius transformation with two disjoint fix points in $\hat{\mathbb{C}}$. Show that there exists a Möbius transformation τ and $s \in \mathbb{C} \setminus \{0, 1\}$ such that
$$(\tau^{-1} \circ f \circ \tau)(z) = sz$$
5. Let $f(z) = sz$ with $s \in \mathbb{C} \setminus \{0, 1\}$ and $z_0 \in \mathbb{C} \setminus 0$. Let $(z_n)_{n \in \mathbb{N}}$ be defined by $z_n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}(z_0)$. Determine the behavior (convergence, divergence, periodicity) of the sequence (z_n) depending on the parameter s.

Exercise 3 **5 points**



Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ four disjoint points on a circle as shown in the figure on the right.

a) Show that the cross-ratio $\operatorname{cr}(z_1, z_2, z_3, z_4)$ is negative, whereas the cross-ratio $\operatorname{cr}(z_1, z_3, z_2, z_4)$ is positive.

b) Verify that $|\operatorname{cr}(z_1, z_2, z_3, z_4)| + 1 = |\operatorname{cr}(z_1, z_3, z_2, z_4)|$ and deduce the following equality for the lengths of the edges and the diagonals of the circular quadrilateral:

$$|z_1 - z_3||z_2 - z_4| = |z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_4 - z_1|.$$

Hint: Apply a suitable Möbius transformation to the circle that maps it to the real axis.