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http://www3.math.tu-berlin.de/geometrie/Lehre/SS12/ComplexAnalysis SS 12

Complex Analysis - Exercise Sheet 4

- Cauchy's theorem, integral formula and Applications -

due 16-17.05.2012

Tutorial

Exercise 1

Find an explicit parametrization mapping a rectangle to an annulus (the region between two non-intersecting circles).

Exercise 2

Let $I : \mathbb{R} \to \mathbb{C}$, $I(a) = \int_{-\infty}^{\infty} e^{-(x+ia)^2} dx$. Show that $I(a) = I(0)$ for all $a \in \mathbb{R}$ by integrating $f(z) = e^{-z^2}$ along the boundary of a suitable rectangle.

Exercise 3

a) Calculate the following integrals:

$$\int_{|z+1|=1} \frac{1}{(z+1)(z-1)^2} dz \quad \text{and} \quad \int_{|z|=1} \frac{e^z}{z^n} dz \text{ for } n \in \mathbb{N}.$$

b) Let $\mathbb{D} = \{\, z \in \mathbb{C} \mid |z| < 1 \,\}$ and $G \supset \overline{\mathbb{D}}$ open. Further let $f : G \to \mathbb{C}$ be a holomorphic map. Show that

$$F : \mathbb{C} \setminus \partial\mathbb{D} \to \mathbb{C}, \quad F(z) = \int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

is holomorphic.

Exercise 4

Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Then f is constant in all of the following cases:

a) There exists $c \in \mathbb{R}$ with $|f(z)| \geq c > 0$ for all $z \in \mathbb{C}$.

b) $\overline{f(\mathbb{C})} \neq \mathbb{C}$.

Homework

Exercise 1 **8 points**

a) Calculate the integral $\int_{|z|=2} \frac{1}{z^2+1} dz$ using Cauchy's integral formula.

b) Let $z_1, z_2 \in \mathbb{C}$, $R > \max\{|z_1|, |z_2|\}$ and $f : \mathbb{C} \to \mathbb{C}$ a holomorphic function. Show:

$$\int_{|z|=R} \frac{f(z)}{(z-z_1)(z-z_2)} dz = 2\pi i \frac{f(z_2) - f(z_1)}{z_2 - z_1}.$$

c) Show that for $a, b > 0$:

$$\int_0^{2\pi} \frac{1}{a^2 \cos^2(t) + b^2 \sin^2(t)} dt = \frac{2\pi}{ab}$$

by integrating $\frac{1}{z}$ for a suitable ellipse.

Exercise 2 **6 points**

Let p be a polynomial of degree n and $R > 0$ such that $|z_0| < R$ for all $z_0 \in \mathbb{C}$ with $p(z_0) = 0$. Prove:

$$\int_{|z|=R} \frac{p'(z)}{p(z)} = 2\pi i n.$$

Exercise 3 **6 points**

Use the Cauchy Theorem to calculate the following integrals:

$$\int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2} \qquad \int_0^\infty \frac{\cos(x) - e^{-x}}{x} dx = 0.$$

Hint: Consider the function

$$f(z) = \frac{e^{iz}}{z}$$

and the figure on the right.