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http://www3.math.tu-berlin.de/geometrie/Lehre/SS12/ComplexAnalysis SS 12

Complex Analysis - Exercise Sheet 7

- Singularities, Logarithm -

due 06-07.06.2012

Tutorial

Exercise 1

a) Let $U \subset \mathbb{C}$ open, $w_0 \in U$ and $f : U \setminus \{w_0\} \to \mathbb{C}$ be holomorphic with a pole of order m at w_0. Show that there exist neighborhoods H of w_0 and G of 0 and a biholomorphic map $h : G \to H$ such that:

$$(f \circ h)(z) = \frac{1}{z^m}.$$

b) Let $G, H \subset \mathbb{C}$ open, $w_0 \in H$ and $f : H \setminus \{w_0\} \to \mathbb{C}$ holomorphic with an isolated singularity at w_0. Further let $h : G \to H$ be a holomorphic function with $h(z_0) = w_0$ and $h'(z_0) \neq 0$.

Show that f has the same kind of singularity at w_0 as $f \circ h$ at z_0.

c) Let $f : \{\, z \in \mathbb{C} \mid 0 < |z| < \varepsilon \,\} \to \mathbb{C}$ holomorphic for some $\varepsilon > 0$ and let 0 be a pole of f. Let g be an entire function that is not a polynomial.

- What kind of singularity does $g \circ f$ have at 0?
- What happens if g is a polynomial?

Exercise 2

Let $G \subset \mathbb{C}$ be a region and $f : G \to \mathbb{C}$ holomorphic. Every function $g : G \to \mathbb{C}$ with $e^{g(z)} = f(z)$ is called *holomorphic logarithm* of f in G. Prove the following:

a) If f has a holomorphic logarithm, then f does not have any zeroes.

b) Let f be a function without zeroes. Then f has a holomorphic logarithm if and only if $\frac{f'}{f}$ has an anti–derivative in G.

c) Let f be a function without zeroes and G be simply-connected. Then f has a holomorphic logarithm.

Homework

Exercise 1 **8 points**

Let $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}$.

a) Let $f : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ a holomorphic map and $|f(z)| \leq M|z|^t$ for some $M > 0$, $t > -1$, and all $z \in \mathbb{D} \setminus \{0\}$. Show that f has a removable singularity at $z = 0$.

b) Let $g : \mathbb{D} \to \mathbb{C}$ be holomorphic with $g(0) = 0$ and $g'(0) \neq 0$. Show using part a) that there exists no holomorphic function $h : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ with $h^2 = g$.

Exercise 2 **8 points**

Let $R > 0$, $G_R = \{z \in \mathbb{C} \mid |z| > R\}$ and $f \colon G_R \to \mathbb{C}$ holomorphic. Then $z_0 = \infty$ is called a *removable singularity* (*pole*, *essential singularity*) of f, if the function $g(z) = f\left(\frac{1}{z}\right)$ has a removable singularity (pole, essential singularity) in $z = 0$.

a) Let f be an entire function. Show that f has a removable singularity at infinity if and only if f is constant.

b) Let f be an entire function. Show that f has a pole of order m at infinity if and only if f is a polynomial of degree m.

c) Let $R > 0$ and $G_R = \{ z \in \mathbb{C} \mid R < |z| \}$ and $f, g : G_R \to \mathbb{C}$ be two holomorphic functions with removable singularities at infinity. Show that $f = g$ if $f((n+2)R) = g((n+2)R)$ for all $n \in \mathbb{N}$.

Exercise 3 **2 points**

Find a rational function f with poles at the points z_1, z_2 ($z_1 \neq z_2$), that has a holomorphic anti–derivative on $\mathbb{C} \setminus \{z_1, z_2\}$.

Exercise 4 **2 points**

Find an example of a holomorphic function $f \colon G \to \mathbb{C}$ without zeros, that has no holomorphic logarithm.