Technische Universität Berlin
Institut für Mathematik
Bobenko // Gunn/Knöppel
http://www3.math.tu-berlin.de/geometrie/Lehre/SS12/ComplexAnalysis SS 12

Complex Analysis - Exercise Sheet 6

- Singularities, Laurent series -

due 30-31.05.2012

Tutorial

Exercise 1

Classify the singularities of the following functions:

$$a) \quad f(z) = \frac{\sin(z)}{z^n}, \qquad b) \quad g(z) = \frac{1-\cos(z)}{\sin(z)}, \qquad c) \quad h(z) = \cos(z + \frac{1}{z}).$$

Exercise 2

Determine the Laurent series of the function $f(z) = \frac{2}{z^2-4z+3}$ in the following three annuli:

$$a) \quad 0 < |z| < 1, \qquad b) \quad 1 < |z| < 3, \qquad c) \quad 3 < |z|.$$

Exercise 3

Let f be an entire function with Taylor series $\sum a_n z^n$. Show that $f(\frac{1}{z})$ has an essential singularity at 0 if and only if infinitely many a_n are non-zero.

Homework

Exercise 1 **8 points**

Determine and classify the singularities of the following functions:

$$a)\ f(z) = \frac{z^2-1}{z^3-1}, \quad b)\ g(z) = z\sin(\frac{1}{z}), \quad c)\ h(z) = \frac{z}{e^z-1}, \quad d)\ k(z) = \frac{1}{e^{\frac{1}{z}}}.$$

Exercise 2 **6 points**

Let $f, g : G \to \mathbb{C}$ be two meromorphic functions, $z_0 \in G$ and $n = \mathrm{Ord}(f, z_0)$ and $m = \mathrm{Ord}(g, z_0)$ the orders of f resp. g at z_0. Prove the following:

1. $\mathrm{Ord}(fg, z_0) = m + n$,
2. $\mathrm{Ord}(\frac{f}{g}, z_0) = n - m$, and
3. if $m \neq n$ then $|\mathrm{Ord}(f \pm g, z_0)| \leq \max\{|n|, |m|\}$.

Exercise 3 **6 points**

Let $f : \mathbb{C} \setminus \{0, -2i\}$ with $f(z) = \frac{2}{z^2+2iz}$.

1. Let $A_1 = \{\, z \in \mathbb{C} \mid 0 < |z| < 1 \,\}$ and $A_2 = \{\, z \in \mathbb{C} \mid 0 < |z + 2i| < 1 \,\}$ be two annuli. Determine the Laurent series of f on A_1 resp. A_2. What is the maximal domain of convergence of the series?
2. Use part a) to determine the values of the following integrals:

$$\int_{|z|=1} \frac{1}{z^2+2iz} dz \quad \text{and} \quad \int_{|z+2i|=1} \frac{1}{(z+2i)^2(z^2+2iz)} dz$$