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http://www3.math.tu-berlin.de/geometrie/Lehre/SS12/ComplexAnalysis SS 12

Complex Analysis - Exercise Sheet 9

- Residue Theorem, Rouché's Theorem -

due 20-21.06.2012

Tutorial

Exercise 1

Let $G \subset \mathbb{C}$ be a region, $a \in G$ and $f : G \setminus \{a\} \to \mathbb{C}$ a holomorphic function.

a) If f has a pole of order one in a then

$$res(f, a) = \lim_{z \to a}(z - a)f(z).$$

b) If f has a pole of order k in a then

$$res(f, a) = \lim_{z \to a} \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial z^{k-1}}((z - a)^k f(z)).$$

c) Calculate $res(\frac{\cot z}{z^2(z+1)}, \pi)$.

Exercise 2

Let $f, g : G \to \mathbb{C}$ be two holomorphic functions. Can you express the residue of $h = \frac{f}{g}$ at z_0 in terms of derivatives of f and g? What happens if g has a root of order 2 at z_0?

Exercise 3

Calculate the integral $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}$ using the residue theorem.

Exercise 4

Determine the number of roots (including multiplicity) of the polynomial $p(z) = z^8 - 5z^3 + z - 2$ contained in the unit disc.

Homework

Exercise 1 **6 points**

Calculate the following integrals:

$$(i)\ \int_0^{\infty} \frac{x^2}{x^4+6x^2+5}dx, \qquad (ii)\ \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2}dx \text{ with } a \in \mathbb{R},\ a>0.$$

Exercise 2 **4 points**

Calculate $res(e^{z+1/z}, 0)$.

Exercise 3 **10 points**

Let $\mathbb{D} = \{\, z \in \mathbb{C} \mid |z| < 1 \,\}$.

a) Determine the number of roots of the following polymonial in the unit disk (including their multiplicity):

$$p(z) = z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1.$$

b) Let $U \subset \mathbb{C}$ be an open set containing $\bar{\mathbb{D}}$. Let $f : U \to \mathbb{C}$ be a holomorphic function with $f(\bar{\mathbb{D}}) \subset \mathbb{D}$. Show that f has exactly one fixpoint in $\mathbb{D}$.

c) Let $\lambda \in \mathbb{R}$ with $\lambda > 1$. Show that the equation $e^{z-\lambda} = z$ has exactly one solution in $\mathbb{D}$. Show further that this solution is real.