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http://www3.math.tu-berlin.de/geometrie/Lehre/SS12/ComplexAnalysis SS 12

Complex Analysis - Exercise Sheet 8

- Analytic continuation, fundamental group, homology -

due 13-14.06.2012

Tutorial

Exercise 1

Let $[-i,i] = \{ z \in \mathbb{C} \mid z = ix, x \in [-1,1] \subset \mathbb{R} \}$ and $G = \mathbb{C} \setminus [-i,i]$. Let $z_0 = 1$ and γ_z an arbitrary curve in G from z_0 to z. Prove that:

$$F : G \to \mathbb{C}, \quad F(z) = \int_{\gamma_z} \frac{1}{1+\zeta^2} d\zeta$$

is independent of the choice of γ_z and is holomorphic.

Exercise 2

Let $c : [0,1] \to \mathbb{C}$ be a closed curve with only finitely many self-intersections. Let $p = c(t_0)$ be a regular point of the curve and $\gamma : [-\varepsilon, \varepsilon] \to \mathbb{C}$ a differentiable curve intersecting c in p transversally, i.e. $\det(\dot{c}(0), \dot{\gamma}(t_0)) \neq 0$. If $a = \gamma(-\varepsilon)$ and $b = \gamma(\varepsilon)$, how do $n(c,a)$ and $n(c,b)$ differ? How can you use this to calculate the index of an arbitrary point?

Exercise 3

“Dog on the leash”-Theorem.

Informal: How do you prevent your dog from winding around a tree? Well – keep him on a leash shorter than the distance to the closest tree.

Mathematical: Let $c_1, c_2 : [0,1] \to \mathbb{C}$ be two closed curves and $z_0 \in \mathbb{C} \setminus (|c_1| \cup |c_2|)$. Further assume that for all $t \in [0,1]$:

$$|c_1(t) - c_2(t)| < |c_1(t) - z_0|.$$

Show that $n(c_1, z_0) = n(c_2, z_0)$. Calculate $n(c, 0)$ for the following curves $c : [0, 2\pi] \to \mathbb{C}$, $\varphi \mapsto e^{n\varphi i} + re^{k\varphi i}$, where $n, k \in \mathbb{Z}$, $r \in (0, \infty)$, $r \neq 1$.

Homework

Exercise 1 **12 points**

Let G and $F : G \to \mathbb{C}$, $F(z) = \int_{\gamma_z} \frac{1}{1+\zeta^2} d\zeta$ as in Tutorial Exercise 1. Let $g : G \to \mathbb{C}$, $g(z) := \frac{\pi}{4} + F(z)$.

a) Let $\gamma_1, \gamma_2 : [0, 2\pi] \to \mathbb{C}$, $\gamma_1(t) = 2e^{it}$, and $\gamma_2(t) = re^{it} + i$ with $r \in (0, 2)$. Show that:

$$\int_{\gamma_1} \frac{1}{1+\zeta^2} d\zeta = 0 \quad \text{and} \quad \int_{\gamma_2} \frac{1}{1+\zeta^2} d\zeta = \pi.$$

b) Is it possible to extend g onto a region H with $G \subsetneqq H$?

Consider $g(z+\varepsilon)$ and $g(z-\varepsilon)$ for $z \in [-i, i]$ and $\varepsilon > 0$ and use part a).

c) Consider $x > 0$ in $\mathbb{R}$ and show that $(\tan)'(x) = 1 + \tan^2(x)$ and $g(x) = \arctan(x)$ resp. $\tan(g(x)) = x$.

Hint: Calculate $F(x)$ by explicit integration.

d) Show that $\tan(g(z)) = z$ for all $z \in G$.

Hint: Use the identity theorem and part c).

Exercise 2 **3 points**

Let c be a simple closed piecewise continuously differentiable curve. Show that $\mathbb{C} \setminus |c|$ has at least two components.

Exercise 3 **5 points**

Let $U \subset \mathbb{C}$ open and bounded such that $\mathbb{C} \setminus U$ is connected. Show that all closed continuous curves in U are homologous to a constant curve, i.e. null-homologous.