Technische Universität Berlin
Institut für Mathematik
Bobenko // Gunn/Knöppel

http://www3.math.tu-berlin.de/geometrie/Lehre/SS12/ComplexAnalysis SS 12

Complex Analysis - Exercise Sheet 3

- Linear transformations, line integrals -

due 09-10.05.2012

Tutorial

Exercise 1

For each two disjoint circles there exists a linear transformation which maps them to a pair of concentric circles.

Exercise 2

Let $g : \mathbb{C} \supset D \to W \subset \mathbb{C}$ be holomorphic, $f : \mathbb{C} \supset W \to \mathbb{C}$ continuous, γ a curve in D. Show that:

$$\int_{g\circ\gamma} f(z)dz = \int_{\gamma} f \circ g(z)g'(z)dz.$$

What does the formula look like, if g is only real differentiable?

Exercise 3

Let $f : \mathbb{C} \supset U \to \mathbb{C}$ be real differentiable and γ be a curve in $U \subset \mathbb{C} = \mathbb{R}^2$.

a) Determine the real part R and and imaginary part I of $\int_\gamma f(z)\,dz$.

b) Determine two vector fields $W_1 : U \to \mathbb{R}^2$ and $W_2 : U \to \mathbb{R}^2$ such that

$$I = \int_a^b \langle W_1(\gamma(t)), \gamma'(t)\rangle\, dt, \qquad R = \int_a^b \langle W_2(\gamma(t)), \gamma'(t)\rangle\, dt.$$

c) When are the vector fields W_1 and W_2 antidrivatives, i.e. the gradients of two differentiable functions $f_1, f_2 : U \to \mathbb{R}$? What does that mean for f? Check that W_1 and W_2 are orthogonal with respect to each other.

Homework

Exercise 1 **6 points**

Let $f : \mathbb{C} \supset D \to \mathbb{C}$ be continuous. Show that:

a) $\overline{\int\limits_{\gamma} f(z)dz} = \int\limits_{\overline{\gamma}} \overline{f(\overline{z})}dz$ for every curve γ in D,

b) $\overline{\int\limits_{|z|=1} f(z)dz} = -\int\limits_{|z|=1} \overline{f(z)}\frac{dz}{z^2}$, if $\{z \in \mathbb{C} \mid |z| = 1\} \subset D$.

Exercise 2 **6 points**

a) Let $r \in (0, \infty)$ and $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = re^{it}$. Calculate $\int_\gamma \mathrm{Re}(z)dz$.

b) Let $\mathbb{R}_0^- = \{ x \in \mathbb{R} \mid x < 0 \} \subset \mathbb{C}$ and $a \in \mathbb{C} \setminus \mathbb{R}_0^-$. Let γ be the curve consisting of the segment $[1, |a|]$ and the circular arc from $|a|$ to a not intersecting $\mathbb{R}_0^-$.

   Draw a picture of γ and find a piecewise continuously differentiable parametrization of γ. Calculate the integral $\int_\gamma \frac{1}{z}dz$.

Exercise 3 **8 points**

Let $G \subset \mathbb{C}$ be an open convex set and let $f : G \to \mathbb{C}$ be holomorphic with continuous derivative f'. Assume further that

$$|f'(z) - 1| < 1 \quad \text{for all } z \in G.$$

Show that f is injective on G.

Hint: Integrate the function $f'(z) - 1$ along the segment $[z_0, z_1] \subset G$ and use an estimate by curve length and maximum.