Technische Universität Berlin
Institut für Mathematik
Bobenko // Gunn/Knöppel
http://www3.math.tu-berlin.de/geometrie/Lehre/SS12/ComplexAnalysis SS 12

Complex Analysis - Exercise Sheet 5

- Identity theorem, Maximum principle, Schwarz' Lemma -

due 23-24.05.2012

Tutorial

Exercise 1 (Reflection principle)

Let U be a region which is symmetric with respect to the involution $z \mapsto \bar{z}$. Denote the open upper half plane by $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and set

$$U_0 := U \cap \mathbb{R}, \quad U_1 := U \cap H \quad \textit{and} \quad U_2 := U \cap \bar{H}.$$

Let $f\colon U_0 \cup U_1 \to \mathbb{C}$ be continuous, real–valued on U_0 and holomorphic on U_1. Then

$$\tilde{f} : U \to \mathbb{C}, \qquad z \mapsto \begin{cases} f(z) & for \quad z \in U_0 \cup U_1, \\ \overline{f(\bar{z})} & for \quad z \in U_2 \end{cases}$$

is holomorphic.

Exercise 2

Let $G \subset \mathbb{C}$ be a region and $f : G \to \mathbb{C}$ a holomorphic map. Is it possible that $|f|$ attains its maximum or minimum on G? What about extrema of $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$?

Exercise 3

Let $\mathbb{D}$ be the open unit disc and $f : \mathbb{D} \to \mathbb{D}$ holomorphic and $z_0 = 0$ a zero of order $n \geq 1$. Show that:

$$\left|f^{(n)}(0)\right| \leq n! \quad \text{and} \quad |f(z)| \leq |z|^n \text{ for all } z \in \mathbb{D}.$$

Show further that if $\left|f^{(n)}(0)\right| = n!$ or $|f(z_0)| = |z_0|^n$ for some $z_0 \neq 0$ then there exists $a \in \mathbb{C}$ with $|a| = 1$ such that $f(z) = az^n$.

Exercise 4

Show that there exists no entire function $f : \mathbb{C} \to \mathbb{C}$ with

$$f(\frac{1}{n}) = \frac{n}{2n-1} \quad \text{for all } n \in \mathbb{N}.$$

Homework

Exercise 1 **8 points**

Let G be a region and $f : G \to \mathbb{C}$ a holomorphic map. Let $M := \{ z \in G \mid f(z) \in \mathbb{R} \}$ and $z_0 \in M$. Prove the following statements:

1. If $f'(z_0) \neq 0$ then there exists a neighborhood V of z_0 in G such that $M \cap V$ is the image of a curve $\gamma :]0,1[\to G$.
2. If $f'(z_0) = 0$ and $\mathrm{Ord}(f - f(z_0), z_0) = n \geq 2$ then there exists a neighborhood V of z_0 in G such that $M \cap V$ is the union of n curves. Furthermore, the intersection angles of these curves at z_0 are multiples of $\frac{2\pi}{n}$.

Calculate and draw the set M for the following maps: exp, $z \mapsto z^3$ and sin.

Exercise 2 **6 points**

Let $f : \mathbb{D} \to U = f(\mathbb{D}) \subset \mathbb{C}$ a biholomorphic map. Show that

$$|f'(0)| \geq dist(f(0), \partial U),$$

where $dist(z_0, \partial U) = \inf\{ |z - z_0| \mid z \in \partial U \}$.
Hint: Use Schwarz' Lemma for a suitable map.

Exercise 3 **6 points**

If $f(z)$ is an entire function such that $f(\mathbb{R}) \subset \mathbb{R}$ and $f(i\mathbb{R}) \subset i\mathbb{R}$, show that f is odd: $f(-z) = -f(z)$.